1 Production Planning with Time-Varying Demand

In this lecture we present a few key results in production planning with known time-varying requirements under convex or concave production costs. Convex production costs arise when we need to utilize increasingly more expensive resources to achieve higher levels of production. As an example, consider a factory with a regular workforce. If demand is beyond the capacity of the regular workforce, management may employ overtime at a higher cost, and, if needed, it may subcontract production at an even higher cost. Convex production costs therefore reflect *diseconomies* of scale. The main insight, when production costs are convex, is that production smoothing is optimal. This means that the production rate (production per period) should be kept as even as possible. Thus, production plans under convex production costs mitigate or smooth out variability in demand. On the other hand, concave production costs reflect *economies* of scale in production. For example, if the production costs are fixed (setup cost) plus linear (variable cost), as in the EOQ, then the unit cost decreases with production volume and this acts as an incentive to order/produce in large batches. As we shall see, under concave production costs, optimal plans tends to *amplify* the demand variability as production is concentrated in a few periods over the planning horizon. Under concave production costs, optimal production plans tend to *propagate* demand variability through the supply chain. Thus, a retailer may experience fairly leveled customer demand, but because of economies of scale he or she may order in batches from the warehouse, which in turn may aggregate orders from different retailers and order in even larger batches from the factory. This phenomena is known as the bullwhip effect. Later in the course we will talk about supply chain management and how increased efficiencies may be obtained by coordinating orders and communicating demand information across the supply chain to mitigate the bullwhip effect.

1.1 Convex Production Costs

Convex production costs arise when we need to utilize increasingly more expensive resources to achieve higher levels of production. The main result of this section is based on the following observation: If $g(\cdot)$ is an increasing convex function, then an optimal solution to the problem

$$\min \sum_{i=1}^{n} g(x_i)$$

subject to

$$\sum_{i=1}^{n} x_i = r,$$

is given by

$$x_i = r/n, \ i = 1, \ldots, n.$$

Thus, if r denotes the total requirements over n periods then a minimum cost production plan is to produce a constant amount each period. Notice that this result depends on the convexity of $g(\cdot)$ and not on its precise form! A proof of this result is based on Lagrangian Relaxation and setting it to zero. This reveals that $g'(x_1) = g'(x_2) = \ldots = g'(x_n)$, a condition that is satisfied by $x_1 = x_2 = \ldots x_n = r/n$. Table 1.1 illustrates this principle for the case $n = 2$, $r = 10$, and two choices of convex functions: $g(x) = x^2$ and $f(x) = x + x\delta(x - 5)$, where as usual $\delta(x) = 0$ on $x \leq 0$ and 1 on $x > 0$.

Unfortunately this smoothing result cannot be used as stated because it ignores inventories and whether or not they are sufficiently large to cover demand as it occurs. Nevertheless, the principle

x_1	x_2	$g(x_1)$	$g(x_2)$	$g(x_1)+g(x_2)$	$f(x_1)$	$f(x_2)$	$f(x_1)+f(x_2)$
0	10	0	100	100	0	15	15
1	9	1	81	82	1	13	14
2	8	4	64	68	2	11	13
3	7	9	49	58	3	9	12
4	6	16	36	52	4	7	11
5	5	25	25	50	5	5	10
6	4	36	16	52	7	4	11
7	3	49	9	58	9	3	12
8	2	64	4	68	11	2	13
9	1	81	1	82	13	1	14
10	0	100	0	100	15	0	15

Table 1: Production Smoothing with Convex Costs

is to smooth out production and to deviate from the leveled production plan only to the extend necessary to satisfy the known requirements and storage capacity constraints.

