

# 1 Multi-Index Models

Factor models or index models assume that the return on a security is sensitive to the movements of various factor or indices. Multiple-factor models are potentially more useful than a single index model based on a market index because it appears that actual security returns are sensitive to more than movements in a market index.

We can think of index models as return-generating processes. That is, a statistical model that describes how the return of a security is produced. As a return generating process a factor model attempts to capture the major economic forces that systematically move the prices of all securities. Implicit is the assumption that the returns on two securities will be correlated only through common reactions to one or more factors specified in the model. Any aspect of a security's return unexplained by the factor model is assumed to be unique or specific to the security and therefore uncorrelated with the unique elements of returns on other securities.

As with the case of single index model, multi-index factor models can be used to:

1. Supply the information needed to calculate  $r$  and  $V$ .
2. It can be used to characterize a portfolio's sensitivity to movements in the factors and to decompose risk in several ways
3. Analyze current portfolio risk
4. Design future portfolios
5. Evaluate past performance

We will discuss three approaches to factor models: time-series, cross-sectional, and factor analytic.

## 1.1 Time-series approach

The time-series approach is perhaps the most intuitive to investors. The model begins with the assumption that he or she knows in advance the factors that influence security returns. The model builder collects information concerning the values of the factors and security returns from period to period. Using these data, the model builder can estimate the unknown parameters, i.e., the  $\alpha$ s,  $\beta$ s, and  $\tau$ s.

The standard form of the multi-index model can be written as follows:

$$R_i = \alpha_i + \sum_{k=1}^K \beta_{ik} I_k + \epsilon_i \quad (1)$$

where  $I_1, \dots, I_K$  are the factors,  $\alpha_i$  is a constant and  $\beta_{ik}$  is the sensitivity of security  $i$  to factor  $k$ . In this model,  $I_1, \dots, I_K$  are random variables and so is  $\epsilon_i$ . The constants  $\alpha_i$  and the sensitivities  $\beta_{ik}$  are unknown constants that need to be estimated, usually by linear regression.

Let  $\tau_i^2$  denote the variance of  $\epsilon_i$  and let  $\sigma_{I_k}^2$  denote the variance of index  $I_k$ . By construction we can and do select  $E[\epsilon_i] = 0$ ,  $\text{Cov}(I_k, I_l) = 0$  for  $k \neq l$ , and  $\text{Cov}(\epsilon_i, I_k) = 0$  for all  $i, k$ . The main assumption of this model is that  $\text{Cov}(\epsilon_i, \epsilon_j) = 0$  for all  $i \neq j$ .

Under these assumptions we can proceed to compute the mean:

$$\bar{R}_i = \alpha_i + \sum_{k=1}^K \beta_{ik} \bar{I}_k,$$

the variance

$$\sigma_i^2 = \sum_{k=1}^K \beta_{ik}^2 \sigma_{I_k}^2 + \tau_i^2,$$

and the covariance

$$\sigma_{ij} = \sum_{k=1}^m \beta_{ik} \beta_{jk} \sigma_{I_k}^2$$

between securities.

Potential factors that explain security returns include:

1. Gross domestic product (GDP)
2. Inflation
3. The level of interest rates
4. The level of oil prices
5. Economic sectors

Example: Table 1 presents a time series of GDP, Inflation, and the return of a specific security.

Year	GDP	Inflation	Return
1	5.7%	1.1%	14.3%
2	6.4%	4.4%	19.2%
3	7.9%	4.4%	23.4%
4	7.0%	4.6%	15.6%
5	5.1%	6.1%	9.2%
6	2.9%	3.1%	13.0%

Table 1: Data for Time Series Model

Regressing the return of the security against GDP and inflation we obtain

$$\bar{R} = 5.80 + 2.17I_1 - 0.68I_2$$

where  $I_1$  is GDP and  $I_2$  is Inflation. A similar regression is needed for each security.

### 1.1.1 Industry Models

A natural extension to the single index model, and a good example of a multi-index model, is to add industry factors resulting in the model

$$R_i = \alpha_i + \beta_{im} I_m + \sum_{k=1}^K \beta_{ik} I_k + \epsilon_i$$

where  $I_m$  is the market index and  $I_k$   $k = 1, \dots, K$  are industry indices that are uncorrelated with the market and with each other. If we further assume that a firm  $i$  is industry  $k$  the equation simplifies to

$$R_i = \alpha_i + \beta_{im} I_m + \beta_{ik} I_k + \epsilon_i$$

resulting in expected return

$$\bar{R}_i = \alpha_i + \beta_{im} \bar{I}_m + \beta_{ik} \bar{I}_k.$$

The covariance between securities  $i$  and  $j$  can be written as

$$\beta_{im} \beta_{jm} \sigma_m^2 + \beta_{ik} \beta_{jk} \sigma_{I_k}^2$$

for firms in the same industry ( $k$ ) and

$$\beta_{im} \beta_{jm} \sigma_m^2$$

for firms in different industries.

## 1.2 Cross-sectional approach

In the cross sectional approach the model builder starts with estimates of securities' sensitivities to certain factors. Then in a particular time period, the values of the factors are estimated based on securities' returns and their sensitivities to the factors. The process is repeated over multiple time periods, thereby providing an estimate of the factors' standard deviations and their correlations.

