Bounds, Heuristics, and Approximations for Distribution Systems

Guillermo Gallego
Columbia University

Özalp Özer
Stanford University

Paul Zipkin
Duke University

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Abstract

This paper develops simple approximate methods to analyze a two-stage stochastic distribution system consisting of one warehouse and multiple retailers. We consider local and central control schemes. The main ideas are based on relaxing and or decomposing the system into more manageable newsvendor type subsystems. We provide bounds on optimal echelon base-stock levels and on the optimal expected cost. We show that one of the heuristics is asymptotically optimal in the number of retailers. The results of this paper provide practicaly useful techniques as well as insights into stock-positioning issues and the drivers of system performance.

1 Introduction

The formulations and the methodologies developed in multi-echelon production and distribution systems are often computationally intractable and difficult to explain to non-mathematically oriented students and practitioners. Users are more likely to embrace decision tools when they understand what is in the black box. In addition, data fed to these tools are not always accurate. Systems and people have limitations. In this paper, we aim to provide efficient control mechanisms for two level infinite horizon distribution system by developing *easy-to-describe*, *close-to-optimal* and *robust* heuristics some of which can be implemented on a spreadsheet. We also develop approximations that require less data, have closed-form expressions and reveal important relationships for stock positioning.

In particular, we consider a two-level distribution system: Goods enter the system from an outside source and proceed first to the warehouse. The warehouse in turn supplies J retailers, where customer demands occur. Each such shipment requires a leadtime, but there are no economies of scale. Demands are stochastic; those that cannot be filled immediately are backlogged. There is an inventory holding cost at each location and a backorder penalty cost at each retailer. The horizon is infinite, all data are stationary, and the objective is to minimize total average cost. In this paper, we focus on a continuous review system and study a class of replenishment policies known as base-stock policies that use local or centralized stock information. Local control is also known as decentralized control (Eppen and Schrage 1981). However, this terminology is different than its recent use in contracting literature where the emphasis is on multiple decision makers with different objective functions (Chen 2003).

Clark and Scarf (1960) initiated the analysis of the distribution system under centralized information. They correctly pointed out that an optimal policy, if it exists, would be very complex. Since then the research on distribution systems has shifted towards identification of close-to-optimal heuristics and evaluation of plausible classes of policies. A comprehensive review can be found in Axsäter (1993) for local control policies and in Federgruen (1993) for central control. Often local control is referred to as a pull system and central control as a push system. Here, we present heuristics, approximations and bounds for both local and central control policies.

Under local control, the policy space is *restricted* to *local* policies such that each location monitors its own inventory. The warehouse manager replenishes from an outside supplier and ships to the retailer based on local information. Similarly, the retailers replenish through based on their local information. For local control we assume that the warehouse satisfies retailers requests on a first-come-first-serve basis. The literature on local base-stock control policy begins with the METRIC approximation of Sherbrooke (1968). We know how to compute the best local base-stock policy in any particular instance (Graves 1985 and Axsäter 1990). The exact computational method is not easy to communicate to managers who are interested in easy-to-use, near-optimal policies. In this paper, we provide simple heuristics and approximations that are based on newsvendor-type solutions. Some of these heuristics are related to similar methods in Gallego and Zipkin (1999). The first heuristic, the restriction and decomposition (RD) heuristic, applies three sub-heuristics and selects the best. In particular, it restricts the warehouse to hold one of the three levels of inventory: zero inventory (as in cross-docking), zero safety stock or the maximum possible inventory. In our numerical study (based on more than 750 cases) the percentage difference from the exact solution for RD heuristic is 1.37% on average. These sub-heuristics provide simple rules to set the total system stock and its division among locations, and to estimate overall system performance. For example, the superior performance of zero safety stock sub-heuristic suggests that the major part of risk pooling benefit can be realized even without holding safety stock at the warehouse. We also show that the RD heuristic is asymptotically optimal in the number of retailers. In other words, the RD heuristic captures the behavior of the system even better for distribution systems with many retailers.

We also develop two approximations, normal and maximal approximation that accurately predict the systems performance. The maximal approximation, for example, does not require one to estimate demand distributions, hence it is *robust* with respect to demand parameters. For these approximations, we employ the first two moments of leadtime demand distribution. Graves (1985) also uses first two moments but he approximates the number of outstanding orders at each retailers and fits a negative binomial distribution. This approximation works well. However, it is difficult to communicate and it does not provide closed-form solution. Graves also establishes that the METRIC approximation is unreliable. The METRIC approximation takes the stochastic sojourn time from outside supplier to the retailer as deterministic by replacing it with its mean. For a comprehensive review of these two approximations, we refer the reader to Axsäter (1993).

Another value of the RD heuristic and approximations is due to their computational tractability. The scale of these problems in practice could be very large. For example, General Motor's service parts organization manages more than 4 Million stock keeping units. GM revisits stock allocation problem every day and solve these problems repeatedly. [1] Computational improvements are therefore very important for implementation.

Under central control, the manager oversees the entire system. She decides on how much to order from outside supplier; how much to withdraw from the warehouse and how to allocate the withdrawn quantity to retailers so as to minimize the total average holding and penalty cost. Note that the difference between local and central control policy is due to the allocation rule at the warehouse. Under local control the warehouse allocates stock to the retailers on a first-come-first-served bases. Under central control the warehouse allocates to the retailer based on additional information, such as inventory levels and costs at the retailers. On the other hand replenishment policies, not allocation policies, are the same under both local and echelon base-stock control. For example, Axsater and Rosling (1993) show that installation and echelon reorder point policies are equivalent. However, due to the different allocation rule at the warehouse, the system performs differently under local and central control as illustrated in the numerical section.

For the central control problem, we do not restrict the policy space to a certain class of policies. Instead we use two approaches to simplify the problem and provide close-to-optimal heuristics. Our first approach is based on *relaxing* some constraints on the control variables. This relaxation leads to a simpler problem whose solution provides a lower bound on the optimal average cost. The relaxation approach yields a problem that resembles a series system for which the optimal policy is an echelon base stock policy (Gallego and Zipkin 1999). Combined with an allocation strategy at the warehouse, this approach provides a heuristic for solving the central control case. The percentage difference between the cost of this heuristic and the lower bound, called the optimality gap, was on average 9.44% in our experiments. For periodic review systems a similar relaxation approach is used by Federgruen and Zipkin (1984), Aviv and Federgruen (2001) and Özer (2003). The second approach is based on *decomposition*. It decomposes the warehouse into J warehouses. This approach provides bounds for the best echelon base stock levels. Using these bounds, we search for the best echelon base stock levels that result in an optimality gap of 1.78% on average. This approach is related to one proposed by de Kok and Visshers (1999) for assembly-distribution systems.

Current trends in logistics point in opposite directions: Wal-Mart's cross docking system, a central part of its overall strategy (Stalk et. al. 1992), operates with little warehouse stock. On the other hand, Apple Computers consolidate inventories at distribution hubs in Ireland

[1] Dr. Jeffrey Tew of GM R&D provided this information during a private communication.

for Europe and in Shenzen for China [2]. The recent resurgence of mail-order retailing (Dell, Land's End, Amazon, Medco, etc.) also attests to the strategic value of centralized stocks. The effectiveness of a given strategy depends on the control policies used. Our heuristics and numerical studies provide a foundation that sheds light on current developments in practice.

Through a numerical study, we test our heuristics and address issues that arise in the design of production and distribution systems. To understand and quantify the role of centralization, we compare the system's performance under local control to the performance under central control. We illustrate, for example, that the value of central control increases in the warehouse's leadtime, its holding cost, and the retailers' penalty cost.

Note that we could as well discuss the issues here under the setting of multi-item production system with a common intermediate product. This interpretation underlies the literature on postponed product differentiation; see Lee et al. (1993) and Aviv and Federgruen (2001). Under this scenario, the warehouse represents the differentiation point and the retailers represent the differentiated products. The value of risk pooling corresponds to the value of postponement or delayed differentiation. Our results apply to this setting as well.

The rest of the paper is organized as follows. In §2, we introduce the notation and describe the system dynamics in detail. In §3, we study the distribution system under local control. We provide some preliminaries followed by an easy-to-use heuristic and two approximations. In §4, we study the distribution system under central control and provide heuristics and an approximation to obtain echelon base-stock levels. In §5, we conduct numerical study of several problem instances. First, we report on performance measures, such as optimality gap and the computational requirement. Next, we provide insights for distribution system design issues and the value of centralization. In §6, we conclude and suggest directions for future research.

2 Distribution System

Consider a two level distribution system. All items enter the system from an external supplier and proceed first to location $j = 0$, called the warehouse. The warehouse in turn supplies J retailers, where the customer demands occur, indexed by $j = 1, \ldots, J$. Shipments from the external supplier arrive to the warehouse after time L_0. Shipments arrive to retailer j

[2] Emily Choi, a senior manager in charge of world wide new product introductions at Apple, provided this information

after time L_j. The retailers satisfy the customer demand from on-hand inventory, if any. Unsatisfied demand at retailer j is backordered at a linear penalty cost rate b_j. All locations are allowed to carry inventory. The local (installation) holding cost is h_j per unit at retailer j. Holding inventory at the retailer is assumed to be more expensive than holding it at the warehouse $h_j \geq h_0$ for $j > 0$. This could be due to more expensive storage space, overhead or the value added operations carried out at the downstream. On the other hand, inventory located closer to the end customer enables a quick response, hence reduces the possibility of a backorder at each retailer. We assume that the demand at each retailer j follows a Poisson process, $\{D_j(t), t \geq 0\}$ and it is independent across retailers. The question is where to locate the inventory and how to control the distribution system so as to minimize the long run average holding and penalty cost.

2.1 Cost

Let $E[\cdot]$ denote the expectation, $V[\cdot]$ the variance and $[x]^+ = \max\{0, x\}$. The state of the distribution system just after all the decisions are made is summarized by

For each retailer j:

$$
\begin{aligned}
I_j &: \text{ on-hand inventory at retailer } j, \\
B_j &: \text{ backorders at retailer } j, \\
IN_j &= I_j - B_j \\
&: \text{ net inventory at retailer } j, \\
IT_j &: \text{ inventory in transit to retailer } j, \\
ITP_j &= IT_j + IN_j, \\
&: \text{ inventory transit position of retailer } j, \\
IO_j &: \text{ inventory on order by retailer } j, \\
IOP_j &= IO_j + IN_j, \\
&: \text{ inventory order position of retailer } j, \\
ITP_r &= \sum_{j>0} ITP_j, \\
&: \text{ sum of inventory transit positions at each retailers}
\end{aligned}
$$

For the warehouse:

$$
\begin{aligned}
I_0 &: \text{ on-hand inventory at the warehouse}, \\
IN_0 &= I_0 + ITP_r,
\end{aligned}
$$

$$B_{0j} \quad : \quad \text{warehouse backorders due to the orders from retailer } j,$$

$$ITP_0 \quad = \quad IT_0 + IN_0.$$

We use D_j to denote the leadtime demand for location j in equilibrium, a generic random variable that has the distribution of $D_j(t, t+L_j] = D_j(t+L_j) - D_j(t)$. Notice that $IO_j \geq IT_j$ for $j > 0$ since retailer j can be replenished only when warehouse has inventory. Also inventory order and transit positions are the same for the warehouse since it orders from an external supplier with ample stock. We use the superscript $^-$ to refer to the state space right before any decision. For example, ITP_j^- refers to the inventory transit position at retailer j before any decision is made. Under *any* policy, the total average cost can be expressed as

$$\begin{aligned} C &= h_0 E[I_0] + h_0 \sum_{j>0} E[IT_j] + \sum_{j>0} (h_j E[I_j] + b_j E[B_j]) && (1) \\ &= H_0 E[IN_0] + \sum_{j>0} (H_j E[IN_j] + (b_j + h_j) E[B_j]), && (2) \end{aligned}$$

where $H_j = h_j - h_0$ and $H_0 = h_0$ are the echelon holding costs. The first equation above is based on the local cost accounting while the second is based on the echelon cost accounting. We use the former to establish our *local* control policies and the latter to develop *central* control mechanisms. We use the lower case c to denote the resulting cost under a local control policy and the capital C to denote the cost under a central control policy. For series systems the cost and the optimal policy under local and central control are the same (Axsäter and Rosling 1993, Chen and Zheng 1994, and Gallego and Zipkin 1999), but this is not the case for distribution systems.

