Lecture Plan

1. The EOQ and Extensions
2. Multi-Item EOQ Model

1 The EOQ and Extensions

This section is devoted to the Economic Production Quantity (EPQ) model, its specialization to the Economic Order Quantity (EOQ) model and its extension to allow backorders. The next section deals with multiple products. The primary driver to hold inventories is the presence of a fixed production/ordering cost that is incurred every time a positive number of units are produced/ordered. We wish to determine the number of units to produce/order every time the fixed production/ordering cost is incurred. Producing/ordering in large quantities reduce the average fixed production/ordering cost. What prevents the production/ordering quantities from becoming too large is that it costs money to hold inventories. Inventory costs include the cost of storage and insurance and the cost of capital tied up in inventory. The EPQ balances the average fixed ordering cost (that decreases with large orders) and the average inventory holding cost (that increases with large orders).

We will assume that demand is constant and continuous over an infinite horizon. We will also assume a constant, continuous, production rate. The objective is to minimize the average cost.

Data:

K set up or ordering cost,
c unit production cost,
h holding cost,
λ demand rate,
μ production rate.

The holding cost is often model as $h = Ic$ where I represents the inventory carrying cost per unit per unit time and includes the cost of capital per unit per unit time. Let $\rho = \lambda/\mu$. We assume $\rho \leq 1$, since otherwise the production capacity is not enough to keep up with demand.

Since we have assumed that production and demand rates are constant and costs are stationary, it makes intuitive sense to produce/order in cycles of equal lengths and to start/end each cycle with zero inventory. This last property is know as the Zero Inventory Property (ZIP) and can be formally verified. Let T denote the cycle length. The cycle length is the time between consecutive orders; it is also called the order/reorder interval. We want to express the average cost $a(T)$ as a function of T and then select T to minimize $a(T)$. We could have also chosen the production/order quantity $Q = \lambda T$ as the decision variable. The advantage of using a time variable is apparent when dealing with multiple items, where time provides a unified variable.

Let $I(t)$ denote the inventory at time t. We will assume that $I(0) = 0$, and by design we want $I(T) = 0$. At time zero we start production at rate μ. We need to determine when to stop production so that $I(T) = 0$. Since λT units are demanded over the interval $[0, T]$, it takes $\frac{\lambda T}{\mu} = \rho T$ units of time to produce λT units at rate μ. Because demand is constant at rate λ, inventory accumulates at rate $\mu - \lambda$ over the interval $[0, \rho T]$ reaching a peak equal to $(\mu - \lambda)\rho T = \lambda(1 - \rho)T$ at time ρT. Since production stops at time ρT, inventory decreases at rate λ over the interval $[\rho T, T]$ reaching zero at time T.

Let us now compute the average inventory over the interval $[0, T]$. Formally, this is equal to

$$\frac{1}{T}\int_0^T I(t)dt.$$

The function $I(t)$ forms a triangle that repeats ever T units of time. The base of each triangle is T and the height is $\lambda(1 - \rho)T$. It follows that

$$\frac{1}{T}\int_0^T I(t)dt = \frac{1}{T}\frac{1}{2}\lambda(1 - \rho)T^2 = \frac{1}{2}\lambda(1 - \rho)T.$$

Consequently, the average holding cost per unit time is equal to $\frac{1}{2}h\lambda(1-\rho)T$.

Figure 1: **Inventory as a function of time.**

Since we incur the fixed ordering cost K once every T units of time, the average ordering cost is simply $\frac{K}{T}$. Consequently, the average ordering and holding cost per unit time is:

$$a(T) = \frac{1}{2}h\lambda(1-\rho)T + \frac{K}{T}. \tag{1}$$

Notice that the unit production cost, say c, is not included in (1) because it does not depend on T. Thus $a(T)$ is the relevant function to study although $c\lambda + a(T)$ is the actual average cost per unit of time.