We now generalize this idea to more general cost functions where lower and upper bounds are imposed on inventories. To this end, consider an n period production planning problem where the cost of producing x_i units in period i is

$$d_i g(x_i/d_i)$$

where $g(x)$ is an increasing convex function. The cost function $d_i g(x_i/d_i)$ arises in several ways, for example if period i consists of d_i days and x_i/d_i units are produced per day at a daily cost of $g(x_i/d_i)$, the cost in period i is $d_i g(x_i/d_i)$. A similar cost function arises if d_i is the number of workers or the number of machines used in period i, each producing x_i/d_i units at cost $g(x_i/d_i)$. In either case, the total cost over n periods is given by

$$f(x) = \sum_{i=1}^{n} d_i g(x_i/d_i).$$

Before moving on, to deal with the problem of how to deal with inventories, let us consider the unconstrained problem of minimizing $f(x)$ subject to $\sum_{i=1}^n x_i = r$. From the smoothing principle, x_i/d_i should be a constant, say θ, across all periods. Then $r = \sum_{i=1}^n x_i = \sum_{i=1}^n d_i\theta$. Let $D_i = \sum_{j=1}^i d_j$ and $X_i = \sum_{j=1}^i x_j$ for $j = 1, \ldots, n$. Then,

$$x_i = \frac{d_i r}{D_n} \quad i = 1, \ldots, n,$$

and

$$X_i = \frac{D_i r}{D_n} \quad i = 1, \ldots, n.$$

If we plot the points (D_i, X_i), $i = 1, \ldots, n$, they lie on a straight line from the origin $(0, 0)$ to (D_n, r). We call this the perfect smoothing solution.

Let $R_i = \sum_{j=1}^i r_j$ be the cumulative requirements up to period i. Then, if I_o is the initial inventory, then the inventory at the end of period i is given by

$$I_i = I_o + X_i - R_i.$$

A sensible objective is to keep the ending inventories within desirable sets. Thus, we may specify a_i, b_i, $i = 1, \ldots, n$ and impose the constraint $a_i \leq I_i \leq b_i$. The case $a_i = 0$ and $b_i = c$ may arise if shortages are not allowed and if c is the warehouse capacity. If the bounds on inventories are fairly tight, then the perfect smoothing solution may no longer be feasible. As an example, consider the case where most of the requirements come early. In this case, the perfect smoothing solution may result in negative inventories violating the inventory constraints lower bound when $a_i = 0$ for all i.

We are now interested in minimizing the production cost subject to the constraint that the inventories are kept within desirable ranges. Mathematically, the problem is

$$\min f(x) = \sum_{i=1}^{n} d_i g(x_i/d_i)$$

subject to

$$I_{i-1} + x_i - r_i - I_i = 0, \qquad i = 1, \ldots, n,$$
$$a_i \le I_i \le b_i, \qquad i = 1, \ldots, n,$$

where r_i denotes the requirements in period i and a_i and b_i represent bounds on the inventory at the end of period i. Here we will assume that $I_0 = 0$ and that $I_n = a_n = b_n = 0$, i.e., that the initial and the ending inventories are zero. These assumptions can be relaxed by decreasing (resp., increasing) the requirements at the beginning (resp., end) of the planning horizon. Under these assumptions, the constraints can be written as

$$a_i \ \le\ X_i - R_i \ \le b_i, \quad i = 1, \ldots, n$$

or equivalently as

$$E_i \ \le\ X_i \ \le\ F_i, \quad i = 1, \ldots, n,$$

where $E_i \equiv a_i + R_i$ and $F_i \equiv b_i + R_i$.

Let $x = (x_1, \ldots, x_n)$ be a feasible solution, and recall that the cost function depends on the ratios x_i/d_i $i = 1, \ldots, n$. One convenient way of graphically representing these ratios is to plot the points $(D_i, X_i),\ i = 1, \ldots, n$ since the slope of the line joining the points (D_{i-1}, X_{i-1}) and (D_i, X_i) is precisely x_i/d_i. Notice also that to be feasible the points (D_i, X_i) must lie between (D_i, E_i) and $(D_i, F_i),\ i = 1, \ldots, n$.