3 Local Control With Base Stock Policy

Under a continuous review local control mechanism each location monitors its own inventory and its local base stock level. Whenever the inventory order position, IOP_j at retailer j falls below the local base-stock level s_j, its manager orders from the warehouse to raise it up to this level. The sum of the retailers' orders constitute the warehouse's demand process. The warehouse manager replenishes its own inventory from an outside supplier with ample stock whenever its inventory order position is below s_0 and satisfies the retailers' requests on a first-come-first-served basis. Notice that the information and the control are decentralized or localized in that each location sees its own demand and monitors its own inventory-order position.

Assume that the initial on-hand inventory at each location is s_j units and that there are no inventories in the pipeline. Arrival of demand to any location prompts its manager to place an order from the warehouse. This in turn triggers an order at the warehouse from the outside supplier. The warehouse fills retailers' orders sequentially. If the demand process at retailer j is Poisson with mean λ_j, then the warehouse's demand process is also Poisson with mean $\lambda_0 = \sum_{j>0} \lambda_j$. The standard exact analysis of this system is due to Graves (1985).

3.1 Preliminaries

3.1.1 The Analysis

Following a top down approach, that is, analyzing first the warehouse then the retailers we have

$$\begin{aligned} B_0 &= [D_0 - s_0]^+, & (3)\\ I_0 &= [s_0 - D_0]^+, & (4)\\ B_j &= [B_{0j} + D_j - s_j]^+ \text{ for } j > 0, & (5)\\ I_j &= [s_j - B_{0j} - D_j]^+ \text{ for } j > 0, & (6)\end{aligned}$$

where B_{0j} represents the number of backorders due to retailer j. Notice that B_{0j} and D_j are independent. Recall that the warehouse faces independent Poisson arrivals from each retailer and satisfies the retailer's request in order. Consequently $(B_{0j}|B_0)$ is binomial with parameters B_0 and $\theta_j = \frac{\lambda_j}{\lambda_0}$. Given the base stock levels one can construct numerically the distribution of state equations (3)-(6) and obtain performance measures such as $E[B_j]$. This enables us to evaluate the total average cost of a base stock policy from equation (1);

$$\begin{aligned} c(s_0, s_1, \ldots, s_J) &= h_0 E[I_0] + \sum_{j>0} c_j(s_0, s_j), & (7)\\ c_j(s_0, s_j) &= h_j E[I_j] + b_j E[B_j]. & (8)\end{aligned}$$

We omit the holding cost of shipments in transit since it is a constant equal to $h_0 \sum_{j>0} \lambda_j L_j$. Let $s^* = (s_j^*)_{j=0}^J$ denote the *optimal* base stock policy that achieves the minimum total average cost c^*. Given s_0 the average cost c_j in (8) is a newsvendor type (single-location) subsystem. Hence, its minimizer is easy to compute. Notice also that c_j is a function of s_0 due to the backlog B_{0j}. Let

$$\begin{aligned} \Delta c_j(s_0, s_j) &\equiv c_j(s_0, s_j + 1) - c_j(s_0, s_j) = h_j - (b_j + h_j) P(B_{0j} + D_j > s_j) & (9)\\ s_j^*(s_0) &\equiv \min\{s_j : \Delta c_j(s_0, s_j) > 0\}. & (10)\end{aligned}$$

3.1.2 Exact Solution

This exact solution originally discussed in Graves (1985) and Axsäter (1990). Here, we outline the exact solution for an easy reference and point out the challenges involved. Later we compare this algorithm to the proposed easy-to-use, close-to-optimal heuristics and closed-form approximations.

Given a base stock level s_0, the total average cost in (7) separates into J convex functions and the cost associated to the warehouse (that is a constant). Notice that to compute these convex functions numerically, we need to obtain the distribution of B_j and I_j for all j. To do so, the exact algorithm requires one to carry out numerical convolutions. Another challenge is that the total average cost $c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0))$ is not convex in s_0. Finding the optimal s_0, therefore, requires an exhaustive search over all possible values. To simplify this search we later provide an upper bound for s_0^* (see Proposition 1). Let UB denote such an upper bound. Next, we describe an algorithm (projection method) to obtain c^* and s^*:

FOR $s_0 = 0$ to UB

Let $c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) = 0$.

FOR $j = 1$ to J

$s_j^*(s_0) \leftarrow \min\{y : c_j(s_0, y) \le c_j(s_0, x) \text{ for all } x \neq y\}$

$c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) \leftarrow c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) + c_j(s_0, s_j^*(s_0))$

END

$c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) \leftarrow h_0 E[I_0] + c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0))$

END

$c^* \leftarrow \min_{s_0 \in [0, UB]} c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0))$.

Notice from equation (3) that B_0 the backlog at the warehouse decreases as we increase s_0. This in turn reduces the stock out probability and hence the optimal base stock level at the retailers. This observation reduces the computational requirement since the search for $s_j^*(s_0)$ for $j > 0$ is over a smaller set as we increase s_0.

Next, we present several heuristics and approximate methods to analyze the system. Some of these approximations and heuristics are in closed form that involves only the original problem data. They are easy to describe, compute and implement. These heuristics also enable us to study the effect of system parameters on optimal cost and policy.

3.2 A Heuristic: Restriction and Decomposition (RD)

This approach involves restriction of the policy space and decomposition of the resulting model into independent single-location, newsvendor type subsystems. The overall RD heuristic applies three sub-heuristics and selects the best. The first sub-heuristic restricts the warehouse to carry zero inventory. The second one restricts the warehouse to carry maximal inventory. The third sub-heuristic restricts the warehouse not to carry any safety stock. Next, we discuss these sub-heuristics in detail.

3.2.1 Cross-Docking (CD) Sub-heuristic

The first sub-heuristic restricts s_0 to 0. We call the restriction to no warehouse inventory *cross-docking*. This phrase refers to a warehouse-management approach that, among other things, utilizes little warehouse inventory. For a general discussion of managerial advantages of cross-docking, we refer the reader to Rosenfield and Pendorck (1980). Our restricted system can be viewed as a stylized representation of this practical approach. The warehouse can be considered as a repackaging or bulk-breaking center with zero inventory, that is $I_0 = 0$ and $B_0 = D_0$. In this case B_{0j} has the Poisson distribution with mean $L_0\lambda_j$. The average cost is

$$c(0, s_1, \ldots, s_J) = \sum_{j>0} c_j(0, s_j) = \sum_{j>0}(h_j E[s_j - B_{0j} - D_j]^+ + b_j E[B_{0j} + D_j - s_j]^+),$$

where $B_{0j} + D_j$ is Poisson with mean $(L_0 + L_j)\lambda_j$. In other words each retailer j operates as an independent newsvendor subsystem with total leadtime $L_0 + L_j$. The optimal base stock level for each retailer is given by $s_j^*(0) \equiv \min\{y : c_j(0, y) \leq c_j(0, x) \text{ for all } x \neq y\}$.

3.2.2 Stock-Pooling (SP) Sub-heuristic

The second sub-heuristic decomposes the system into $J + 1$ newsvendor subsystems. For any policy, from (5), we have $B_j \leq B_{0j} + [D_j - s_j]^+$ for all $j > 0$ so we have

$$\begin{aligned} c(s_0, s_1, \ldots, s_J) &= h_0 E[I_0] + \sum_{j>0}(h_j E[I_j] + b_j E[B_j]) \\ &\leq h_0 E[s_0 - D_0]^+ + \sum_{j>0}(h_j E[s_j - D_j]^+ + b_j E[B_{0j}] + b_j E[D_j - s_j]^+) \qquad (11) \\ &= (h_0 E[s_0 - D_0]^+ + (\sum_{j>0} b_j\theta_j)E[D_0 - s_0]^+) + \sum_{j>0}(h_j E[s_j - D_j]^+ + b_j E[D_j - s_j]^+) \\ &= c_0(s_0) + \sum_{j>0} c_j(\infty, s_j), \end{aligned}$$

where we have defined $c_0(s_0) = h_0 E[s_0 - D_0]^+ + b_0 E[D_0 - s_0]^+$ with shortage cost $b_0 = \sum_{j>0} b_j \theta_j$. Each term in this expression is the cost of an independent, newsvendor subsystem, one for each location, and we can optimize each one separately. Let $s_j^*(\infty) \equiv \min\{y : c_j(\infty, y) \leq c_j(\infty, x) \text{ for all } x \neq y\}$. The optimal base stock level for the warehouse is given by $s_0^u \equiv \min\{y : c_o(y) \leq c_0(x) \text{ for all } x \neq y\}$. We call this sub-heuristic as *stock-pooling* because as we will show it uses maximal warehouse inventory, that is s_0^u. The model can be considered as a stylized version of the practical method of using minimal inventory at the retailers and maximal inventory at the warehouse. This discussion leads us to the special case of stock-pooling sub-heuristic next.

Notice the special case of the stock-pooling sub-heuristic when $s_j = 0$. In this case, all retailers act as order processing centers. Call centers, e-retailers and dealers of luxury goods are examples of retailers that carry inventory, if at all, only for display purposes. Hence, the inequality in equation (11) leads to an equality because $B_j = B_{0j} + D_j$. Hence, the average cost is

$$\begin{aligned} c(s_0, 0, \ldots, 0) &= h_0 E[s_0 - D_0]^+ + b_0 E[D_0 - s_0]^+ + \sum_{j>0} b_j L_j \lambda_j \\ &= c_0(s_0) + \sum_{j>0} c_j(\infty, 0), \end{aligned}$$

This is also a newsvendor subsystem. Intuitively, the retailers are charged only for the backorder due to the demand during the retailer leadtime and the warehouse incurs the shortage cost of b_0 per backorder if he cannot satisfy retailer's order requests.

Since $c_j(\infty, s_j^*(\infty)) \leq c_j(\infty, 0)$, allocating $s_j = 0$ to retailers results in an upper bound to the second sub-heuristic. The difference will be null only when the solution to the second sub-heuristic results in zero base stock level at the retailers. Another observation is that the second sub-heuristic gives us a lower bound on the cost of treating the retailers as call centers carrying no inventory. This special case also enables us to characterize an upper bound for warehouse base stock level and provides insights for a system with stock-less retailers.

3.2.3 Zero-Safety-Stock (ZS) Sub-heuristic

The third sub-heuristic sets $s_0 = E[D_0]$ and then optimize over $s_j, j > 0$ for J single location problems. If $E[D_0]$ is a fractional number we round it down to the nearest integer. The ZS sub-heuristic, in contrast to the other sub-heuristics, uses a moderate value of warehouse stock. Note that ZS sub-heuristic requires us to compute convolutions for B_j and I_j whereas CD and SP sub-heuristics do not.

3.2.4 Properties of RD Heuristic

In summary, the RD heuristic restricts the policy space, decomposes the problem into newsvendor type subsystems and selects the best of the three sub-heuristics. Note that it chooses among two extreme levels of warehouse stock (none or maximal inventory) and an intermediate value (some inventory). Next, we provide monotonicity results for the base stock levels. We also show that choosing minimum of CD or SP sub-heuristics, the two extreme levels of warehouse stock, is near-optimal when the distribution system has a large number of retailers. In other words, RD heuristic is asymptotically optimal.

Proposition 1

1. $0 \leq s_0^* \leq s_0^u$,
2. $s_j^*(\infty) \leq s_j^*(s_0^*) \leq s_j^*(0)$,
3. $\sum_{j>0} c_j(\infty, s_j^*(\infty)) \leq c^* \leq \min\{c(0, s_1^*(0), \ldots, s_J^*(0)), c_0(s_0^u) + \sum_{j>0} c_j(\infty, s_j^*(\infty))\}$.

Axsäter (1990) provides a proof for the first two parts of this proposition. Here, we refine their proof and provide an intuitive argument. The base stock level s_0^u is an upper bound because to obtain this value during the third sub-heuristic we restrict $s_j = 0$ for $j > 0$. Increasing s_j would only reduce the required warehouse safety stock. The base stock level $s_j^*(\infty)$ is a lower bound since the second sub-heuristic assumes infinite supply at the warehouse and ignores the backlog B_{0j}. Similarly, $s_j^*(0)$ is an upperbound since the first sub-heuristic assumes that the warehouse has zero inventory. [3] To see why Part 3 is correct, notice that the newsvendor cost for the retailer's $c_j(\infty, s_j^*(\infty))$ are lower bounds to the optimal retailer costs since they ignore the warehouse backorders. If we sum these lower bounds and exclude the cost for the warehouse then this sum would be a lower bound for the optimal cost c^*.

