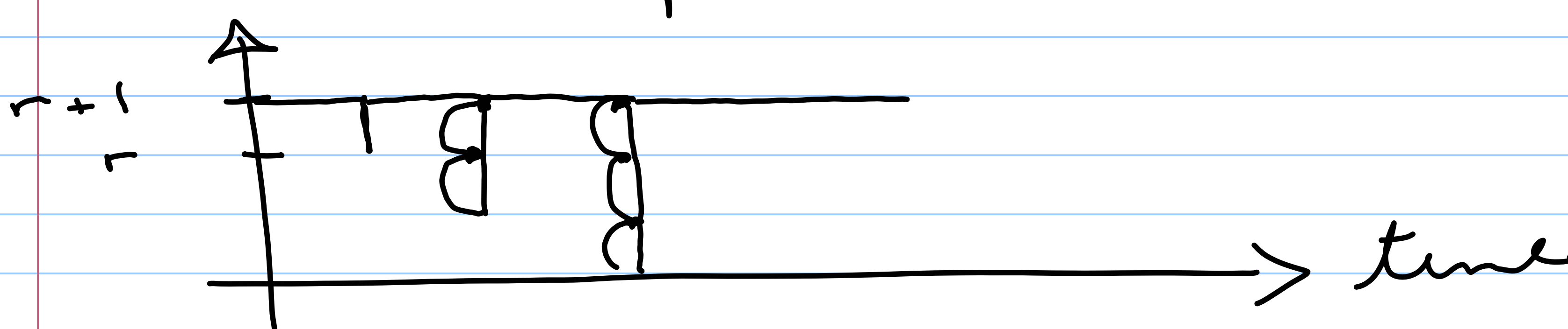
Note Title 11/18/2004

(Q, r) Model

- $Q = 1$
- $s = r$, $S = r + Q = r + 1$
- Both (Q, r) & (s, S) are equivalent



$$r+1 = S = \underline{I}P(t) = IN(t) + \underline{I}O(t)$$

net inv.

inv. position

inv on order

$$IN(t) = S - \underline{I}O(t)$$

$$\underline{I}O(t) = D(t-L, t) = D(t|L)$$

= lead time demand

In Poisson case $D(t|L) \sim \text{Poisson}(\lambda L)$

$$D(t|L) = D$$

In steady state

$$\therefore \quad IN = S - D$$

$$I = IN^+ = (S-D)^+$$

$$B = IN^- = (D-S)^+$$

$$\boxed{G(S) = h\,E(S-D)^+ + b\,E(D-S)^+}$$

Newsvendor Problem

An optimal base stock level S^* is given by the smallest integer such that

$$\Pr(D \le S) \ge \frac{b}{b+h}.$$

$$\begin{aligned} A &= \Pr(B > 0) \\ &= \Pr((D-S)^+ > 0) \\ &= \Pr(D > S) \\ &= 1 - \Pr(D \le S) \\ &\le \frac{h}{b+h}. \end{aligned}$$

$Ex \quad b = 19h \Rightarrow A \le \frac{1}{20},$

$b = 99h \Rightarrow A \le \frac{1}{100}.$

k	$Pr(D = k)$	$Pr(D \ge k)$	$E(D-k)^+$
0	$e^{-\lambda}$	$1 - e^{-\lambda}$	λ
1	$e^{-\lambda}\frac{\lambda}{1}$	$1 - e^{-\lambda} - e^{-\lambda}\frac{\lambda}{1}$	$\lambda - (1-e^{-\lambda}$

k	$Pr(D = k)$	$Pr(D \ge k)$	$E(D-k)^+$
$k+1$	$Pr(D = k) \cdot \frac{\lambda}{k+1}$	$Pr(D \ge k) - \bigcirc$	$E(D-k)^+ - \bigcirc$