By taking the first derivative of (1), setting it to zero, and solving for T, we find

$$T^* = \sqrt{\frac{2K}{h\lambda}}\sqrt{\frac{1}{1-\rho}}. \tag{2}$$

It is easy to see, by looking at the sign of the second derivative of $a(T)$, that (1) is convex and consequently (2) is the optimal order interval. As a consequence, the economic production quantity (EPQ) is equal to

$$Q^* = \lambda T^* = \sqrt{\frac{2K\lambda}{h}}\sqrt{\frac{1}{1-\rho}}. \tag{3}$$

Substituting (2) into (1), we find that optimal average cost is given by

$$a^* \equiv a(T^*) = \sqrt{2Kh\lambda}\sqrt{1-\rho} \tag{4}$$

Example: Suppose $K = 100$, $h = 5$, $\lambda = 200$, $\mu = 1000$. Then $\rho = 0.2$, the optimal order interval is $T^* = 0.50$, the optimal order quantity is $Q^* = 100$, and the optimal average cost is $a^* = \$400$.

We now explore how T^* and a^* change with ρ. Notice that T^* increases and a^* decreases as ρ increases. Intuitively, inventories accumulate at a slower rate when ρ increases. As a result, we can use longer reorder intervals and reduce costs. It is interesting to explore what happens in the limit as $\rho \to 1$. In this case, the production rate and demand rate are in perfect balance. Thus, after an initial set up, we simply produce to meet demand and keep doing that forever. Thus, $T^* \to \infty$, and $a^* \to 0$. Notice that for fixed $\mu < \infty$ the average cost is zero when $\lambda = 0$ or $\lambda = \mu$. The average cost is the highest when $\lambda = \mu/2$, i.e., when only half of the production capacity is utilized. Thus, when selecting the production rate μ to satisfy demand λ, it is better to avoid $\mu = 2\lambda$ as lower average costs will result with either a lower or a higher production rate μ.

Example: Suppose $K = 100$, $h = 5$, $\lambda = 500$, $\mu = 1000$. Then $\rho = 0.5$, the optimal order interval is $T^* = 0.40$, the optimal order quantity is $Q^* = 200$, and the optimal average cost is $a^* = \$500$.

The average cost a^* is increasing concave in the setup cost K. This means that the savings that accrue from reducing the fixed cost K become larger the more we reduce K. Table 4 makes this clear. This is an example of changing the givens. Japanese manufacturers, such as Toyota, were quick to recognize the value of reducing setup costs.

K	a^*	savings
100	500.00	0
90	474.34	25.66
80	447.21	27.13
70	418.33	28.88
60	387.30	31.03
50	353.55	33.74
40	316.23	37.33
30	273.86	42.37
20	223.61	50.25
10	158.11	65.49
0	0	158.11

Table 1: *Savings from reducing fixed costs.*

1.1 The Economic Order Quantity (EOQ) Model

The infinite production rate case is often used to model the situation where the product is ordered rather than produced and corresponds to $\rho = 0$. The infinite production rate case is also known as the Economic Order Quantity (EOQ) model. This optimal order interval for this model is given by

$$T^* = \sqrt{\frac{2K}{h\lambda}}. \tag{5}$$

The optimal order quantity for this model is known as the EOQ and is given by

$$Q^* = \lambda T^* = \sqrt{\frac{2K\lambda}{h}}. \tag{6}$$

Finally, the optimal average cost is given by

$$a^* \equiv a(T^*) = \sqrt{2Kh\lambda} \tag{7}$$

1.2 Sensitivity Analysis.

Using the results from previous sections, we can write the average cost as

$$a(T) = HT + \frac{K}{T}.$$

Here K is the fixed ordering cost, and $H = \frac{1}{2}h\lambda(1-\rho)$ for the finite production rate case and $H = \frac{1}{2}h$ for the infinite production rate case.