To better understand the cross-sectional approach, imagine collecting returns and the values of certain attributes such as dividend-yield, size, momentum, etcetera for a number of different stocks in a given time period. The goal is to see how the returns are related to the attributes. Notice that in contrast to the time series approach, the cross-sectional approach is based on the sensitivity of stock returns to the attributes over one period of time. To quantify the relationship we postulate the model

$$R_i = \alpha + \sum_{k=1}^K \beta_{ik} I_k + \epsilon_i$$

across a large number of stocks over one period of time rather than across time for a single stock. For these models we know the exposures or sensitivities  $\beta_{ik}$  and the objective is to estimate the value of  $\alpha$  and the value of the factors  $I_k$ ,  $k = 1, 2, \dots, K$ .

For example, in a two factor model with dividend-yield and size<sup>1</sup> as factors,  $\beta_{i1}$  is the dividend yield of security  $i$  and  $\beta_{i2}$  is the size of security  $i$ . The objective is to find out how these factors affect returns. Regression will give us the values of  $\alpha$ ,  $I_1$  and  $I_2$  over the period. The value  $\alpha$  represents the expected return on a typical stock with a dividend yield of zero and size zero. The slope  $I_1$  represents the increase in expected return for each percent of dividend yield and  $I_2$  represents the increase in expected return for each percent increase in size.

If we obtain

$$\bar{R}_i = 7 + .4b_{i1} - .3b_{i2}$$

from regression, then the zero factor is 7% meaning that a stock with zero dividend yield and zero size would have been expected to have a return of 7%. During this time period, higher-dividend yields and smaller sizes were both associated with larger returns.

The procedure is repeated over several periods to obtain the mean and variance of  $I_1$  and  $I_2$  and the covariance of  $I_1$  and  $I_2$ . This can then be used to obtain estimates of the covariance between securities.

Suppose that the cross-sectional approach is followed. Let  $\bar{I}_k$  denote the mean of  $I_k$ , and let  $\nu_{kl} = \text{cov}(I_k, I_l)$ . Let  $\tau_i^2 = \text{Var}[\epsilon_i]$  and suppose that  $\epsilon_i$  is uncorrelated with the factors and with  $\epsilon_j$  for all  $j \neq i$ . Under these conditions we can estimate the mean and variance of the securities as follows:

$$\bar{R}_i = \alpha + \sum_{k=1}^K \beta_{ik} \bar{I}_k,$$

$$\sigma_i^2 = \sum_{k=1}^K \beta_{ik}^2 \nu_k^2 + \tau_i^2,$$

where  $\nu_k^2 = \nu_{kk}$  is the variance of factor  $I_k$ , and finally

$$\sigma_{ij} = \sum_{k=1}^K \sum_{l=1}^K \beta_{ik} \beta_{jk} \nu_{kl}$$

for  $i \neq j$ . In vector notation we can write the vector of expected returns as

$$r = \alpha e + B\bar{I},$$

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<sup>1</sup>The size attribute that is often computed by taking the logarithm of the total market value of the firms' outstanding equity measured in millions. This convention is based on the empirical observation that the impact of the size factor on a security with a large total market value is likely to be twice as great as that on a security with one-tenth the value.

where  $B = (\beta_{ik})$  is an  $n \times K$  matrix and  $\bar{I}$  is the column vector of the expected factor values. In matrix notation we can write the variance–covariance matrix of security returns as

$$V = BFB' + T$$

where  $F$  is the  $K \times K$  variance-covariance matrix of the factors and  $T$  is an  $n \times n$  diagonal matrix with elements  $\tau_1^2, \dots, \tau_n^2$  in the diagonal.

### 1.3 Factor Analytic Approaches

Here the model builder knows neither the factor values nor the securities' sensitivities to those factors. A statistical technique called factor analysis, also known as principal component analysis, is used to extract factors and securities' sensitivities based simply on a set of securities' past returns. Factor analysis takes the returns over many time periods on a sample of securities and attempts to identify one or more statistically significant factors that could have generated the covariances of returns observed within the sample. Unfortunately, factor analysis does not specify what economic variables the factors represent.

To see how this works assume that  $R_1, \dots, R_n$  denote the returns of  $n$  securities and let  $V$  be the corresponding variance-covariance matrix. A vector  $x$  is an eigenvector of  $V$  if

$$Vx = \lambda x$$

for some constant  $\lambda$  called an eigenvalue of  $V$ . It is easy to see that if  $x$  is an eigenvector of  $V$  with eigenvalue  $\lambda$ , and  $c$  is a constant, then  $y = cx$  is also an eigenvector of  $V$  with eigenvalue  $\lambda$ . Consequently, if  $x$  is an eigenvector with  $e'x \neq 0$  then  $\frac{1}{e'x}x$  is also an eigenvector with the same eigenvalue. This shows that after scaling we can find eigenvectors with the portfolio property  $e'x = 1$ . Suppose  $x$  is such an eigenvector. The return of portfolio  $x$  is the random variable

$$R_1x_1 + R_2x_2 + \dots + R_nx_n,$$

and it is known as a principal component (corresponding to eigenvalue  $\lambda$ ). The *first* principal component is the one corresponding to the *largest* eigenvalue of  $V$ . A good candidate for the factor in a single factor model is the first principal component. In a similar way, other principal components can also be used as factors.

### 1.4 Summary

Not surprisingly, multi-index models are better at explaining historical covariances than the single index model, but they are not necessarily better in predicting the covariance structure of future returns. In part this is due to the tendency of multi-factor models to fit noise. On the other hand, key factors change over time as the effect of energy prices on security markets in the 1970s, during the Persian Gulf war, and more recently due to oil shortages. As we have seen by the tests done by Elton and Gruber parsimonious models, such as the single-index and the constant correlation model tend to be better at predicting future returns than many multi-index models. These reasons have not kept people from trying to build ever more sophisticated multi-index models. Some of these models are proprietary, and a number of companies are in the business of selling estimates of  $r$  and  $V$ .