

1 Capital Asset Pricing Model

The CAPM is a cornerstone of financial economics. It answers a puzzle that had intrigued many economists. What is the risk return relationship for specific assets and for portfolios of assets?

We know that when short sales and borrowing and lending at the risk free rate is allowed efficient portfolios have expected return given by

$$\bar{R} = R_f + S_q \sigma,$$

where S_q is the Sharpe ratio of the tangency portfolio x_q . Individual securities and inefficient portfolios will lie below this line. From an earlier assignment we know that if x_q is the tangency portfolio and $\beta = \frac{V x_q q}{\sigma_q^2}$ then for any portfolio x_p we have

$$q_p = x_p' q = \beta_p q_q.$$

To see this, notice that

$$\beta_p = \frac{\sigma_{qp}}{\sigma_{qq}} = \frac{x_q' V x_p}{x_q' V x_q}.$$

Now notice that

$$x_q' V = \frac{q'}{e' V^{-1} q},$$

which implies that

$$\beta_p = \frac{q' x_p}{q' x_q} = \frac{q_p}{q_q}.$$

The result follows from multiplying by q_q . The equation $q_p = \beta_p q_q$ translates into

$$\bar{R}_p = R_f + \beta_p (\bar{R}_q - R_f),$$

and in particular to

$$\bar{R}_i = R_f + \beta_i (\bar{R}_q - R_f).$$

The CAPM goes further by concluding (under some assumptions to be stated shortly) that in equilibrium the tangent portfolio is the “market portfolio¹,” resulting in

$$\bar{R}_i = R_f + \beta_i (\bar{R}_m - R_f)$$

where \bar{R}_m is the expected return of the market portfolio.

Thus, the two main conclusions of the CAPM are:

- (i) The market portfolio is efficient (in fact it has the highest Sharpe ratio), and
- (ii) The CAPM gives a pricing formula that relates the excess return of a security or a portfolio to that of the excess return of the market.

Before we state the assumptions and the equilibrium proof of the CAPM we will discuss utility theory and its relationship to mean variance analysis.

¹The market portfolio is a portfolio consisting of all securities where the proportion invested in each security corresponds to its relative market value. The relative market value of a security is simply equal to the aggregate market value of the security divided by the sum of the market value of all securities.

1.1 Utility Theory

While it clear that \$5,000,000 is better than \$4,000,000, comparing random outcomes is more complicated. To see this, try to decide what you would prefer among the following two choices for your total wealth:

A \$10,000,000 with probability 1/2 and \$0 with probability 1/2.

B \$4,000,000 with probability 1.

Although choice A has an expected value of \$5,000,000, most people would prefer choice B.

Economists have developed a theory to help investors make decision under uncertainty. Under reasonable assumptions this theory shows that every investors has a utility function relating its future wealth to the utility the investor derives from this wealth. It doesn't matter how utility is measured. What is important is that the existence of a utility function allows the investor to rank different investment opportunities, and this is done by selecting the investment opportunity that maximizes expected utility. If U is the utility function of an investor than he or she will prefer wealth X to wealth Y if

$$EU(X) \geq EU(Y).$$

For all investors U is increasing. This means that all investors prefer more wealth. For most investors U is also concave, which means that there is a diminishing marginal utility of wealth. Investors with increasing concave utility functions are said to be risk-averse. This means that these investors are willing to pay money to avoid risk. Indeed, by Jensen's inequality

$$EU(X) \leq U(EX)$$

which means that $EU(EX) \geq EU(X)$, so a risk-averse investor will prefer EX to wealth X .

Example: Suppose $U(x) = \sqrt{x}$ and X is a random variable taking value 10,000 with probability .05 and value 40,000 with probability .95. Then

$$EU(X) = .05(100) + .95(200) = 195.$$

On the other hand, $EX = 500 + 38,000 = 38,500$, so

$$U(EX) = \sqrt{38,500} = 196.21.$$

Here we see that $EU(X) < U(EX) = EU(EX)$ so an investor with initial wealth of \$40,000 will be willing to pay at least \$1,500 to avoid the risk of losing \$30,000 with probability 0.05.