Notice that $HT^* = \frac{K}{T^*} = \frac{1}{2}a(T^*)$, so we can write

$$HT = HT^* \frac{T}{T^*} = \frac{1}{2}a(T^*)\frac{T}{T^*}$$

and

$$\frac{K}{T} = \frac{K}{T^*}\frac{T^*}{T} = \frac{1}{2}a(T^*)\frac{T^*}{T}.$$

Consequently,

$$\begin{aligned} a(T) &= HT + \frac{K}{T} \\ &= HT^*\left(\frac{T}{T^*}\right) + \frac{K}{T^*}\left(\frac{T^*}{T}\right) \\ &= a(T^*)\frac{1}{2}\left(\frac{T}{T^*} + \frac{T^*}{T}\right) \\ &= a(T^*)e\left(\frac{T}{T^*}\right) \end{aligned}$$

where $e(x) \equiv \frac{1}{2}(x + 1/x)$.

This allow us to write

$$\frac{a(T)}{a(T^*)} = e\left(\frac{T}{T^*}\right). \tag{8}$$

Notice that $e(x) \geq 1$ and that it achieves its minimum at $x = 1$

There are two immediate applications of this sensitivity result.

1.2.1 Powers-of-Two Policies

Let T^{POT} denote the minimizer $a(T)$ subject to $T \in \{2^k\beta : k \in \mathcal{Z}\}$, where β is a base planning period and $\mathcal{Z}$ denotes the set of integers. It can be shown that

$$\frac{a(T^{POT})}{a(T^*)} \leq e(\sqrt{2}) \simeq 1.06.$$

This tells us that if we restrict the order interval to be a power-of-two multiple of a base planning period, the resulting level of sub-optimality is at most 6%.

1.2.2 Parameter Estimation

Consider an inventory system where the holding cost parameter H has been estimated correctly but where the ordering cost parameter K is incorrectly estimated as K'. Let T' be the "optimal" order size resulting from using K'. What is the actual average cost as a function of K'/K? From equation (8) we have

$$\frac{a(T')}{a(T^*)} = e\left(\frac{T'}{T^*}\right) = e\left(\sqrt{K'/K}\right).$$

From this we can see that the cost of overestimating K is lower than that of underestimating K. To see this more clearly, suppose we overestimate K by 40%, then $K' = 1.40K$ resulting $e(1.40) = 1.057$.Conversely, if we underestimate K by 40%, then $K' = 0.60K$ resulting in $e(0.60) = 1.133$.

1.3 The Backorder Case.

Backorders can be completely avoided in a deterministic setting. However, if customers are willing to accept backorders, and backorders are charged at a linear rate, say b, per unit per unit time, then it is optimal to incur some backorders during each cycle. Thus, in addition to a tradeoff with setup costs, there is a tradeoff between holding and backorder costs. You can think of backorder cost rate b as the cost rate, per unit of time, of keeping a customer waiting for one unit of inventory. On the other hand, you can think of the holding cost rate h as the cost rate, per unit of time, of keeping a unit of inventory waiting for a customer. It turns out that the optimal balance between holding and backorder costs results in us holding inventory $100b/(b+h)\%$ of the time. Thus, if $b = 9h$, then $100b/(b+h)\% = 90\%$ so it is optimal to keep inventories 90% of the time and incur backorders 10% of the time. Thus, customers will find inventory on hand 90% of the time. This interpretation will be useful later when we discuss models with random demands.

Because we hold less inventory when backorders are allowed, the costs will be lower. We will later show that allowing backorders is mathematically equivalent to reducing the holding cost rate by the factor $b/(b+h)$ so that the modified holding cost rate is equal to $hb/(h+b)$. Using this modified holding cost rate results in the optimal order interval

$$T^* = \sqrt{\frac{2K}{h\lambda}}\sqrt{\frac{b+h}{b}}\sqrt{\frac{1}{1-\rho}}, \tag{9}$$

