

These lecture notes summarize, and at times complement the material in the textbook:

Advanced Engineering Economics by Park and Sharpe-Bette.

The course is about industrial projects and investments. In this course an industrial project is any candidate project that involves the outlay of cash in exchange for an anticipated return flow of future benefits. In contrast, an investment is a funded project. It involves the sacrifice of something now for the prospect of something later.

We distinguish between real (physical) and financial investments (financial instruments). Some characteristics that distinguish real from financial investments are:

- Real investments are not easily scalable
- Real investments are often irreversible
- Real investments shape the future of the firm

We will deal mostly with real investments, but at times we will need to talk about financial investments as well.

The course is concerned with how to allocate capital to projects within a firm to maximize the value of the firm to its shareholders.

Course objective: Taking into account the cost of capital, budgets, taxes, depreciation, inflation, and uncertainty, decide which projects, if any, to undertake, reject, or postpone.

An investment project can be described by its cash flow profile, i.e., the amount and timing of costs and benefits in the planning horizon.

We will start by ignoring taxes, inflation, depreciation, and uncertainty and later build these features into the analysis.

### Notes on Chapter 1: Accounting Income and Cash Flows.

**Investment** (a sacrifice of something now for the prospect of something later).

Factors involved: *time* and *risk*. Type of investments: *financial* (financial instrument) and *real* (physical assets).

**Economic Analysis** consists of economic evaluation of real investments.

**Corporate Investment Framework:** We will assume that the objective of any firm is to maximize its value to its shareholders. From the economic analysis perspective a firm must make three major decisions:

(1) the *investment decision* (Capital Budgeting is the allocation of capital to investment projects whose benefits are to be realized in the future),

(2) the *financing decision* (sources of funds: sales of stocks, retained earnings, the sale of bonds, and short term borrowing from financial institutions) and

(3) the *dividend decision*.

In practice these decisions are interrelated. For convenience in economic analysis the investment and financial activities are usually separated. First the selection procedure for investment projects are analyzed and then the choice of the financing sources is considered. After this, modifications are made to the investment models to allow the capital budget to become a decision variable.

*Accounting information* is often needed in economic analysis and the student should be familiar with the *balance sheet*, the *income statement* and the *funds flow statement*.

The **balance sheet** is a statement of the financial position of a company as of a reporting date. It shows the sources from which the current operating funds have been obtained (liabilities and owner's equity) and the types of property and property rights in which the funds are currently locked up (assets). Liabilities are separated into current and other liabilities. Assets are generally divided into current, fixed, and other assets. *Working capital* is defined as current assets minus current liabilities

The **income statement** summarizes the revenue items, the expense items, and the difference between them for an accounting period. The two principal methods of reporting income are the cash basis method and the accrual method. Most businesses use the accrual method.

The **funds flow statement** details how the company acquired funds (sources) and how the funds were used (uses).

A company's cash inflows are approximately equal to its net income plus depreciation plus any taxes it has deferred, and all three of these quantities depend on the depreciation method the company uses for tax and reporting purposes.

Many companies use one *depreciation* method for *tax purposes* (usually the method that minimizes the current tax liability) and another for *financial reporting purposes* (usually the method that most accurately reflects the decline in value of its assets over time). It is common practice to use an *accelerated* depreciation method to compute the actual tax liability and a *straight-line* depreciation method for financing reporting. The difference between the computed taxes and the actual tax liability is called *deferred taxes*.

**Cash Flows** are used in evaluating the economic value of a project. Projects usually start with an investment, which is a cash flow from the company to the project. Cash *outflows* that move from the company to the project are arbitrarily given a *negative* sign. Conversely, cash *inflows* that move from the project to the company are given a *positive* sign. Each flow is usually assumed to occur at the end of the year. The *net cash* flow is the difference between the cash inflows and the cash outflows. It is also equal, see figure 1.2 in textbook, to the net income from operations, the noncash depreciation expense, the net proceeds of equipment disposal, and new borrowing minus the purchase of new equipment, the repayment of debt, and the net change in working capital. See Example 1.8.

## Notes on Chapter 2: Interest and Equivalence

An investment project can be described by its **cash flow profile**, i.e., the *amount* and *timing* of expected costs (disbursements) and benefits (receipts) in the planning horizon. The term **payment** is used to denote the receipts less the disbursements that occur at the same point in time.

We classify project cash flows in two categories:

- (1) *discrete-time cash flows*, and
- (2) *continuous-time cash flows*.

Notation:

$F_n$  = discrete payment occurring at period  $n$ ,

$F_t$  = continuous payment occurring at time  $t$ .

Money has a *time* value because:

- (i) it has a potential earning power,
- (ii) users may have different utility of consumption of dollars at different times, and
- (iii) the buying power of the dollar changes through time.

### Types of Interest: Simple and Compound.

If an amount of money is deposited in a financial institution, interest accrues at regular time intervals. Each time interval represents an **interest period**. Two approaches are used to calculate the earned interest: **simple interest** and **compound interest**. The first approach assumes that interest earned is a linear function of time. Let  $i$  be the interest rate per period,  $P$  = amount borrowed,  $F_N$  = amount paid after  $N$  periods. Then, the interest payment is  $I = F_N - P = NPi$ , so

$$F_N = P + I = P(1 + Ni).$$

The compound interest approach assumes that the earned interest is not withdrawn at the end of an interest period and is automatically redeposited with the original amount. If  $P$  is the initial amount and  $i$  is the interest rate per period, then

$$F_1 = P(1 + i)$$

is the balance after 1 period,

$$F_2 = F_1(1 + i) = P(1 + i)^2$$

is the balance after 2 periods, and

$$F_N = P(1 + i)^N$$

is the balance after  $N$  periods. The total interest earned over  $N$  periods is

$$I = F_N - P = P[(1 + i)^N - 1].$$

The additional interest earned with the compound interest is

$$\Delta_N = P[(1 + i)^N - (1 + Ni)].$$

*Example:*  $P = \$1,000$ ,  $i = 9\%$  and  $N = 10$ . Then, with a simple interest rate  $F_{10} = \$1,900$ ; while with a compound interest rate  $F_{10} = \$2,367.36$ . The additional interest earned by the compounded interest method is  $\Delta_{10} = \$467.36$ .

### Nominal and Effective Interest Rates.

In economic analysis a year is usually used as the interest period. In financial transactions, however the interest period may be of any duration. We use the terms **nominal** interest rate and **effective** interest rates to describe more precisely the nature of compounding schemes.

Even though financial institutions may use more than one interest period per year in compounding the interest, they usually quote the interest on an annual basis. For example, a year's rate at 1.5% compounded monthly is typically quoted as "18% compounded monthly". When stated in this fashion, the 18% is called a **nominal** interest rate or annual percentage rate. The **effective** interest rate represents the actual interest earned or charged for a specified time period. The effective interest rate based on a year is referred to as the **effective** annual interest rate  $i_a$ . The effective interest rate based on the payment period is called the **effective interest rate per payment period**  $i$ .

*Example:* Suppose a bank charges a rate of 12% compounded quarterly (four times per year). This means that 3% is charged every quarter. The interest per dollar accrued at the end of the year is  $(1.03)^4 - 1 = 0.1255$ . Thus the effective annual interest rate is  $i_a = 12.55\%$ . If the interest is compounded monthly (12 times per year) then the effective annual interest rate is  $i_a = (1.01)^{12} - 1 = 0.1268 = 12.68\%$ . In general

$$i_a = \left(1 + \frac{r}{m}\right)^m - 1$$

where  $i_a$  is the effective annual interest rate,  $r$  is the nominal interest rate per year,  $m$  is the number of interest periods per year,  $r/m$  is the effective interest rate per interest period.

As  $m$  goes to infinity we obtain:

$$i_a = \lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1.$$

The above formula can be generalized to compute the effective interest rate in any payment period. If  $c$  is the number of interest periods per payment period, then the effective interest rate per payment period is given by

$$\begin{aligned} i &= \left(1 + \frac{r}{m}\right)^c - 1 \\ i &= \left(1 + \frac{r}{ck}\right)^c - 1, \end{aligned}$$

where  $k$  is the number of payment periods per year, and  $r/k$  is the nominal interest rate per payment period.

*Example:* Let  $r = 12\%$ ,  $m = 12$ ,  $c = 3$ , so  $k = 4$ . Then  $i_a = 12.68\%$ , and  $i = (1.01)^3 - 1 = 3.03\%$ .

### Useful Sums.

Geometric Series:

$$\sum_{n=0}^N x^n = 1 + x + \dots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

where  $x \neq 1$ . If  $-1 < x < 1$ , then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Arithmetic-Geometric Series:

$$\sum_{n=0}^N nx^n = 0 + x + 2x^2 + \dots + Nx^N = \frac{x [1 - (N+1)x^N + Nx^{N+1}]}{(1-x)^2}$$

where  $x \neq 1$ .

If  $-1 < x < 1$ , then

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

### Discrete Compounding.

Consider the cash flow  $F_0, F_1, \dots, F_N$  where  $F_n$  represents the flow of cash at time  $n$ , with the convention that positive values represent inflows and negative values represent outflows. What amount now is equivalent to cash flow? We can answer this question very simply if we know the current value of a dollar at time  $n = 0, 1, \dots, N$ . Let  $\beta_n$  denote the value of a dollar at time  $n$ . Then the current, or present value of the cash flow is

$$P = F_0 + F_1\beta_1 + F_2\beta_2 + \dots + F_n\beta_n,$$

since by definition  $\beta_0 = 1$ .

Suppose that the interest rate in period  $n$  is  $i_n$ . How are the  $\beta$  factors related to the interest rates? At interest rate  $i_1$  a dollar now will be worth  $1 + i_1$  dollars a year from now, so we need to deposit  $\beta_1 = 1/(1 + i_1)$  dollars now to obtain a dollar a year from now. Thus, at interest rate  $i_1$ ,  $\beta_1$  dollars now are equivalent to 1 dollar at the end of year 1. Similarly, it takes  $\beta_2 = \beta_1/(1 + i_2)$  dollars now to obtain a dollar after two years by earning  $i_1\%$  in year one and  $i_2\%$  in year two. In general, we have

$$\beta_n = \frac{1}{(1 + i_1) \dots (1 + i_n)}.$$

Of special interest is the case when  $i_1 = i_2 = \dots = i_N = i$ , i.e., the case where the interest rate is constant over time. In this case we have

$$\beta_n = 1/(1 + i)^n = (1 + i)^{-n}.$$

The book has formulas, as well as tables, for the present value of cash flows of the form  $F_n = A$ ,  $n = 1, \dots, N$ ,  $F_n = (n - 1)G$ ,  $n = 1, \dots, N$ , and  $F_n = F_1(1 + g)^{n-1}$ . The first case is known as the uniform series, the second as the arithmetic gradient series, and the third as the geometric gradient series. In what follows, we will develop formulas for the present and future value of these series. We start by considering the case in which the payment periods are identical to the compounding periods.

**Single Sum.** The value  $F$  at time  $N$  of a single payment  $P$  invested at interest rate  $i$  is:

$$F = P(1 + i)^N$$

The factor  $(1 + i)^N$  is called the **single-payment compound amount factor**, and is denoted by

$$(F/P, i, N).$$

This factor represents the value of one dollar after  $N$  years when it is invested at interest rate  $i$ . Values of  $(F/P, i, N)$  are tabulated in tables at the back of the book.

*Example:* If  $P = \$24$ ,  $i = 6\%$ , and  $N = 369$ , then

$$F = \$24(F/P, 6\%, 369) = \$24(2, 177, 029, 343) = \$52, 248, 704, 230.$$

□.

By inverting the above formula, we obtain

$$P = F(1 + i)^{-N}.$$

The factor  $(1 + i)^{-N}$  is called the **single-payment present-worth factor**, and is denoted by

$$(P/F, i, N).$$

This factor represents the amount we need to invest now for  $N$  years at interest rate  $i$  to obtain one dollar at time  $N$ .

**Uniform Series.**

An amount  $A$  deposited at the end of each period, for  $N$  periods, in an account paying  $i\%$  per period would grow to:

$$F = A \sum_{n=1}^N (1+i)^{N-n} = A \left[ \frac{(1+i)^N - 1}{i} \right].$$

The term in brackets is called the **uniform-series compound amount factor**, or **equal series compound amount factor** and is represented by

$$(F/A, i, N) = \left[ \frac{(1+i)^N - 1}{i} \right].$$

*Example:* An annual deposit of  $A = \$1,000$ , for  $N = 30$  years at  $i = 10\%$  grows to

$$F = \$1,000(164.49402) = \$164,494.02.$$

□.

The inverse relationship yields the **uniform-series sinking-fund factor** designated by

$$(A/F, i, N) = \left[ \frac{i}{(1+i)^N - 1} \right].$$

*Example:* To build a fund of  $F = \$18,000$ , in  $N = 18$  years at  $i = 5\%$  compounded annually requires

$$A = \$18,000(A/F, 5\%, 18) = \$18,000(0.0356) = \$640.80$$

at the end of each year.

□.

Multiplying  $(P/F, i, N)$  by  $(F/A, i, N)$  we obtain

$$(P/A, i, N) = \left[ \frac{(1+i)^N - 1}{i(1+i)^N} \right].$$

The bracketed term is the **uniform-series present worth factor**, designated by  $(P/A, i, N)$ .

*Example:* What is the present worth of an annuity that pays  $A = \$12,000$  at the end of each year for  $N = 10$  years at  $i = 8\%$ .

$$P = \$12,000(P/A, 8\%, 10) = \$12,000(6.710081397) = \$80,520.98.$$

□.

The inverse factor is called the **capital recovery factor**, and is denoted by  $(A/P, i, N)$ .

$$(A/P, i, N) = \left[ \frac{i(1+i)^N}{(1+i)^N - 1} \right].$$

*Example:* Consider a commercial mortgage at  $i = 8\%$  over  $N = 20$  years with a loan principal of  $P = \$1,000,000$ . The annual payment is

$$A = \$1,000,000(A/P, 8\%, 20) = \$1,000,000(0.10185) = \$101,850.$$

What is the balance at the end of 5 years?

$$P = \$101,850(P/A, 8\%, 15) = \$871,785.00.$$

□.

**Linear Gradient Series.**

Consider the series of payments:

$$F_n = (n - 1)G,$$

for  $n = 1, 2, \dots, N$ . The present value of this series is:

$$P = \sum_{n=1}^N (n - 1)G(1 + i)^{-n}$$

$$P = G \left[ \frac{1 - (1 + Ni)(1 + i)^{-N}}{i^2} \right].$$

The factor in brackets is called the **gradient series present worth factor** designated  $(P/G, i, N)$ .

*Example:* If  $G = 100$ ,  $N = 10$ , and  $i = 8\%$ , we have

$$P = \$100(P/G, 8\%, 10) = \$100(25.9768) = \$2,597.68.$$

□.

A **uniform series equivalent to the gradient series** is easily obtained by the relationship between  $P$  and  $A$ . Resulting in the factor :

$$(A/G, i, N) = \left[ \frac{1}{i} - \frac{N}{(1 + i)^N - 1} \right].$$

The **future-worth equivalent of a gradient series** can be easily obtained by multiplying the **uniform series equivalent to the gradient series** by the  $(F/A, i, N)$  factor, resulting in

$$(F/G, i, N) = \frac{G}{i} [(F/A, i, N) - N].$$

*Example:* Let  $F_n = \$100 + \$6(n - 1)$ ,  $N = 8$ ,  $i = 10\%$ . Then,

$$F = \$100(F/A, 10\%, 8) + \$6(F/G, 10\%, 8) = \$1,349.76.$$

□.

**Geometric Series.**

Consider the geometric series of payments:

$$F_n = F_1(1 + g)^{n-1},$$

for  $n = 1, 2, \dots, N$ . Then, the present value is given by

$$P = \frac{F_1}{1+g} \sum_{n=1}^N \left( \frac{1+g}{1+i} \right)^n$$

$$= F_1 \left[ \frac{1 - (1+g)^N (1+i)^{-N}}{i-g} \right]$$

for  $i \neq g$ .

Using  $g' = \frac{1+i}{1+g} - 1$ , we also obtain

$$P = \frac{F_1}{1+g} (P/A, g', N).$$

*Example:* If  $F_1 = \$39,600$ ,  $g = 0.1$ ,  $N = 5$ ,  $i = 0.15$ , then

$$P = \$39,600 \left[ \frac{1 - (1.1)^5 (1.15)^{-5}}{.05} \right] = \$157,839.18.$$

□.

### Compounding period different from payment period

Whenever the payment period and the compounding period do not correspond, we approach the problem by finding the *effective interest rate* based on the *payment period* and then use this rate in the formulas with the appropriate accounting of the number of payments.

*Example:* What is the present worth of *quarterly* payments of  $A = \$1,000$  that extends over a period of  $N = 5$  years if the interest rate is  $r = 8\%$  compounded monthly? The effective interest rate per payment period (quarter) is  $i = (1 + 0.08/12)^3 - 1 = 2.0133\%$ . There are  $N = 20$  quarters in 5 years, so

$$P = \$1,000(P/A, 2.0133\%, 20) = \$1,000(16.33047) = \$16,330.47.$$

□.

### Continuous Compounding.

#### Discrete Payments

Here again, we compute the effective interest rate  $i = e^r - 1$ , and use it as above. This gives rise to a whole new set of interest formulas. For instance, if we are interested in the present value of a **uniform series of payments**  $A$  over  $N$  years at interest rate  $r$  compounded continuously, we can use the previously discussed factor  $(P/A, i, N)$  with  $i = e^r - 1$ . Alternatively, we can directly use the continuous factor  $(P/A, r, N)$ , where

$$(P/A, r, N) = \left[ \frac{e^{rN} - 1}{e^{rN}(e^r - 1)} \right],$$

and so on.

*Example:* What is the present worth of a uniform series of year-end payments of  $A = \$500$  each for  $N = 10$  years if the interest rate is  $r = 8\%$  compounded continuously?

The effective rate is  $i = e^{0.08} - 1 = 8.33\%$ . Then

$$P = \$500(P/A, 8.33\%, 10) = \$3,305.85.$$

□.

*Example:* A series of equal quarterly payments of  $A = \$1,000$  each extends for  $N = 5$  years. What is the present worth at  $r = 8\%$  compounded continuously?

The interest per quarter is  $i = e^{0.02} - 1 = 2.02\%$ ,

$$P = \$1,000(P/A, 2.02\%, 20) = \$16,319.70.$$

□.

#### Continuous Cash Flows.

When the cash flows occur very frequently we can approximate them by a continuous flow. Remember that we are assuming continuous compounding. For instance, a company that pays \$100,000 a week in salaries may approximate the annual expense as if the total amount, say \$5,200,000, was paid continuously throughout the year. The approximation is done by letting  $F_t$  denote the flow rate at time  $t$ , and by integrating instead of summing. For present value, this results in the formula

$$P = \int_0^N F_t e^{-rt} dt,$$

and for future value, this results in

$$F = \int_0^N F_t e^{r(N-t)} dt.$$



For example, if  $F_t = \bar{A}$ , then

$$P = \bar{A} \int_0^N e^{-rt} dt = \bar{A} \left( \frac{1 - e^{-rN}}{r} \right),$$

and

$$F = \bar{A} \int_0^N e^{rt} dt = \bar{A} \left( \frac{e^{rN} - 1}{r} \right).$$

These factors are called the **funds flow present-worth** factor, denoted by  $(P/\bar{A}, r, N)$ , and the **funds flow compound amount** factor, denoted by  $(F/\bar{A}, r, N)$ .