Since the optimal production plan is independent of which convex function we use, we find it convenient to use the convex function $g(x) = \sqrt{x^2 + 1}$. In this case $d_i g(x_i/d_i) = \sqrt{x_i^2 + d_i^2}$, so the function to minimize is $f(x) = \sum_{i=1}^{n} d_i g(x_i/d_i) = \sum_{i=1}^{n} \sqrt{d_i^2 + x^2}$. Notice that this function measures the distance from $(0, 0)$ to (D_1, X_1) to $(D_2, X_2) \ldots (D_n, X_n)$, so an optimal solution can be found by tying a string to a pin at the origin, thread it between the pairs (D_i, E_i) and (D_i, F_i) for $1 \le i \le n$ and pull the string taut at (D_n, E_n). This is known as the taut-string solution and it traces out the desired solution *for all convex* functions $g(x)$. To summarize, if the production cost function is convex an optimal production plan is a feasible plan where the ratios $x_i/d_i,\ i = 1, \ldots, n$ are as even as possible as given by the taut-string solution.

Example:

Suppose the initial inventory is $I_o = 5$, that the requirements are given by the vector $(12, 3, 9, 17, 4)$, that the bounds on inventory are given by the vectors $a = (0, 0, 0, 0, 3)$ and $b = (5, 3, 5, 5, 3)$. The first step is to transform this problem to one with zero initial and ending inventory. To do this we reduce the requirement of the first period by 5 units and increase the requirement of the last period by 3 units. This result in the requirement vector $(7, 3, 9, 17, 7)$, and the modified inventory bounds $a = (0, 0, 0, 0, 0)$ and $b = (5, 3, 5, 5, 0)$. We can now construct the bound vectors $E = (0, 7, 10, 19, 36, 43)$ and $F = (0, 12, 13, 24, 41, 43)$ and find the tight string solution: $x_1 = 7$, $x_2 = 6$, $x_3 = 11$, $x_4 = 12$ and $x_5 = 7$. This results in actual ending inventories equal to $I_1 = 0$, $I_2 = 3$, $I_3 = 5$, $I_4 = 0$ and $I_5 = 3$.

2 Concave Production and Holding Costs

In practice it often happens that the cost functions exhibit economies of scale, i.e., they are concave. Under these conditions, optimal production schedules tend to *amplify* rather than smooth out fluctuations in requirements. In this lecture we analyze a number of inventory models with a concave cost structure. We begin by giving a brief background on minimizing a concave function subject to linear constraints, discuss the implications for a simple capacitated production planning problem, and for a general un-capacitated production planning problem with concave production and inventory costs. Finally, we discuss a dynamic programming procedure to solve the famous Wagner-Whitin model.

2.1 Preliminaries

From a mathematical viewpoint, the problem of finding minimum-cost production schedules in the presence of scale economies often reduces to the problem of finding a vector x that minimizes a concave function $c(\cdot)$ subject to a system of linear constraints. Let P denote the set of solutions satisfying the system of linear constraints. Such a set P is said to be a polyhedral set.

A well known result is that the minimum of a concave function $c(\cdot)$ over a polytope, i.e., a bounded polyhedral set, is attained at an extreme, or corner, point of P. Mathematically, x is an extreme point of P if x cannot be written as $0.5(x' + x'')$ where x' and x'' are two distinct points in P. In what follows we denote by E the set of extreme points of X.

2.2 A Capacitated Production Planning Problem with Concave Costs

Let $r_1, \ldots, r_n$ denote the requirements for periods $i = 1, \ldots, n$. Assume that there are no storage costs, and that the production cost function is concave and time invariant. The production capacity is c units per period. The problem is to

$$\begin{aligned}
\min c(x) &= \sum_{i=1}^{n} g(x_i) \\
\text{subject to:} & \\
0 \le \quad x_i &\le c \\
\sum_{j=1}^{i} x_j &\ge \sum_{j=1}^{i} r_j \qquad i = 1, \ldots, n-1 \qquad (1) \\
\sum_{j=1}^{n} x_j &= \sum_{j=1}^{n} r_j.
\end{aligned}$$

Let I_j denote the inventory at the end of period j. The above formulation assumes that the $I_0 = I_n = 0$ and that no backorders are allowed, but this assumptions can be easily modified as in the case of convex costs. Let us consider first the problem where we ignore the constraints (1). This restricted problem is feasible if and only if $\sum_{j=1}^{n} r_j \le nc$. In this case, it is possible to find an integer integer $k \le n$ and a real number $q \in [0, c)$ such that $\sum_{j=1}^{n} r_j = kc + q$. It can be shown that the extreme points are given by solutions of the form $x_i = c$ in k periods and $x_i = q$ on a single period. Since all solutions of this form have the same cost, they are all optimal for the restricted problem. Returning to the original problem, we see that an optimal solution is to produce $x_i = c$ in periods $1, \ldots, k$ and $x_i = q$ in period $k + 1$, provided, of course, that the constraints (1) are satisfied.