and optimal average cost

$$a^* = \sqrt{2Kh\lambda}\sqrt{\frac{b}{b+h}}\sqrt{1-\rho}. \tag{10}$$

It is instructive to view the optimal order interval (9) and the optimal average cost (10) as functions of ρ and b. In particular it is interesting to compare (9) and (10) to the case where production rates are infinite, $\rho = 0$, and backorders are not allowed ($b = \infty$). Note that

$$T^*(\rho, b) = T^*(0, \infty)\sqrt{\frac{b+h}{b}}\sqrt{\frac{1}{1-\rho}} \tag{11}$$

$$a^*(\rho, b) = a^*(0, \infty)\sqrt{\frac{b}{b+h}}\sqrt{1-\rho} \tag{12}$$

Here $T^*(0, \infty) = \sqrt{\frac{2K}{h\lambda}}$ and $a^*(0, \infty) = \sqrt{2Kh\lambda}$ are respectively the optimal cycle length and the optimal average cost for the EOQ, i.e., the case of infinite production run and no backorders allowed.

Notice that with finite production rates and finite backorder costs, the cycle length increases while the average cost decreases, always satisfying

$$a^*(\rho, b)T^*(\rho, b) = 2K. \tag{13}$$

That is, the optimal cost per cycle is *independent* of the production rate, of the backorder cost, and of the holding cost; and is always equal to twice the ordering or setup cost.

1.3.1 Derivation of the Backorder Cost Formula

Remember that without backorders the function $I(t)$ was always non-negative and drops to zero at the time of replenishment. When backorders are allowed, we expect the inventory to be negative during some portion of the cycle. We can find the optimal proportion of time that the inventory should be negative by shifting the inventory curve down by some $s \geq 0$ to $I(t) - s$ and by optimizing over s for a fixed order interval T. With this convention we can write the average cost per unit time as

$$a(T, s) = \frac{1}{T}\left\{K + h\int_0^T (I(t) - s)^+ dt + b\int_0^T (I(t) - s)^- dt\right\}, \tag{14}$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.

The integrals in (14) represent the holding and backorder costs. Clearly $a(T, s)$ is convex in s for fixed T; equating the partial derivative, of $a(T, s)$ with respect to s, to zero, we obtain

$$\frac{1}{T}\int_0^T 1\{t : I(t) > s\}dt = \frac{b}{b+h}. \tag{15}$$

From (15), we see that s should be selected so *the proportion of time the inventory is positive* is $b/(b + h)$. This is a result that holds for more general functions[1] $I(t)$, not just the triangle shaped one we have been discussing.

By similar triangles

$$s^*(T) = \frac{h}{b+h}\lambda(1-\rho)T \tag{16}$$

solves (15). Substituting $s^*(T)$ into $a(T, s)$ we obtain

$$a(T) = \frac{1}{2}\frac{hb}{h+b}\lambda(1-\rho)T + \frac{K}{T}. \tag{17}$$

Clearly, $a(T)$ is convex in T so the optimal cycle length is given by equation (9) and the optimal cost by equation (10).

1.4 Some useful relationships

It can be shown that the optimal average cost is equal to the holding cost rate times the maximum inventory, say I_{max} reached through the cycle, and also equal to the backorder cost rate times the absolute value of the minimum inventory, say I_{min} reached through the cycle.

These relationships are useful since it is often easier to collect information on $Imax, Imin$ and h, that it is on K and b. Suppose that inventories are managed optimally. Given $Imax, Imin$ and h, the imputed values for K and b are given by

$$b = \left|\frac{I_{max}}{I_{min}}\right| h \tag{18}$$

$$K = \frac{b+h}{2(1-\rho)}|I_{max}I_{min}|. \tag{19}$$

2 Multi-item EOQ Models

Our concern here is with inventory problems involving a variety of items. We start with a discussion of exchange curves for a family of items with similar ordering and inventory carrying costs, next we study a heuristic that calls for using the same reorder interval for each item. Finally, we discuss the joint replenishment problem where a major setup cost is incurred every time an order is placed regardless of the number of different items in the order.