1.1.1 Certainty Equivalence

What certain wealth W would make the investor indifferent between W and random wealth X ? We need to find W so that the expected utility of W , i.e., $EU(W) = EU(EW) = U(EW) = U(W)$ is equal to the expected utility of X , i.e., $EU(X)$. The equation is

$$U(W) = EU(X)$$

and the solution is

$$W = U^{-1}(EU(X)).$$

In the above example, we need to solve for

$$U(W) = EU(X) = 195$$

or

$$\sqrt{W} = 195$$

or

$$W = 195^2 = 38,025.$$

This tells us that this investor is willing to pay $EX - W = 475$ to remove risk.

1.1.2 Utility Theory and Mean Variance Analysis

Let X and Y be two random variables with the same expected return μ and assume that $\sigma_x^2 = \text{Var}[X] < \sigma_y^2 = \text{Var}[Y]$. Under what conditions would an investor with utility function U prefer X over Y ?

Let X and Y be two random variables with the same variance σ^2 and assume that $\mu_x = E[X] > \mu_y = E[Y]$. Under what conditions would an investor with utility function U prefer X over Y ?

It can be shown that the mean variance criterion is consistent with maximizing expected utility if either:

- The utility function is the increasing concave part of a quadratic function.
- The random variables are normally distributed.

In addition, mean variance analysis is consistent with the second order Taylor approximation of strictly risk-averse utility functions. Indeed, suppose that U is twice differentiable and $U'' < 0$. Then

$$U(X) \simeq U(\mu) + (X - \mu)U'(\mu) + \frac{1}{2}(X - \mu)^2 U''(\mu).$$

Taking expectations we find

$$EU(X) \simeq U(\mu) + \frac{1}{2}U''(\mu)\sigma^2$$

If X and Y have the same mean but difference variance we see that

$$EU(X) - EU(Y) \simeq \frac{1}{2}U''(\mu)[\sigma_x^2 - \sigma_y^2]$$

and the right hand side is positive if and only if $\sigma_x^2 < \sigma_y^2$.

A similar argument can be used to show that investors prefer the security with highest expected return among those with the same standard deviation.

1.1.3 Indifference Curves

Suppose that mean variance analysis is consistent with expected utility maximization. Given a random variable X with mean μ and standard deviation σ we can find its certainty equivalent, which plots as $(0, W)$ in standard deviation expected return space. An indifference curve in this space is the set of all (σ, μ) pairs with certainty equivalent equal to W . This is an increasing convex function reflecting the fact the risk-averse investors require increasingly more compensation for taking more risk.

For example, if the utility derived from a (σ, μ) security is $\mu - \frac{\sigma^2}{\tau}$ then the utility derived from $(0, W)$ is W so the certainty equivalent is $W = \mu - \frac{\sigma^2}{\tau}$. Solving for μ we find

$$\mu = W + \frac{\sigma^2}{\tau}$$

which is increasing convex in σ indicating that the risk averse investor requires increasingly more return to take more risk.

1.2 Assumptions of Capital Asset Pricing Model

CAPM1 There are no transactions costs

CAPM2 Assets are infinitely divisible

CAPM3 Absence of personal income tax

CAPM4 Individuals cannot affect prices by buying and selling

CAPM5 Individuals make decisions in terms of means and variances

CAPM6 Short sales are allowed

CAPM7 Borrowing and lending at risk-free rate

CAPM8 All investors have same one-period horizon

CAPM9 All investors have same r and V

“The relevant question to ask about assumptions of a theory is not whether they are descriptively realistic, for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions.” (Milton Friedman, Nobel Prize in Economics 1976)

1.3 Equilibrium Argument of CAPM

Under these assumptions all investors would determine the composition of the same tangency portfolio, and all investors will have portfolios involving combinations of the agreed-upon tangency portfolio and either risk free lending or borrowing. As all investors face the same efficient set, the only reason they will choose different portfolios is that they have different indifference curves. Note, however, that although the chosen portfolios will be different, each investor will choose the same combination of risky securities.

In equilibrium each security must have a nonzero proportion in the composition of the tangency portfolio. If a security has zero weight in x_q then nobody is investing in this security and its price must fall, thereby causing the expected returns of these security to rise until the resulting tangency portfolio has a nonzero proportion.

On the other hand, if each investor concludes that the tangency portfolio should include x_{qi} of security i but there are not enough shares outstanding to meet the demand, orders to buy this share will raise the price in search of sellers.

Ultimately, everything balances out. When all the price adjusting stops, the market will have been brought into equilibrium:

- Each investor will want to hold a certain positive amount of each security
- The current market price of each security will be at a level where the number of shares demanded equal the number of shares outstanding.
- The risk-free rate will be at a level where the total amount borrowed equals the total amount of money lent.