[3] A more formal argument is as follows. We first show that $s_j^*(s_0+1) \leq s_j^*(s_0)$ for any s_0, which implies Part 2. To do so, we show that $\Delta c_j(s_0+1, s_j) - \Delta c_j(s_0, s_j) \geq 0$. This implies $s_j^*(s_0+1) \leq s_j^*(s_0)$ due to the definition of the base stock level (10) and the convexity of $c_j(s_o, s_j)$ with respect to s_j. From equation (9) $\Delta c_j(s_0+1, s_j) - \Delta c_j(s_0, s_j) = (b_j + h_j)[P(B_{0j}(s_0) + D_j > s_j) - P(B_{0j}(s_0+1) + D_j > s_j)]$. To show that this difference is greater or equal to zero, it sufficies to show $P(B_{0j}(s_0+1) > y) \leq P(B_{0j}(s_0) > y)$ for any y since D_j is a nonnegative random variable. From equation (3), we have $B_0(s_0+1) \leq_{st} B_0(s_0)$. Note that $P(B_{0j}(s_0+1) > y) = E1_{\{B_{0j}(s_0+1)>y)\}} = E[E1_{\{B_{0j}(s_0+1)>y)\}}|B_0(s_0+1)] = E1_{\{\sum_{j=1}^{B_0(s_0+1)} X_j > y)\}} \leq E1_{\{\sum_{j=1}^{B_0(s_0)} X_j > y)\}} = P(B_{0j}(s_0) > y)$, where $1_{\{\cdot\}}$ is an indicator function and X_j's are independent Bernoulli random variables with parameter θ_j.

Recall also that all of the optimal costs for the sub-heuristics are upper bounds to the true optimal cost c^* because we restrict the policy space and minimize within this space.

From the bounds in Proposition 1, we have

$$\sum_{j>0} c_j(\infty, s_j^*(\infty)) \le c^* \le c(s) \le c_o(s_o^u) + \sum_{j>0} c_j(\infty, s_j^*(\infty))$$

Proposition 2 *If* $\min_{j>0}\{c_j(\infty, s_j^*(\infty)\} > 0$*, then the RD heuristic is asymptotically optimal in the number of retailers.*

Notice that

$$\frac{c(s)}{c^*} \le 1 + \frac{c_o(s_o^u)}{\sum_{j>0} c_j(\infty, s_j^*(\infty))} \le 1 + \frac{\sqrt{h_0 b_0 L J(\max_j\{\lambda_j\})}}{J \min_{j>0}\{c_j(\infty, s_j^*(\infty))\}} \underbrace{\rightarrow}_{J\rightarrow\infty} 1,$$

where $c_o(s_o^u) \le \sqrt{h_o b_0 L \sum_j \lambda_j}$ is the distribution-free upper bound (Gallego and Moon 1993).

3.3 Approximations

3.3.1 Normal Approximation (NA)

The standard normal approximation for newsvendor systems extends readily to this multi-location system. Axsäter (2002) independently develops a similar but more intricate method for the more general case of (r,q) policies. Let ϕ denote the standard normal density function, Φ^0 the standard normal complementary cumulative distribution function, Φ^1 the standard normal loss function, and Φ^2 the standard normal second-order loss function, that is,

$$\begin{aligned}
\Phi^0(z) &= \int_z^\infty \phi(x)dx \\
\Phi^1(z) &= \int_z^\infty \Phi^0(x)dx = \int_z^\infty (x-z)\phi(x)dx = -z\Phi^0(z) + \phi(z) \\
\Phi^2(z) &= \int_z^\infty \Phi^1(x)dx = \frac{1}{2}[(z^2+1)\Phi^0(z) - z\phi(z)].
\end{aligned}$$

Notice that although there is no closed-form expressions for these integrals, all standard software packages (including MS Excel) today have a built in function to compute $\Phi^0(z)$. The mean and also the variance of $D_j, j \ge 0$ are $L_j\lambda_j$. The normal approximation at the warehouse yields

$$\begin{aligned}
E[B_0] &= \Phi^1(z_0)\sqrt{L_0\lambda_0}, \\
E[B_0(B_0-1)] &= 2\Phi^2(z_0)\ L_0\lambda_0, \\
E[I_0] &= \Phi^1(-z_0)\sqrt{L_0\lambda_0},
\end{aligned}$$

where $z_0 = (s_0 - L_0\lambda_0)/\sqrt{L_0\lambda_0}$. Moreover,

$$\begin{aligned} E[B_{0j}] &= \theta_j E[B_0], \\ V[B_{0j}] &= \theta_j(1-\theta_j)E[B_0] + \theta_j^2 V[B_0], \text{ for } j > 0. \end{aligned} \tag{12}$$

We obtain the variance by using the conditional variance formula. Now, we approximate each $B_{0j} + D_j$ by a normal distribution with mean and variance,

$$\begin{aligned} \hat{\mu}_j &= E[B_{0j}] + L_j\lambda_j \\ \hat{\sigma}_j^2 &= V[B_{0j}] + L_j\lambda_j. \end{aligned}$$

Recall that B_{0j} and D_j are independent. The normal approximation for retailer j yields

$$\begin{aligned} E[B_j] &= \Phi^1(z_j)\hat{\sigma}_j, \\ E[I_j] &= \Phi^1(-z_j)\hat{\sigma}_j, \end{aligned}$$

where $z_j = (s_j - \hat{\mu}_j)/\hat{\sigma}_j$. We now have all the elements needed to evaluate the average cost c. Furthermore, if we set s_j optimally given s_0, that is $s_j^*(s_0) = \hat{\mu}_j + z_j^*\hat{\sigma}_j$ then the average cost for retailer j is

$$c_j(s_0, s_j^*(s_0)) = (b_j + h_j)\phi(z_j^*)\hat{\sigma}_j,$$

where z_j^* solves $\Phi^0(z) = h_j/(b_j+h_j)$. Notice also that the cost function depends on s_0 through $\hat{\sigma}_j$. The total average cost in equation (7), therefore, reduces to a function of one variable.

$$\min_{s_0} c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) = \min_{s_0}\{h_0 E[I_0] + \sum_{j>0}(b_j + h_j)\phi(z_j^*)\hat{\sigma}_j\}. \tag{13}$$

This function is not convex, in general. However, one can show that it has a unique local minimum. Thus, it is easy to optimize numerically. Once the optimal base stock levels are obtained, we truncate the fractional values.

Next, we apply the normal approximation to the RD heuristic and provide approximations to the bounds in Proposition 1. The resulting approximate bounds are in closed form and they provide insight into the system's performance. To do so, we approximate D_j and $B_{0j} + D_j$ by normal distributions with mean and variance $L_j\lambda_j$ and $(L_0 + L_j)\lambda_j$, respectively. We have

$$\sum_{j>0}(h_j + b_j)\phi(z_j^*)\sqrt{L_j\lambda_j} \le c^* \le \min\{\sum_{j>0}(h_j + b_j)\phi(z_j^*)\sqrt{(L_0 + L_j)\lambda_j}, \sum_{j\ge 0}(h_j + b_j)\phi(z_j^*)\sqrt{L_j\lambda_j}\}$$

To gain some insights into the system performance under local control, we consider the normal approximation for identical retailers having parameters λ, h, b and L. Notice that $\theta_j = \frac{1}{J}$. Notice first that from equation (12), we have $\hat{\sigma}_j = \sqrt{V[B_{0j} + D_j]} =$

$\sqrt{V[B_0] + (J-1)E[B_0] + J^2\lambda L}/J$. From equation (13), the total average cost is

$$c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0)) = h_0 E[I_0] + (b+h)\phi(z^*)\sqrt{V[B_0] + (J-1)E[B_0] + J^2\lambda L},$$

where z^* solves $\Phi^0(z) = h/(b+h)$. Let's examine the joint impact of s_0 and the number of retailers. First notice that when $s_0 = 0$. If we fix the total mean demand λ_0 then the average cost is $(b+h)\phi(z^*)\sqrt{J(L_0+L)\lambda_0}$, which is proportional to $\sqrt{J}$. But increasing s_0 increases $E[I_0]$ and reduces $V[B_0]$, whose effects are independent of the number of retailers. On the other hand, increasing s_0 reduces $E[B_0]$ and this has a coefficient of J. This suggests that the overall cost depends roughly on the square root of the number of retailers, but the strength of this dependence declines with s_0.

Hence base stock at the warehouse level has two purposes: (1) it serves to pool some of the retailers' demand uncertainty; this inventory can be used to fill an order from *any* retailer and (2) it may be cheaper to carry inventory at the warehouse than at the retailer. But notice that under local control the manager could not fully utilize the benefit of risk pooling. The warehouse manager lacks the authority to take corrective action even if he observes that some retailers need more inventory than the retailer that is scheduled to receive an order according to the first-in-first-serve rule.

3.3.2 Maximal Approximation (MX)

Using Gallego and Moon's (1993) distribution free bound, we provide a closed-form expression for the base stock levels.

$$s_j^m = L_j\lambda_j + \frac{1}{2}\sqrt{L_j\lambda_j}\left(\sqrt{\frac{b_j}{h_j}} - \sqrt{\frac{h_j}{b_j}}\right) \text{ for } j \geq 0.$$

We also truncate any fractional base stock level. These base stock levels are optimal against the worst possible distribution with mean and variance equal to $L_j\lambda_j$ and the corresponding closed-form upper bound on the cost of the optimal policy is

$$c^* \leq \sum_{j\geq 0} \sqrt{h_j b_j}\sqrt{L_j\lambda_j}.$$

Notice that a better approximation is to set the retailers' base stock level to $\max(s_j^*(\infty), s_j^m)$ and the warehouse's base stock level to $\min(s_0^u, s_0^m)$. Nevertheless, in our numerical study we investigate the stand alone performance of s_j^m.

4 Central Control with Echelon Base Stock Policy

Notice that the local control mechanism relies on the history rather than the current status of the system to replenish the retailers' request. The warehouse fills the orders sequentially; that is, on a first-come-first-serve basis. Consider what happens when the warehouse runs out of stock in a system with identical retailers. Suppose the first demand is observed at retailer j while the next five demands are observed at retailer k. At this point the warehouse receives a unit from its supplier. The warehouse allocates this unit to retailer j under local control even though retailer k needs this unit more. It is here where we expect that a central control mechanism to be beneficial. Under central control a unique decision maker collects the information about the distribution system and decides on (1) how much to *order* from an outside supplier to replenish the warehouse inventory, (2) how much to *withdraw* from the warehouse and (3) how to *allocate* the withdrawn quantity to the retailers who then satisfy the random demand.

4.1 The Analysis

Under any policy the average cost is given by equation (2). There is nothing that one can do at time t to affect retailer j's *on-hand* inventory before time $t+L_j$. Hence the system manager should protect the retailer against the demand uncertainty faced during the replenishment leadtimes. From this observation, we have $IN_j = ITP_j - D_j$. Below we re-write the average cost in equation (2) and define C_j for an easy reference.

$$C = H_0 EIN_0 + \sum_{j>0} C_j(ITP_j), \tag{14}$$

$$C_j(y) = H_j E[y - D_j] + (b_j + h_j)E[y - D_j]^-. \tag{15}$$

The form of the optimal policy that minimizes this above cost function under centralized control is unknown. We use C^* to denote the optimal cost under *central* control. Note that the above average cost includes the shipments in transit. During our numerical study we subtract the in-transit holding cost $h_0 \sum_{j>0} E[IT_j]$ from equation (14) to be consistent with local control case. Under any policy the shipment quantity to the retailers should be nonnegative and it should not exceed what is available at the warehouse. In other words $ITP_j \geq ITP_j^-$ and $ITP_r - \sum_{j>0} ITP_j^- \leq I_0^-$. We present two heuristics and an approximation to solve this problem.

4.2 Heuristics

4.2.1 Relaxation-Based (RB) Heuristic

Using a *relaxation* approach we construct a lower bound to the average cost in (14). Based on this bound we propose a heuristic that yields a feasible solution and, hence, an upper bound to the true optimal average cost. Imagine that at any time we can ship any amount to the retailers as long as the sum of inventory transit positions is equal to whatever is available at the warehouse. We keep the constraint $\sum_j ITP_j = ITP_r$ but relax $ITP_j \geq ITP_j^-$. Under this scenario, since we can shift stock among the retailers at any time in the future, we can focus solely on the current cost rate. In other words, warehouse can always allocate to achieve equal fractile at the retailers. The cost rate under this relaxation is

$$H_0 IN_0(t) + C_r(ITP_r),$$

where $C_r(ITP_r) = \{\min_{(y_1,\ldots,y_J)} \sum_{j>0} C_j(y_j) \text{ s.t. } \sum y_j = ITP_r\}$. A recursive algorithm similar to the series system (as in Chen and Zheng 1994 or Gallego and Zipkin 1999) solves this problem:

$$\begin{aligned} C_r(x) &= \{\min_{(y_1,\ldots,y_J)} \sum_{j>0} C_j(y_j) \text{ s.t. } \sum y_j = x\} \\ S_r &= \min\{y : C_r(y) \leq C_r(x) \text{ for all } x \neq y\} \\ C_0(y) &= H_0 E(y - D_0) + EC_r(\min\{y - D_0, S_r\}) \\ S_{0,r} &= \min\{y : C_0(y) \leq C_0(x) \text{ for all } x \neq y\} \end{aligned} \tag{16}$$

Notice that $C_r(x)$ is convex and has a finite minimizer. Let s_j^* be the minimizer of $C_j(y)$, then the minimizer of $C_r(\cdot)$ is given by $S_r = \sum_{j>0} s_j^*$. Note that $C_0(S_{0,r})$ is also a lower bound to C^* since it is obtained by relaxing the constraint set $ITP_j \geq ITP_j^-$.