2.1 Exchange Curves for a Family of Products

Consider n items, where item i has unit cost c_i and fixed cost K_i, and demand per unit time equal to $\lambda_i \quad i = 1, \ldots, n$. Assume further that the K_i cannot be determined explicitly, but that it is reasonable to assume that K_i is proportional to the number of standard hours devoted to the setup of the order. That is, $K_i = s_i a$ where s_i is the number of standard hours and a is the rate in dollars per setup hour. We also assume that the holding cost rate h_i is of the form $h_i = Ic_i$ where I is the

[1]It is enough for the function to be integrable and nowhere flat for this result to hold.

per dollar inventory carrying charge. In what follows, we assume for simplicity the case of infinite production rates with no backorders allowed, but the arguments to be made can be easily extended to encompass those cases.

The economic order interval for item i is thus

$$T_i^* = \sqrt{\frac{2s_i a}{I c_i \lambda_i}} \quad i = 1, \ldots, n.$$

Then, the average cycle stock (ACS), i.e., the average number of dollars invested in inventory is equal to

$$ACS = \frac{1}{2}\sum_{i=1}^{n} c_i \lambda_i T_i^* = \sqrt{\frac{a}{I}}\frac{1}{\sqrt{2}}\sum_{i=1}^{n}\sqrt{c_i \lambda_i s_i}, \tag{20}$$

while the average number of setup hours per year is

$$N = \sum_{i=1}^{n} s_i \frac{1}{T_i} = \sqrt{\frac{I}{a}}\frac{1}{\sqrt{2}}\sum_{i=1}^{n}\sqrt{c_i \lambda_i s_i}. \tag{21}$$

Notice that the product of the number of setup hours per year times the average cycle stock is a constant, so

$$ACS = \frac{(\sum_{i=1}^{n}\sqrt{c_i \lambda_i s_i})^2}{2N} \tag{22}$$

is a hyperbola and is independent of a and I. This tells us that if the family of items is managed according to the EOQ logic then the number of orders per year and the average cycle stock should lie on the hyperbola given by (22). This surprising result allow us to determine whether or not the items are managed effectively *without* knowing the values of a and I. All we need to determine if the items are managed effectively is the unit cost, the demand rate and the setup times. As we shall see, the hyperbola represents different efficient points depending on the ratio of a to I.

Example: Consider the following inventory system consisting of 4 items:

λ	c	s
7200	\$2.00	1
4000	0.90	2
500	5.00	1
100	0.81	3

Table 2: *Data for Exchange Curve Example*

Then the curve of efficient points is given by $ACS = 36569/N$. A few points of this curve are given by the following table:

A manager can determine the average cycle stock and the number of hours spent in setups under the current policy and then see whether or not the point lies on the hyperbola. Suppose, for example, that the items are ordered once a week or 52 times a year. Then $N = 208$ and $ACS = \$396$ which is suboptimal. Suppose that the manager does not know the ratio a/I but feels that it would be a great improvement to move to the point $N = 200$ and $ACS = \$182.85$ in the curve. How can he efficiently obtain the order intervals to achieve this goal? To answer this question first notice that by dividing ACS (20) by N (21) we find

$$\frac{ACS}{N} = \frac{a}{I}. \tag{23}$$

This equation makes intuitive sense. The hyperbola gives us the set of efficient points for different values of a/I. Equation (23) tell us where in the hyperbola we should be for each value of a/I. If a/I

N	ACS
50	731.38
100	365.69
150	243.79
200	182.85
250	146.28
300	121.90
350	104.48
400	91.42
450	81.26
500	73.14

Table 3: *A Few Data Points of the Exchange Curve*

is high, then the line will intersect the curve at a point where ACS is high and N is low. Conversely, if a/I is low, the line will intersect the curve at a point where ACS is low and N is high.