*Example:* The end of year equivalent payment to 52 payments of \$100,000 a year at 8% compounded continuously is

$$F = \$5,200,000(F/\bar{A}, 8\%, 1) = \$5,413,659.40.$$

It is instructive to compare this to the exact calculation via

$$F = \$100,000(F/A, 8\%/52, 52) = \$5,409,328.36.$$

□.

### Equivalence Of Cash Flows.

#### Concept of Equivalence.

Two cash flows are equivalent at interest rate  $i$  if we can convert one cash flow into the other by using proper compound interest factors.

If we deposit  $P = \$100$  in a bank for  $N = 3$  years at  $i = 8\%$  we will accumulate  $F = \$125.97$ . We may say that at  $i = 8\%$ ,  $P = \$100$  at time zero is equivalent to  $F = \$125.97$  at time  $N = 3$ .

The notion of equivalence can also be posed when the balancing interest rate is unknown. For example, consider the following two options. Option I is to receive a lump sum of \$1,000 now; option II is to receive \$600 at the end of each year for two years. Finding the rate at which these two options are equivalent can be done by equating the future value of the two options, i.e., by solving the equation

$$\$1000(1 + i)^2 = \$600(1 + i) + \$600.$$

The tables can be used to find that  $i = 13\%$ . Of course, we could have also used present value or annual worth.

### Effect Of Inflation On Cash Flow Equivalence.

The dollar is an imperfect unit of measure because its value changes with time. Inflation (resp., deflation) is a measure of the decline (resp., increase) in purchasing power of the dollar.

#### Measures of Inflation:

CPI(Consumer Price Index) based on market basket,  
 GNPIPD(Gross National Product Implicit Price Deflator),  
 PPI(Producer Price Index),

Most inflation indexes are computed by monitoring the price changes of a basket of goods. This tends to overstate inflation for the following reasons:

- Improvements in quality are not taken into account.
- Opportunities for substitution are not taken into account.

**Average Inflation.**

Suppose that inflation in year one is 5%, and inflation in year 2 is 10%, then the overall inflation is  $(1.05)(1.10) - 1 = 15.5\%$ . The average inflation  $f$  satisfies  $(1 + f)^2 = 1.155$ , so  $f = 7.47\%$ . Notice that the average inflation is *less* than the arithmetic average. This is due to the geometric-arithmetic inequality.

$$\left( \prod_{j=1}^N x_j \right)^{1/N} \leq \frac{1}{N} \sum_{j=1}^N x_j.$$

**Explicit and Implicit Treatments of Inflation in Discounting.**

**Actual Dollars.** Represent the out of pocket dollars received or expended at any point in time. (Current dollars, future dollars, inflated dollars, nominal dollars).

**Constant Dollars.** Hypothetical purchasing power of future receipts and disbursements in terms of the purchasing dollars in some base year. (Real dollars, deflated dollars, today's dollars)

**Market Interest rate**  $i$  represents the opportunity to earn as reflected by the actual interest rates available in financial markets.

**Inflation-free interest rate**  $i'$  represents the earning power of money isolated from the effects of inflation. (Real interest rate, true interest rate, constant dollar interest rate.)

Note that  $i'$  can be negative.

**General inflation rate**  $f$  represents the average annual percentage of increases in prices of good and services.

**Relationship between  $i$ ,  $i'$  and  $f$ .**

Amounts in constant dollars will be primed. The relationship between real and constant dollars is through the inflation rate. That is  $F'$  dollars at time  $n$  are equivalent to  $F = F'(1 + f)^n$  actual dollars at time  $n$ .

Now assume that the market interest rate is  $i$ . Then the present value of  $F$  actual dollars  $n$  years from now is given by

$$P = F(1 + i)^{-n}.$$

On the other hand, the present value of  $F'$  real dollars  $n$  years from now is obtained via the inflation-free interest rate. Thus

$$P = F'(1 + i')^{-n}.$$

From the above we deduce that

$$F(1 + i)^{-n} = F'(1 + i')^{-n} = F(1 + f)^{-n}(1 + i')^{-n}.$$

So

$$(1 + i) = (1 + i')(1 + f)$$

or

$$i = i' + f + i'f.$$

We can also write  $i'$  in terms of  $i$  and  $f$  as

$$i' = \frac{i - f}{1 + f}.$$

Note that  $f > 0$  implies  $i' < i - f$ .

*Example:* Suppose that  $i = 6\%$  and  $f = 4\%$ , then  $i' = .02/1.04 = .0193 = 1.93\%$ .

Some economists view inflation as a form of taxation on the holding of money. Indeed, if you were to hold a dollar for a year its purchasing power at the end of the year would be  $(1 + f)^{-1}$  for a loss of

$$\frac{f}{1 + f}.$$

This is often called the “inflation” tax. For example, if  $f = 4\%$ , the “inflation” tax is equal to 3.84%.

Most textbook examples assume that all cash flows respond equally to inflationary trends. This is not so in real life. For instance depreciation schedules are not affected by inflation.

### Home Ownership Analysis:

A question asked by many families is whether they should invest in buying a house. While there may be many non economic reasons to own a house, we will analyze home ownership from a purely economic view point.

Our objective is to find the present cost of owning a house over a reasonable time horizon, say  $N$  years. This cost can be used for instance, to compare against the cost of renting.

Let us assume that a house of price  $H$  is purchased now by a conventional loan. That is, we give a 20% down payment and take a 30 year fixed loan at nominal rate  $r$  compounded monthly. In addition, let us assume that there are closing fees of the form  $A + 0.008Hj$ , where  $A$  includes the cost of inspection, appraisal, title insurance and lawyers fees and  $j\%$  represents the points charged to secure the loan. Let  $T_n$  denote property taxes due at the end of year  $n$ , and assume that  $T_n$  is of the form  $T_n = T(1 + f_T)^{n-1}$ ,  $n = 1, \dots, N$ , where  $T$  represents the amount due at the end of the first year. Interest paid on the mortgage and property taxes are tax deductible; we assume a tax rate  $t$ . Let  $M_n$  denote the maintenance cost in year  $n$ ,  $n = 1, \dots, N$ . We assume that utilities and insurance cost are included in  $M_n$ . Assume that this costs increase at rate  $f_M$  so that  $M_n = M(1 + f_M)^{n-1}$ ,  $n = 1, \dots, N$ , where  $M$  is the maintenance cost during the first year. Let us assume that the house increases in value at the rate of  $f_H$ . Thus, we estimate that after  $N$  years the house will be sold at  $H(1 + f_H)^N$ . Assuming a sales commission of 6%, it is easy to write a spreadsheet to compute the present value of the resulting cash flow at market interest rate  $i$ . This present value can then be compared to the cost of renting.

To do the above calculations, we need to compute the present value of the sequence of interest payments on an equal installment loan. This calculation is of importance since interest on a home mortgage is tax deductible. We next show how to accomplish this task. Let

- $B$  amount borrowed,
- $i_b$  interest rate per period,
- $N$  number of periods,
- $A = B(A/P, i_b, N)$  payment per period.

The unpaid balance  $B_n$  at the end of period  $n$  satisfies the difference equation

$$B_n = B_{n-1}(1 + i_b) - A, \quad n = 1, \dots, N,$$

with  $B_0 = B$ . Solving the difference equation, we find that

$$B_n = B(1 + i_b)^n - A \left[ \frac{(1 + i_b)^n - 1}{i_b} \right].$$

The interest paid in period  $n + 1$ , say  $I_{n+1}$  is equal to  $i_b B_n$ . So, we have

$$I_{n+1} = i_b B_n = A - (A - B i_b)(1 + i_b)^n.$$

Now, suppose we want to find the present value of the cash flow  $I_n$ ,  $n = 1, \dots, N$  at rate  $i$ . We see that  $I_n$  is the difference of a uniform series of payments and a geometric series of payments. Thus,

$$\begin{aligned} PV(i) &= \sum_{n=1}^N I_n (1 + i)^{-n} \\ &= A(P/A, i, N) - \frac{(A - B i_b)}{1 + i_b} (P/A, j, N), \end{aligned}$$

where  $j = \frac{i - i_b}{1 + i_b}$ .

### Notes on Chapter 3: Transform Techniques in Cash Flow Modeling

#### Z-transforms and Discrete Cash Flows.

Up to now we have been able to compute the present value of a limited number of cash flows: constant, linear gradient and geometric gradient. Often we need to compute the present value of more complex cash flows. A technique that comes in handy is the use of  $Z$ -transforms.

Consider an infinite cash flow  $\{f(n)\}$ . Its present value can be written as

$$PV(i) = \sum_{n=0}^{\infty} f(n)(1+i)^{-n}.$$

Letting  $z = 1 + i$ , we see that the present value,  $PV(i)$ , can be written as

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}.$$

$F(z)$  is called the  $Z$ -transform of  $f(n)$  and this relationship is denoted by writing  $F(z) = Z\{f(n)\}$ .

One of the most important and useful properties of  $Z$ -transforms is that there is a *one to one* correspondence between the infinite cash flows  $f(n)$  and their  $Z$ -transforms  $F(z)$ . Another property of  $Z$ -transforms is that linear combinations of cash flows correspond to the same linear combinations of their  $Z$ -transforms. For instance, if  $f(n) \leftrightarrow F(z)$  and  $g(n) \leftrightarrow G(z)$ , then  $\alpha f(n) + \beta g(n) \leftrightarrow \alpha F(z) + \beta G(z)$ .

These properties plus the fact that  $F(z)$  has been pre-computed for a large number of cash flows  $f(n)$  is what makes the technique useful in practice. See Tables 3.1, 3.2, 3.3 and 3.4 in your book.

*Example:* If  $f(n) = 3n + 7(1+j)^n - 4e^{-jn} + 8$ ,  $n = 0, 1, \dots$ , then

$$F(z) = 3 \left[ \frac{z}{(z-1)^2} \right] + 7 \left[ \frac{z}{z-(1+j)} \right] - 4 \left[ \frac{z}{z-e^{-j}} \right] + 8 \left[ \frac{z}{z-1} \right],$$

on substituting for  $z = 1 + i$  we obtain

$$F(z) = 3 \left[ \frac{1+i}{i^2} \right] + 7 \left[ \frac{1+i}{i-j} \right] - 4 \left[ \frac{1+i}{1+i-e^{-j}} \right] + 8 \left[ \frac{1+i}{i} \right].$$

Most realistic cash flows extend over finite time horizons, e.g.  $\{f(n)\}$ ,  $n = 0, 1, \dots, k$ . Here we cannot directly apply  $Z$ -transforms, but we can try writing the present value as the difference of two  $Z$ -transforms. Indeed, if

$$PV(i) = \sum_{n=0}^k f(n)z^{-n}.$$

Then

$$PV(i) = \sum_{n=0}^{\infty} f(n)z^{-n} - z^{-(k+1)} \sum_{n=0}^{\infty} f(n+k+1)z^{-n},$$

so that

$$PV(i) = F(z) - z^{-(k+1)}G(z)$$

where  $F(z) = Z\{f(n)\}$  and  $G(z) = Z\{f(n+k+1)\}$ .

More generally, if the cash flow is of the form  $\{f(n)\}$ ,  $n = h, \dots, k$ , we can write

$$PV(i) = z^{-h}H(z) - z^{-(k+1)}G(z),$$

where  $H(z) = Z\{f(n+h)\}$ .

*Example:* If  $f(n) = A$ ,  $n = 1, \dots, N$ , then  $F(z) = G(z) = H(z) = Cz/(z - 1)$ , so

$$PV(i) = A \left[ \frac{z}{z-1} \right] \left[ z^{-1} - z^{-(N+1)} \right] = A \left[ \frac{1 - (1+i)^{-N}}{i} \right],$$

which recovers the  $(P/A, i, N)$  factor.

### Laplace Transforms and Continuous Cash Flows

For continuous cash flows, the present value formulas

$$PV(r) = \int_0^{\infty} f(t)e^{-rt} dt \equiv F(r),$$

correspond to the *Laplace* transforms. The techniques used to compute present values of complex continuous cash flows, using Laplace transforms, are similar to those used for complex discrete cash flows using  $Z$ -transforms.

## Notes on Chapter 4: Depreciation and Corporate Taxation.

This is a brief treatment of the major aspects of the U.S. corporate law. The emphasis is on the depreciation and tax treatment of assets used for production and distribution in a trade or business.

### Corporate Tax Rates.

#### Tax Structure for Corporations.

The government levies a variety of taxes on corporations. Here we concentrate on federal income tax levied against taxable income.

Taxable Income = Gross Income - Allowable Deductions.

See pp. 130 for a list of allowable deductions. The federal tax rate is on a graduated structure. See table 4.1 in the book. Notice that the marginal tax rate goes from 15% to 25% to 34% to 39% and then back to 34%. The reason that the marginal tax rate is not monotone increasing is that the government wants a low tax rate (15%, 25%) for small corporations and a higher tax rate (34%) for large corporations. The tax surcharge (39%) is designed to gradually take away the subsidy as the taxable income increases. Notice that the average tax rate *is* monotonically increasing.

#### Effective and Marginal Income Tax Rates.

The *marginal* tax rate of a project, for a corporation, is the ratio of the additional income taxes levied against the corporation as a consequence of engaging in the project to the additional taxable income generated by the project.

The *effective* tax rate is the ratio of the total income taxes to the total taxable income.

Economic analysis should be done on the basis of marginal tax rates, because this is the rate at which the projects under consideration would be taxed.

Notice that large, profitable corporations have a marginal tax rate of 34% and an effective federal tax rate close to 34%. Small corporations and those that fluctuate between losses and profits will have marginal tax rates that vary, being lower in low income years and higher in high income years.

#### How to Combine Federal and State Taxes.

State taxes are an allowable deduction in computing federal income taxes. In some states the amount paid on federal taxes are an allowable deduction in computing state income taxes. What follows is an analysis that determines the overall tax rate in either case. Let

- $t_f$  federal tax per dollar of taxable income,
- $t_s$  state tax per dollar of taxable income,
- $f$  federal tax per dollar of before state tax income,
- $s$  state tax per dollar of before federal tax income,
- $y = 1$  if state taxes are deductible,
- $y = 0$  if state taxes are not deductible.

Clearly

$$\begin{aligned} f &= t_f(1 - s) \\ s &= t_s(1 - yf), \end{aligned}$$

or

$$\begin{aligned} f + t_f s &= t_f \\ y t_s f + s &= t_s. \end{aligned}$$

Solving the system of equations, we obtain,

$$\begin{aligned} f &= \frac{t_f - t_f t_s}{1 - y t_s t_f}, \\ s &= \frac{t_s - y t_f t_s}{1 - y t_s t_f}, \\ f + s &= \frac{t_f + t_s - (1 + y) t_s t_f}{1 - y t_s t_f}. \end{aligned}$$

*Example:* If  $t_f = 34\%$  and  $t_s = 7\%$ , then the overall tax rate is 37.12% or 38.62% depending on whether federal taxes are deductible or not.

### Taxes and Variability of Income

Consider the incomes of two firms A and B over a four year period. Firm A has a constant income of \$100,000.00 per year, for a total 4 year income of \$400,000.00. Firm B has an income of \$500,000.00 in years 1 and 3, and a loss of \$300,000.00 in years 2 and 4 for a total income of \$400,000.00. Assume that the tax rate is 40%. Then firm A pays \$40,000.00 per year in taxes, for a total of \$160,000.00. On the other hand, if taxes are only paid on income, firm B pays \$200,000.00 in years 1 and 3, and pays no taxes on years 2 and 4, for a total of \$400,000.00.

Since this method of computing taxes penalizes firms with variable income, the tax law allows losses to be carried forward up to five years and backward up to 3 years.

### Depreciation and its Relation to Income Taxes.

The cost of an asset that will be useful to a company for a number of years can not be written in its entirety as an allowable deduction in the year it was purchased. Most fixed assets have a limited useful life and consequently their value decreases with time, i.e., they depreciate. The federal government sets accounting guidelines as to how the cost of an asset can be depreciated, i.e., gradually converted into an allowable deduction. Notice that depreciation is not a cash expense, but being an allowable deduction it reduces taxable income and hence reduces taxes.

The best way to understand the effect of depreciation on after tax income is to ignore for the moment the government guidelines and focus on the objective of a profitable firm, i.e., maximizing the present value of future operations. Let

- $P$  installed cost of an asset,
- $F$  estimated salvage value,
- $N$  be the lifetime used for depreciation,
- $t_n$  tax rate in year  $n$ ,
- $D_n$  depreciation taken in year  $n$ ,
- $I_n$  taxable income before depreciation allowance in year  $n$ .

We assume that the installed cost  $P$  is paid in full at time zero. Later we will discuss the effect of financing the asset. The cash flow at time  $n = 1, \dots, N$  is

$$I_n - (I_n - D_n)t_n = I_n(1 - t_n) + t_n D_n.$$

Notice that the cash flows depend on the depreciation  $D_n$  taken in year  $n$ . In the absence of government regulations, one would select the depreciation schedule  $\{D_n\}_{n=1}^N$  maximizing the present value of the resulting cash flows at certain interest rate  $i$ . This is equivalent to

$$\begin{aligned} \max \quad & \sum_{n=1}^N D_n t_n (1 + i)^{-n} \\ \text{s.t.} \quad & \sum_{n=1}^N D_n = P - F. \end{aligned}$$

This problem is very easy to solve. We find the period with largest discounted tax rate

$$k = \operatorname{argmax}\{n : t_n(1+i)^{-n}\}$$

and set  $D_k = P - F$  and  $D_n = 0$  for  $n \neq k$ .

An important special case is when the tax rate is independent of  $n$ , say  $t_n = t$  for all  $n$ , and the interest rate  $i$  is positive. Then clearly  $k = 1$ , and therefore the optimal depreciation schedule is  $D_1 = P - F$ , and  $D_n = 0$ ,  $n \geq 2$ .

As mentioned before, the government does not allow this. The second best thing to do when the tax rate is constant is to depreciate an asset as quickly as possible. For this reason tax accountants have come up with a number of depreciation methods: straight line, S.O.Y.D, declining balance, etc. We briefly mention them here. In what follows  $BB_n$  stands for book balance at the end of year  $n = 1, \dots, N$ , with  $BB_0 = P$ . Then, the book balance is updated by the formula

$$BB_n = BB_{n-1} - D_n,$$

for  $n = 1, \dots, N$ .

#### **Straight-Line Method.**

$$\begin{aligned} D_n &= (P - F)/N, & n = 1, \dots, N, \\ BB_n &= P - n(P - F)/N, & n = 1, \dots, N. \end{aligned}$$

This method may be used to depreciate any property; the lifetime used for depreciation is taken from government ADR (asset depreciation range) tables.