2.3 Uncapacitated Production Planning with Concave Production and Inventory Costs

Consider now the uncapacitated problem with concave period dependent production and inventory costs:

$$\begin{aligned}
\min c(x) &= \sum_{i=1}^{n} [g_i(x_i) + h_i(I_i)] \\
\text{subject to:} &
\end{aligned}$$

$$\begin{aligned} x_i &\geq 0 \\ I_i &= \sum_{j=1}^{i} x_j - \sum_{j=1}^{i} r_j \geq 0 \quad i = 1, \ldots, n-1 \\ I_n &= \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} r_j = 0. \end{aligned}$$

Notice that $c(x)$ is a sum of concave functions and is therefore concave. Thus a minimum will occur at an extreme point of the polytope P of feasible solutions. Our problem is to characterize the set E of extreme points of P. For that purpose we need a few definitions.

Let $x \in P$. Period t is called a *regeneration point* of x if $I_t = 0$. Period t is called a *production point* of x if $x_t > 0$.

Theorem 1 *$x \in E$ if and only if between every pair of production points of x there is a regeneration point of x.*

Proof

Let i and $j > i$ be two consecutive production points, i.e., $x_i > 0, x_t = 0$ for $t = i+1, \ldots, j-1$ and $x_j > 0$. We need to show that $I_t = 0$ for some $t \in \{i, i+1, \ldots, j-1\}$. Suppose for a contradiction that $I_t > 0$ for all $t \in \{i, i+1, \ldots, j-1\}$, then

$$\epsilon = \min\{x_i, x_j, I_i, \ldots, I_{j-1}\} > 0.$$

Consider now the production plans

$$y = x + \epsilon(e_i - e_j)$$

and

$$z = x - \epsilon(e_i - e_j),$$

where e_k denote the unit vector in the direction of the k^{th} component.

Clearly both y and z are feasible production plans and $x = 0.5(y+z)$ so x is not an extreme point contradicting the hypothesis that $x \in E$. Thus, x must have a regeneration point in the set $\{i, i+1, \ldots, j-1\}$.

Conversely, assume that $x_i > 0, x_t = 0$ for $t = i+1, \ldots, j-1,\ x_j > 0$ and that $I_t = 0$ for some $t \in \{i, i+1, \ldots, j-1\}$. We want to show that $x \in E$. Suppose that $x \notin E$, so there exists $y, z \in E$ such that $x = 0.5(y+z)$ where y and z are different from x only at periods i and j. Thus, we can and do assume that y and z satisfy $y_i < x_i < z_i$ and that $z_j < x_j < y_j$. However, under the production plan y, the inventory at time $t \in \{i, \ldots, j-1\}$ is given by $I_t - (x_i - y_i) \geq 0$ so $I_t \geq x_i - y_i > 0$ contradicting the hypothesis that $I_t = 0$ thereon. Thus, $x \in E$ must be an extreme point.

$\square$

Corollary 2 *If $x \in E$ then there exists at most one production point between every consecutive pair of regeneration points.*

A production plan is said to satisfy the exact requirements if for all m production points $i_k,\ k = 1, \ldots,\ m \leq n$,

$$x_{i_k} = \sum_{j=i_k}^{i_{k+1}-1} r_j \quad k = 1, \ldots, m,$$

and

$$x_i = 0 \quad \text{for} i \notin \{i_1, \ldots, i_m\}.$$

Proposition 3 *If $x \in E$ then x satisfy the exact requirements. Moreover, if x is feasible and satisfies exact requirements then $x \in E$.*

Theorem 4 *If $x \in E$ then $I_{t-1}x_t = 0$ for all $t = 1, \ldots, n$.*

Proof
Assume that there is a period b with $I_{b-1} > 0$ and $x_b > 0$ and let $a \leq b-1$ be the latest production period prior to b. Thus, $I_a > 0,\ I_{a+1} > 0, \ldots, I_{b-1} > 0$, but this contradicts our earlier result which states that there must be a regeneration point between a and b.