We use this lower bound to decide on how much to *order* from an outside supplier and how much to *withdraw* from the warehouse. In particular, the system manager orders from outside supplier to bring the system wide inventory-transit position ITP_0 up to $S_{0,r}$ whenever it is below $S_{0,r}$. Next, she withdraws (as much inventory as available) from the warehouse inventory to bring ITP_r up to S_r whenever it is below S_r. Finally, she *allocates* the withdrawn quantity to the retailers based on the solution of a problem similar to $C_r(x)$ in equation (16) but with the additional constraint set $y_j \geq ITP_j^-$.

In our numerical study, we use a greedy algorithm to solve this allocation problem. We start with $y_j = ITP_j^-$ and allocate one unit at a time to j-th retailer that has the smallest current value of the first difference, (that is, $\min_j\{\Delta C_j(y_j)\}$) until all withdrawn quantity is allocated.

We mention in passing that another approach for obtaining a lower bound is based on *aggregation*. Note that

$$\begin{aligned} C &= H_0 IN_0 + \sum_{j>0}\{H_j E[ITP_j - D_j] + (b_j + h_j)[ITP_j - D_j]^-\} \\ &\geq H_0 IN_0 + \min_{j>0}\{H_j\} E[\sum_{j>0} ITP_j - \sum_{j>0} D_j] + \min_{j>0}\{b_j + h_j\} \sum_{j>0}[ITP_j - D_j]^- \\ &\geq H_0[ITP_0 - D_0] + H_a E[ITP_r - D_a] + (b_a + h_a)[ITP_r - D_a]^- \end{aligned}$$

where $H_a = \min_{j>0}\{H_j\}$, $b_a = \min_{j>0}\{b_j\}$, $h_a = \min_{j>0}\{h_j\}$ and $D_a = \sum_{j>0} D_j$. The last inequality is due to $\sum_j \max(0, -x_j) \geq \max(0, -\sum_j x_j)$. D_0 is the leadtime demand observed at the warehouse. This is precisely the cost rate of a two stage serial system in which the leadtime demand is D_0 at the upstream stage and D_a at the downstream stage. The second stage can be interpreted as an aggregate retailer that is facing aggregate leadtime demand D_a. The optimal policy can be found through solving a serial system as in equation (16) to obtain echelon base stock levels S_a and $S_{0,a}$. Through our numerical study, we observed that the performance of this lower bound together with the warehous allocation policy described above was inferior to that of the relaxation based heuristic.

4.2.2 Direct-Search (DS) Heuristic

One can obtain the best echelon base stock levels (S_0^*, S_r^*) by enumeration. For all possible values of (S_0, S_r) together with the allocation policy described in § 4.2.1, one can simulate the system and chose the policy that yields the smallest cost. Next, we provide bounds for the best echelon base stock levels to limit our search.

To obtain these bounds we first ignore the risk pooling effect by restricting the warehouse to maintain separate safety stock for each retailer. In this case, the warehouse manager orders from an outside supplier to replenish the stock that serves as a buffer for retailer j. This restriction *decomposes* the distribution system into J independent 2-Stage serial systems and hence ignores the risk pooling effect. Let

IN_{0j}	:	Echelon net inventory at the warehouse for retailer j
ITP_{0j}	:	Echelon inventory transit position at the warehouse for retailer j.

It is well known that the echelon base stock policy is optimal for a serial system (Clark and Scarf 1960). The optimal echelon base stock levels can be found through solving the following recursive optimization:

$$\begin{aligned}
C_j(y) &= E\{H_j[y - D_j] + (h_j + b_j)[y - D_j]^-\} \\
S_j &= \min\{y : C_j(y) \leq C_j(x) \text{ for all } x \neq y\} \\
C_{0j}(y) &= E\{H_0(y - D_{0j}) + C_j(\min[y - D_{0j}, S_j])\} \\
S_{0j} &= \min\{y : C_{0j}(y) \leq C_{0j}(x) \text{ for all } x \neq y\},
\end{aligned} \tag{17}$$

where D_{0j} is the leadtime demand, Poisson with mean $\lambda_j L_0$, at the warehouse due to retailer j. The optimal system wide average cost under a decomposed warehouse mechanism is $\sum_{j>0} C_{0j}(S_{0j})$. Notice that this is an upper bound to the true optimal cost C^* since it is obtained by restricting the policy space. Similarly $\sum_{j>0} S_{0j}$ is an upper bound to the optimal echelon base stock level S_0^* since this would ignore the gains from risk pooling. (This assumes that echelon safety stocks $S_{0j} - \lambda(L_o + L_j)$ are positive for each j). Also $\sum_{j>0} S_j$ is a lower bound to the optimal echelon base stock level S_r^* since it ignores the stock out possibility at the warehouse due to other retailers' orders. Now consider restricting the warehouse not to hold any inventory. In this case, the system manager has to protect each retailer against the demand that would occur during $L_0 + L_j$ periods. For each retailer the manager faces the problem of minimizing the newsvendor cost with demand $D_{0j} + D_j$. Let S_j^u be the minimizer of this newsvendor cost. Notice that this is an upper bound on the true $S_r^* \leq \sum_{j>0} S_j^u$.

Proposition 3 *We have* $0 \leq S_0^* \leq \sum_{j>0} S_{0j}$ *and* $\sum_{j>0} S_j \leq S_r^* \leq \sum_{j>0} S_j^u$.

4.3 An Approximation: Normal Approximation (NA)

Next, we provide a closed-form expression for the cost function $C_r(x)$ defined in equation (16), an approximation to solve the central control problem and a closed form lower bound for the cost of every policy. This normal approximation to the distribution system is also discussed in Zipkin (2000). Under identical retailer assumption and when we approximate the leadtime demand D_j by a Normal distribution with mean and variance equal to $\lambda_j L_j$, the second stage cost function in equation (16) simplifies to $C_r(x) = \sum_{j>0} C_j(y_j^*) = (b+h)\phi(z) \sum_{j>0} \sqrt{\lambda_j L_j}$, or equivalently to $C_r(y) = E[H(y - D_r) + (b+h)(y - D_r)^-]$, where D_r is normal with mean $\mu_r = \sum_{j>0} \lambda_j L_j$ and standard deviation $\sigma_r = \sum_{j>0} \sqrt{\lambda_j L_j}$. If the costs are not identical we set h_j and p_j equal to the minimum of all holding and penalty costs, respectively. Together with the warehouse the retailer constitute a serial system. Hence, a similar recursive algorithm as in equation (16) can be used to obtain S_N and $S_{0,N}$.

Next suppose that $h_0 = h$, the local holding cost at the warehouse and at the downstream representative retailer are equal. For this case, there is no incentive to carry inventory at the warehouse instead one can ship all the inventories to the representative retailer. Then, the solution to this two-stage system has $S_{0,N} = 0$. The system reduces to a single-stage system with leadtime demand $\tilde{D} = D_0 + D_r$, having mean $\tilde{\mu} = \sum_{j\geq 0} \lambda_j L_j$ and variance $\tilde{\sigma}^2 = \lambda_0 L_0 + (\sum_{j>0} \sqrt{\lambda_j L_j})^2$. Thus, the optimal cost becomes

$$(b+h)\phi(z^*)\tilde{\sigma}, \tag{18}$$

where z^* solves $\Phi(z) = b/(b+h)$. Note that this is a lower bound, up to the normal approximation, on the cost of every policy, not just base-stock policies. In the case of identical retailers, with $\lambda_j = \lambda$ and $L_j = L$ for $j > 0$, $\tilde{\mu} = J\lambda(L_0 + L)$ and $\tilde{\sigma}^2 = J\lambda(L_0 + JL)$. Thus, for fixed total leadtime $L_0 + L$, $\tilde{\sigma}$ and hence the cost estimate decline as L_0 increases and L decreases.

5 Numerical Results

This section presents first the performance of the suggested heuristics and approximations. Next, we provide managerial insights into stock positioning issues in a distribution system under both local and central control. We conclude by illustrating the value of centralization.

5.1 Performance Measures

For the local control case, we compare the exact solution, which is based on the projection algorithm of section 3.1.2, to the heuristics. We measure the percentage error $\epsilon_i\% = \frac{c_i - c^*}{c^*}$ for $i \in \{\text{RD, NA, MX}\}$. A small percentage indicates that the heuristic is close to optimal.

For the central control case, we measure the difference between the lower bound and the upper bound $\epsilon_i\% = \frac{UB_i - LB}{LB}$ for $i \in \{RB, NA, DS\}$. To estimate the cost of the upper bounds, we use discrete event system simulation and simulate the distribution system under the proposed heuristics. We run several replications to have a 95% confidence interval. Notice that a small ϵ indicates that both the heuristic is close to optimal *and* the lower bound is accurate.

For all heuristics and the exact algorithm, we carry out a total of 750 experiments, the details of which is described in the following subsection. We report the average and the standard deviation for the above measures as well as the computational time required to solve them. For some problem instances, we also report the specific details.

For approximations, we are also interested in whether they accurately predict the performance of the system. In other words, we investigate whether a change in any of the parameters, such as leadtimes, would change the system cost under these approximations in the same proportion as the cost under optimal policy. To do so, we carry out regression analysis.

5.2 Experiments: Parameters and Structures

We study two sets of experiments. The first experiment covers identical retailers scenario. The second experiment covers non-identical retailers and some parameters are generated randomly.

The first experiment includes a total of 144 cases with local holding cost $h_j = 1$ for $j > 0$. Within this group, we fix $L_0 = L_j \in \{0.10, 0.25\}$ and consider 96 instances with $J \in \{2, 4, 8, 16, 32, 64\}$; $\lambda_0 \in \{16, 64\}$; $b_j \in \{9, 39\}$ that corresponds to fill rates of 90% and 97.5%; $h_0 \in \{0.3, 0.9\}$ that corresponds to adding 30% and 90% of the value to the item at the warehouse. In the remaining 48 instances we consider distribution systems with $L_0 \in \{0.1, 0.9\}$ with $L_j = 1 - L_0$ for $j > 0$ so as to address systems with different degrees of risk pooling at the warehouse.

The second experiment includes a total of 200 cases and allows for non-identical retailers with randomly drawn parameters. In particular, we set $h_0 = 0.3, h_j = 1$, $J = \{2, 4, 8, 16, 32\}$, $L_0 = \{0.1, 0.25\}$ and $\lambda_0 = \{16, 32\}$. The parameters, b_j and L_j are generated randomly, according to independent uniform distributions. For each retailer j, we draw b_j randomly from a uniform distribution with parameters $[9, 39]$ and L_j from $[0.1, 0.25]$. We consider all possible combination, a total of 20 cases. For each case, we generate ten independent replications, for a total of 200 problem instances. Note that unequal leadtimes have the same effect as unequal demand rates. Similarly, unequal penalty costs mean unequal holding to penalty cost ratio at each retailer.

5.3 Performance: Local Control Case

For the first experiment, the RD heuristic's average error term is 1.17% and its standard deviation is 1.48%. In Table 1, we summarize some statistics for the RD heuristic. The

Table 1: Summary statistics for RD Heuristic when retailers are identical

	# Cases	Average	σ_{RD}		# Cases	Average	σ_{RD}
$J = 2$	24	1.90%	1.79%	$\lambda_0 = 16$	96	1.27%	1.71%
$J = 64$	24	0.22%	0.31%	$\lambda_0 = 64$	48	0.96%	0.82%
$b = 9$	72	0.93%	1.24%	$h_0 = 0.3$	72	1.24%	1.36%
$b = 39$	72	1.40%	1.66%	$h_0 = 0.9$	72	1.09%	1.59%

Table 2: Summary statistics for RD heuristic when retailers are nonidentical

	Average	σ_{RD}		Average	σ_{RD}		Average	σ_{RD}
$J = 2$	1.90%	1.57%	$\lambda_0 = 16$	1.67%	1.01%	$L_0 = 0.10$	0.98%	0.78%
$J = 32$	0.84%	0.26%	$\lambda_0 = 32$	1.40%	1.06%	$L_0 = 0.25$	2.07%	0.97%

performance of the RD heuristic is better for a distribution system with shorter warehouse leadtime L_0, lower penalty cost b and higher retailer demand rate λ_j. The heuristic is relatively less sensitive to the changes in the warehouse's holding cost. These observations are also supported by the asymptotic optimality of the RD heuristic. From the proof of Proposition 2, RD heuristic converges to the optimal solution faster when L_0, b, λ_j are smaller. Everything else being equal, increasing the number of retailers reduces the error term for RD heuristic, but the rate of reduction is problem specific.