#### **Sum of Year Digits (SOYD) Method.**

Notice that

$$\sum_{n=1}^N n = \frac{N(N+1)}{2}.$$

The idea is to depreciate the fraction

$$\frac{2(N+1-n)}{N(N+1)}$$

in year  $n$ ,  $n = 1, \dots, N$ . Thus

$$D_n = \left[ \frac{2(N+1-n)}{N(N+1)} \right] (P - F),$$

$n = 1, \dots, N$ .

#### **Declining Balance Method.**

The declining balance method allocates each year a given fraction of the book balance at the end of the previous year.

$$D_n = \alpha BB_{n-1} = \alpha P(1 - \alpha)^{n-1}.$$

The DDB (double declining balance) sets  $\alpha = 2/N$ , the 150% DB (one and a half declining balance) sets  $\alpha = 1.5/N$ . A problem with this method is that, if unchecked, it is unlikely that  $BB_N = F$ . If  $BB_N > F$ , a switch to straight line is made as soon as the straight line allowance is larger than the declining balance allowance. Otherwise depreciation terminates as soon as the  $BB_n < F$ , by appropriately adjusting the depreciation during year  $n$ .

#### **Tax Reform of 1986.**

Assets are fully depreciated, i.e.,  $F = 0$ .

Only two possible methods can be used; either the MACRS (Modified Accelerated Cost Recovery System) or Straight Line.

To apply the MACRS we must first classify personal property by determining the ADR (Asset Depreciation Range) class life by using a U.S. Treasury Department Publication.



The MACRS is DDB with optimal switching to straight-line and mid year convention for assets in the 3,5,7 and 10 year classes; 150%DB with optimal switching and mid year convention for assets in the 15 and 20 year classes and straight line with mid month convention for property in the 27.5 and 31.5 year classes. See table 4.8.

*Example:* To construct table 4.8 for  $N = 5$ , note that the double declining balance applies, so  $\alpha = 0.4$ . Because of the mid year convention, we allow

$$\begin{aligned} D_1 &= 0.2 \text{ so } BB_1 = 0.8, \\ D_2 &= \alpha BB_1 = 0.32 \text{ so } BB_2 = 0.48, \\ D_3 &= 0.192 \text{ etc.} \end{aligned}$$

The tables include the timing of the optimal switching to straight line.

### After Tax Cash Flow Analysis.

If the acceptance of a project is likely to change the company's marginal tax rate, then we can estimate the after tax cash flow of the project by comparing the after tax cash flow with and without the project.

For large, profitable corporations, the acceptance or rejection of a project is not likely to change the marginal tax rate. In this case, we can use *generalized cash flows*. This approach consists of identifying items that increase or decrease the after tax cash flow. Items that increase the after tax cash flow are:

$$\begin{aligned} &\text{after tax proceeds from sale of investment at time } n, \\ &\text{bank loans at time } n, \\ &t_m(\text{depreciation at time } n,) \\ &(1 - t_m)(\text{revenues at time } n). \end{aligned}$$

Items that decrease the after tax cash flow are:

$$\begin{aligned} &\text{investment at time } n, \\ &\text{loan principal repayment at time } n, \\ &(1 - t_m) \text{ expenses at time } n, \end{aligned}$$

where  $t_m$  represents the marginal tax rate and expenses include labor, materials and interest.

**Impact of Inflation.** Assume that an asset worth  $P$  dollars is purchased at time 0 and that it is 100% financed with interest payments  $\{IP_n\}_{n=1}^N$ , and principal payments  $\{PP_n\}_{n=1}^N$ . Assume further that the asset will generate the string of revenues  $\{R_n\}_{n=1}^N$ ; that it will be depreciated according to the schedule  $\{D_n\}_{n=1}^N$ . The tax rate is assumed to be  $t$  throughout the horizon, and the inflation rate is assumed to be zero. According to the above formula for generalized cash flows, we have

$$\begin{aligned} F_n &= tD_n + (1 - t)R_n - PP_n - (1 - t)IP_n \\ &= (1 - t)R_n + t(IP_n + D_n) - (PP_n + IP_n) \\ &= (1 - t)R_n + t(IP_n + D_n) - TP_n, \end{aligned}$$

for  $n = 1, \dots, N$ , where  $TP_n = IP_n + PP_n$ . The present value of the cash flow is

$$PV(i) = (1 - t) \sum_{n=1}^N R_n(1 + i)^{-n} + \sum_{n=1}^N [(IP_n + D_n)t - TP_n](1 + i)^{-n}.$$

Now, assume a similar scenario where the inflation rate  $f$  is positive and the revenues are responsive to inflation. That is,  $R'_n = R_n(1 + f)^n$ . The actual dollar cash flows are therefore

$$G'_n = R'_n(1 - t) + (IP_n + D_n)t - TP_n,$$

$n = 1, \dots, N$ . Are we better or worst off? To answer this question we first convert the actual dollar cash flows to constant dollar cash flows, and then obtain the present value of the cash flows. The constant dollar cash flows are

$$G_n = R_n(1 - t) + [(IP_n + D_n)t - TP_n](1 + f)^{-n},$$

$n = 1, \dots, N$ . Taking the present value we obtain

$$PV(i) = (1 - t) \sum_{n=1}^N R_n(1 + i)^{-n} + \sum_{n=1}^N [(IP_n + D_n)t - TP_n](1 + f)^{-n}(1 + i)^{-n}.$$

Comparing the present value with and without inflation, we see that the revenue term is not affected, since revenues are responsive to inflation. Consequently, the answer depends on the second term involving the interest payments, the principal payments and the depreciation schedule. If  $(IP_n + D_n)t - TP_n < 0$  for all  $n = 1, \dots, N$ , as is usually the case when an asset is financed, then we are better off with positive inflation. On the other hand, if the asset is paid in full at time zero, then  $IP_n = TP_n = 0$ ,  $n = 1, \dots, N$ , so we are better off without inflation.

## Chapter 5: Selecting a Minimum Attractive Rate of Return

The interest rate used for discounting cash flows has a direct effect on the outcome of project evaluation and comparisons. A change in the interest rate can change the accept-reject decision for an individual project, and it can alter the choice of the “best” from among several projects. The rate is usually designated as the **interest rate**, or the **discount rate**; it is also known as the **minimum attractive rate of return, MARR**.

### Investments and Borrowing Opportunities.

Faced with a number of projects and financial sources the decision makers problem is to select now the projects that will add to the firm’s revenues and at the same time maintain the firm’s financial strength so that it will be able to undertake profitable investments in the future.

If all the investment opportunities available to a firm were ranked in decreasing order of their investment efficiency criterion, and were plotted against their cumulative capital requirements we would obtain the **investment opportunity curve**.

On the other hand the firm can sort its borrowing opportunities by their interest rate and plot them against the cumulative capital availability. This results in the **borrowing opportunity curve**.

It is not hard to argue that in a static environment, the intersection of the two curves represent the point at which the firm should operate, accepting all the projects whose investment efficiency is higher than the efficiency at the intersection of the two curves.

These curves will change from year to year. If this changes are not drastic, we can use the information in the curve to develop the MARR. Otherwise we need to use more sophisticated techniques.

Here we have ignored (i) the short term and long term effects, (ii) the effect of making investments on the costs of borrowing and (iii) the issue of project divisibility.

Uncertainties in profit projections may lead managers to “play safe” and operate somewhere to the left of the intersection. Expectations of better investment opportunities in future years, with corresponding cash needs may be another valid reason for not operating at the intersection point. Idle funds are usually invested in relatively risk-free, liquid investments.

More typically, to set the MARR, the cost to the corporation of each type of financing is computed independently. The proportion from debt and equity sources is weighted to estimate the actual interest rate paid for capital investment. The resulting rate is called the weighted average *cost of capital*. The MARR is often set relative to this cost, sometimes equal to it, but usually higher, depending upon the risk that must be taken when any of the available projects are accepted for investment. More appropriately, the MARR should be set relative to the weighted *marginal* cost of capital, which is typically higher than the weighted average cost of capital.

To compute the weighted average cost of capital, we must determine the after tax cost of debt capital, say  $k_b$ , and the cost of equity capital, say  $i_e$ , and weight each by the relative amount of debt and equity capital. Thus, if  $B$  denotes the market value of debt capital, and  $S$  represents the market value of equity capital, then the weighted average cost of capital is

$$k = \frac{B}{V}k_b + \frac{S}{V}i_e$$

where  $V = B + S$ .

Often corporations need to raise additional capital to fund projects. An important question is how to raise additional capital. Debt financing increases risk and decreases the ability of the firm to raise debt capital in future years. On the positive side debt financing can increase the return on equity if the capital is invested profitably. On the other hand, equity financing decreases risk and preserves the ability to borrow in the future. Equity financing is, however, often regarded as bad news which reduces the market value of current stockholders. Finding the best debt to equity ratio is a non-trivial problem which we will only briefly discuss here.

Rearranging, we obtain

$$i_e = k + \frac{B}{S}(k - k_b).$$

We can think of  $k$  as the return on assets, of  $k_b$  as the after tax return on bonds, and  $i_e$  as the return on equity. This equation suggests that if  $k > k_b$ , the return on equity increases as we increase the debt to equity ratio  $B/S$ . However, increasing, a highly leveraged company is perceived as more risky by bondholders and after tax cost of debt capital is likely to rise reducing  $i_e$ . Also, a highly leveraged firm magnifies losses to equity holders in years where  $k < k_b$ .

Of more importance to us is the marginal cost of capital as a minimal attractive rate of return. Suppose that engaging in a project that requires  $C$  units of capital changes the average cost of capital from  $k_0$  on  $V$  to  $k_1$  on  $V + C$ . Let  $r$  be the return of the project. Then, we are better off by engaging in the project if and only if

$$Cr \geq (V + C)k_1 - Vk_0$$

or equivalently if and only if

$$r \geq k_1 + \frac{V}{C}(k_1 - k_0).$$

Since raising additional capital usually results in a higher average cost of capital, we have  $k_0 < k_1$  and consequently  $r > k_1$ .

As an example consider the following situation:  $k_0 = 14.5\%$ ,  $k_1 = 15\%$ ,  $V = \$1,000,000$ , and  $C = \$100,000$ . Then,

$$r \geq 0.15 + \frac{\$1,000,000}{\$100,000}(.15 - .145) = 0.20.$$

To continue the example, consider what happens if  $r = 15\%$  and the investment is accepted. Then,

$$Cr < (V + C)k_1 - Vk_0$$

or

$$\$15,000 > \$165,000 - \$145,000 = \$20,000.$$

This example illustrates that caution should be exercised on determining the minimum attractive rate of return.

### Cost of Capital from Individual Sources.

The bulk of financing for large corporations is achieved with bonds and common stocks, referred as **debt** and **equity**, respectively. For simplicity we will treat borrowing as consisting of these two types of opportunities. In what follows we will learn to compute the cost to the corporation of debt and equity.

#### Debt Capital

Recall that interest paid on loans is a tax-deductible expense. So the after-tax cost of loans and bonds is less than the before-tax loan interest rate and bond yield rate.

**Bank Loans:** There are several cases to consider, depending on the number of interest payments and tax payments per year. In the simplest case both are paid annually resulting in an after tax rate

$$k_b = (1 - t_m)i_b$$

where  $i_b$  is the before-tax nominal interest rate per year and  $t_m$  is the marginal corporate income tax rate. See example 5.1

Most corporations, however, pay income taxes more frequently, say quarterly, based on estimated income statements. So the tax benefit of the interest expense is taken quarterly. To obtain the after tax rate, we first compute the quarterly rate by solving

$$1 + (i_b t_m / 4)(P/A, k_{bp}, 4) - (1 + i_b)(P/F, k_{bp}, 4) = 0,$$

and then converting to the cost per year

$$k_b = (1 + k_{bp})^4 - 1.$$

Here  $k_{bp}$  represents the quarterly interest rate. See example 5.2

A third possibility is when the interest payments are actually made quarterly. Then the appropriate after-tax rate is

$$k_b = [1 + i_{bp}(1 - t_m)]^4 - 1.$$

Notice that the corresponding formula, 5.4, in the book is incorrect. See example 5.3

When there are **discount points** on a loan, the borrower does not receive the full amount of the loan, but payments are calculated on the full amount. The effective loan rate is thereby increased. For tax purposes the discount is **prorated** over the life of the loan. The exact method for finding the after-tax rate requires setting up a loan schedule and computing the after tax cash flow.

The loan schedule method may be time consuming if an electronic spreadsheet that computes the loan schedule and the after tax cash flow is not available. An alternative method, is to compute the interest rate that makes the present value of the borrower's before-tax cash flow equal to zero, and then multiplying it by one minus the tax rate ( $1 - t_m$ ). See example 5.5.

**Bonds:** Before discussing the after-tax cost of bonds, we will define and classify bonds. In addition we will discuss how to compute the present value of a bond at a given interest rate, and how to compute the yield of a bond at a given price.

A **bond** is a note issued by a corporation or government entity for the purpose of obtaining needed capital for financing projects. The repayment conditions are specified at the time the bonds are issued. These conditions include the bond **face** value, the bond **interest rate**, the bond interest **repayment period**, and the bond **maturity date**.

The bond face values is usually a multiple of \$1000. The face value:

- (i) represents the sum paid at maturity date,
- (ii) determines the interest payment.

Let

- $F$  bond face value,
- $r$  bond interest rate,
- $m$  number of payments per period.

Then, the interest paid per period, is  $I = Fr/m$ . Often a bond is purchased at a **discount** (less than face value) or at a **premium** (more than face value).

In addition to U.S. government bonds, bonds are usually classified as **mortgage**, **debenture**, or **municipal** bonds. *Mortgage* bonds are backed by mortgages or specified assets. There are several types of mortgage bonds including first and second mortgage bonds. In the event that of foreclosure by the bondholders, the first-mortgage bonds take precedence during liquidation.

*Debenture* bonds are not backed by any form of collateral. The reputation of the company is important for attracting investors to this type of bond. They can be convertible, non-convertible, subordinate or junk bonds. Convertible bonds can be converted into common stock at a fixed rate as long as the bonds are outstanding. Subordinate debentures represents debt that ranks behind other debt in the event of liquidation or reorganization. Junk bonds are debenture bonds rated BBB or lower by Standard and Poor's bond rating.

Finally, *municipal* bonds are income tax free bonds. They can be either general obligation or revenue bonds. See handout for a more complete classification of bonds.

What is the present worth of a bond with face value  $F$  and coupon rate  $r$ , if it pays interest  $m$  times per year, it matures in  $N$  years, and its cash flow is discounted at a nominal interest rate of  $i\%$  per year?

$$P = \frac{Fr}{m}(P/A, i/m, mN) + F(P/F, i/m, mN).$$

*Example:* If  $F = \$10,000$ ,  $r = 6\%$ ,  $m = 2$ ,  $N = 15$ , and  $i = 8\%$ , then

$$P = 300(P/A, 4\%, 30) + \$10,000(P/F, 4\%, 30) = \$8,270.60.$$

The rate of return, or **yield**, of a bond with face value  $F$ , that matures in  $N$  years, with coupon rate  $r$ , that pays interest  $m$  times per year, and sells for  $P$  dollars, is the interest rate that makes the present value of the cash flow equal to zero. In other words, the yield is the interest rate at which the cash flow generated by the bond is equivalent to its current price. Thus, the yield, is the interest rate that solves

$$P = \frac{Fr}{m}(P/A, i/m, mN) + F(P/F, i/m, mN).$$

The solution to the above equation is the nominal yield. The effective yield is

$$(1 + i/m)^m - 1.$$

*Example:* If  $F = \$1,000$ ,  $r = 4\%$ ,  $m = 2$ ,  $N = 20$ , and  $P = \$800$ , then the nominal yield is 5.74% and the effective yield is 5.86%.

The **after-tax cost** of bonds depends on the timing of interest and tax payments. In addition, there are issuing expenses and discounts (or premiums) to be considered. The selling expense  $S_e$  must be prorated over the maturity period  $N$  for tax purposes. The annual expenses  $A_e$  are tax-deductible each year. The discount or difference between the face value  $F$  and the selling price  $P$ , must be spread over the maturity period by prorating (bonds issued before 7-01-82) and by the yield method (bonds issued after 7-01-82). We will assume quarterly tax payments.

To find the after tax cost for bonds issued before 7-01-82 we must first find the after tax cost per quarter by solving equation 5.5 in the book. The annual after tax rate is then obtained by using equation 5.3. Alternatively, an approximate method can be used by solving equation 5.6 followed by 5.3. See example 5.6.

To find the after tax cost for bonds issued after 7-01-82 we must find the yield to maturity of the bond. We solve for  $i_b$  using

$$P - (Fr)(P/A, i_b, N) - F(P/F, i_b, N) = 0.$$

Then, for tax purposes, we treat the bond as a loan at interest rate  $i_b$ , with negative amortization and a balloon payment. The after tax cash flow is obtained by subtracting from the actual interest payments, the interest tax benefit, the selling tax benefit, and the annual expense tax benefit. The effective after tax cost of the bond is computed by finding the interest rate that makes the present value of the cash flow equal to zero. See Table 5.1. In fact, this is only an approximate method. To be more precise, we would need to consider quarterly payments of taxes, and semi-annual payments of interest.

The above method, of finding the yield of the bond, developing a loan schedule, computing the after tax cash flow, and finding the interest rate that makes the present value of the after tax cash flow may be time consuming, if an electronic spreadsheet that computes the yield, the loan schedule, and the after tax cash flow is not available. As for the case of a loan, an alternative method is to compute the interest rate that makes the present value of the borrower's before-tax cash flow equal to zero, and then multiplying it by one minus the tax rate  $(1 - t_m)$ . Notice that the selling expenses

and the annual expenses should be taken into account in determining the before tax cash flow. See page 178, for an example of this alternative method.

### Equity Capital.

Issuance of **preferred stock** carries with it an obligation to pay a stated dividend annually. The cost of capital is therefore the stated dividend divided by the price of the stock. However, the stock is often sold at a slight discount, so the actual proceeds from the stock, rather than the stated price, should be used as the denominator. For example, a 10% preferred stock with a value of \$200, which is sold for \$190 per share (5% discount), has a cost of  $10/0.95 = 10.53\%$ . The dividends are paid from after-tax corporate earnings.

**Common Stock:** The simplest valuation model for common stock considers only dividend payments. The value of a share to a stockholder immediately after the year 0 dividend is

$$P_0 = \sum_{n=1}^{\infty} DPS_n(1 + i_e)^{-n},$$

If, the dividends are constant, this results in

$$P_0 = \frac{DPS}{i_e}.$$

#### Growth model:

If  $DPS_n = DPS_1(1 + g)^{n-1}$ , and  $g < i_e$  then

$$P_0 = \frac{DPS_1}{i_e - g}$$

resulting in

$$i_e = \frac{DPS_1}{P_0} + g.$$

The growth rate can be estimated by using the ratio of retained earnings to current share price

$$g = (EPS_1 - DPS_1)/P_0$$

so

$$i_e = \frac{EPS_1}{P_0}.$$

*Example* A corporation plans to raise capital for a new plant through the sale of \$2,500,000 worth of common stocks valued at \$20 each. If a \$1 dividend is planned for the first year and an appreciation of 9% per year is desired for future dividends. Estimate the cost of capital from this stock issue.

$$i_e = \frac{1}{20} + 0.09 = 0.14.$$

□.