$$E(D-k)^+ = \sum_{j=k}^{\infty} Pr(D > j)$$

$$G(k) = h\,E(k-D)^+ + b\,E(D-k)^+$$

$$= h(k-\lambda) + (h+b)\,E(D-k)^+$$

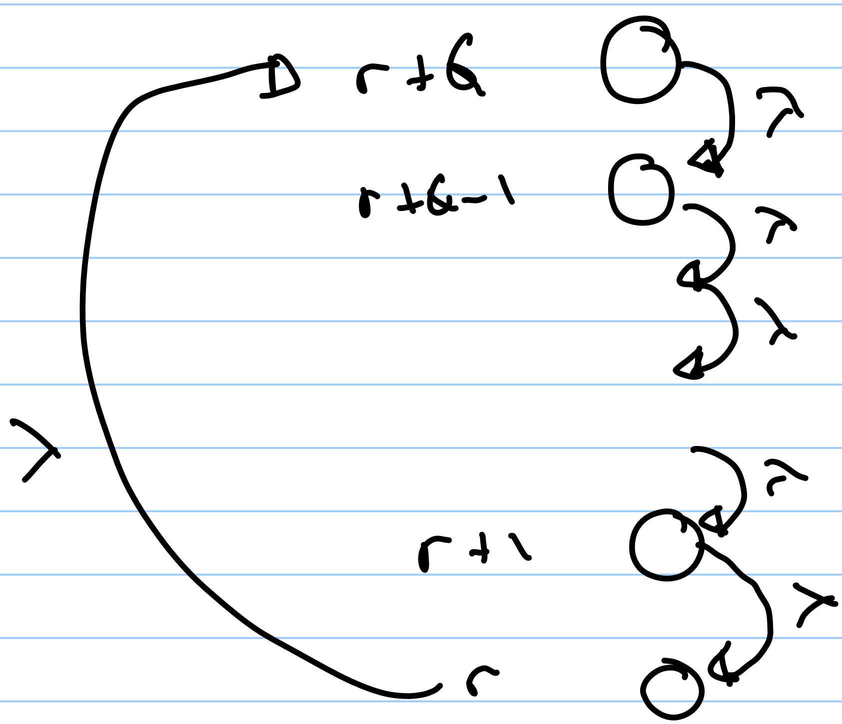
$$E(k-D)^+ = k-\lambda + E(D-k)^+$$

59 12.75907

$$S^* = 59, \quad G(59) = 12.76$$

What if Q is positive?

$IP \sim \mathcal{U}\{r+1, \ldots, r+Q\}$



$$\Pr(IP = r+k) = \frac{1}{Q}$$

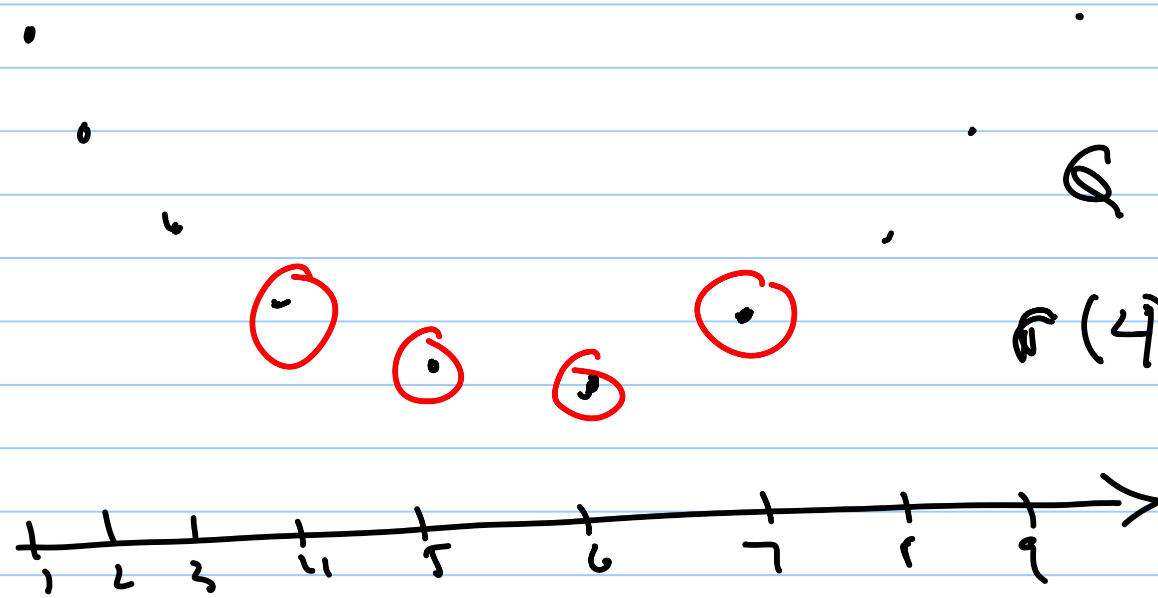
$k = 1, 2, \ldots, Q$

Holds very generally

- $\Pr(D=1) > 0$
- $D \geq 0$
- Integer values

How does this helps?