As a result, in equilibrium the proportions of the tangency portfolio will correspond to the proportion of what is known as the market portfolio. That is $x_m = x_q$.

In theory, the market portfolio x_m consists not only of common stocks but also bonds, preferred stocks, real estate, etc.. In practice, some people restrict x_m to common stocks.

1.4 The Capital Market Line

Identifying x_m with x_q and S_m with S_q we see that efficient portfolios have expected return

$$\bar{R} = R_f + S_m \sigma.$$

This is known as the capital market line (CML). Individual securities and other inefficient portfolios will have expected returns that lie below the CML.

1.5 The Security Market Line

The security market line gives the expected returns of all portfolios (efficient and inefficient). Identifying x_m with x_q we define beta as

$$\beta = \frac{Vx_m}{x_m' Vx_m} = \frac{Vx_m}{\sigma_m^2}.$$

For any portfolio x_p let $\beta_p = \beta' x_p$. In particular security i , represented by portfolio e_i has

$$\beta_i = e_i' \beta = \frac{\sigma_{im}}{\sigma_m^2},$$

where we have used the fact that $\sigma_{im} = e_i' Vx_m$.

Now, from

$$q_p = \beta' x_p q = \beta_p q,$$

(recall assignment two), we have for any portfolio x_p

$$\bar{R}_p - R_f = \beta_p (\bar{R}_m - R_f).$$

In particular, for security i we have

$$\bar{R}_i = R_f + \beta_i (\bar{R}_m - R_f). \quad (1)$$

Equation (1) is known as the security market line (SML), and can be written in vector notation as

$$r = R_f e + (\bar{R}_m - R_f) \beta,$$

where $\beta = \frac{Vx_m}{\sigma_m^2}$. As described above, the SML works for portfolios as well as for individual securities.

1.6 What does the CAPM say about risk?

The CAPM suggests that

$$R_i = R_f + \beta_i (R_m - R_f) + \epsilon_i.$$

It is easy to see that

$$\begin{aligned} \sigma_{im} &= \text{Cov}(R_i, R_m) = \beta_i \sigma_m^2 + \text{Cov}(\epsilon_i, R_m) \\ &= \sigma_{im} + \text{Cov}(\epsilon_i, R_m), \end{aligned}$$

so the CAPM implies that $\text{Cov}(\epsilon_i, R_m) = 0$.

This implies that

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \tau_i^2.$$

The first part $\beta_i^2 \sigma_m^2$ is known as systematic risk. The second part τ_i^2 is known as unique risk. Notice that the SML (1) tell us that the market compensates you only for bearing systematic risk. Since

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \rho_{im} \frac{\sigma_i}{\sigma_m},$$

the systematic risk is given by

$$\beta_i^2 \sigma_m^2 = \rho_{im}^2 \sigma_i^2$$

and the unique risk by

$$\tau_i^2 = (1 - \rho_{im}^2) \sigma_i^2.$$

These formulas decompose the total risk of a security into systematic risk and unique risk as a function of the correlation coefficient of the security with the market. Can you extend this formula for portfolios? If so, how?

1.7 Informal Derivation of the SML

The above derivation of the SML is completely formal. An informal (but perhaps more intuitive) derivation often presented to business school students goes as follows. Since

$$\sigma_m^2 = x_m' V x_m = \sum_{i=1}^n x_{mi} \sigma_{mi},$$

$$\begin{aligned} \bar{R}_m &= R_f + \frac{S_m}{\sigma_m} \sigma_m^2 \\ &= R_f + \frac{S_m}{\sigma_m} \sum_{i=1}^n x_{mi} \sigma_{mi}, \end{aligned}$$

and

$$\bar{R}_m - R_f = \sum_{i=1}^n x_{mi} (\bar{R}_i - R_f).$$

Then

$$\sum_{i=1}^n x_{mi} (\bar{R}_i - R_f) = \sum_{i=1}^n x_{mi} \frac{S_m}{\sigma_m} \sigma_{mi},$$

which suggests that

$$\bar{R}_i - R_f = \frac{S_m}{\sigma_m} \sigma_{mi}$$

or equivalently

$$\bar{R}_i - R_f = \frac{\bar{R}_m - R_f}{\sigma_m^2} \sigma_{mi},$$

or

$$\bar{R}_i - R_f = \frac{\sigma_{mi}}{\sigma_m^2} (\bar{R}_m - R_f),$$

which is equivalent to

$$\bar{R}_i = R_f + \beta_i (\bar{R}_m - R_f).$$

1.8 What does the CAPM tells us about Sharpe Ratios?

As a final note, we relate Sharpe ratios of portfolios to their correlation with the market.