The performance of the RD heuristic is better when the warhouse's leadtime is shorter than the retailer leadtimes. For example, see Table 5, the average error term is 2.43% when $L_0 = 0.9, L_j = 0.1$ and it is 0.20% when $L_0 = 0.1, L_j = 0.9$. When the warehouse leadtime is longer, the system gains more from risk pooling. The RD heuristic takes advantage of risk pooling relatively less than the optimal algorithm.

The observations are similar under the second experiment for which the retailers are nonidentical. The RD heuristic's average error is 1.53% and its standard deviation is 1.04%. In Table 2, we report some statistics for the RD heuristic. Among 200 cases, the ZS sub-heuristic yields the lowest cost in 144 cases and the SP sub-heuristic yields the lowest cost in the remaining 46 cases. The CD sub-heuristic is not cheaper because the warehouse leadtime is not significantly shorter than the rest of the retailer leadtimes.

In summary, the RD heuristic performs very well because its sub-heuristics complement each other. In particular, the cross docking sub-heuristic performs better than the other two sub-heuristic when the warehouse leadtime is significantly shorter than the retailers leadtimes and most of the value add, (the holding cost) is done (incurred) by the retailers. The stock-pooling sub-heuristic performs better when the warehouse leadtime is long relative to retailer leadtimes and the holding cost is low at the warehouse. The zero safety stock sub-heuristic performs better when the warehouse leadtime and holding cost are similar to those for the retailers (see Tables 5 and 6).

The error terms for the normal and maximal approximations are large for both experiments. The average absolute error term based on the first experiment is 20.97% for the normal approximation and it is 30.2% for the Maximal approximation. As expected Normal approximation works poorly whenever $L_j\lambda_j$ are small. We use approximations, not to describe the system dynamics in detail but rather to predict performance. Hence, we carry out a number of regressions of the normal and maximal approximations to the actual cost for different parameter values and report the R^2 in Table 7. We fix all parameters except the parameter for which we investigate its effect on the optimal cost and the approximations. Note that R^2 is close to 1 for all factors. This suggests that both NA and MX approximations can safely be used to investigate the impact of any parameter changes on the cost of managing a distribution system.

The computational time required for the exact algorithm of § 3.1.2 is increasing in the number of retailers, warehouse leadtime, and the total demand rate as observed by the warehouse. In particular, exact algorithm requires one to compute on the order of J many convolutions for a given base stock level s_0. The upper bound s_o^u is also proportional to J. Hence, the computational requirement for the optimal algorithm is $O(J^2)$. For the RD heuristic, we compute one convolution per retailer only for the ZS sub-heuristic. The other sub-heuristics do not require us to compute any convolutions. Hence, the RD heuristic's computational complexity is O(J).

5.4 Performance: Central Control Case

For the first experiment, the average error term and the standard deviation are 12.86%, 25.18% for the RB heuristic; 18.99%, 20.84% for the normal approximation (NA); and 2.77%, 3.20% for the DS heuristic. The heuristics are better when the number of retailers in the system are less than 32. In this case, the average error term and the standard deviation are 7.75%, 10.33% for the RB heuristic; 11.89%, 11.60% for the NA; and 3.07%, 3.23%for the

Table 3: Summary statistics for heuristics when retailers are identical and $J < 32$

	RB	NA	DS		RB	NA	DS
$J = 2$	2.73%	4.89 %	1.59%	$\lambda_0 = 16$	10.03%	14.18%	3.22%
$J = 16$	13.50%	20.43%	2.93%	$\lambda_0 = 64$	4.33%	7.85%	1.98%
$b = 9$	8.31%	7.98%	3.11%	$h_0 = 0.3$	3.41%	9.89%	1.32%
$b = 39$	6.77%	15.54%	2.44%	$h_0 = 0.9$	11.68%	13.63%	4.24%

Table 4: Summary statistics when retailers are nonidentical

	RB	NA	DS		RB	NA	DS		RB	NA	DS
$J = 2$	1.80%	6.08 %	1.28%	$\lambda_0 = 16$	6.20%	25.40%	2.12%	$L_0 = 0.10$	7.03%	20.39%	1.98
$J = 16$	8.5%	28.8%	1.81%	$\lambda_0 = 32$	6.46%	17.18%	1.81%	$L_0 = 0.25$	7.13%	19.11%	3.05

DS heuristic. In Table 3, we summarize our numerical results for $J < 32$. The RB heuristic performs better under low holding cost and shorter leadtime at the warehouse and larger penalty cost at the retailers. For example, the average error term for the RB heuristic is 0.99% when $h_0 = 0.3, b = 39, L_0 < 0.25, J < 32$ (there are 12 such cases) whereas it is 10.71% when $h_0 = 0.9, b = 9, L_0 \geq 0.25, J < 32$ (there are 12 such cases). In Table 8, we report some of the problem instances.

For the second experiment (nonidentical retailer case), the average error term and the standard deviation are 6.97%, 4.54% for the RB heuristic; 20.01%, 14.70% for the normal approximation (NA); and 1.78%, 1.12% for the DS heuristic. In Table 4, we report some statistics for the second experiment.

The computational time for each heuristic is proportional with the length of the simulation horizon and the confidence interval (or equivalently the number of replications). Hence, for a given simulation horizon and the number of replication the time required for RB and NA heuristics are quite similar. In each case, we simulate the system for a given echelon base stock level at the warehouse and at the aggregate retailer. The longest time required to solve the RB or NA heuristics among all problem instances was less than one minute with a Pentium III processor. However, the computational time required for the DS heuristic is longer because we simulate the system as many times as the size of the upper bound for the warehouse's echelon base stock level, that is $\sum_{j>0} S_{0j}$. The longest time among all problem instances to solve the DS heuristic was less than fifteen minutes.

5.5 Insights

We describe our insights through numerical studies in three subgroups. The first group of observations is based on the local control case, the second group is based on the central control case. Finally, we conclude by comparing the two control strategies.

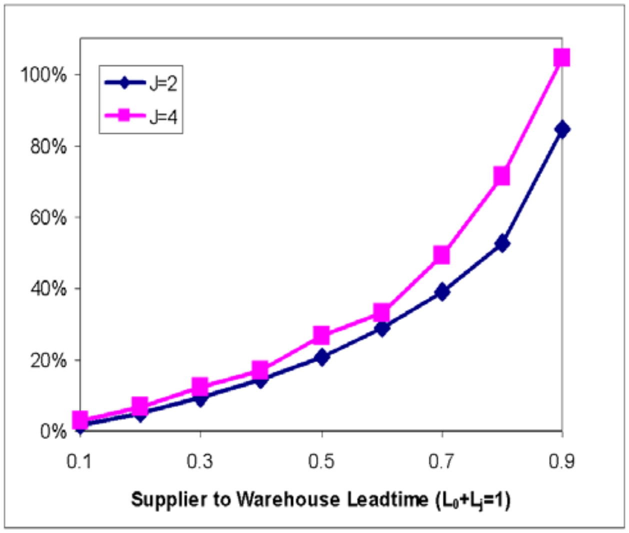
5.5.1 Cost of Cross Docking under Local Control

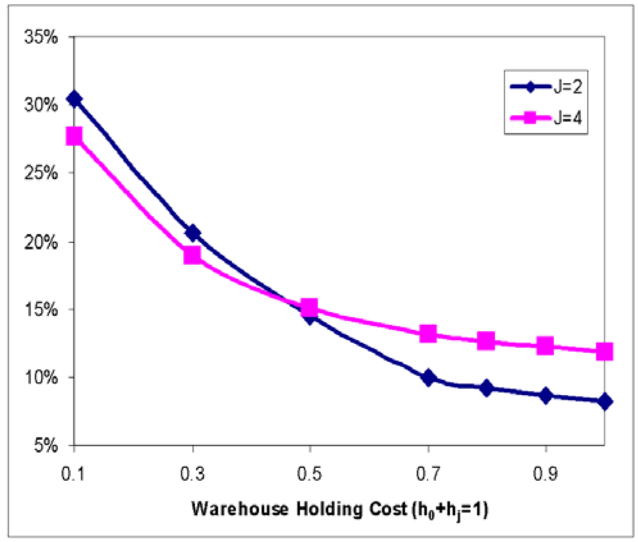
Now consider the special case where the warehouse cannot or does not hold inventory. We investigate how a cross-docking strategy affects the system performance under *local* control. In this case, the retailers pull inventory from the warehouse which carries zero inventory and replenishes the orders in a first come first served bases. The CD sub-heuristic outlined in section 3.2 is a stylized model for this strategy. Hence, the impact of a cross-docking strategy on inventory related costs can be measured by the percentage difference in costs between this sub-heuristic and the optimal policy. The percent difference can be interpreted as the potential savings forgone due to not risk pooling. In Figure 1 (a), we fix $L_0+L_j = 1$ and change L_0. We observe that the percentage difference increases with warehouse's replenishment leadtime and the number of retailers. The system sacrifices larger gains due to risk pooling by not carrying inventory at the warehouse. On the other hand, an increase in warehouse holding cost reduces the percent difference as illustrated in Figure 1 (b). Note that even when the retailer holding costs are zero, the system gains from holding inventory at the warehouse. An approximation for this percentage difference is given by the difference between the CD sub-heuristic and the minimum of SP or CD sub-heuristics. Both of these sub-heuristics are newsvendor type calculations which makes it easy to carry out such analysis on a simple spread sheet. In conclusion, cross-docking may be a viable strategy when it is too expensive to carry inventory at the warehouse, the penalty cost is high at the retailers and the warehouse leadtime is short relative to the retailer leadtimes.

5.5.2 Value of Safety Stock under Local Control

To investigate the value of safety stock at the warehouse under local control, we plot the percentage difference between the ZS sub-heuristic and the optimal solution with respect to supplier-to-warehouse leadtime and holding cost in Figure 2 (a) and (b), respectively. The value of safety stock increases with an increase in the leadtime; that is, the longer the warehouse waits for the its order, the more valuable is the safety stock at the warehouse. Comparing this figure with Figure 1 (a), we observe that a large portion of risk pooling

Figure 1: Cost of Cross-docking under Local Control when $\lambda_0 = 16, b = 9$ for $J = 2, 4, 8$ w.r.t. (a) Warehouse Lead Time when $L_0 + L_j = 1, h_j = 1$ (b) Warehouse Holding Cost when $L_0 = L_j = 0.25$



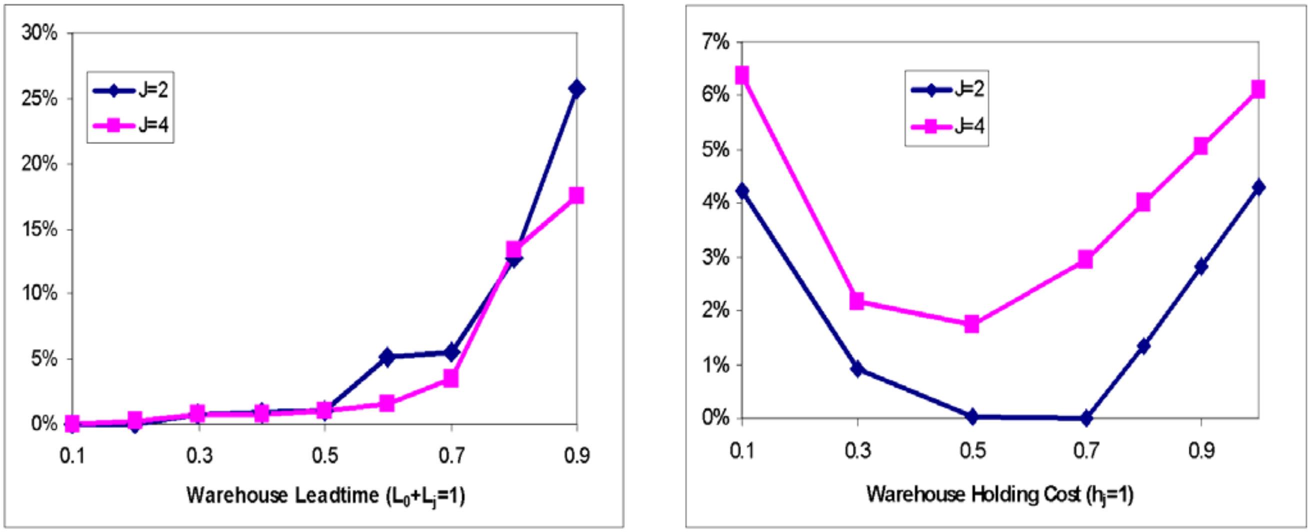


benefit could be materialized by holding ED_0 units at the warehouse. In Figure 2 (b), we observe that the safety stock value is convex with respect to the warehouse holding cost. The convexity is intuitive. At very low levels of warehouse holding cost it is optimal to carry significant safety stocks and there is a large penalty for not doing so. As we increase the holding cost rate at the warehouse it is optimal to carry less safety stock so the benefit of doing so deminishes. When the warehouse holding cost rate is very high it is optimal to carry negative safety stocks so it becomes expensive to carry zero safety stocks.