An alternative valuation model for common stocks, called the capital asset pricing model (CAPM), consists of a regression analysis of the historical returns of a specific stock to the performance of the market as a whole. The return required for a stock issue is the cost of equity capital  $i_e$ . The CAPM states that

$$i_e = i_f + \beta(i_m - i_f)$$

where  $i_f$  is the return of a risk free asset such as US treasury bills,  $i_m$  is the return on stocks in a defined market portfolio by a prescribed index, and  $\beta$  is the volatility factor of the company's stock. If  $\beta < 1.0$ , the stock is less volatile than the market. Conversely if  $\beta > 1.0$ , the stock is more volatile than the market.

*Example:* A computer manufacturer has an historic  $\beta$  value of 1.7, the market premium is 5% and treasury bills are paying 7%. Estimate the cost of the common stock capital.

$$i_e = 0.07 + 1.7(0.05) = 0.155.$$

□.

### Use of a Weighted-Average Cost of Capital.

We assume:

- The ratio of debt financing to equity financing remains constant;
- The costs of individual financing sources remain the same;
- The ratio of dividend growth remains constant;
- The marginal tax rate remains the same.

For simplicity we assume annual cash flows. The general formula for computing a weighted-average cost of capital is

$$k = \frac{B}{V}k_b + \frac{S}{V}i_e,$$

where  $B$  represents debt capital,  $S$  represents equity capital, and  $V = B + S$  represents total capital. Notice that  $\frac{B}{V}$  is the fraction financed by debt, and  $\frac{S}{V}$  is the fraction financed by equity. If there are several types of debt, the formula is expanded accordingly.

When the debt consists of bonds issued at par and there are no selling expenses or annual expenses, we can write

$$k = \frac{B}{V}i_b(1 - t_m) + \frac{S}{V}i_e.$$

See example 5.9

Much has been written about the optimal debt to equity ratio. To understand the basic concepts let  $EBIT$  represent earnings before interest and taxes. Then return on assets is given by  $r = EBIT(1 - t)/V$ , and return on equity is given by  $ROE = (EBIT(1 - t) - Bk_b)/S$ , which can be written as

$$\begin{aligned} ROE &= \frac{EBIT(1 - t)}{V} \frac{B + S}{S} - \frac{B}{S}k_b \\ &= r \frac{B + S}{S} - \frac{B}{S}k_b \\ &= r + \frac{B}{S}(r - k_b). \end{aligned}$$

Thus a large debt to equity ratio  $B/S$  magnifies the return on equity when  $r > k_b$ . Notice that a high debt to equity ratio increases the risk taken by the firm. What happens if the return on assets  $r$  is smaller than the after tax cost of debt capital  $k_b$ ?

### Net Equity Flows.

When the cash flow computations reflect interest, taxes, and debt repayment, what is left is called net equity flow. The proper rate for discounting net equity flow, when using PV, is  $i_e$ , the equity interest rate. See example 5.10

### After-Tax Composite Flows.

We can also evaluate expressions without explicitly treating the debt flows by using  $k$ , the weighted average cost of capital. The debt financing is treated implicitly, this method is particularly appropriate when debt financing, such as bonds, is not identified with individual investments but rather enables the company to engage in a set of investments. See table 5.3

Both methods will yield the same solution if the debt-equity ratio remains constant. We recommend the after-tax composite flow method.



### Setting the MARR Relative to Cost of Capital

As discussed before, the MARR is usually set above the cost of capital when the measure of risk is taken into account. There are no set rules to determine the MARR. However, it is common to separate project alternatives into risk categories and establish the MARR relative to the cost of capital for each category. For example, if the weighted average cost of capital is 12%, a corporation may use a MARR of 13-17% for low risk projects such as productivity improvement, a MARR of 18-24% for medium risk projects such as capacity expansion or the implementation of new, but accepted technology, and a MARR of 25%-30% on high risk projects such as new product development, international joint ventures, etc.

## Notes on Chapter 6: Measures of Investment Worth—Single Project

Here we focus on evaluating single projects. Investment projects and securities (stocks, bonds, etc.) require initial outlays in order to provide a later sequence of cash receipts. Investments, however, are not marketable securities.

We will study several different criteria to evaluate a single-project. For all the criteria we assume:

MARR is known to the decision maker,  
 stable, perfect capital market, can borrow or lend at MARR,  
 complete certainty about investment outcomes,  
 cash flows represent the net after tax cash flows.

Notation:

$n$  index of time period,  
 $i$  interest rate used for discounting, e.g., MARR or market rate,  
 $F_n$  net cash flow at the end of period  $n$ ,  
 $N$  project life.

**Present Value Criterion:** The present value at interest rate  $i$  is

$$PV(i) = \sum_{n=0}^N F_n(1+i)^{-n}.$$

More generally, if  $i = (i_n)_{n=1}^N$ , and  $i_n$  is the interest rate in period  $n$ , then

$$PV(i) = F_0 + \sum_{n=1}^N F_n [\prod_{k=1}^n (1+i_k)]^{-1}.$$

The criterion is to accept the project if  $PV(i) > 0$ , remain indifferent if  $PV(i) = 0$  and reject if  $PV(i) < 0$ . A positive PV represents a surplus and the project should be accepted if sufficient funds are available for it.

**Future Value Criterion:** Compute the future value  $FV(i)$  and accept the project if  $FV(i) > 0$ , remain indifferent if  $FV(i) = 0$  and reject if  $FV(i) < 0$ .

**Annual Equivalence Criterion:** Compute the annual equivalence  $AE(i)$  and accept the project if  $AE(i) > 0$ , remain indifferent if  $AE(i) = 0$  and reject if  $AE(i) < 0$ .

All of the above criteria are equivalent.

**Project Balance Concept:** Let  $PB_0 = F_0$  then the project balance at the end of year  $n$  at interest rate  $i$  is given by

$$PB_n = PB_{n-1}(1+i) + F_n, n = 1, \dots, N.$$

Sometimes we write  $PB(i)_n$  to emphasize that dependence on  $i$ . We can think of  $PB_n$  as the value at time  $n$  of the truncated cash flow  $\{F_k : k = 0, \dots, n\}$ . In particular  $PB_N = FV$ . If  $PB_N > 0$ , we can say that the firm recovers the initial capital plus any interest owed, with an end of horizon profit equal to  $PB_N$ . If  $PB_N = 0$  the firm recovers exactly its investment plus interest owed. If  $PB_N < 0$  the proceeds from the project are not enough to recover the investment plus the interest owed, so the firm ends up with a loss.

If a project is accepted and the estimates of the cash flows are accurate, then  $PB_n$  indicates the balance that we would observe at the end of period  $n$ . A negative value of  $PB_n$  indicates that the investment plus the interest owed are yet unrecovered. If  $PB_n$  is positive at time  $n$ , then the firm has recovered the initial investment plus interest owed and has an over-recovered balance. Notice that the formula that computes  $PB_{n+1}$  from  $PB_n$  assumes that  $PB_n$  can be invested at rate  $i$ .

*Example:* Consider the cash flow  $\{-1600, 10000, -10000\}$ . Assume  $i = 30\%$ . Then  $PV(30\%) = 175.15 > 0$ , so according to the present value criterion the project should be accepted. The project

Figure 1: Present Value of the Pump Problem

balances are  $PB_0 = -1,600$ ,  $PB_1 = -1,600(1.3) + 10,000 = 7,920$ ,  $PB_3 = 7,920(1.3) - 10,000 = 296 > 0$ . Notice that  $FV = PB_3 = PV(1.3)^2$ . Notice also that at the end of year one, the project balance is positive and that there is a negative cash flow at the end of year 2. Since the project will not generate more cash, we will need to invest  $PB_1$  outside the project. The computations leading to  $PB_2$  assume that we can invest the money at 30%. If we cannot invest the money outside the project at a sufficiently high interest rate, we may end up losing money.  $\square$ .

This example indicates that one must use the PV criterion with care and must ask: What can be done with over-recovered balances when they occur? Of course, if the company can actually reinvest money outside the company at rate  $j = i$ , then the present value criterion will be correct. Often however, the reinvestment assumption does not hold. When this happens, we must be very careful in using traditional criteria such as present value, future value, annual equivalence. These criteria assume that we can invest over-recovered balances at the same rate that we use to discount the project's cash flow. If the rate at which we can invest over-recovered balances is smaller than the discount rate, these traditional criteria may lead us to incorrectly accept a project; a mistake that may have dire financial consequences.

The safest way out, is to pre-specify a conservative rate, say  $j$ , at which we can invest over-recovered balances. This rate may be the weighted average cost of capital, because we can always use over-recovered balances to pay back debt, and to repurchase outstanding stock. Then, if a rate  $i$  is to be charged to the project, we can compute the project balances as:

$$PB(i, j)_0 = F_0$$

and

$$PB(i, j)_n = \begin{cases} PB(i, j)_{n-1}(1 + i) + F_n & \text{if } PB(i, j)_{n-1} \leq 0 \\ PB(i, j)_{n-1}(1 + j) + F_n & \text{if } PB(i, j)_{n-1} > 0. \end{cases}$$

Notice that  $PB(i, j)_N$  is the amount of money (possibly negative) that the firm will have at the end of the project after repaying the investment and the interest owed. The obvious, and correct

criterion, is to accept the project only if  $PB(i, j)_N > 0$ . This is the only criterion that I will recommend. However, because most people use one or more of the traditional criteria that we have discussed before, we need to make an effort to determine the conditions under which we can trust these criteria.

It is easy to show that  $PB(i, j)_n$  is decreasing in  $i$  and increasing in  $j$  for all  $n$  (the terms increasing and decreasing are to be interpreted in the non-strict sense). This is true, in particular, for  $PB(i, j)_N$ . The intuition behind this assertion is that the final balance, at time  $N$ , after repaying the capital and the interest owed will decrease as the interest rate charged to the project increases. Likewise, the final balance increases as the interest earned on over-recovered balances increases.

Because  $F_0 < 0$ , the initial balance is negative. If the interest rate charged to the project is very large, then all the balances will be non-positive, and we will never have an occasion to invest at rate  $j$ . Consequently, for large values of  $i$ , the balances  $PB(i, j)_n$  are independent of  $j$ . Let  $i_{min}$  be the smallest value of  $i$  for which  $PB(i, j)_N$  is independent of  $j$ . Thus for  $i < i_{min}$ , there will be at least one period with an over-recovered balance, whereas if  $i > i_{min}$  all the balances, except perhaps the last one, will be unrecovered. We divide the interval  $[0, \infty)$  into two regions:  $[0, i_{min})$  and  $[i_{min}, \infty)$ . The first region is called the mixed region, while the second region is called the pure region. If  $i$  (the interest rate that is charged to the project) falls into the pure region, there are no over-recovered balances, and so the decision to accept or to reject the project based on present value, future value, or annual equivalence will be correct. On the other hand, if  $i$  falls into the mixed region, the present value, the future value, and the annual equivalence method will yield an incorrect value unless  $j = i$ . These incorrect values can lead us to accept a bad project. In summary, the present value, the future value and the annual value criteria can be safely applied only if either  $j = i$  or if  $i > i_{min}$ .

#### The Internal Rate Of Return Criterion:

The “internal” rate of return of a project with cash flow  $\{F_n\}_{n=0}^N$  is “the” rate  $i^*$  such that  $PV(i^*) = 0$ . Notice that  $PV(i^*) = 0$  implies  $AE(i^*) = FV(i^*) = 0$ . The criterion is to accept the project if  $i^* > MARR$ , reject if  $i^* < MARR$  and to remain indifferent if  $i^* = MARR$ .

The reason we quote the word “the” in the definition is that a root satisfying  $PV(i)$  does not always exist, and even when it does, it is not always unique. Later we will see why we also have the word “internal” under quotes. Finding “the” internal rate of return of the cash flow is equivalent to finding the roots of a polynomial in  $x = (1 + i)^{-1}$ . This can be done by a number of methods including bisection and Newton’s method.

When is there a unique positive  $i^*$  such that  $PV(i^*) = 0$ ? To answer this question we need a few definitions. An *investment* is a cash flow with  $F_0 < 0$ ; a *conventional* (or simple) investment is an investment with only one change in the sign of the cash flow. A *potentially profitable* investment has a positive sum of net cash flows, i.e.,  $PV(0) = \sum_{n=1}^N F_n > 0$ .

**Proposition 1** *A potentially profitable conventional investment has a unique positive root.*

*Proof:* Because the cash flow is potentially profitable, we have  $PV(0) > 0$ ; because the cash flow represents an investment we have  $PV(\infty) = F_0 < 0$ ; consequently there is a sign change on  $PV(i)$ . Finally, because  $PV(i)$  is a continuous function of  $i$ , there must be a positive interest rate, say  $i^*$ , at which the present value vanishes. To show that  $i^*$  is unique note that because the investment is conventional the cash flow has one sign change, and from Descartes’ rule of signs we know that  $PV(i)$  can have at most one sign change.  $\square$ .

Because of the above proposition, we can confidently apply the IRR criterion to a potentially profitable conventional investment. One way to interpret the internal rate of return is to think of it as the largest interest rate at which you can borrow money to finance the project and still break even. Alternatively, you can think of the internal rate of return as the interest rate earned on the unrecovered project balance of an investment. You should not, however, think of the internal rate of return as the interest earned on the initial investment.

Can we extend the class of investments for which the IRR criterion is meaningful? The answer lies on whether or not the project has over-recovered balances at the largest root that makes the present

value equal to zero. If there are no over-recovered balances, then we can meaningfully interpret that rate as the IRR and apply the criterion. Otherwise, if there are over-recovered balances, we need to know at what rate we can invest these balances before we can say something meaningful.

Let us call an investment *pure*, if it has no over-recovered balances at its largest root. Thus an investment is pure if there exist an interest rate, say  $i^*$ , such that  $PV(i^*) = 0$ ,  $PV(i) \neq 0 \forall i \geq i^*$ , and  $PB_n(i^*) \leq 0 \ n = 0, \dots, N - 1$ . An investment that is not pure is called mixed. It is easy to show that all conventional investments are pure.

An alternative way of deciding whether a project is pure or mixed is to find

$$i_{\min} \doteq \inf\{i : PB(i)_n \leq 0, \ n = 1, \dots, N - 1\},$$

and to evaluate  $PB(i_{\min})_N$ . If  $PB(i_{\min})_N > 0$  then  $i^* > i_{\min}$ , and the investment is pure, if  $PB(i_{\min})_N = 0$  then  $i^* = i_{\min}$ , and the investment is pure, if  $PB(i_{\min})_N < 0$  then  $i^* < i_{\min}$ , and the investment is mixed.

As mentioned above, once a project is classified as mixed, we cannot say something meaningful unless we specify a reinvestment rate. Let  $j$  be the rate at which over-recovered balances can be invested, and let  $i(j)$  be the interest rate at which the ending balance is zero. That is,  $i(j)$  is defined implicitly by the formula:

$$PB(i(j), j)_N = 0.$$

We can think of  $i(j)$  as the largest rate at which the firm can borrow money to invest in the project and still break even. Notice that if  $i$  is the interest rate charged to the project, then the project is profitable only if  $i < i(j)$ . This is because

$$PB(i, j)_N > PB(i(j), j)_N = 0.$$

The above gives rise to the criterion:

$$\begin{aligned} i(j) > MARR & \text{ accept,} \\ i(j) = MARR & \text{ remain indifferent,} \\ i(j) < MARR & \text{ reject.} \end{aligned}$$

**Proposition 2** *The rate  $i(j)$  exists and is unique. Furthermore  $i(j)$  is increasing in  $j$ .*

*Proof:* Notice that  $PB(0, j)_N > \sum_n F_n > 0$ , and that  $F_0 < 0$  implies that for sufficiently large interest rate the ending balance is negative, i.e.,  $\lim_{i \rightarrow \infty} PB(i, j)_N < 0$ . Because  $PB(i, j)_N$  is continuous, there must be a positive rate  $i(j)$  such that  $PB(i(j), j)_N = 0$ .

To see that this rate is unique, just note that for fixed  $j$ ,  $PB(i, j)_N$  is strictly decreasing in  $i$ , so it cannot admit two different roots.

Finally, to see that  $i(j)$  is increasing in  $j$  note that

$$\frac{\partial i(j)}{\partial j} = -\frac{\frac{\partial PB(i, j)_N}{\partial j}}{\frac{\partial PB(i, j)_N}{\partial i}} > 0.$$

□.

We can obtain bounds on  $i(j)$  by observing  $PB(i_{\min}, j)_N$ . If  $PB(i_{\min}, j)_N \geq 0$ , then, since  $PB(i, j)_N$  is decreasing in  $i$  and is negative in the limit, we must have  $i(j) > i_{\min}$ . Recall that projects for which  $PB(i_{\min}, j)_N > 0$  are called pure. In this case we can identify  $i(j)$  with the largest root of  $PV(i) = 0$ , and the internal rate of return criterion makes sense. However, if  $PB(i_{\min}, j)_N < 0$ , then the root  $i(j)$  must occur to the left of  $i_{\min}$ , and therefore the project has over-recovered balances. These projects are called mixed, and for mixed projects the use of the internal rate of return criterion can lead us to accept a bad project.

In addition to the Return on Investment Capital Criterion, there are a number of “modified rate of return” methods that can be shown to be consistent with present value. These methods make

consistent assumptions about the interest rate at which intermediate *receipts* from projects may be invested. The reinvestment could be either in other projects or in the outside market.

### Solomon's Average Rate of Return (ARR)

This method assumes  $F_0 = -C_0 < 0$ ,  $F_n \geq 0, n = 1, \dots, N$ , and that the net revenues are reinvested at rate  $i$ , letting them accumulate until time  $N$ . It then seeks the rate, say  $s$ , which  $C_0$  needs to earn to reach the same accumulated value by time  $N$ . Thus

$$C_0(1+s)^N = \sum_{n=1}^N F_n(1+i)^{N-n}$$

so

$$s = \left[ \frac{\sum_{n=1}^N F_n(1+i)^{N-n}}{C_0} \right]^{1/N} - 1.$$

The criterion is

If  $s > i = \text{MARR}$ , accept,  
 if  $s = i$ , remain indifferent,  
 if  $s < i$ , reject.

It is easy to see that this criterion is consistent with present value. See example 6.8.