Returning to our example, if the manager wants to operate at the point $N = 200$ and $ACS =$ \$182.85 all he needs to do is compute the ratio $ACS/N = 182.85/200 = 0.91423$ and use this value for a/I in the formulas for the reorder intervals, e.g., $T_i^* = \sqrt{2s_i/c_i\lambda_i}\sqrt{.91423}$, $i = 1, \ldots, 4$. The resulting reorder intervals will be such that $N = 200$ and $ACS =$ \$182.5. See the Exchange-curve.xls file.

2.2 A Simple Inventory Coordination Heuristic

Our previous analysis suggests that we should operate on the exchange curve by ordering items according to their economic order interval. For management, however, it may be easier and more convenient to order all the items in the family at the same time. We now investigate, in the worst case sense, the relative loss of optimality that results from using a common reorder interval for a family of items rather than using the economic reorder interval for each item.

Let

$$a_i(T) = \frac{K_i}{T} + \frac{1}{2}h_i\lambda_i T$$

be the average cost of using reorder interval T for item i. Then, the average cost of using a common reorder interval T for all the items in the group is given by

$$a(T) = \sum_{i=1}^{n} a_i(T).$$

The common reorder interval that minimizes $a(T)$ is

$$T^* = \sqrt{\frac{2\sum_{i=1}^n K_i}{\sum_{i=1}^n h_i\lambda_i}},$$

and the overall average cost is

$$a(T^*) = \sum_{i=1}^{n} a_i(T^*) = \sqrt{2\sum_{i=1}^{n} K_i \sum_{i=1}^{n} h_i\lambda_i}.$$

In contrast, if we *individually* minimize each item's average cost

$$a_i(T_i) = \frac{K_i}{T_i} + \frac{1}{2}h_i\lambda_i T_i$$

by using the reorder intervals

$$T_i = \sqrt{\frac{2K_i}{h_i\lambda_i}}, \quad i = 1, \dots, n,$$

the overall average cost is given by

$$\sum_{i=1}^{n} a_i(T_i^*) = \sum_{i=1}^{n} \sqrt{2K_i h_i \lambda_i}.$$

Now, let us consider the ratio

$$\frac{\sum_{i=1}^n a_i(T^*)}{\sum_{i=1}^n a_i(T_i^*)}$$

of the average cost of using a the best common reorder interval for all the items to the average cost of using the optimal reorder interval for each item. Obviously the ratio is greater or equal to one, and a number much larger than one would indicate that using a common order interval is a bad idea. On the other hand, if the ratio is very close to one, say 1.03, we can have the benefits of order coordination at a relatively small cost.

In order to obtain an *upper bound* on the cost ratio we order the items according to the ratios $\frac{K_i}{h_i\lambda_i}$, so that items with small ratios come first. Let

$$r = \frac{K_n h_1 \lambda_1}{K_1 h_n \lambda_n}$$

be the ratio of the largest to the smallest ratio. It is possible to show that

$$\frac{\sum_{i=1}^n a(T^*)}{\sum_{i=1}^n a_i(T_i^*)} \leq \frac{1}{2}(\sqrt[4]{r} + 1/\sqrt[4]{r}).$$

Thus, if $r \leq 4$, then the cost ratio is at most 1.06. This analysis indicates that using a common order interval is a good idea as long as the items are not too different as measured by r.

For the following data set, we found that $r = 3.19$, so the ratio of $a(T^*)$ to $\sum_{i=1}^4 a_i(T_i^*)$ is at most $\frac{1}{2}(\sqrt[4]{3.19}+1/\sqrt[4]{3.19}) = 1.042$. Actual calculations show that $\frac{a(T^*)}{\sum_{i=1}^4 a_i(T_i^*)} = 1.018$ so the relative cost of using a common reorder interval to jointly managing the items is less than 2%.

λ	h	K
1550	0.4	7
1725	0.35	9
1875	0.5	14
1110	0.25	10

Table 4: *Data for Coordinating Heuristic.*