$$c(Q,r) = \frac{K\lambda}{Q} + \frac{1}{Q}\sum_{y=r+1}^{r+Q} G(y)$$



$Q = 4$

$r(4) = 3$

$Q = 2$ $\quad r(Q) = 4$ $\quad Q = 3,\ r(3) = 5$

$$C(2,6) = \frac{K\lambda}{2} + \frac{1}{2}\sum_{y=7}^{8} G(y)$$

To find optimal (Q,r) policy.

First minimize over r, e.g. obtain $r(Q)$ and then optimize over Q.

$$G_K = K\text{th smallest value of } G(y)$$

$$C(Q) = C(Q, r(Q)) = \frac{K\lambda}{Q} + \frac{1}{Q}\sum_{K=1}^{Q} G_K$$

$$C(Q+1) = \frac{K\lambda}{Q+1} + \frac{1}{Q+1}\sum_{K=1}^{Q+1} G_K$$

$$C(Q+1) = \frac{Q}{Q+1}\left[\frac{K\lambda}{Q} + \frac{1}{Q}\sum_{k=1}^{Q} G_k\right] + \frac{1}{Q+1} G_{Q+1}$$

$$\boxed{C(Q+1) = \frac{Q}{Q+1} C(Q) + \frac{1}{Q+1} G_{Q+1}}$$

$$C(Q+1) < C(Q)$$

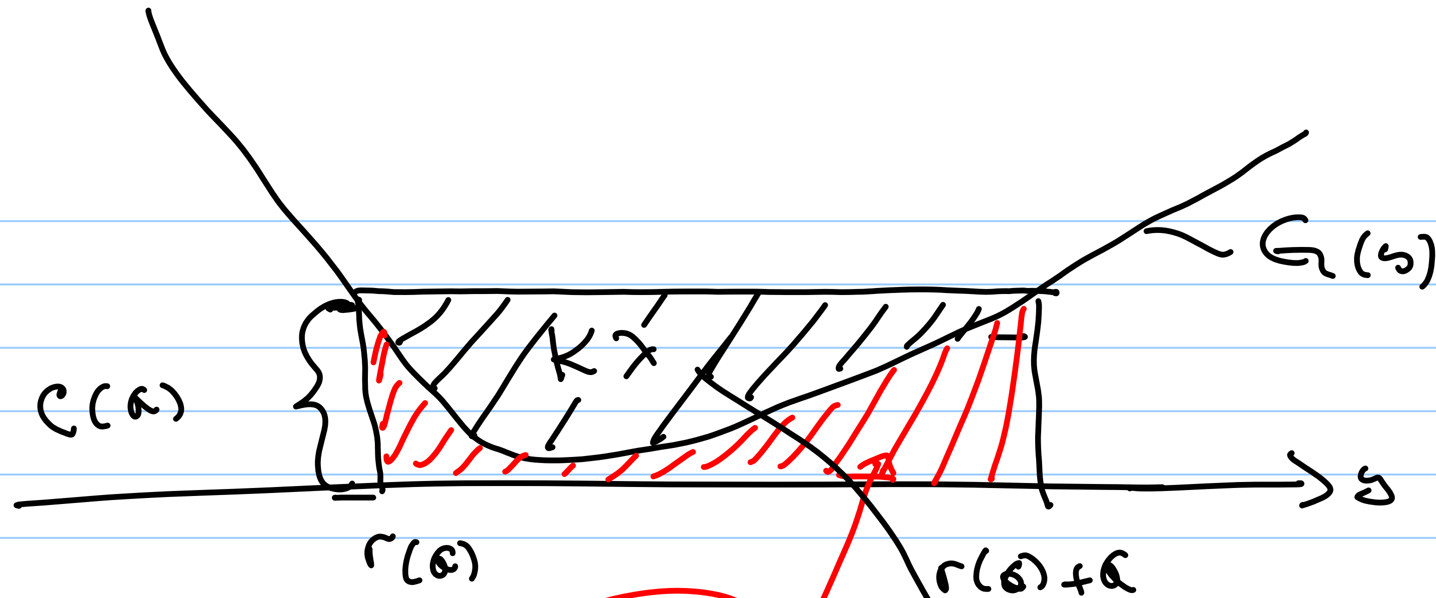
if and only if $G_{Q+1} < C(Q)$

$\uparrow Q$ while $C(Q) \searrow$

stop when $C(Q) \uparrow$ for first time.