□

2.3.1 The Wagner-Whitin Problem

As an example of the above production inventory problem, consider the case where the production cost function is $g_i(x) = K_i + cx$ if $x > 0$ and $g_i(x) = 0$, if $x = 0$ and the inventory cost function is $h_i(I_i) = h_i I_i$. This model was first studied by Harvey M. Wagner and Thomas M. Whitin in their famous paper "Dynamic Version of the Economic Lot Size Model" written in 1958. Wagner and Whitin's model is essentially a dynamic demand, finite horizon, version of the economic lot size model.

As before let $r = (r_i)_{i=1}^n$ denote the vector of non-negative requirements. Since the production cost function is concave and the holding cost function is linear, the total cost is concave in the production plan x. Let $\delta(x) = 1$ if $x > 0$ and $\delta(x) = 0$ otherwise. Since $\delta(\cdot)$ is concave on $\Re_+$ and the set P is a polytope, it follows that there is an optimal production plan in E, the set of extreme points of P. Moreover $x \in E$ implies that $I_{t-1}x_t = 0$ for all $t \in \{1, \ldots, n\}$. In other words there is an optimal production plan x that satisfy the exact requirements. We now take advantage of this fact to write a dynamic programming formulation of the Wagner-Whitin problem.

Let c_{ij} denote the cost of ordering at period i to satisfy the demands in periods $i, i+1, \ldots, j-1$. This means that we order or produce $\sum_{k=i}^{j-1} r_k$ units in period i, carry $\sum_{l=i+1}^{j-1} r_l$ units into period $i+1$ at a cost of $h_i \sum_{l=i+1}^{j-1} r_l$, carry $\sum_{l=i+2}^{j-1} r_l$ units into period $i+2$ at a cost of $h_{i+1} \sum_{l=i+2}^{j-1} r_l$, etcetera until we reach period j with no inventory. The total cost c_{ij} is therefore given by

$$c_{ij} = K_i \delta\left(\sum_{k=i}^{j-1} r_k\right) + \sum_{k=i}^{j-2} h_k \sum_{l=k+1}^{j-1} r_l.$$

Let f_i denote the minimum cost of satisfying demand for periods $\{i, \ldots, n\}$ starting with no inventory at the beginning of period i. If we define $f_{n+1} \doteq 0$, then by the principle of optimality we have

$$f_i = \min_{i < j \leq n+1} \{c_{ij} + f_j\}.$$

The f_i's can be computed recursively starting from period n down to period 1. At each i we need to record the period, say $j(i)$, at which the next production will take place. The optimal production plan is recovered as follows: The first production period is of course period $i_1 = 1$. The next production period is $i_2 = j(i_1)$ followed by $i_3 = j(i_2)$ etc. This straightforward implementation of the dynamic lot sizing problem has complexity $O(n^2)$. Recently a number of authors have developed improved dynamic programming implementations with complexity $O(n \log n)$.

Example: Consider the data

From the data we can construct the following cost table:

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} & 3 & 4 & 6 & 36 \\ & & 2 & 3 & 28 \\ & & & 2 & 22 \\ & & & & 1 \end{pmatrix} \end{array}$$

i	r_i	K_i	h_i
1	3	3	1
2	1	2	1
3	1	2	4
4	5	1	1

Table 2: Data for Wagner Whiting Problem

We can now start the DP with $f_5 = 0$. This yields $f_4 = 1$, $f_3 = 3$, $f_2 = 4$, and $f_1 = 7$. There are several paths that lead to the minimal cost $f_1 = 7$ including the path $c_{12} + c_{24} + c_{25} = 3 + 3 + 1 = 7$ and the path $c_{13} + c_{34} + c_{45} = 4 + 2 + 1 = 7$.