### Modified Internal Rate of Return (MIRR)

This method allows the  $F_n$ 's to be of any sign. For any number  $x$ , let  $x_+ = \max(x, 0)$ , and  $x_- = \max(-x, 0)$ , note that  $x = x_+ - x_-$  and also that  $x_+ \geq 0$ ,  $x_- \geq 0$ . Given a cash flow  $\{F_n\}$ , let  $G_n = (F_n)_+$  and  $H_n = (F_n)_-, n = 0, \dots, N$ . Thus,  $F_n = G_n - H_n, \forall n$  so the cash flow  $\{F_n\}$  can be written as the difference of  $\{G_n\}$  and  $\{H_n\}$ . Notice that the cash flow  $\{G_n\}$  corresponds to net *inflows*, and the cash flow  $\{H_n\}$  correspond to net *outflows*.

The MIRR method first computes the single outflow at time zero, say  $H$ , that is equivalent to the cash outflows  $\{H_n\}$  at rate  $i$ , and the single inflow at time  $N$ , say  $G$ , that is equivalent to the cash inflows  $\{G_n\}$  at rate  $i$ . Thus  $H = \sum_{n=0}^N H_n(P/F, i, n)$ , and  $G = \sum_{n=0}^N G_n(F/P, i, N-n)$ . Once  $H$  and  $G$  are computed, we ask for the rate, say  $j$ , at which  $H$  needs to be invested at time zero to accumulate to  $G$  by time  $N$ . That is,

$$H(1+j)^N = G,$$

so

$$j = \left( \frac{G}{H} \right)^{1/N} - 1.$$

The MIRR criterion is

If  $j > i = \text{MARR}$ , accept,  
 if  $j = i$ , remain indifferent,  
 if  $j < i$ , reject.

Again, it can be shown that MIRR is consistent with present value. The fact that ARR and MIRR are consistent with present value does not preclude the possibility of, as we have seen, PV itself being wrong when the re-investment assumption at  $i$  does not hold.

**Benefit-Cost Ratios** Another way to express the worthiness of a project is to compare the "benefits" with the "costs" and accept the project if the "benefits" are larger than the "costs". There are at least three different benefit-cost ratio methods, and they differ on what is meant by "benefit" and what is meant by "costs".

Assume that  $F_n = b_n - c_n$  is the net after tax cash flow, where  $b_n$  (resp.,  $c_n$ ) represents the revenue (resp., expense) at the end of period  $n$ . Now, let

$$B = \sum_{n=0}^N b_n(1+i)^{-n}$$

and

$$C = \sum_{n=0}^N c_n(1+i)^{-n}.$$

So,  $B$  (resp.,  $C$ ) represents the present value of the cash inflows (resp., outflows).

The **Aggregate B/C Ratio** is defined as

$$R_A = \frac{B}{C}.$$

An alternative expression for the benefit-cost ratio, is to separate the present value of the cash outflows into two parts. The first part, say  $I$ , is considered to be the “initial” investment while the second part, say  $O = C - I$ , consists of annual operating and maintenance costs. Notice that  $I$  may consist of the present value of cash outflow during several periods.

The **Netted B/C Ratio** is defined as

$$R_N = \frac{B - O}{I}.$$

Finally, the **Lorie-Savage Ratio** is defined as

$$L - S = \frac{B - C}{I}.$$

All of the above benefit-cost ratios are equivalent to the PV criterion for *single* project acceptance.

### Payback Period

A popular rule-of-thumb method for evaluating projects is to determine the number of periods needed to recover the original investment. The method favors projects with small payback periods. It can be shown through examples, that a blind application of the payback period method may lead us to reject good projects. This is, in part, because the payback period method ignores the time value of money, but mainly because it ignores the cash stream after the investment is initially recovered. However, the payback period may serve as a secondary criterion. For instance, it may be better to select a project with present value equal to \$1,000,000 with a 2 year payback period, than a project with present value equal to \$1,001,000 with a 30 years payback period.

### Time-Dependent Measures of Investment Worth

Recall the definition of  $PB(i, j)_n$ , the project balance at time  $n$  where an interest rate  $i$  is charged to the project, and over-recovered balances are invested at rate  $j$ . Among projects that have the same life  $N$ , and the same ending balance  $PB(i, j)_N$ , we may prefer those for which the *area of negative balance*

$$\text{ANB} \equiv \sum_{n=0}^{N-1} PB(i, j)_n^-$$

is smallest, or those for which

$$\text{APB} \equiv \sum_{n=0}^{N-1} PB(i, j)_n^+$$

is largest. Here  $x^+ = \max(x, 0)$ , and  $x^- = \max(-x, 0)$ .

Such projects tend to give the investor greater flexibility, in that they tend to tie-up the investment for a shorter period of time, or to generate positive cash balances for a longer period of time.

Notice, that our definitions of ANP and APB generalizes the one given in the textbook when  $j = i$  and  $PB(i, j)_n$  has a unique change in sign as  $n$  increases.



## Notes on Chapter 7: Decision Rules for Selecting among Multiple Alternatives

In the last few lectures, we have been concerned with different criteria to determine the worthiness of a single-project. We now turn our attention to the problem of selecting one among a set of mutually exclusive alternatives. Each alternative consists of a subset of the projects in consideration and, as a rule, the do-nothing alternative is always included among the set of alternatives.

For example, if projects  $A$ ,  $B$  and  $C$  are being considered, then the full set of alternatives consists of the 8 subsets of  $\{A, B, C\}$ . Sometimes, however, we can reject some of the alternatives at the outset because projects may exhibit some dependencies. For example, if  $B$  represents a die that can only be used with machine  $A$ , then it does not make sense to consider subsets of  $\{A, B, C\}$  where project  $B$  appears without project  $A$ . As a second example, if  $A$  is a machine, and  $B$  and  $C$  are projects that require the full use of  $A$ , then it may not make sense to consider alternatives where both projects  $B$  and  $C$  are present.

Once the set of feasible alternatives is considered, our problem is to select one and only one of them. Before we go ahead and prescribe a way to select an alternative let us note that, in this chapter, the decision is made without budgetary considerations. In addition, the decision is made under the assumption that we can earn money at the rate we use to discount the after tax cash flows. Under these conditions, if it is deemed appropriate to use the same discount rate for all the alternatives, it is not difficult to show that, the best alternative is the one with the largest present value.

As we have discussed earlier, the above assumptions do not always hold in practice. The results of this section may still be valid, however, if the chosen alternative does not violate any existing budget constraints, and if there are no over-recovered balances throughout its life.

Some of the criteria developed in Chapter 6 are harder to apply, e.g., rate of return methods and benefit-cost ratio methods. The reason is that these methods are *relative* measures of investment worth. To see this, suppose that your cost of money is 18% and you have to select between the following two investment alternatives. Alternative 1 requires an initial investment of \$1,000, and pays back \$1,500 at the end of one year. On the other hand, alternative 2 requires an investment of \$10,000 and pays back \$12,000 at the end of one year. The internal rate of return of alternative 1 is 50%, while the internal rate of return of alternative 2 is only 20%. Clearly both alternatives are desirable if the cost of money is 10%; moreover, alternative 1 seems to be more attractive from the internal rate of return point of view. Yet, present value calculations yield \$363.63 for alternative 1 and \$909.09 for alternative 2. Thus, we have an example of *opposite ranking*. One reason for this is that the alternatives cannot be scaled. If we could take alternative 1 10 times, then we would invest \$10,000 and would receive \$15,000 at the end of one year. In that case, both the present value and the rate of return methods would agree. However, as mentioned above, mutually exclusive alternatives cannot be scaled. A way to obtain results that are consistent with present value is to evaluate the alternatives in some specific order. For potentially profitable conventional investments, we may sort the alternatives in non-decreasing order of the sum of their cash flows. For the above example, alternative 1 comes before alternative 2. Alternative 1 is accepted because its internal rate of return (50%), is higher than the cost of money (10%). Should we give up alternative 1 to take alternative 2? That is equivalent to investing an additional \$9,000 to earn \$10,500 at the end of one year. The internal rate of return of this marginal investment is 11.67%, and since this rate of return is larger than the cost of money we should give up alternative 1 and take alternative 2 instead.

A similar incremental approach can be used with other rate of return methods, as well as with the benefit-to-cost ratio methods. The ordering of the projects that should be used with the different methods is given in page 246 of your book. Our recommendation, is to stay away from incremental methods.

If the lifetime of each alternative is finite, and a reinvestment assumption is made, then the project balance method can be used. The criterion is to select the alternative with largest  $PB(i, j)_N$  where  $N$  represents the last period in which an alternative has a non-zero cash flow.

Some projects may have a very long life, and may necessitate periodic replacement of an asset.

It is common to assume that the asset will be replaced by an identical asset at the same cost. This is the repeatability assumption. Implicit in this assumption, is that we are working with constant, as opposed to real, dollars. Therefore, it is appropriate to use an inflation-free discount rate. It is also possible to model technological changes. See the book of Canada & White for a few of these models.

On the other hand, a project may have a shorter life than the asset useful life. In this case, careful consideration of salvage values are important.

## Notes on Chapter 8: Deterministic Capital Budgeting Models

The objective of the capital budgeting process is the selection of an optimal portfolio of investments from a set of alternative investment proposals. An **optimal portfolio of investments** is defined as the set of investments that makes the greatest possible contribution to the achievement of the organization's goals, given its constraints. The constraints faced by a corporation in the capital budgeting process can include limited supplies of capital or other resources (personnel, for example) and dependencies between investment proposals. A dependency occurs if two projects are mutually exclusive—acceptance of one requires the rejection of the other—or if one project can be accepted only if another is accepted. Assuming that the organizational goals and constraints can be formulated as linear functions and that the future cash flows associated with the investment proposals can be forecast, the optimal portfolio of investments can be found using linear programming (LP).

An advantage of linear programming is that it leads to better solutions than alternative methods. Ranking is the most common alternative method. Some of the limitations of ranking methods include the inability to handle either multiple-resource constraints or interrelationships between the proposed investment alternatives.

A limitation of LP models for capital budgeting is portfolios of investment proposals which may contain fractional investment proposals. The optimal solution may, for example, include buying a third of an oil tanker. If the optimal set of investments identified through linear programming is infeasible because of fractional projects, integer programming should be used. Despite the “fractional projects” limitations, solutions to LP problems have meaningful interpretations which we will discuss later on.

As an example, consider Corporation ABC. ABC must determine the most profitable mix between two competing investment projects subject to budgetary constraints. Let  $x_1$  and  $x_2$  be the number of units of Investment 1 and 2 to be funded. Suppose that the present value per unit of Investment 1 is \$6 and the net present value per unit of Investment 2 is \$3. Suppose that Investment 1 requires \$2 of cash per unit in year 1, and \$4 per unit in year 2, while Investment 2 requires \$3 of cash per unit in year 1, and \$1 per unit in year 2. The total funds available in year 1 are \$12, and the total amount of cash available for investment in year 2 is \$8. Allied wants to maximize net present value. The linear program can be written as

$$\begin{aligned}\max Z &= 6x_1 + 3x_2 \\ 2x_1 + 3x_2 &\leq 12 \\ 4x_1 + x_2 &\leq 8 \\ x_1 \geq 0, x_2 &\geq 0.\end{aligned}$$

The optimal solution is  $Z^* = 6(1.2) + 3(3.2) = \$16.8$ . If fractional projects are not allowed then we must require  $x_1$  and  $x_2$  to be integers, and solve the corresponding integer programming problem. In this case the optimal solution is  $Z^* = 6(1) + 3(3) = \$15$ .

To state the problem in more general terms, the objective of the linear programming in capital budgeting is the allocation of limited resources among competitive activities (capital investments) in such a way that some measure of effectiveness involving these activities is optimized. Because the available resources are limited in quantity, the problem faced by the decision maker is the selection of the most effective mix of activities.

The choice of the objective function depends on the type of data available regarding the investment proposals under consideration. For example, if quantifiable data are available regarding the covariance between expected returns from the investment proposals, then minimization of the expected risk should be considered as the objective function. In that case, however, a related technique called quadratic programming should be used. If such data are unavailable, as in most capital budgeting problems, the objective function should be maximization of either the total net present value

or the total value of the investment proposals at a specific time in the future. If we can estimate how long we will be involved with the investment proposals, we should maximize the total value of the investment proposals at a future time (that is, at the end of our expected involvement with the investment proposals). If we cannot reasonably estimate how long we are expected to be involved with the investment proposals, we should choose to maximize total net present value.

The limits on both the availability of resources and on the possible relationships among some of the decision variables are represented by equations or inequalities called constraints. The problem is to find the values for the decision variables that result in maximization of the objective function subject to the constraints.

To be an LP problem, the objective function and the constraints all must be linear functions. Stated mathematically, the purpose of the LP model is to determine the value of the decision vector  $x$  that maximizes

$$\begin{aligned} z &= c'x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0. \end{aligned}$$

Linear programs can be solved by a variety of methods, including the so called interior point methods. The most common method, however, is the simplex method; an algorithm that moves from vertex to vertex of the polyhedron  $\{x : Ax \leq b, x \geq 0\}$ , while improving the objective value, until no further improvement is possible.

Efficient versions of the simplex method capable of solving moderate size problems are now available in spreadsheets such as Quattro Pro 6.0 for Windows and Excell 6.0 for Windows. Alternatively, commercial codes such as LINDO, and GAMS are also available. You should learn how to use one of these codes to solve the homework problems that I will be assigning.

In a typical constrained capital budgeting problem, there are limits on the amounts of certain resources available. Examples of such constraints are limits on trained manpower available, limits on certain materials, and limits on the amount of capital available.

Capital rationing is a frequently encountered constrained capital budgeting problem in which the amount of capital available for investment is limited. We will first study the pure capital rationing with no lending or borrowing allowed. We then present a model that allows both lending and borrowing. The intermediate models that allow borrowing but not lending or viceversa, can be obtained from the later model, by eliminating the lending or borrowing opportunities.

#### **Pure Capital Rationing, with No Lending or Borrowing Allowed**

It should be noted that many managers actually operate in a pure capital rationing environment once the budget has been determined during the planning stage. Therefore, pure capital rationing is a realistic assumption in many cases.

The linear programming problem is

$$\begin{aligned} Z &= \max \sum_{j=1}^J p_j x_j \\ \text{subject to} \quad & - \sum_{j=1}^J a_{nj} x_j \leq M_n, \quad n = 0, \dots, N \\ & x_j \leq 1, \quad j = 1, \dots, J \\ & 0 \leq x_j \quad j = 1, \dots, J, \end{aligned}$$

where

- $p_j$  is the present value of project  $j = 1, \dots, J$ ,
- $x_j$  project selection variable,
- $a_{nj}$  cash flow for project  $j$  at time  $n$ ,

$M_n$  budget limit at time  $n$ ,  
 $N$  end of the planning period.

It can be shown that there exist an optimal solution to this LP with at most  $N + 1$  fractional projects, i.e., at most  $N + 1$  of the  $x$  variables will be different from zero or one. Of course, if all the variables are non-fractional, then the solution is optimal. Even if there are some fractional projects, the LP solution may still be useful in a number of ways. First, the optimal value of the objective function, say  $Z^*$ , is an upper bound on the value of the optimal integer solution. Second, valuable information can be obtained showing the impact on the objective function of making small changes to the budget values. Such information is obtained by looking at the solution to the “dual” linear program. Such a solution can be readily obtained from the solution of the original, or “primal” problem, and it is part of the standard output of most commercial codes. The dual information may indicate, for example, that the budget for year one is larger than needed, and that there is a high return associated with an increased budget in year 2. On the basis of the dual solution, management may decide that it is in its best interest to revise the budget. Thus, the solution to the linear program, and its associated dual, may be a very helpful device to management in deciding the budget levels.

The dual of the LP can be written as:

$$\begin{aligned} v &= \min \sum_{n=0}^N \rho_n M_n + \sum_{j=1}^J \mu_j \\ \text{subject to} & \quad - \sum_{n=0}^N a_{nj} \rho_n + \mu_j \geq p_j, \quad j = 1, \dots, J \\ & \quad \rho_n \geq 0, \quad n = 0, \dots, N \\ & \quad \mu_j \geq 0 \quad j = 1, \dots, J, \end{aligned}$$

Here the  $\rho_n$  is the dual variable corresponding to the budget constraint  $-\sum_{j=1}^J a_{nj} x_j \leq M_n$ , and  $\mu_j$  is the dual variable associated with the constraint  $x_j \leq 1$ .

At the optimal solution, say  $x^*, \rho^*, \mu^*$ , the complementary slackness conditions must hold. These conditions state that

$$\begin{aligned} \rho_n^* \left( M_n + \sum_{j=1}^J a_{nj} x_j^* \right) &= 0, \quad \forall n, \\ x_j^* \left( \mu_j^* - p_j - \sum_{n=0}^N a_{nj} \rho_n^* \right) &= 0, \quad \forall j, \\ \mu_j^* (1 - x_j^*) &= 0, \quad \forall j, \end{aligned}$$

The economic interpretation of the dual variable  $\rho_n^*$ , is the rate by which the value of the objective function increases, at the optimal solution, as the budget in year  $n$  is increased. The economic interpretation of the dual variable  $\mu_j^*$  is the rate at which the value of the objective function increases at the optimal solution as we relax the constraint  $x_j \leq 1$ , and effectively allow more than one unit of alternative  $j$  to be taken.

Notice that if at the optimal solution not all the budget for year  $n$  is used, then by complementary slackness  $\rho_n^* = 0$ . This implies that a small increase in the the budget in year  $n$  has no impact on the value of the objective function. Conversely, if  $\rho_n^* > 0$ , then all the budget in year  $n$  is used in the optimal solution. Notice that  $x_j^* > 0$ , and  $\mu_j^* \geq 0$ , imply that

$$p_j + \sum_{n=0}^N a_{nj} \rho_n^* \geq 0.$$

If a project is not fully accepted, i.e., if  $x_j^* < 1$ , then  $\mu_j^* = 0$ , so

$$p_j + \sum_{n=0}^N a_{nj} \rho_n = 0.$$

Finally, for rejected projects,  $\mu_j = 0$ , and in view of the constraint

$$- \sum_{n=0}^N a_{nj} \rho_n + \mu_j \geq \mu_j$$

we have

$$p_j + \sum_{n=0}^N a_{nj} \rho_n \leq 0.$$

The economic interpretation is that the present value of a project plus the sum of the cash flows over the horizon “discounted” by the dual variables is non-negative for a fully accepted project, is exactly zero for a partially accepted project, and it is non-positive for a rejected project.

Notice that constraints on scarce resources other than capital can easily be added to the above LP formulation.

### Capital Budgeting

The pure capital rationing problem was developed under the assumption that the firm had no access to external markets, that is, it could neither borrow nor lend money. Including borrowing opportunities, without including lending opportunities, while keeping present value as the objective function may lead to situations where the firm borrows money at a rate that is higher than the discount rate, see example 8.2. Borrowing at a rate higher than the discount rate may happen when a highly profitable project is not fully taken because of tight budget constraints as shown in the following example.