$$C(Q) = \frac{K\lambda + \sum_{k=1}^{Q} G_k}{Q}$$

Additional Intuition:



$$C(a) = \frac{1}{a}\int_{r(a)}^{r(a)+a} G(s)\,ds \qquad \frac{K\lambda}{a}$$

$$= \frac{1}{a}\left[\text{area of rect}\right]$$

$$\frac{C(Q)}{C(Q^*)} \leq \frac{1}{2}\left(\frac{Q}{Q^*} + \frac{Q^*}{Q}\right)$$

less sensitive than the EOQ ($=$).

Bounds:

$$C_d = \sqrt{\frac{2K\lambda}{\frac{hb}{h+b}}}$$

cost of EOQ

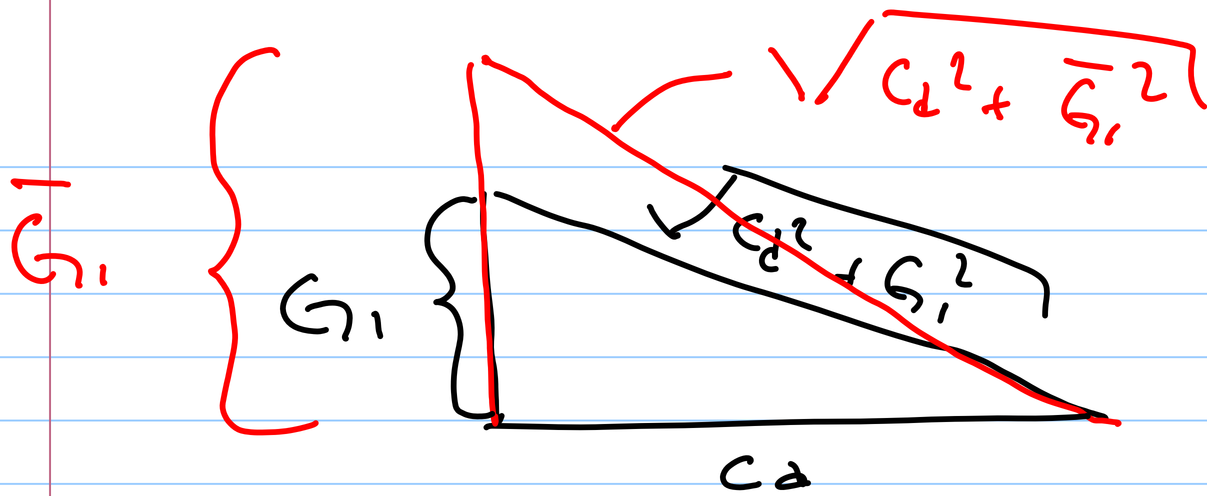
$$G_1 = \min_y \{ G(y) \}$$

$= \text{newsvendor cost}$

$$\min_y G(y) \leq \sigma\sqrt{h b} = \overline{G_1}$$

Scarf's Upper Bound.

$$\sqrt{C_d^2 + G_1^2} \leq C(\theta^*) \leq \sqrt{C_d^2 + \overline{G_1}^2}$$



Quick & Dirty Heuristic

$$Q = \underbrace{\sqrt{2}\, Q_d}_{EOQ}$$

$$\frac{C(\sqrt{2}\, Q_d)}{C(Q^*)} \le 1.06 \quad \blacksquare$$

Next time

- (s, S) policies
- Multi-echelon models.