5.5.3 Cost of Cross-Docking under Central Control

To investigate the cost of cross-docking under *central* control, we first obtain the best echelon base stock levels (S_0^*, S_r^*) and the resulting average cost using the direct search heuristic described in Section 4.2.2 for each problem instances. Next, we force the warehouse echelon base stock level $S_r = S_0^*$ and calculate the resulting cost C. This corresponds to carrying zero inventory at the warehouse and shifting all the inventory to the retailers. In Figure 3, we plot the percentage difference in cost between the two. Similar to our observations for the local control case, here the cost of cross docking is also higher the longer the warehouse leadtime and the smaller the warehouse holding cost are. Note also that one can approximate the above analysis by using the lower bound, which is simpler to calculate, instead of the

Figure 2: Value of Safety Stock under Local Control when $\lambda_0 = 16, b = 9$ for $J = 2, 4$ w.r.t. (a) Warehouse Lead Time when $L_0 + L_j = 1, h_j = 1$ (b) Warehouse Holding Cost when $L_0 = L_j = 0.25$



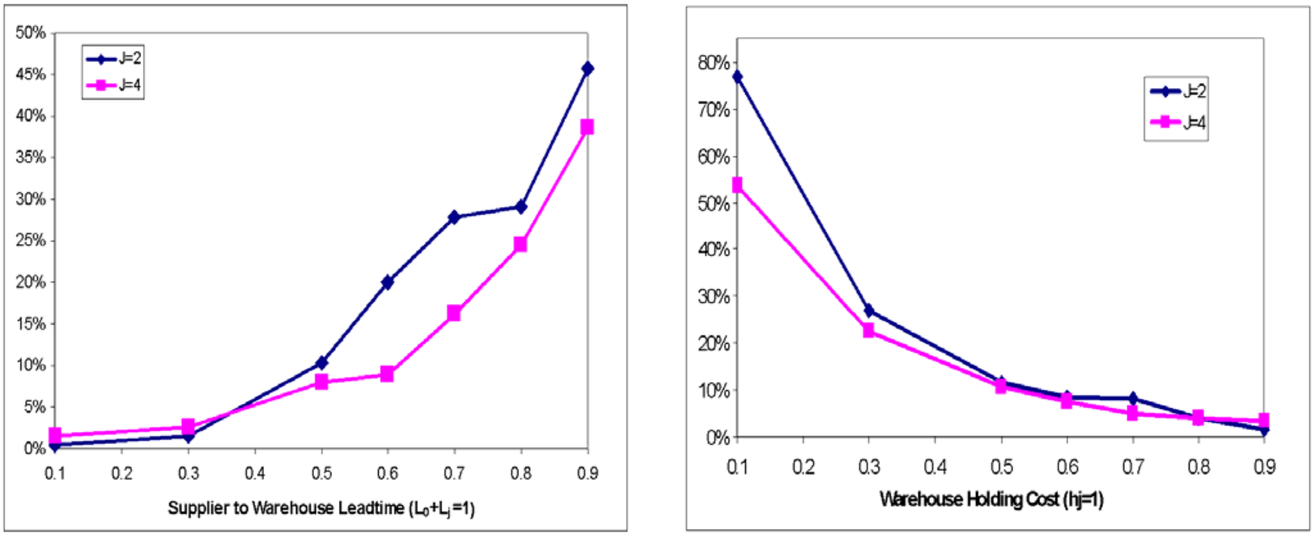
upper bound.

5.5.4 Local versus Central Control

To quantify the benefit of central control, we first obtain the optimal local base stock levels s^* and the average cost c^* for each problem instances and by using the optimal algorithm in Section 3.1.2. Next, we set the echelon base stock levels to $S_r = \sum_{j=1}^{J} s_j^*$ and $S_0 = s_0^* + S_r$ and simulate the system under central control to obtain the average cost C. The percentage difference between the two (that is, $\frac{c^*-C}{c^*}$) gives us the value of central control that is, the cost reduction due to centralization. Note that one can compare the cost of the optimal local control with that of the *best* echelon base stock policy obtained by direct-search heuristic (instead of C above). This results in a percentage difference that is even higher than what we report here (hence a larger gain due to centralization). We report the lower ratio to address only the value of centralization and not the value gained by a better heuristic.

In Table 10, we report some results for a system with $L_0 + L_j = 1, h_j = 1$ and $\lambda_0 = 16$. We observe that the value of central control increases with an increase in leadtime and holding cost at the warehouse and the penalty cost at the retailers. For example, for a two retailer distribution system with $\lambda_0 = 16, h_0 = 0.9, L_0 = 0.9$ the optimal local base stock levels are $(s_0^*, s_j^*) = (20, 5)$ and the optimal average cost is $c^* = 15.28$. If one uses a central

Figure 3: Cost of Cross-docking under Central Control when $\lambda_0 = 16, b = 9$ for $J = 2, 4$ w.r.t. (a) Warehouse Lead Time when $L_0 + L_j = 1, h_j = 1$ (b) Warehouse Holding Cost when $L_0 = 0.8, L_j = 0.2$



control with echelon base stock levels $(S_0, S_r) = (30, 10)$, the average cost under this policy is $C = 14.22 \pm 0.049$. This results in a cost reduction of $6.93\% = (\frac{15.28-14.22}{15.28})$. Note also that increasing the number of retailers in the system does not change $\lambda_0 = 16$. Hence the larger the number of retailers, the smaller is the mean demand at each retailer. In this case the warehouse seldom runs out of stock. When this is the case, the local control is equivalent to central control. The two control differs only when the warehouse runs out of stock. Recall that the local control allocates using the FIFO rule, while the centralized policy uses an allocation policy based on individual retailer's inventory and cost information. These observations suggest that local control can be used safely when (1) the warehouse holding cost is low, (2) the penalty cost is low, (3) the demand rate seen by the retailers are small and (4) the warehouse leadtime is shorter relative to the retailer leadtimes.

6 Conclusions

We established simple bounds, heuristics and approximation for distribution systems under local and central control. We show that the RD heuristic is asymptotically optimal as the number of retailers increases. The ZS subheuristic, while simple, is fairly effective, except for the cases where the leadtime for the warehouse is significantly larger than the leadtime to the retailers. The RD heuristic provides a powerful tool to manage distribution systems.

The normal and maximal approximations are easy to compute and provide a mechanism to perform sensitivity analysis. The thorny problem of managing a distribution system under central control is approached through a relaxation scheme and a decomposition scheme. Our numerical studies shed light into the value of risk-pooling and into the value of centralized control. Our model also enables one to quantify the value of centralizing control by comparing the average holding and penalty cost under each control.

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Table 5: Performance of RD Heuristic when Retailers are Identical

Parameters		Optimal		CD-Subheuristic		SP-Subheuristic		ZS-Subheuristic		
	J	s_0/s_j	c^*	s_j	c_{CD}	s_0^u/s_j^l	c_{SP}	s_0/s_j	c_{ZS}	RD %
$L_0 = 0.1$	2	2/11	10.40	12	10.60	4/11	11.09	2/11	10.40	0.00
$L_j = 0.9$	4	2/6	14.92	7	15.39	4/6	15.47	2/6	14.92	0.00
$b = 9$	8	1/4	21.83	4	22.01	4/4	22.62	2/4	21.88	0.22
$\lambda_0 = 16$	16	3/2	30.85	2	32.58	4/2	31.26	2/2	30.92	0.22
$H_0 = 0.3$	32	3/1	46.39	1	50.09	4/1	46.65	2/1	46.75	0.14
	64	2/1	65.22	1	66.43	4/1	65.66	2/1	65.22	0.00
$L_0 = 0.1$	2	1/11	10.57	12	10.60	3/11	12.43	2/11	10.84	0.28
$L_j = 0.9$	4	1/6	15.26	7	15.39	3/6	16.81	2/6	15.35	0.59
$b = 9$	8	1/4	21.96	4	22.01	3/4	23.96	2/4	22.32	0.26
$\lambda_0 = 16$	16	2/2	31.35	2	32.58	3/2	32.60	2/2	31.35	0.00
$H_0 = 0.9$	32	2/1	47.18	1	50.09	3/1	47.99	2/1	47.18	0.00
	64	2/1	65.66	1	66.43	3/1	67.00	2/1	65.66	0.00
$L_0 = 0.1$	2	2/13	14.29	14	14.55	5/13	15.19	2/13	14.29	0.00
$L_j = 0.9$	4	2/8	20.76	8	21.38	5/8	21.72	2/8	20.76	0.00
$b = 39$	8	2/5	30.39	5	31.20	5/5	31.29	2/5	30.39	0.00
$\lambda_0 = 16$	16	3/3	44.60	3	46.94	5/3	45.29	2/3	44.74	0.31
$H_0 = 0.3$	32	3/2	65.89	2	68.90	5/2	66.52	2/2	66.13	0.37
	64	4/1	110.76	2	117.89	5/1	111.12	2/1	112.40	0.11
$L_0 = 0.1$	2	0/14	14.55	14	14.55	5/13	17.24	2/13	14.73	0.00
$L_j = 0.9$	4	1/8	20.99	8	21.38	5/8	23.76	2/8	21.20	0.99
$b = 39$	8	1/5	30.70	5	31.20	5/5	33.33	2/5	30.83	0.42
$\lambda_0 = 16$	16	2/3	45.18	3	46.94	5/3	47.33	2/3	45.18	0.00
$H_0 = 0.9$	32	2/2	66.57	2	68.90	5/2	68.57	2/2	66.57	0.00
	64	3/1	111.96	2	117.89	5/1	113.17	2/1	112.84	0.79
$L_0 = 0.9$	2	20/2	5.74	12	10.60	22/2	6.30	15/3	7.22	4.28
$L_j = 0.1$	4	20/1	7.52	7	15.39	22/1	7.95	15/2	8.84	2.23
$b = 9$	8	19/1	9.85	4	22.01	22/1	10.64	15/1	11.50	4.25
$\lambda_0 = 16$	16	17/1	16.60	2	32.58	22/0	17.14	15/1	16.90	1.79
$H_0 = 0.3$	32	22/0	17.14	1	50.09	22/0	17.14	15/0	26.06	0.00
	64	22/0	17.14	1	66.43	22/0	17.14	15/0	26.06	0.00
$L_0 = 0.9$	2	16/3	8.10	12	10.60	20/2	10.06	15/3	8.32	2.64
$L_j = 0.1$	4	16/2	9.92	7	15.39	20/1	11.71	15/2	9.94	0.16
$b = 9$	8	17/1	12.08	4	22.01	20/1	14.39	15/1	12.60	4.17
$\lambda_0 = 16$	16	15/1	18.00	2	32.58	20/0	20.90	15/1	18.00	0.00
$H_0 = 0.9$	32	20/0	20.90	1	50.09	20/0	20.90	15/0	27.16	0.00
	64	20/0	20.90	1	66.43	20/0	20.90	15/0	27.16	0.00
$L_0 = 0.9$	2	21/3	7.79	14	14.55	24/3	8.71	15/5	10.62	5.14
$L_j = 0.1$	4	21/2	10.28	8	21.38	24/2	11.25	15/3	13.51	4.43
$b = 39$	8	22/1	15.20	5	31.20	24/1	15.84	15/2	18.03	1.43
$\lambda_0 = 16$	16	21/1	19.97	3	46.94	24/1	20.95	15/1	26.46	2.36
$H_0 = 0.3$	32	19/1	34.04	2	68.90	24/1	35.42	15/1	36.43	2.42
	64	17/1	64.72	2	117.89	24/0	65.85	15/1	65.16	0.69
$L_0 = 0.9$	2	18/4	11.16	14	14.55	22/3	14.06	15/5	11.72	4.80
$L_j = 0.1$	4	19/2	13.63	8	21.38	22/2	16.61	15/3	14.61	6.72
$b = 39$	8	17/2	18.66	5	31.20	22/1	21.20	15/2	19.13	2.46
$\lambda_0 = 16$	16	19/1	23.24	3	46.94	22/1	26.30	15/1	27.56	5.67
$H_0 = 0.9$	32	17/1	36.49	2	68.90	22/1	40.78	15/1	37.53	2.76
	64	15/1	66.26	2	117.89	22/0	71.21	15/1	66.26	0.00