*Example:* Suppose that the discount rate is 20%, and that there is a single project which requires an initial investment of \$1000 and pays back \$1300 at the end of one year. Suppose further that the investment budget is \$900, but that borrowing at 25% is possible. Should the firm borrow the \$100 needed to take the full project? If the firm borrows \$100 to take the project it will collect \$1,175 at the end of the year. This results in a present value of  $\$1,175/1.2 = \$979.16$  which is larger than the present value of taking 90% of the project and also larger than the present value of the do nothing alternative. The explanation lies in the fact that the total cost of securing \$1000 for one year is at most  $\$205 = (\$180 = \$900(.2)) + (\$25 = \$100(.25))$ , while the project returns \$300.  $\square$

There are, however, some inconsistencies in present value models that involve borrowing and lending that are not easy to overcome. See the discussion in section 8.3.2. Thus, when borrowing and lending opportunities are present, it is best to look at the *horizon* models. The horizon, is typically the number of periods for which the firm plans to be involved in the projects. In these models the objective function is the value, at the horizon, of the cash flows occurring *after* the horizon of the selected projects plus any cash that is available for lending minus any debt at that horizon. The budget constraints, are now cash balance equations that take into account borrowing and lending during the period as well as collection and payment of earlier lending and borrowing activities.

The simplest horizon model has a borrowing rate equal to the lending rate. See Table 8.6 for the detailed formulation, and Table 8.8 for the dual formulation. One peculiarity of this model is that the set of accepted projects is independent of the budget levels. In fact, under this model we would accept all projects with positive PV, and reject all projects with negative present value regardless of the budget levels. This is not surprising given that we can borrow and lend at the same rate. Notice however, that the value of the objective function is non-decreasing in the budget levels.

One extension is to have borrowing rates higher than lending rates and to impose upper bounds on borrowing during each budget period. This would involve adding a constraint of the form

$$w_n \leq W_n, \quad \forall n,$$

where  $W_n$  is the limit on borrowing during year  $n$ , that is, the firm's credit.

Another extension is to allow different borrowing rates during each period. The firm will first borrow from the least expensive source, and will then borrow at a higher rate once this source is exhausted. The cost of borrowing is assumed to have the shape of a step function; the larger the amount borrowed, within limits, the higher the interest rate. Let  $r_i$  be the interest rate that applies to borrowing an amount greater than  $W_{i-1}$  and less than or equal to  $W_i$ . A firm will borrow at interest rate  $r_i$  if it exhausts the limits placed on its borrowing at lower interest rates. Let  $v_n$  be the lending amount from time period  $n$  to  $n+1$ . The LP formulation is thus

$$\begin{aligned} Z &= \max \sum_{j=1}^J \hat{a}_j x_j + v_N - \sum_{i=1}^I w_{iN} \\ &\quad - \sum_{j=1}^J a_{0j} x_j + v_0 - \sum_{i=1}^I w_{i0} \leq M_0 \\ &\quad - \sum_{j=1}^J a_{nj} x_j - (1+r)v_{n-1} + v_n + \sum_{i=1}^I (1+r_i)w_{i,n-1} - \sum_{i=1}^I w_{in} \leq M_n \quad \forall n \\ &\quad w_{in} \leq W_{in} \quad \forall i, \quad \forall n, \\ &\quad 0 \leq x_j \leq 1. \end{aligned}$$

Notice that the model with only one borrowing and one lending rate, see Section 8.5.2, corresponds to the case  $r = r_l < r_1 = r_b$ , with  $W_{1n} = \infty \quad \forall n$ .

For the case with only one borrowing source with unlimited capacity we can obtain the following results by studying the dual problem and the complementary slackness conditions. For a project that is partially selected ( $0 < x_j < 1$ ) we must have

$$\hat{a}_j + \sum_{n=0}^N a_{nj} \rho_n = 0.$$

For a project that is rejected ( $x_j = 0$ ) we have

$$\hat{a}_j + \sum_{n=0}^N a_{nj} \rho_n \leq 0.$$

Finally, for a fully accepted project ( $x_j = 1$ )

$$\hat{a}_j + \sum_{n=0}^N a_{nj} \rho_n = \mu_j \geq 0.$$

We can interpret the quantity  $\hat{a}_j + \sum_{n=0}^N a_{nj} \rho_n$  as the value of the cash flow at time  $N$  where the value of a dollar at time  $n \leq N$  is valued at  $\rho_n$  dollars at time  $N$ , and a dollar at time  $n > N$  is valued at  $(1+i)^{N-n}$  at time  $N$ .

As for the dual variables we have  $\rho_N = 1$  and for  $n = 0, 1, \dots, N-1$

$$(1+r_l)\rho_{n+1} \leq \rho_n \leq (1+r_b)\rho_{n+1}.$$

Moreover, during periods of borrowing  $w_n > 0$  we have  $\rho_n = (1+r_b)\rho_{n+1}$  and during periods of lending  $v_n > 0$  we have  $\rho_n = (1+r_l)\rho_{n+1}$ . It then follows, from the complementary slackness conditions that if  $r_l < r_b$  then  $v_n > 0$  implies  $w_n = 0$  and similarly  $w_n > 0$  implies  $v_n = 0$ .

Finally, if there is a limit  $W_n$  on the amount we can borrow at time  $n$  then the bound on  $\rho_n$  is modified to

$$(1 + r_l)\rho_{n+1} \leq \rho_n \leq (1 + r_b)\rho_{n+1} + \beta_n,$$

where  $\beta_n$  is the dual variable corresponding to the constraint  $w_n \leq W_n$ .

So far we have avoided the issue of integer solutions in capital budgeting problems. The reason for this is that the LP solution, and its dual, is a useful tool to set budget limits. Once the budget limits are firmly set, we can no longer accept solutions with fractionally accepted projects. Notice that the LP formulation of the basic horizon model with equal borrowing and lending rates always admits an optimal integer solution. This is because, under this formulation, all projects with positive present value are accepted. Thus, the integrality constraints are of no concern when the borrowing and lending rates are equal. This is not the case, however, when the borrowing and lending rates are different, so more work is required to find an optimal integer solution. One technique that is commonly used to solve integer programs is called *Branch-and-Bound*. The technique starts with an optimal solution, say  $x^*, z^*$  for the linear program. Here  $x^*$  represents the project selection vector, and  $z^*$  is the value of the objective function. If all the components of  $x^*$  are either zero or one we are done. Otherwise, we pick a component, say  $j$ , of  $x^*$  such that  $x_j^*$  is fractional and create two subproblems, one with  $x_j = 0$  added as a constraint, and another with  $x_j^* = 1$  added as a constraint. We then proceed to solve these subproblems by an LP algorithm. For each of the subproblems, we repeat the procedure of identifying a fractional component, if such exists, and of creating new subproblems by adding constraints setting the fractional variable to zero and to one. This is the *branching* part of the algorithm. Conceivably we could follow the branches out of each subproblem until all the components are zero or one, and then pick the integer solution with the largest objective value. But this is working too hard! A better way is to record the solution of the subproblems and the corresponding value of the objective function. Remember that the value of the objective function of an LP solution is an upper bound on the value of the objective function of an optimal integer solution. If at any point we find that the LP solution of a subproblem has an objective value that is *smaller* than a previously obtained *integer* solution, we know immediately that we need not look any further into this branch. This is sometimes called *fathoming* the branch. This is the *bound* part of the algorithm. Different implementations of branch and bound differ in the way they look at the branches. The most common is *depth first search* in which a branch is followed until an integer solution is obtained, or until the branch can be fathomed.



### Notes on Chapter 9: Utility Theory

At the beginning of the course I asked the class to select between \$10 for certain, and the opportunity of earning \$25 with probability 50%. Most of the class recognized that the gamble is better in expectation, and the majority decided to take the risk. However, when I scaled the stakes so that the choice was between \$1,000,000 for certain, and an opportunity to earn \$2,500,000 with probability 50%, most of you took the \$1,000,000 in spite of the fact that the gamble had a higher expectation.

To further illustrate this behavior, consider the following game. A fair coin is tossed until the first time a head occurs. If it takes  $n$  tosses to obtain the first head the payoff to the player is  $2^n$  dollars. The probability that it will take  $n$  tosses to obtain the first head is  $\frac{1}{2^n}$ . Consequently, the expected payoff is

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1 = \infty.$$

We have shown that this game has an expected reward equal to plus infinity. What is the maximum amount that you would be willing to pay to play this game?

Why is it, that rational people do not always act to maximize expected rewards? One explanation is that many opportunities can only be taken once, so that we do not have a chance to observe the expected value. This is true, but it does not account for the fact that scale effects may reverse decisions as in the first example above.

Economist have tried to account for this behavior by postulating that decision makers are *risk averse*. A risk averse individual will always prefer less risk, and he may even pay a premium to reduce the risk. A homeowner that takes fire insurance is risk averse because he knows that he is being charged more than the expected cost of repairing his home. Those of you who prefer a \$1,000,000 for certain than a gamble that pays \$2,500,000 with probability one half are also risk averse. One difficulty with this theory is that it does not explain why people who are risk averse are willing to bet in a casino where the odds are against them. One explanation is that most risk averse individuals will only bet relatively small amounts and they see the expected losses from playing as a random cost for their entertainment.

To make things formal, two famous economist, Von Neumann and Morgenstern, came up with a set of axioms of behavior that lead to the existence of a utility function. Once the utility function has been found, they claim, the decision maker can make decisions based on maximizing expected utility.

Risk averse investors have an increasing *concave* utility function. For example,  $u(x) = \ln(x)$  and  $u(x) = \sqrt{x}$  are two example of increasing concave utility functions. The increasing concavity reflects a decreasing incremental value from each incremental dollar of wealth.

If  $X$  is a random variable, and  $u(\cdot)$  is the utility function, then the expected utility is

$$Eu(X).$$

By Jensen's inequality, we know that if  $u(\cdot)$  is concave, then

$$Eu(X) \leq u(EX).$$

This means that an individual with a concave utility function  $u(\cdot)$ , would prefer to have  $EX$ , for certain, than to face the random variable  $X$ . In other words, the individual will take less than  $EX$  for certain, rather than facing  $X$ . The exact amount that they will take for certain is called the certainty equivalent,  $CE$ . This quantity is defined by

$$u(CE) = Eu(X),$$

or by

$$CE = u^{-1}(Eu(X)).$$

Notice that  $CE$  is well defined if  $u(\cdot)$  is increasing, since  $u(\cdot)$  can be shown to be continuous. The difference between  $EX$  and  $CE$  is called the risk premium. For example, if  $X$  is a random variable taking value \$0 with probability 0.5, and \$1,000,000 with probability 0.5 then  $EX = \$500,000$ . If  $u(x) = \sqrt{x}$ , then

$$Eu(X) = \$500 = \sqrt{CE},$$

so  $CE = \$250,000$ , and the risk premium is  $RP = EX - CE = \$250,000$ . In this example, the investor would prefer to take any amount larger or equal to \$250,000 than the random variable  $X$  which has expectation \$500,000.

**Homework:** What is the expected utility of the coin tossing game if the utility function is  $u(x) = b \ln(x/a)$ ? What is the certainty equivalent? Does it depend on  $a$ , and  $b$ ?

It is clear that the shape of the utility function depends on the initial wealth. Buying fire insurance on a \$250,000 house makes more sense for an individual with a net wealth of \$250,000, than for an individual with net wealth of \$10,000,000. The latter can probably afford self insurance.

To measure the degree to which risk averseness depends on wealth,  $W$ , economist have come up with the risk aversion function

$$r(W) = \frac{-u''(W)}{u'(W)}$$

which is defined for utility functions with first and second derivatives. It is clear that  $u'(W) > 0$ , since we have postulated  $u(\cdot)$  to be increasing. As  $W$  increases, we would expect  $u(\cdot)$  to flatten out, so  $r(W)$  would decrease. If this is the case, then we say that the individual has *decreasing risk aversion*. For example, if  $u(W) = \ln(W)$ , then  $r(W) = 1/W$ , and  $r'(W) < 0$ , so an individual with utility function  $u(W) = \ln(W)$  is risk averse, with decreasing risk averseness. It can be shown that an individual with such a utility function, pays a decreasing risk premium as his wealth increases.

An individual's utility function, and his behavior can sometimes tell us something about his subjective probability distribution of a random variable. For example, suppose an individual with initial wealth of \$100,000 has a utility function  $u(x) = \sqrt{x}$  and is willing to bet \$10,000 on a ball game. Let  $p$  be his probability of winning. If he wins his wealth will be \$110,000, if he losses his wealth will be \$90,000. Since he prefers to bet, then

$$p\sqrt{110,000} + (1-p)\sqrt{90,000} \geq \sqrt{100,000},$$

so

$$p \geq \frac{\sqrt{100,000} - \sqrt{90,000}}{\sqrt{110,000} - \sqrt{90,000}} = 51.25, \%$$

so his subjective probability of winning must be at least 51.25%.

Suppose that an individual has initial wealth  $W$ , and an increasing concave utility function  $u(\cdot)$ . He is concerned about a catastrophic event that can occur with probability  $p$ , and would cause him a loss of  $a$  dollars. What is the maximum amount that this individual would be willing to pay for an insurance that will cover his losses if the catastrophic event occurs? Let us assume that cost of insurance is  $b$  dollars. If he takes insurance, his wealth will be  $W - b$ , whether the catastrophic event occurs or not, and the utility that associate with such an action is simply  $u(W - b)$ . If he does not take insurance, then he faces a random variable taking value  $W - a$  with probability  $p$ , and value  $W$  with probability  $1 - p$ . The expected utility of this action is  $pu(W - a) + (1 - p)u(W)$ . If the individual acts to maximize his expected utility then he will take insurance as long as

$$u(W - b) \geq pu(W - a) + (1 - p)u(W),$$

or equivalently as long as

$$b \leq W - u^{-1}(pu(W - a) + (1 - p)u(W)) \doteq \bar{b}.$$

We conclude that the maximum amount that the individual will pay is  $\bar{b}$ .

*Homework:* Show that if  $u(\cdot)$  is concave then  $\bar{b} > pa$ . □.

*Example:* Let  $W = \$1,000,000$ ,  $a = \$200,000$ ,  $p = 0.005$ , and  $u(x) = \sqrt{x}$ . Then  $\bar{b} = \$1,055.45$ .

*Homework:* Compute  $\bar{b}$  for  $W = \$200,000$ , and for the  $W = \$10,000,000$ . □.

By solving the homework problem, you will find that as the wealth of the individual increases, the maximum amount that the individual is willing to pay for insurance decreases. This is true for the class of utility functions satisfying  $r'(W) < 0$  for all  $W > 0$ . At certain level of wealth, the maximal amount that the individual is willing to pay for insurance will not be enough to cover the insurance company operating costs. Thus, above this level of wealth, the individual will prefer not to take *full* coverage.

Now suppose that the individual is allowed to take partial coverage at a premium of  $r$  dollars per dollar insured. If the individual decided to take  $B$  dollars of coverage, the insurance company will collect  $rB$ , but will have to pay out  $B$  dollars if the catastrophic event occurs. If the insurance company is to make a profit, it is necessary to have

$$p(rB - B) + (1 - p)rB = B(r - p) > 0,$$

so  $r > p$ . Given  $r > p$ , the individual must decide how much coverage to buy. If he buys  $B$  dollars of coverage, then his wealth will be equal to  $W - Br$  if the catastrophic event does not occur, and will be  $W - Br - a + B$  if the catastrophic event occurs. His expected utility, as a function of  $B$ , is

$$g(B) \doteq pu(W - Br - a + B) + (1 - p)u(W - Br).$$

The problem is to maximize  $g(B)$  over the region  $0 \leq B \leq a$ . It can be shown that  $g(\cdot)$  is concave. If there is an interior point, say  $B^*$ , satisfying  $g'(B^*) = 0$ , then  $B^*$  is optimal. Taking the derivative of  $g(\cdot)$  and equating it to zero results in

$$\frac{u'(W - Br - a + B)}{u'(W - Br)} = \frac{r(1 - p)}{p(1 - r)} > 1.$$

From this, it is clear that if  $g'(\cdot)$  vanishes at  $B^*$ , then

$$u'(W - B^*r - a + B^*) > u'(W - B^*r),$$

but then, because  $u(\cdot)$  is concave,  $u'(\cdot)$  must be decreasing, so that

$$W - B^*r - a + B^* < W - B^*r.$$

From this analysis, we conclude that

$$B^* < a$$

for all increasing concave utility functions, and for all  $W$ ,  $a$ ,  $r > p$ . This implies, that for all levels of wealth, and for all increasing concave utility functions, it is *never* optimal to take full insurance.

*Example:* Let  $W = \$1,000,000$ ,  $a = \$200,000$ ,  $p = 0.005$ ,  $r = 0.0051$ , and  $u(x) = \sqrt{x}$ . Then the optimal coverage  $B^*$  satisfies

$$\frac{\sqrt{1000000 - 0.0051B^*}}{\sqrt{800000 + 0.9949B^*}} = \frac{(0.0051)(0.995)}{(0.005)(0.9949)},$$

by solving, we find that the optimal coverage is  $B^* = \$161,007.63$ .

*Homework:* Compute  $B^*$  for  $W = \$200,000$ , and for the  $W = \$10,000,000$ . □.

Suppose that an investor has utility function  $u(\cdot)$ , and acts to maximize his expected utility. Then, the investor will prefer the random portfolio  $X$  to the random portfolio  $Y$ , if and only if

$Eu(X) > Eu(Y)$ . Notice, that this choice is not altered if his utility function is replaced by  $au(\cdot) + b$ , with  $a > 0$ ; hence utility functions are equivalent up to affine transformations. By taking  $a = \frac{EX}{u(EX)}$ , and  $b = 0$ , if necessary, we can assume without loss of generality that  $u(EX) = EX \doteq \mu$ . Expanding the utility function around  $\mu$ , we obtain

$$u(x) \simeq \mu + (x - \mu)u'(\mu) + 0.5(x - \mu)^2u''(\mu).$$

Now, for any portfolio  $Y$  with mean  $EY = \mu$ , we have

$$Eu(Y) \simeq \mu + 0.5u''(\mu)Var(Y).$$

Based on this second order approximation, the decision maker prefers the random portfolio  $X$  to  $Y$ , if

$$0.5u''(\mu)Var(X) > 0.5u''(\mu)Var(Y).$$

If the investor is risk *averse*, he will prefer  $X$  to  $Y$ , with  $EX = EY$  if  $Var(X) < Var(Y)$ . This result is part of what is called *mean-variance* analysis. In words, risk averse investors prefer the portfolio with the smallest variance among all portfolios with the same expected return. Hence the variance and its square root, the standard deviation, are taken to be measures of risk.

The other part of the *mean-variance* analysis, states that *all* investors will prefer the portfolio with the highest expected return among all portfolios with the same variance. This part of the mean variance is less transparent than the first, yet it can be derived without making any assumption on the sign of the second derivative of  $u(\cdot)$ . Let  $\mu_x = EX$ ,  $\mu_y = EY$ , and assume that  $Var(X) = Var(Y) = \sigma^2$ . Now, expanding the utility function around  $\mu \doteq 0.5(\mu_x + \mu_y)$ , we find that

$$Eu(X) \simeq u(\mu) + (\mu_x - \mu)u'(\mu) + 0.5(\sigma^2 + (\mu_x - \mu)^2)u''(\mu),$$

notice, that by the choice of  $\mu$ , the last term vanishes when we take the difference of the expected utility, so

$$Eu(X) - Eu(Y) \simeq u'(\mu)(\mu_x - \mu_y).$$

Thus, since  $u'(\mu) > 0$ , this second order approximation indicates that  $Eu(X) > Eu(Y)$  if  $\mu_x > \mu_y$ .

To summarize, a second order approximation of the utility function indicates that *risk averse* investors will select the portfolio with the smallest variance among all portfolios with the same mean, and that *all* investors will select the portfolio with highest return among all portfolios with the same variance.

The above is helpful in comparing portfolios with the same variance, and portfolios with the same expected return. More generally, we can ask for the set of portfolios that have the same expected utility as measured by the second order approximation. Plotting the set we obtain iso-utility curves in the mean-variance space, or in the mean-standard deviation space. This iso-utility curves have a positive slope indicating that a higher return is necessary to compensate a higher risk.

### Notes on Chapter 10: Measures of Investment Worth Under Risk

We have discussed ways of computing means and variances of functions of random variables, i.e., weighted sums, products and quotients. Once we obtain the mean and the variance, we can make some probabilistic statements about the random variable in question. To be more precise, if  $X$  is a random variable with mean  $\mu$  and variance  $\sigma^2$ , and that is all we know about  $X$ , then we can state the following three inequalities:

$$Pr\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2},$$

$$Pr\{X \leq \mu - k\sigma\} \leq \frac{1}{1 + k^2} \quad \text{for } k \geq 0,$$

and

$$P(X \leq \mu + k\sigma) \geq \frac{k^2}{1 + k^2} \quad \text{for } k \geq 0.$$

These inequalities are due respectively to Tchebycheff, Cantelli and Marshall. If the  $X \geq 0$ , we have Markov's inequality

$$P(X \geq \lambda) \leq \frac{\mu}{\lambda}.$$

If in addition, it is known that the distribution of  $X$  can be written as  $F(x) = \int_{-\infty}^x f(s)ds$  for  $x < \mu$ , and  $F(x) = F(\mu) + \int_{\mu}^x f(s)ds$  for  $x \geq \mu$ , with  $f(\cdot)$  non-decreasing on  $x < \mu$ , and non-increasing on  $x > \mu$ , then Tchebycheff's inequality can be sharpened to yield Gauss's inequality:

$$Pr\{|X - \mu| \geq k\sigma\} \leq \frac{4}{9k^2},$$

for  $k \geq \sqrt{4/3}$ , and to

$$Pr\{|X - \mu| \geq k\sigma\} \leq 1 - \frac{k}{\sqrt{3}},$$

for  $k \leq \sqrt{4/3}$ .

## Notes on Chapter 11: Methods for Comparing Risky Projects

We have already discussed mean-variance analysis in the context of utility theory. The two mean-variance rules are: (i) Select the project with the lowest variance among projects with the same expected return, and (ii) Select the project with the largest return among projects with the same variance. Recall that both rules were obtained by approximating the utility function by its second degree Taylor expansion, under the assumption that individuals act to maximize expected utility. Rule (i) was established under the additional assumption of risk averseness, while rule (ii) did not require risk averseness.

If we have a set of projects under consideration, and we plot their means and their variances, we will be able to use rules (i) and (ii) to eliminate some of the projects. Those, that we cannot eliminate are said to lie in the *efficient frontier*. The ultimate choice between projects in the efficient frontier depends on the tradeoff between risk and reward for the decision maker. A very conservative decision maker will not be willing to take substantial additional risk to increase the expected return, so he will tend to select low return, low variance projects in the efficient frontier. In contrast, a more aggressive investor will be willing to incur substantial additional risk to increase the expected return, so he will tend to select high return, high variance projects in the efficient frontier. If the utility function of the investor is known, then the decision maker selects the project with the largest utility function. Graphically, this is the project that intersects the *largest* iso-utility curve.

Recall that  $\sigma^2 = E(X - \mu)^2$ , as such the variance is a measure of risk that penalizes deviations from the mean value on *both* directions. The *semi-variance* is an alternative measure of risk that penalizes deviations below a certain value  $h$ . The semi-variance  $S_h$  is defined to be

$$S_h = E[(X - h)^-]^2,$$

where values below  $h$  are considered bad. Typical values of  $h$  are 0 and  $\mu$ . Not surprisingly, the knowledge of the semi-variance may help us reach a decision where the mean variance analysis does not. For example selecting between Project 6 with mean 2 and variance 176, and Project 7 with mean 28 and variance 356 in Chapter 11 is not possible by the rules of mean variance analysis. However, once we realize that all of the variance of Project 7, is *good* variance, and measure risk through the semi-variance we see that Project 7 *dominates* Project 6 under the mean semi-variance rules. The mean semi-variance rules are similar to the mean variance rules except that the semi-variance is used instead of the variance as a measure of risk. Notice that there is no theoretical justification for these rules.

A third criterion, called the *safety-first* criterion favors the alternative with the smallest probability of loss, where loss is usually defined as a negative PV or as an IRR below MARR.

The above criteria often do not indicate a clear choice among competing alternatives. A concept that is useful to decide among competing alternatives is called *stochastic dominance*. Several types of stochastic dominance exist. The definition of *first degree* stochastic dominance is given next.

Given two random variables  $X$  and  $Y$  with probability functions  $F(\cdot)$  and  $G(\cdot)$ , we say that  $X$  *dominates*  $Y$  or  $FDG$  [ $F$  dominates  $G$ ], if

$$Eu(X) \geq Eu(Y)$$

for every non-decreasing function  $u(\cdot)$  for which the expectation exists and if the inequality is strict for at least one non-decreasing function  $u(\cdot)$ .

Suppose that  $X$  and  $Y$  represent the outcomes of two projects. Since decision makers act to maximize expected utility it is clear  $X$  is preferred to  $Y$ , if  $Eu(X) \geq Eu(Y)$ . Thus, to say that  $X$  dominates  $Y$  is to say that *all* decision makers having non-decreasing utility functions will prefer  $X$  to  $Y$ .

The above is a strong statement, because it essentially says that all decision makers (with nondecreasing utility functions) prefer  $X$  to  $Y$ . An example of first degree stochastic dominance is given by Projects 2 and 3 in Chapter 11. Letting  $PV_2$  and  $PV_3$  denote the random present value of projects

2 and 3 respectively, we can write

$$Eu(PV_3) = Eu(PV_2) + 0.1(u(20) - u(10)) \geq Eu(PV_2).$$

Notice that the above holds for all nondecreasing utility functions. So all investors with nondecreasing utility functions will prefer Project 3 over Project 2. Notice that the mean-variance analysis was not capable of deciding between Projects 2 and 3.

While the condition defining first degree stochastic dominance has an intuitive interpretation, it is not always easy to verify it directly because it involves the set of *all* non-decreasing functions  $u(\cdot)$ . Fortunately, checking  $X \geq Y$  [ $FDG$ ] can be done directly, just by looking at the cumulative distribution functions. Indeed, a necessary and sufficient condition is that

$$F(x) \leq G(x) \quad \forall x,$$

and that strict inequality holds for at least one value of  $x$ .

To see that this is true, one first shows that  $F(x) \leq G(x)$  implies  $EX \geq EY$ , and then show that  $F(x) \leq G(x)$  implies  $F(u(x)) \leq G(u(x))$  for all non-decreasing functions  $u(\cdot)$ . Consequently,  $F(x) \leq G(x)$  implies  $Eu(X) \geq Eu(Y)$  for all non-decreasing function  $u(\cdot)$  for which the expectation exists. Conversely, if  $Eu(X) \geq Eu(Y)$  for all non-decreasing  $u(\cdot)$  for which the expectation exists, we take  $u(x) = 1\{x > a\}$  and obtain

$$1 - Pr\{X \leq a\} = Pr\{X > a\} = Eu(X) \geq Eu(Y) = Pr\{Y > a\} = 1 - Pr\{Y \leq a\},$$

so we conclude that  $F(a) \geq G(a), \forall a$ .

As mentioned before, first degree stochastic dominance is a very strong condition, and it is by no means true that  $X \geq Y$  or  $Y \geq X$  will hold. In fact, in most cases neither condition holds. One way of enlarging the class of projects for which a dominance relationship holds is to reduce the set of functions  $u(\cdot)$ . A meaningful subclass of the class of non-decreasing functions, is the class of non-decreasing *concave* functions. Given two random variables  $X$  and  $Y$  with probability functions  $F(\cdot)$  and  $G(\cdot)$ , we say that  $X \geq_2 Y$  ( $X$  dominates  $Y$ ) or  $FD_2G$  [ $F$  dominates  $G$ ], if

$$Eu(X) \geq Eu(Y)$$

for every non-decreasing *concave* function  $u(\cdot)$  for which the expectations exists, and if the inequality is strict for at least one non-decreasing concave function  $u(\cdot)$ . Again, while the definition of second degree stochastic dominance is intuitively appealing— $X \geq_2 Y$  if and only if all decision makers with an increasing concave utility function prefer  $X$  to  $Y$ —it is difficult to check the condition directly. However, an alternative definition states that we can establish  $X \geq_2 Y$  or  $FD_2G$  by looking into the areas under the cumulative distribution functions. Indeed, a necessary and sufficient condition is that

$$\int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds \quad \forall x,$$

and that strict inequality holds for at least one value of  $x$ .

A good example that motivates and illustrates second order stochastic dominance is to compare Projects 3 and 4 in Chapter 11.

### Portfolio Theory

Portfolio theory is a popular topic in finance. The problem is to allocate a fixed sum of money among a set of securities, such as stocks and bonds, to create an efficient portfolio. A portfolio is said to be efficient, in the mean-variance sense, if it is not possible to form another portfolio with the same expected return and smaller variance, or a portfolio with the same variance and larger expected return. Portfolio theory was developed in the 1950's by H. Markowitz. The main tool used in portfolio theory is quadratic programming, where given the first and second moments of the securities, and a desired expected return one seeks to find a minimal variance portfolio. Markowitz recently received a Nobel prize in Economics for his work on portfolio theory.

Let  $R_j$  be the random annual return on a dollar invested in security  $j = 1, \dots, J$ . Formally, one obtains  $R_j$  by computing the ratio of the dividend per share plus the increase per share to the previous share value. Let  $\mu_j$ , and  $\sigma_j^2$ , denote respectively the mean, and the variance of the return on security  $j = 1, \dots, J$ , and let  $\rho_{jk}$  denote the correlation of securities  $j$  and  $k$ . See example 11.8.

Let  $x_j$  denote the number of dollars invested in security  $j = 1, \dots, J$ . Then our portfolio will have a random return equal to

$$R = \sum_{j=1}^J x_j R_j,$$

with mean

$$E(R) = \sum_{j=1}^J x_j \mu_j$$

and variance

$$\text{Var}(R) = \sum_{j=1}^J x_j^2 \sigma_j^2 + 2 \sum_{j < k} x_j x_k \rho_{jk} \sigma_j \sigma_k.$$

Markowitz's quadratic program can be written as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{j=1}^J x_j^2 \sigma_j^2 + 2 \sum_{j < k} x_j x_k \rho_{jk} \sigma_j \sigma_k \\ \text{s.t.} \quad & \sum_{j=1}^J x_j \mu_j \geq rB, \\ & \sum_{j=1}^J x_j \leq B, \\ & 0 \leq x_j \quad \forall j, \end{aligned}$$

where  $r$  is the minimal desired return, and  $B$  is the budget to be allocated. Most formulations take advantage of the fact that both the objective and the constraints are homogeneous of degree 1, so we can assume without loss of generality that  $B = 1$ . With  $B = 1$  we are essentially asking how to allocate \$1 dollar among the securities. If we have  $B > 1$  dollars to invest, then we just scale the allocations.

As presented above the model requires  $\frac{J^2+3J}{2}$  estimates of means, variances and covariances. The amount of historical data that is required to accurately estimate these parameters is enormous. Thus, the original formulation can only be used to allocate portfolios among a small set of securities for which the parameters can be accurately estimated. More frequently, the portfolio model is used to allocate money among classes of securities, rather than among securities themselves. For example, the model may be used to decide how to allocate money among stocks, bonds, and money market accounts.

A different version of the above model, called the *index model*, uses a market index to estimate  $\mu_j$ ,  $\sigma_j^2$ , and  $\rho_{jk}$ . The index mode postulates that

$$R_j = A_j + B_j I + C_j, \quad \forall j,$$

where  $A_j$  and  $B_j$  are constants,  $I$  is a random market index, and  $C_j$  is a random error. Thus, the index model postulates that the return of security  $j$  is linearly related to the performance of the market index  $I$ . Let  $A_{J+1}$ , and  $Q_{J+1}$  denote the expectation and the variance of the random index  $I$ . The random errors are assumed to have mean zero and variance  $Q_j \doteq \text{Var}(C_j)$   $j = 1, \dots, J$ . Using the above notation, we obtain

$$\mu_j = E(R_j) = A_j + B_j A_{J+1}, \quad j = 1, \dots, J,$$



$$\sigma_j^2 = \text{Var}(R_j) = B_j^2 Q_{J+1} + Q_j$$

and

$$\text{cov}(R_j, R_k) = B_j B_k Q_{J+1}.$$

Then our portfolio  $R = \sum_{j=1}^J x_j R_j$  has mean

$$E(R) = \sum_{j=1}^J x_j A_j + \left( \sum_{j=1}^J B_j x_j \right) A_{J+1}$$

and variance

$$\text{Var}(R) = \sum_{j=1}^J x_j^2 Q_j + \left( \sum_{j=1}^J B_j x_j \right)^2 Q_{J+1}.$$

Notice that the index model requires only  $3J+2$  estimates. The index model is much easier to solve, and it provides a reasonable approximation to the full covariance model. It is of course possible to consider models with two rather than one index, e.g., gross national product, and the number of house starts.

In selecting a portfolio of securities, one generally has the opportunity to invest part of one's funds in a risk free asset such as US Treasury securities, with the rest of the funds being invested in some risky portfolio. Let  $R_f$  be the return of the risk free security. We assume that  $R_f$  is a known constant. If we invest a fraction  $\alpha$  of our wealth in the risk free security, and the fraction  $1 - \alpha$  in portfolio with random return  $R_m$ , the combined security is:

$$R_c = \alpha R_f + (1 - \alpha) R_m.$$

This security has expected return

$$\mu_c = ER_c = \alpha R_f + (1 - \alpha) \mu_m,$$

and variance

$$\sigma_c^2 = \text{Var}(R_c) = (1 - \alpha)^2 \sigma_m^2.$$

Thus, by selecting the fraction  $\alpha$ , we can control the risk  $\sigma_c^2$  of the combined portfolio. For any value of  $\sigma_c^2$  we find

$$\alpha = 1 - \frac{\sigma_c}{\sigma_m},$$

and for that  $\alpha$ , we obtain the expected return

$$\mu_c = R_f + \frac{\mu_m - R_f}{\sigma_m} \sigma_c. \quad (1)$$

Given  $R_f$ ,  $\mu_m$ , and  $\sigma_m$ , the above equation for the expected return,  $\mu_c$ , is linear function of the associated with the risk level  $\sigma_c$ .

Since we want to obtain the largest possible return at any level of risk, it is clear that we would select the risky portfolio  $m$ , that maximizes the slope

$$\frac{\mu_m - R_f}{\sigma_m}.$$

Such a portfolio is called the *market portfolio*, and the associated linear equation (1) is called the *Capital Market Line* (CML).

Any risk averse decision maker will select the point where his iso-utility curve is tangent to the CML. That means that all risk averse investors will allocate part of their money to the risk free security and the other part to the market portfolio. If the iso-utility curve is tangent to the CML to

the right of the market portfolio, the investor will be willing to borrow money at the risk free rate  $R_f$  and invest it, together with his own money in the market portfolio. Thus portfolios to the right of the market portfolio involve borrowing, while portfolios to the left of the market portfolio involve lending.

From the capital market line, we see that

$$\frac{\mu_c - R_f}{\sigma_c} = \frac{\mu_m - R_f}{\sigma_m},$$

the above ratios are called *reward to volatility* ratios.

Virtually any individual can invest at the risk free rate. However, most investors cannot borrow at the risk free rate. This requires a modification of the above to incorporate different rates for borrowing and the separation of the risk return decision; the risky-investment decision does not hold any longer over the entire mean-standard deviation space. However, it continues to hold over some region of the mean-standard deviation space.

What happens if we combine a *risky* security, say  $R_j$ , with the market portfolio? We obtain a new portfolio

$$R_p = \alpha R_j + (1 - \alpha) R_m.$$

Then

$$\mu_p = \alpha \mu_j + (1 - \alpha) \mu_m,$$

and

$$\sigma_p^2 = \alpha^2 \sigma_j^2 + 2\alpha(1 - \alpha) \sigma_{jm} + (1 - \alpha)^2 \sigma_m^2.$$

We see that

$$\frac{\partial \mu_p}{\partial \alpha} = \mu_j - \mu_m,$$

and that

$$\frac{\partial \sigma_p}{\partial \alpha} = \frac{\sigma_{jm} - \sigma_m^2}{\sigma_m},$$

at  $\alpha = 0$ . Consequently,

$$\frac{\partial \mu_p}{\partial \sigma_p} = \frac{(\mu_j - \mu_m) \sigma_m}{\sigma_{jm} - \sigma_m^2} = \frac{\mu_m - R_f}{\sigma_m}.$$

Solving for  $\mu_j$  we obtain

$$\mu_j = R_f + \beta(\mu_m - R_f)$$

where

$$\beta = \frac{\sigma_{jm}}{\sigma_m^2}.$$

The above model, called the capital asset pricing model, can be used to price out securities by comparing the actual to the predicted expected returns.

I like to write the CAPM as

$$\mu_j = R_f + \frac{\mu_m - R_f}{\sigma_m} \rho_j \sigma_j \leq R_f + \frac{\mu_m - R_f}{\sigma_m} \sigma_j,$$

where  $\rho_j$  is the correlation between security  $j$  and  $R_m$ , i.e.,  $\rho_j = \sigma_{jm} / (\sigma_j \sigma_m)$ .

### The Average Downside Risk Model

An alternative portfolio selection model that is popular is the so called average down side risk (ADR) model. Here  $R_{ij}$  represents the return of security  $j = 1, \dots, J$  under scenario  $i = 1, \dots, n$ . Scenario  $i$  occurs with probability  $p_i > 0$ , with  $\sum_{i=1}^n p_i = 1$ . Given a portfolio  $x = (x_1, \dots, x_J)$  of nonnegative weights summing to one,  $\sum_{j=1}^J x_j = 1$ , the return under scenario  $i$  by computing

$$R_i = \sum_{j=1}^J x_j R_{ij}.$$

The expected return of the portfolio is given by

$$R = \sum_{i=1}^n p_i R_i.$$

The downside risk under scenario  $i$  is  $DR_i = \max(-R_i, 0)$  and the average downside risk is  $ADR = \sum_{i=1}^n p_i DR_i$ . The formulation calls for minimizing the  $ADR$  subject to a constraint on the return. The down side risk, and the average down side risk, as currently defined, are piecewise linear functions. To convert this into a linear function, we relax the above definition of  $DR_i$  and make  $DR_i$  a decision variable satisfying the constraints  $DR_i \geq 0$  and  $DR_i + R_i \geq 0$ . This relaxation results in a linear program, and since we are minimizing the optimal solution will always have  $DR_i = 0$  if  $R_i \geq 0$  and  $DR_i = -R_i$  if  $R_i < 0$ .