Table 6: Performance of RD Heuristic when Retailers are Not Identical

Parameters			Optimal		RD Heuristic			
$J = 4$	$L_1/L_2/L_3/L_4$	$b_1/b_2/b_3/b_4$	$s_0/s_1/s_2/s_3/s_4$	$c*$	Type	$s_0/s_1/s_2/s_3/s_4$	c	% ϵ
$L_0 = 0.1$	0.12/0.16/0.14/0.18	19.99/20.12/27.90/15.90	3/2/2/2/2	8.61	ZS	2/2/2/2/2	8.92	3.59
λ=16	0.13/0.21/0.11/0.16	12.58/17.17/30.82/29.30	3/2/3/2/2	8.95	ZS	2/2/3/2/3	9.07	1.28
	0.13/0.24/0.21/0.11	34.89/37.95/32.67/19.26	3/2/3/3/2	10.06	ZS	2/2/3/3/2	10.34	2.85
	0.23/0.18/0.14/0.20	38.74/13.33/11.77/18.08	3/3/2/2/2	9.37	ZS	2/3/2/2/3	9.44	0.76
	0.13/0.17/0.22/0.14	12.18/24.30/09.78/36.90	3/2/2/2/2	9.05	ZS	2/2/3/2/3	9.16	1.26
	0.17/0.23/0.12/0.16	31.13/26.99/26.73/20.36	3/3/3/2/2	9.64	ZS	2/3/3/2/2	9.79	1.60
	0.21/0.23/0.24/0.16	29.80/31.32/24.97/15.00	3/3/3/3/2	10.26	ZS	2/3/3/3/2	10.39	1.24
	0.11/0.20/0.22/0.15	09.92/13.52/11.74/11.40	3/1/2/2/2	8.01	SP	4/1/2/2/2	8.11	1.24
	0.11/0.22/0.10/0.21	28.62/12.31/20.54/30.11	3/2/2/2/3	8.93	ZS	2/2/2/2/3	9.08	1.70
	0.19/0.25/0.13/0.15	25.13/18.84/21.28/31.24	3/3/3/2/2	9.76	ZS	2/3/3/2/3	9.85	0.99
$L_0 = 0.25$	0.23/0.25/0.14/0.19	17.22/33.28/35.9/23.12	7/3/3/2/3	11.07	ZS	5/3/3/3/3	11.20	1.15
λ=16	0.25/0.24/0.16/0.14	28.70/36.76/17.77/22.44	7/3/3/2/2	10.60	SP	9/3/3/2/2	10.98	3.61
	0.16/0.15/0.22/0.13	32.88/32.75/09.04/36.49	7/2/2/2/2	9.99	SP	9/2/2/2/2	10.27	2.78
	0.20/0.14/0.22/0.22	37.59/31.63/26.99/18.24	6/3/2/3/3	10.83	ZS	5/3/3/3/3	11.04	1.89
	0.24/0.19/0.20/0.12	25.21/18.83/17.44/33.80	7/3/2/2/2	10.29	ZS	5/3/3/3/2	10.61	3.06
	0.16/0.12/0.16/0.25	11.89/17.34/27.35/14.57	6/2/2/2/3	9.23	ZS	5/2/2/3/3	9.37	1.48
	0.19/0.23/0.16/0.11	35.41/33.89/36.21/23.16	6/3/3/3/2	10.63	ZS	5/3/3/3/2	10.85	2.13
	0.12/0.11/0.19/0.22	11.93/38.88/16.13/13.02	7/2/2/2/2	9.13	ZS	5/2/2/2/3	9.46	3.54
	0.22/0.20/0.15/0.16	31.37/10.17/10.74/11.84	6/3/2/2/2	8.93	ZS	5/3/2/2/2	9.10	1.90
	0.22/0.13/0.19/0.11	11.61/10.48/22.77/38.68	6/2/2/3/2	9.27	ZS	5/3/2/3/2	9.49	2.33
$L_0 = 0.1$	0.23/0.25/0.14/0.19	17.22/33.28/35.9/23.12	5/4/5/4/4	14.00	ZS	4/4/5/4/4	14.07	0.49
λ=32	0.25/0.24/0.16/0.14	28.70/36.76/17.77/22.44	5/5/5/3/3	13.81	ZS	4/5/5/4/3	13.99	1.30
	0.16/0.15/0.22/0.13	32.88/32.75/09.04/36.49	5/4/4/4/3	12.67	ZS	4/4/4/4/4	12.72	0.39
	0.20/0.14/0.22/0.22	37.59/31.63/26.99/18.24	4/5/4/5/4	14.17	ZS	4/5/4/5/4	14.17	0.00
	0.24/0.19/0.20/0.12	25.21/18.83/17.44/33.80	5/5/4/4/3	13.07	ZS	4/5/4/4/3	13.15	0.55
	0.16/0.12/0.16/0.25	11.89/17.34/27.35/14.57	5/3/3/4/4	11.86	ZS	4/3/3/4/4	11.90	0.40
	0.19/0.23/0.16/0.11	35.41/33.89/36.21/23.16	5/4/5/4/3	13.60	ZS	4/4/5/4/3	13.71	0.82
	0.12/0.11/0.19/0.22	11.93/38.88/16.13/13.02	5/3/3/4/4	11.57	ZS	4/3/3/4/4	11.61	0.29
	0.22/0.20/0.15/0.16	31.37/10.17/10.74/11.84	4/5/4/3/3	11.78	ZS	4/5/4/3/3	11.78	0.00
	0.22/0.13/0.19/0.11	11.61/10.48/22.77/38.68	5/4/3/4/3	11.63	ZS	4/4/3/4/3	11.68	0.40
$L_0 = 0.25$	0.20/0.18/0.23/0.18	13.64/29.78/16.32/30.78	11/4/4/4/4	13.73	ZS	9/4/4/4/4	14.22	3.62
λ=32	0.17/0.16/0.19/0.17	30.47/35.75/32.58/18.90	11/4/4/4/4	13.99	ZS	9/4/4/5/4	14.39	2.87
	0.10/0.15/0.14/0.24	22.16/20.47/31.29/37.22	11/3/3/4/5	13.48	ZS	9/3/4/4/5	13.77	2.14
	0.17/0.14/0.12/0.19	19.26/23.69/09.14/17.95	11/4/3/2/4	12.12	ZS	9/4/4/3/4	12.32	1.65
	0.14/0.14/0.25/0.18	10.23/20.14/11.60/35.76	11/3/3/4/4	12.42	ZS	9/3/3/4/4	12.95	4.26
	0.24/0.13/0.21/0.19	38.12/19.49/13.02/37.43	11/5/3/4/4	14.05	ZS	9/5/3/4/5	14.48	3.07
	0.21/0.13/0.23/0.10	16.02/20.10/25.48/25.22	11/4/3/5/3	12.80	ZS	9/4/3/5/3	13.16	2.80
	0.19/0.24/0.13/0.14	11.24/26.45/16.47/34.01	10/4/5/3/4	13.10	ZS	9/4/5/3/4	13.26	1.22
	0.16/0.14/0.20/0.17	38.21/10.35/25.18/22.25	11/4/3/4/4	12.99	ZS	9/4/3/5/4	13.37	2.88
	0.18/0.15/0.14/0.21	20.18/25.91/27.67/36.57	10/4/4/4/5	13.97	ZS	9/4/4/4/5	14.12	1.10

Table 7: Performance of NA and MX under Local Control for $\lambda_0 = 16$

J	c^*	c_{NA}	c_{MX}	L_0	c^*	c_{NA}	c_{MX}	b	c^*	c_{NA}	c_{MX}	h_0	c^*	c_{NA}	c_{MX}
2	8.94	7.51	24.51	0.1	14.29	12.98	37.84	1	3.07	3.27	4.49	0.1	8.23	6.67	22.86
4	12.95	10.27	31.82	0.3	13.30	11.91	37.05	9	6.95	6.09	13.47	0.3	9.37	7.70	28.04
8	18.45	14.20	42.17	0.5	12.11	10.55	34.66	19	8.07	6.94	19.57	0.5	10.29	8.48	31.60
10	20.61	15.78	46.34	0.6	11.32	9.75	32.94	39	9.37	7.70	28.04	0.6	10.74	8.81	33.11
16	30.10	19.74	56.80	0.7	10.33	8.81	30.80	99	10.80	8.58	44.67	0.7	11.11	9.11	34.49
32	38.89	27.56	77.50	0.8	9.37	7.70	28.04	199	11.78	9.19	63.33	0.8	11.47	9.39	35.78
64	65.91	38.60	106.76	0.9	7.79	6.24	24.15	399	12.78	9.76	89.68	0.9	11.83	9.64	37.00
	$R^2 =$	98.51%	98.53%	$J = 2, R^2 =$		99.89%	98.35%	$J = 2, R^2 =$		99.70%	80.52%	$J = 2, R^2 =$		99.83%	99.69%
$L_0 = L_j = 0.25, b = 39, h_0 = 0.3$				$L_0 = 0.8, L_j = 0.2, b = 39, h_0 = 0.3$				$L_0 = 0.8, L_j = 0.2, h_0 = 0.3$				$L_0 = 0.8, L_j = 0.2, b = 39, h_0 = 0.3$			

Table 8: Comparison of NA, RB and DS Heuristics for Central Control: Identical Retailers

	$\mathbf{L_0 = 0.1, L_j = 0.9}$								
	NA Approximation			**RB Heuristic**			**DS Heuristic**		
J	$S_0/S_{0,N}$	C_{NA}	%	S_0/S_r	C_{RB}	%	S_0^*/S_r^*	C^*	%
2	23 / 22	10.45 ± 0.009	1.08	23 / 22	10.42 ± 0.020	0.78	24 / 22	10.34 ± 0.020	0.02
4	26 / 26	15.24 ± 0.014	2.74	26 / 28	15.41 ± 0.034	3.86	26 / 24	14.88 ± 0.019	0.28
8	30 / 30	22.52 ± 0.019	3.63	33 / 32	21.90 ± 0.010	0.77	32 / 30	21.81 ± 0.106	0.37
16	36 / 37	32.82 ± 0.049	6.45	35 / 32	30.85 ± 0.025	0.04	32 / 47	31.71 ± 0.139	2.84
32	44 / 46	50.24 ± 0.097	8.61	35 / 64	49.77 ± 0.082	7.59	32 / 43	49.17 ± 0.567	6.28
64	55 / 59	76.50 ± 0.074	17.41	66 / 64	65.18 ± 0.036	0.04	64 / 64	66.16 ± 0.057	1.54
	$b = 9, \lambda_0 = 16, H_0 = 0.3$								
2	23 / 27	10.59 ± 0.008	2.29	23 / 28	10.52 ± 0.025	1.68	24 / 28	10.35 ± 0.037	0.01
4	26 / 32	15.24 ± 0.014	2.74	26 / 36	15.41 ± 0.034	3.86	27 / 32	14.95 ± 0.017	0.74
8	30 / 39	22.52 ± 0.019	3.62	32 / 48	21.97 ± 0.016	1.10	32 / 40	21.75 ± 0.024	0.09
16	35 / 50	32.73 ± 0.046	5.88	34 / 64	32.61 ± 0.036	5.50	32 / 39	31.99 ± 0.079	3.50
32	43 / 64	50.14 ± 0.098	8.32	35 / 96	49.77 ± 0.082	7.52	32 / 49	49.39 ± 0.156	6.69
64	54 / 85	77.63 ± 0.093	18.50	66 / 128	67.65 ± 0.043	3.27	64 / 78	65.83 ± 0.010	0.49
	$b = 9, \lambda_0 = 16, H_0 = 0.9$								
2	27 / 26	14.47 ± 0.044	1.71	28 / 26	14.43 ± 0.070	1.39	28 / 26	14.25 ± 0.012	0.11
4	31 / 30	21.65 ± 0.064	4.83	33 / 32	20.99 ± 0.106	1.62	34 / 33	20.76 ± 0.028	0.53
8	37 / 37	33.51 ± 0.197	10.51	42 / 40	30.53 ± 0.040	0.66	40 / 38	30.55 ± 0.142	0.74
16	46 / 46	50.28 ± 0.387	12.81	51 / 48	44.84 ± 0.074	0.61	48 / 58	46.03 ± 0.501	3.28
32	58 / 60	83.14 ± 0.939	26.24	67 / 64	65.87 ± 0.158	0.02	67 / 64	65.99 ± 0.451	0.20
64	76 / 78	119.30 ± 1.233	8.18	69 / 128	119.63 ± 0.910	8.48	72 / 67	110.38 ± 0.894	0.09
	$b = 39, \lambda_0 = 16, H_0 = 0.3$								
2	27 / 29	14.68 ± 0.048	2.80	28 / 32	14.58 ± 0.065	2.15	28 / 34	14.33 ± 0.011	0.35
4	31 / 36	21.67 ± 0.051	4.71	33 / 40	21.43 ± 0.131	3.54	32 / 35	20.99 ± 0.166	1.44
8	37 / 44	33.51 ± 0.197	10.11	41 / 56	31.35 ± 0.063	2.99	40 / 49	30.61 ± 0.052	0.57
16	46 / 57	50.28 ± 0.387	12.28	50 / 64	47.24 ± 0.123	5.48	48 / 73	45.68 ± 0.090	2.01
32	58 / 75	83.14 ± 0.939	25.48	66 / 96	69.25 ± 0.243	4.52	64 / 89	67.80 ± 0.840	2.33
64	75 / 99	119.51 ± 1.408	8.37	69 / 128	119.63 ± 0.910	8.48	67 / 64	110.58 ± 0.067	0.28
	$b = 39, \lambda_0 = 16, H_0 = 0.9$								