## Chapter 12: Risk Simulation

For many complex problems, for which analytic techniques are hard to apply, simulation provides a convenient alternative. Simulation, provides an easy way to combine the necessary probabilistic distributions -and in some cases the only possible way.

For example, if the Cash Flow in a given period is a function of a number of random variables such as selling prices, size of the market, investment required, inflation rate, etc., so that it is difficult or impossible to find the distribution of the Cash Flow, we may resort to simulation to answer questions such as: What is the probability that the Cash Flow is greater than zero?

We will give only sufficient detail to present the major aspects of the method and how it can be applied to investment analysis.

**Monte Carlo Sampling.** Suppose you are interested in the present value:

$$PV = F_0 + \frac{F_1}{1+i} + \frac{F_2}{(1+i)^2}.$$

If we can express the random variables by probability distributions, we can simulate the actual state of nature by sampling the state of each random variable from its specific distribution, and then compute the value of PV. These computations are repeated until a sufficient number of PV values is available to tabulate its distribution. Three questions arise: (i) How to sample values of the random variables?, (ii) How many repetitions are required?, (iii) What happens if the variables are correlated? We will address these questions later.

**Using the Simulation Output.** For a single project, we can construct estimates of

$$\Pr(PV \leq x),$$

For multiple projects, we can construct estimates of

$$\Pr(PV_A \geq PV_B),$$

or for any  $\Delta$ , we can compute

$$\Pr(PV_A - PV_B \geq \Delta).$$

### Selecting Input Probability Distributions

To carry out a simulation, we have to specify the probability distribution of the inputs, then the simulation proceeds by generating values of these random variables from appropriate distributions.

If past data is available, we can either (i) fit a theoretical distribution, or (ii) use the empirical distribution. To fit a theoretical distribution, we need to select a distribution, fit the data, and test for goodness of fit. See pp 501-507 for a fit to the Gamma distribution.

If data is *not* available, then the decision maker is asked to select three values of the random variable:  $L, M, H$ . These values are *low*, *most likely*, and *high*, and are educated guesses made by the decision maker. The *low value* is selected so that in “95%” of the realizations of the random variable will be above  $L$ . Similarly, the *high* value is selected so that “95%” of the realizations of the random variable will be below  $H$ . Finally,  $M$  is selected to be the *most likely* value of the random variable. Once,  $L, M, H$  are obtained, several distributions can be used to fit  $L, M$  and  $H$ . Two distributions that are easy to fit, are the triangular, and the beta distributions. See pp 508-509.

### Sampling Procedures for Independent Random Variables

To draw values of the random variable, one first draws a value, say  $u$ , from the uniform  $U(0, 1)$  distribution, and then computes

$$F^{-1}(u) \doteq \inf\{x : F(x) \geq u\},$$

where  $F$  is either the theoretical or the empirical distribution. Note: It can be shown that  $X \doteq F^{-1}(U)$  is a random variable with cumulative distribution  $F$ . So, if  $u_1, \dots, u_n$  are  $n$  values of  $U$ , then  $x_1 = F^{-1}(u_1), \dots, x_n = F^{-1}(u_n)$  are values  $n$  values of  $X$ .

*Example:*

Suppose  $X$  has an exponential distribution with parameter  $\lambda$ . Then

$$P(X \leq x) = 1 - e^{-\lambda x},$$

so  $U = 1 - e^{-\lambda X}$ . Solving for  $X$ , we obtain

$$X = -\frac{1}{\lambda} \ln(1 - U) = -\frac{1}{\lambda} \ln(W),$$

where  $W$  is also uniform  $(0, 1)$ . Suppose  $\lambda = 5$ , and that we draw  $w_1 = 0.32, w_2 = 0.76, w_3 = 0.59$  then  $x_1 = \frac{1}{5} \ln(0.32), x_2 = \frac{1}{5} \ln(0.76), x_3 = \frac{1}{5} \ln(0.59)$ .

We can generate a normal random variable  $X$ , with mean  $\mu$ , and variance  $\sigma^2$  by first writing  $X$  as

$$X = \sigma Z + \mu,$$

where  $Z$  is a standard normal random variable. So

$$X = \sigma \Phi^{-1}(u) + \mu,$$

where  $\Phi(\cdot)$  is the cumulative distribution of the standard normal random variable, and  $u$  is a value drawn from a uniform  $(0, 1)$  random variable. Other ways of generating Normal, and other distributions, are discussed in pp 514.

**Sampling Procedures for Dependent Random Variables** Suppose that  $X$  and  $Y$  are two random variables that are statistically dependent on each other. Dependency can be checked, for example, by looking at a cross plot of  $(X, Y)$ , to observe if there are underlying trends. The problem is to generate representative pairs of  $(X, Y)$ . This can be done, if we know the distribution of  $X$ , say  $F(\cdot)$ , and for each realization  $X = x$ , we know the distribution of  $Y|X = x$ , say  $F_{Y|X=x}(\cdot)$ . Then,

$$(F^{-1}(U_1), F_{Y|X=F^{-1}(U_1)}^{-1}(U_2))$$

is a pair of random variables with the appropriate distribution where  $U_1$  and  $U_2$  are uniform  $(0, 1)$  random variables. Other methods of generating dependent random variables are discussed in section 12.5 of the book.

**Output Data Analysis** How close is an estimator to its true measure? Suppose  $Y$  is a random variable with mean  $\mu$ , then

$$\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n Y_j \quad \text{w.p. 1}$$

where  $Y_j, j = 1, \dots, n$ , are  $n$  replications of  $Y$ . When to stop? Well, suppose that we repeat the experiment  $m$  times, the  $k^{th}$  replication yielding

$$\bar{Y}_k = \frac{1}{n} \sum_{j=1}^n Y_{kj}.$$

Then the  $\bar{Y}_k$ 's are i.i.d. random variables. Let

$$\bar{Y} = \frac{1}{m} \sum_{k=1}^m \bar{Y}_k,$$

and

$$S^2 = \frac{1}{m-1} \sum_{k=1}^m (\bar{Y}_k - \bar{Y})^2,$$

and  $\hat{\sigma} = \frac{S}{\sqrt{m}}$ . Then a  $100(1 - \alpha)\%$  confidence interval for  $\mu$  based on the  $t$ -distribution is given by:

$$\bar{Y} - t_{\alpha/2, m-1} \hat{\sigma} \leq \mu \leq \bar{Y} + t_{\alpha/2, m-1} \hat{\sigma}.$$

**Homework:** Airline companies typically open a flight  $t$  units of time before the schedule departure time;  $t$  is typically one year. Assume that the plane scheduled for the flight has  $n$  seats, and that seats are initially offered at a promotional fare of  $p_1$  dollars. After some time the promotional fare is phased out and the remaining seats are priced at the “regular” fare  $p_2 > p_1$ .

Assume that at fare  $p_i$ , the demand over  $s$  units of time, say  $N_i(s)$ , is a Poisson random variable with parameter  $\lambda_i s$ ,  $i = 1, 2$ , with  $p_1 \lambda_1 > p_2 \lambda_2$ , and  $\lambda_1 t > n > \lambda_2 t$ . Our objective is to maximize expected revenue by selecting the best time to switch from the promotional to the regular fare. Two policies, to be described below are being considered. Your job is to simulate the two policies and to use the outcome of the simulation to decide which policy is better.

Policy 1. Is to keep the promotional fare until the first time  $s \in [0, t]$  such that

$$N_1(s) + \lambda_2(t - s) \geq n.$$

In words policy 1 says to phase out the promotional fare as soon as we expect to fill the remaining seats at the regular fare. To see this, note that at time  $s$ , there are  $n - N_1(s)$  remaining seats, and over the remaining  $t - s$  units of time, the expected demand at the regular fare is  $\lambda_2(t - s)$ .

Policy 2. After the sale, say at time  $s$ , of a seat at the promotional fare, we decide to keep the promotional fare until the next seat is sold if

$$p_1 \Pr(N_1(t - s) \geq n - N_1(s)) \geq p_2 \Pr(N_2(t - s) \geq n - N_1(s))$$

and to phase out the promotional sale otherwise.

Your code should run in either Pascal, Gauss or C. It should allow the user to input the values of  $p_i, \lambda_i, N$  and  $T$ . However, for purposes of the output analysis, you should use the following data:  $T = 360$ ,  $N = 250$ ,  $\lambda_1 = 1.25$ ,  $\lambda_2 = 0.5$ ,  $p_1 = \$200$ , and  $p_2 = \$400$ .

Hint: To set up the simulation initialize  $t = 0$ ,  $R = 0$ ,  $N_1 = 0$ ,  $N_2 = 0$ ,  $\lambda = \lambda_1$ ,  $p = p_1$ .

(i) At time  $t$  generate an exponential random variable by drawing

$$E = -\frac{1}{\lambda} \ln(W)$$

where  $W$  is a uniform  $(0, 1)$  random variable.

(ii) If  $t + E > T$ , or  $N_1 = n$  stop. Else update the variables by

$$t := t + E, \quad R := R + p, \quad N_1 := N_1 + 1.$$

(iii) If the phase out condition for the low fare has not been met go to step(i),

(iv) Set

$$\lambda = \lambda_2, \quad p = p_2.$$

(v) At time  $t$  generate an exponential random variable by drawing

$$E = -\frac{1}{\lambda} \ln(W)$$

where  $W$  is a uniform  $(0, 1)$  random variable.

(vi) If  $t + E > T$ , or  $N_2 = n - N_1$  stop. Else update the variables by

$$t := t + E, \quad R := R + p, \quad N_2 := N_2 + 1,$$

and go to (v).

At the end of the simulation,  $N_1$  is the number of seats sold at the promotional fare,  $N_2$  is the number of seats sold at the regular fare, and  $R = p_1 N_1 + p_2 N_2$  is the total revenue. For both policies, you should report  $R, N_1$ , and  $N_2$ .

### Chapter 13: Decision Analysis

Decision analysis is a commonly used technique for multi-stage decision problems. Let  $r(a, w)$  denote the reward associated with a certain action  $a \in A$ , chosen by the decision maker, and certain state of nature  $w \in \Omega$ , not known to the decision maker at the time he/she makes the decision. It is assumed that  $\Omega$  is a finite or countable set, and that a probability mass function  $p(w) \forall w \in \Omega$  is well defined. Given any action  $a \in A$ , the expected reward is given by

$$r(a) = \sum_{w \in \Omega} r(a, w)p(w).$$

Of course, if our objective is to maximize the expected monetary value we simply select  $a$  to maximize  $r(a)$ . So

$$\text{EMV} = \max_{a \in A} r(a).$$

If we had perfect information, that is, if we knew  $w \in \Omega$  before selecting  $a \in A$ , we would clearly act to select  $a$  to maximize  $r(a, w)$  for every  $w \in \Omega$ . Thus, the expected monetary value under perfect information, EMVPI, is

$$\text{EMVPI} = \sum_{w \in \Omega} p(w) \max_{a \in A} r(a, w).$$

The EMVPI is an upper bound on EMV, and the difference is called the maximal value of additional information MVAI. Thus,

$$\text{MVAI} = \sum_{w \in \Omega} p(w) \max_{a \in A} r(a, w) - \max_{a \in A} \sum_{w \in \Omega} r(a, w)p(w).$$

Typically decision makers collapse the states of nature into three broad states: poor ( $p$ ), moderate ( $m$ ), and good ( $g$ ). Thus  $r(a, p)$  is the reward obtain by selection action  $a$  if the state of nature is  $p$ . Suppose that this is the case, and that a consultant is hired to do a study. At the end of the study, the consultant will predict favorable ( $f$ ), inconclusive ( $i$ ), or unfavorable ( $u$ ) conditions. In order to assess the value of the study, we need to know how good the consultant is in predicting the states of nature. Suppose that from past history you know the conditional probabilities  $p(f|g), p(f|m), \dots, p(u|p)$ . That is for every state of nature you know the probability that the consultant will report  $f, i$ , or  $u$ . By Bayes theorem, we can compute the conditional probabilities  $p(g|f), p(g|i), p(g|u), \dots, p(p|u)$ , as well as the probabilities  $p(f), p(i)$ , and  $p(p)$ . For example,

$$p(g|f) = \frac{p(f|g)p(g)}{p(f|g)p(g) + p(f|m)p(m) + p(f|p)p(p)}.$$

Given these conditional probabilities, we can compute the conditional rewards associated with each action  $a \in A$ . Thus

$$r(a|f) = r(a, g)p(g|f) + r(a, m)p(m|f) + r(a, p)p(p|f).$$

The rewards  $r(a|i)$ , and  $r(a|u)$  can be similarly computed. Of course, for each possible outcome of the study, we select the action  $a \in A$ , that maximizes  $r(a|\cdot)$ . To obtain the expected monetary value taking the study, EMVS, we compute

$$\text{EMVS} = \max_{a \in A} r(a|f)p(f) + \max_{a \in A} r(a|i)p(i) + \max_{a \in A} r(a|u)p(u).$$

Where  $p(f), p(i)$ , and  $p(p)$  are the probabilities that the study predicts  $f, i$ , and  $p$ . The value of the study is then

$$\text{EMVS} - \text{EMV}.$$

See the Retail Convenience Store Problem, page 552 of textbook.

### The Wait and See Option

So far we have ignored from consideration the possibility of postponing the investment decision. To illustrate why this is an important option we will introduce a simple example where waiting may be the best option. Let  $I$  be the cost of investing on a widget factory. We assume an inflation-free scenario, so that if we wait an invest a year from now the cost will still be  $I$ . Let  $i$  be the inflation-free interest rate, and assume that investing in the widget factory results in the revenue stream  $\{P_o, P_1, P_2, \dots\}$  where  $P_n = 1.5P_o, n = 1, \dots$  with probability  $q$  and  $P_n = 0.5P_o, n = 1, \dots$  with probability  $1 - q$ . Suppose we must make the decision at time 0. Assume further that the price of the widget is uncorrelated with the market so that discounting the expected value of the cash flow at the risk-free rate makes sense. Then, the expected net present value is given by

$$\begin{aligned} \text{ENPV}_0 &= -I + P_o + q1.5P_o \sum_{n=1}^{\infty} (1+i)^{-n} + (1-q)0.5P_o \sum_{n=1}^{\infty} (1+i)^{-n} \\ &= -I + P_o \left[ 1 + q\frac{1.5}{i} + (1-q)\frac{0.5}{i} \right]. \end{aligned}$$

The NPV criterion is to invest if this quantity is positive, that is, if

$$I \leq P_o \left[ 1 + q\frac{1.5}{i} + (1-q)\frac{0.5}{i} \right].$$

For example, if  $P_o = 200$ ,  $q = 0.5$ ,  $i = 10\%$  and  $I = 1600$  we have

$$\text{ENPV}_0 = -1,600 + 2,200 = 600.$$

More generally, it is optimal to invest now whenever

$$I \leq 2200.$$

Now consider the option of waiting for one year. The net present value of investing at the end of year one is

$$\frac{1.5P_o}{i} - \frac{I}{1+i}$$

if the price goes up and

$$\frac{0.5P_o}{i} - \frac{I}{1+i}$$

if the price goes down. Clearly, we would only make the investment if the resulting value is positive. So the expected net present value when we wait is given by

$$\begin{aligned} \text{ENPV}_w &= q \left[ \frac{1.5P_o}{i} - \frac{I}{1+i} \right]_+ \\ &+ (1-q) \left[ \frac{0.5P_o}{i} - \frac{I}{1+i} \right]_+ \end{aligned}$$

where  $x_+ = \max(x, 0)$ . Of course, the optimal decision is to wait only when the expected value is larger than investing now. Consequently, the *optimal* expected net present value is given by

$$\text{ENPV}^* = \max(\text{ENPV}_0, \text{ENPV}_w).$$

To illustrate, let us continue with our numerical example with  $P_o = 200$ ,  $q = 0.5$ , and  $i = 10\%$ . For values of  $I \in (\$1, 100, \$3, 300]$  we have

$$\begin{aligned} \text{ENPV}_w &= 0.5 \left[ 3000 - \frac{I}{1.1} \right] \\ &= 1500 - \frac{I}{2.2}. \end{aligned}$$



Comparing to  $2200 - I$  we see that  $ENPV_w > ENPV$  for values of  $I \in (\$1,283.33, \$3,300]$  with the opposite inequality holding for  $(\$1,000, \$1,283.33]$ . This results in  $ENPV^* = ENPV_0 = 2200 - I$  on  $I \leq \$1,283.33$  and  $ENPV^* = ENPV_w = 1500 - I/2.2$  on  $I \in (\$1,283.33, \$3,300]$ .

Thus, for example, if  $I = \$1,600$  we see that  $ENPV_0 = \$600 > 0$ , and yet it is better to wait, since  $ENPV_w = \$772.73$ . Here, we are better off postponing the investment even though the present value is *positive*. On the other hand, suppose that  $I = \$2,500$ . Then,  $ENPV_0 = -\$300$  so the present value is negative, yet  $ENPV_w = \$363.64 > 0$ . Here, if we were forced to make the decision now we would reject the project. Yet, the expected present value of the project is positive when we have the option of waiting!

In summary, we would invest now if  $I < \$1,283.33$  and never invest if  $I > \$3,300$ . For intermediate values it is best to wait one year and invest if the price goes up.

A similar analysis can be carried out to determine values of  $P_0$  for which it is best to invest now, wait, or not invest at all. Indeed, if  $q = 0.5$  then

$$ENPV_0 = -I + P_0(1+i)/i,$$

so  $ENPV_0 \geq 0$  whenever

$$P_0 \geq I \frac{i}{1+i}.$$

Let us now consider the option of waiting. If

$$\frac{1}{1.5} I \frac{i}{1+i} \leq P_0 \leq 2I \frac{i}{1+i},$$

then we would only invest if the price goes up, so

$$ENPV_w = 0.5 \left[ \frac{1.5P_0}{i} - \frac{I}{1+i} \right].$$

Again,

$$ENPV^* = \max(ENPV_0, ENPV_w).$$

Here  $ENPV_0 \leq ENPV_w$  on

$$P_0 \leq 2I \frac{1+2i}{1+i} \frac{i}{1+4i}.$$

Thus if  $i = 10\%$  and  $I = 1600$  waiting is better than investing now if

$$97 \leq P_0 \leq 249.$$

An investing now is better than waiting if

$$P_0 > 249.$$

On the other hand, it is best not to invest at all if

$$P_0 < 97.$$