Table 9: Comparison of NA, RB and DS Heuristics for Central Control: Nonidentical Retailers

Parameters			NA Approximation			RB Heuristic			DS Heuristic		
$J=4$	$L_1/L_2/L_3/L_4$	$b_1/b_2/b_3/b_4$	s_0/s_{0N}	c_{NA}	$\%\epsilon$	s_0/s_r	c_{RB}	$\%\epsilon$	s_0^*/s_r^*	c^*	$\%\epsilon$
$L_0=$	0.18/0.13/0.22/0.19	23.4/19.5/35.9/33.7	11/9	11.10±0.004	13.38	12/11	10.56±0.004	7.90	12/10	10.28±0.088	5.03
0.1	0.21/0.13/0.23/0.21	24.4/18.1/9.4/11.7	10/8	9.63±0.004	11.44	11/9	8.92±0.002	3.25	11/9	8.92±0.002	2.29
$\lambda=16$	0.16/0.12/0.12/0.25	22.4/12.6/9.1/9.3	9/7	8.32±0.002	7.88	10/9	8.40±0.001	8.87	11/8	8.08±0.006	4.77
	0.16/0.18/0.19/0.19	27.2/14/28.9/22.5	10/9	11.09±0.003	18.17	12/11	10.04±0.000	7.00	12/9	9.72±0.089	3.53
	0.15/0.11/0.19/0.22	33.1/24.6/18.1/35.3	10/9	11.32±0.012	19.36	12/11	10.14±0.003	6.97	12/9	9.65±0.010	1.82
	0.21/0.24/0.24/0.18	13.3/22.9/16.1/34.9	11/10	11.46±0.014	16.21	13/12	10.35±0.002	4.93	13/12	10.35±0.002	4.93
	0.13/0.22/0.23/0.25	39/27.3/20.8/17	11/10	12.07±0.006	20.57	13/11	10.32±0.002	3.10	13/11	10.32±0.002	3.13
	0.14/0.23/0.1/0.16	11.8/29.3/10.7/9.3	9/7	8.77±0.002	12.25	10/9	8.39±0.003	7.38	11/8	8.11±0.021	3.72
	0.24/0.14/0.14/0.19	29.7/34.1/30.8/23.5	11/9	10.76±0.005	8.65	12/10	10.45±0.004	5.47	13/9	10.16±0.014	2.55
	0.13/0.21/0.17/0.17	37.5/31.3/12.2/27	10/8	11.12±0.005	18.74	12/10	9.74±0.002	3.96	12/10	9.74±0.002	3.96
$L_0=$	0.12/0.24/0.12/0.24	29.8/18.1/21.8/11.1	13/8	10.51±0.003	15.30	15/10	9.55±0.001	4.75	15/9	9.15±0.066	0.37
0.25	0.24/0.20/0.12/0.23	33.6/26.5/14.7/14.3	14/9	11.40±0.005	14.74	15/11	10.66±0.004	7.26	16/11	10.19±0.011	2.56
$\lambda=16$	0.22/0.17/0.12/0.18	31/21.2/17.4/26.1	14/9	10.52±0.004	8.79	15/11	10.34±0.002	6.87	17/10	10.03±0.039	3.67
	0.20/0.21/0.21/0.17	12.7/20/34/10.1	13/9	11.00±0.007	16.57	15/10	9.81±0.001	3.93	16/10	9.69±0.001	2.60
	0.18/0.20/0.16/0.12	37.5/36.6/25.5/19.4	14/9	11.40±0.010	14.99	15/11	10.60±0.004	6.94	16/10	10.25±0.021	3.40
	0.17/0.16/0.23/0.15	22.7/17.2/38.5/17.9	14/9	10.32±0.016	6.44	15/10	10.12±0.003	4.37	16/9	9.87±0.04	1.85
	0.21/0.18/0.13/0.21	34.2/20.9/24/35.7	14/9	12.35±0.013	20.46	16/11	10.67±0.002	4.10	17/11	10.35±0.008	1.00
	0.10/0.25/0.19/0.11	24.9/14.8/34.3/27.8	13/8	11.04±0.004	19.16	15/10	9.61±0.002	3.75	15/10	9.61±0.002	3.75
	0.20/0.13/0.23/0.12	12.3/31.3/18.4/37.2	13/8	10.58±0.003	13.16	15/9	9.65±0.003	3.18	15/9	9.65±0.003	3.18
	0.14/0.15/0.12/0.21	34/30.2/27/31.4	14/9	10.63±0.005	7.39	15/11	10.66±0.002	7.65	16/9	10.01±0.021	1.14
$L_0=$	0.13/0.21/0.23/0.16	24/35.7/9.8/38.8	16/13	15.93±0.009	25.36	19/16	13.15±0.005	3.49	19/16	13.15±0.005	3.49
0.1	0.19/0.11/0.18/0.13	34.3/27.8/28.7/14.9	16/12	13.99±0.009	16.46	18/14	12.24±0.002	1.84	19/14	12.13±0.005	0.95
$\lambda=32$	0.23/0.12/0.12/0.21	18.4/37.2/17.6/19.1	17/14	12.90±0.003	6.72	18/15	12.79±0.006	5.84	20/14	12.15±0.02	0.58
	0.12/0.21/0.22/0.21	27/31.4/16.6/13.3	17/14	14.24±0.003	13.10	19/16	13.07±0.004	3.79	19/16	13.07±0.004	3.79
	0.10/0.11/0.22/0.23	15.3/12.5/25.6/9.4	15/12	12.24±0.005	11.74	17/15	11.53±0.002	5.23	17/15	11.53±0.002	5.23
	0.12/0.17/0.21/0.20	25.3/11.2/22.1/15.1	16/13	13.14±0.007	12.22	18/14	12.03±0.002	2.71	18/14	12.03±0.002	2.71
	0.20/0.14/0.17/0.14	26.3/25/27.9/13.8	16/13	13.77±0.009	13.50	18/16	12.83±0.003	5.72	18/14	12.47±0.021	2.75
	0.18/0.24/0.20/0.24	14.7/19.1/14.4/38.9	19/16	14.29±0.006	7.77	21/18	13.71±0.004	3.36	20/16	13.49±0.021	1.75
	0.17/0.25/0.12/0.19	11.8/22.1/36.9/10.5	16/13	13.38±0.010	12.77	18/15	12.47±0.001	5.13	19/14	12.09±0.023	1.96
	0.23/0.14/0.13/0.22	21.3/15.1/27.8/27.1	17/14	13.95±0.006	10.40	19/16	13.19±0.005	4.35	20/14	12.79±0.006	1.20
$L_0=$	0.17/0.11/0.21/0.19	27.6/29.7/33.1/13.5	23/13	14.81±0.008	13.78	25/16	13.36±0.004	2.69	25/16	13.36±0.004	2.69
0.25	0.19/0.23/0.24/0.19	30.8/10.3/29/38.3	24/15	16.30±0.011	15.21	27/18	14.42±0.003	1.93	28/18	14.27±0.047	0.91
$\lambda=32$	0.15/0.18/0.15/0.13	12.3/35.1/34.5/31.3	21/12	15.33±0.016	20.86	24/14	13.10±0.005	3.34	25/14	12.77±0.021	0.73
	0.12/0.15/0.11/0.11	28.2/33.6/25.4/22.4	21/11	13.56±0.002	14.85	23/13	12.16±0.003	2.94	24/13	11.91±0.018	0.82
	0.16/0.14/0.17/0.18	13.6/18.7/31.1/18.4	22/13	13.33±0.004	8.90	24/14	12.50±0.002	2.11	24/14	12.50±0.002	2.11
	0.22/0.24/0.23/0.21	18/37.3/12.8/11	25/16	14.55±0.008	6.72	27/17	13.96±0.005	2.38	27/17	13.96±0.005	2.38
	0.22/0.18/0.19/0.24	11.2/35.3/28.6/18.7	24/15	15.06±0.013	10.20	26/17	14.06±0.005	2.93	26/17	14.06±0.005	2.93
	0.12/0.18/0.13/0.14	36.6/25.5/28.9/12.4	21/11	14.25±0.008	17.29	23/14	12.71±0.004	4.62	25/14	12.44±0.007	2.40
	0.17/0.16/0.18/0.22	24.3/20.5/29.6/25	24/15	14.30±0.009	6.54	25/17	14.08±0.002	4.93	27/17	13.79±0.002	2.78
	0.19/0.16/0.10/0.21	12/27.7/34.9/23.7	22/13	14.11±0.005	12.38	24/15	12.97±0.008	3.31	25/15	12.92±0.028	2.91

Table 10: Local versus Central Control when $\lambda_0 = 16$, $h_j = 1$ and $L_0 + L_j = 1$

L_0	s_o^*/s_j^*	c^*	S_0/S_r	C	$+/-$	%
0.1	2 / 13	14.29	28 / 26	14.09	0.016	1.43
0.3	6 / 11	13.30	28 / 22	13.16	0.031	1.02
0.5	10 / 9	12.11	28 / 18	11.80	0.033	2.52
0.6	14 / 7	11.32	28 / 14	11.23	0.059	0.77
0.7	16 / 6	10.33	28 / 12	10.19	0.036	1.39
0.8	18 / 5	9.37	28 / 10	9.18	0.007	2.05
0.9	21 / 3	7.79	27 / 6	7.62	0.015	2.19

$J = 2, h_0 = 0.3, b = 39$

h_0	s_0^*/s_j^*	c^*	S_0/S_r	C	$+/-$	%
0.1	19 / 5	8.23	29 / 10	8.11	0.005	1.46
0.3	18 / 5	9.37	28 / 10	9.18	0.008	2.06
0.5	17 / 5	10.29	27 / 10	10.03	0.019	2.57
0.6	17 / 5	10.74	27 / 10	10.47	0.017	2.52
0.7	16 / 5	11.11	26 / 10	10.72	0.048	3.55
0.8	16 / 5	11.47	26 / 10	11.06	0.045	3.59
0.9	16 / 5	11.83	26 / 10	11.41	0.043	3.61

$J = 2, L_0 = 0.8, b = 39$

b	s_o^*/s_j^*	c^*	S_0/S_r	C	$+/-$	%
1	14 / 2	3.07	18 / 4	3.02	0.000	1.63
9	16 / 4	6.95	24 / 8	6.83	0.010	1.78
19	18 / 4	8.07	26 / 8	7.92	0.014	1.82
39	18 / 5	9.37	28 / 10	9.18	0.008	2.07

J	s_0^*/s_j^*	c^*	S_0/S_r	C	$+/-$	%
2	18 / 5	9.37	28 / 10	9.17	0.078	2.13
4	18 / 3	12.59	30 / 12	12.43	0.034	1.27
5	17 / 3	14.52	32 / 15	14.26	0.012	1.78
8						