$$\begin{aligned} S_p &= \frac{\bar{R}_p - R_f}{\sigma_p} \\ &= \frac{(\bar{R}_m - R_f) \beta_p}{\sigma_p} \\ &= S_m \frac{\beta_p \sigma_m}{\sigma_p} \\ &= S_m \frac{\sigma_{pm}}{\sigma_m \sigma_p} \\ &= S_m \rho_{pm}. \end{aligned}$$

Which implies that

$$\rho_{pm} = \frac{S_p}{S_m}.$$

This implies that the higher the correlation of a portfolio with the market, the higher its return for a given level of risk.

1.9 CAPM as a Pricing Formula

Let V_0 be the current price of an asset or portfolio and let $E[V_1]$ be the expected value at the end of the horizon. The CAPM suggests that

$$\frac{E[V_1] - V_0}{V_0} = R_f + \beta(\bar{R}_M - R_f)$$

where β is the Beta of the asset. This implies that

$$V_0 = \frac{E[V_1]}{1 + R_f + \beta(\bar{R}_M - R_f)}.$$

This tells us that the fair price of the asset today is the expected value of the asset a period from now discounted at rate $R_f + \beta(\bar{R}_M - R_f)$.

Example: You have \$100 and invest 10% at the risk-free rate and 90% in the market. Suppose that $R_f = 7\%$ and $\bar{R}_m = 15\%$. The portfolio has expected value $E[V_1] = 10(1.07) + 90(1.15) = 114.2$. Since the portfolio Beta is 0.9, the present value is

$$V_0 = \frac{E[V_1]}{1 + R_f + 0.9(R_m - R_f)} = \frac{114.2}{1.142} = 100.$$

There is another way we can use the CAPM as a pricing formula. Recall that

$$\begin{aligned} \beta &= \text{Cov}(R, R_m) / \sigma_m^2 \\ &= \text{Cov}\left(\frac{V_1 - V_0}{V_0}, R_m\right) / \sigma_m^2 \\ &= \frac{1}{V_0} \text{Cov}(V_1, R_m) / \sigma_m^2. \end{aligned}$$

Consequently,

$$V_0 = \frac{E[V_1]}{1 + R_f + \frac{1}{V_0} \text{Cov}(V_1, R_m) (\bar{R}_M - R_f) / \sigma_m^2}.$$

Solving for V_0 we obtain

$$V_0 = \frac{E[V_1] - \text{Cov}(V_1, R_m) \frac{\bar{R}_m - R_f}{\sigma_m^2}}{1 + R_f}.$$

Here we are subtracting from $E[V_1]$ the cost of risk and discounting using the risk-free rate.

1.10 Comparison with Single Index Market Model

The Single Index Market Model is not an equilibrium model. The SIMM uses a broad market index, while CAPM uses the market portfolio. In practice, people use broad indices to approximate the market portfolio. Implications: The SIMM postulates

$$\bar{R}_i = \alpha_i + \beta_{iI} \bar{I}$$

while the CAPM postulates

$$\bar{R}_i - R_f = \beta_{im} (\bar{R}_m - R_f).$$

We can rewrite the SIMM as

$$\bar{R}_i - R_f = [\alpha_i - (1 - \beta_{iI})R_f] + \beta_{iI}(\bar{I} - R_f)$$

to make it look like the CAPM. In practice, the CAPM is used with a broad market index I that serves as a proxy for the market portfolio. Used in this way the practical-CAPM states that

$$\bar{R}_i - R_f = \beta_{iI}(\bar{I} - R_f).$$

Thus the practical-CAPM implies that

$$\alpha_i = (1 - \beta_{iI})R_f.$$

If this fails to hold in practice the expected returns forecasted by the practical-CAPM will differ from those forecasted by the SIMM. This opens the possibility for active managers to do better than the market. Caution: This is easier said than done.

1.11 Zero-Beta Form of the CAPM

Assume that there is no lending or borrowing and that short sales are allowed. Now individuals invest in (possibly different) portfolios in the efficient frontier. By repeatedly using the two fund theorem we conclude that weighted sums of efficient portfolios are also efficient. Since the market portfolio is the weighted sum of all efficient portfolios it follows that the market portfolio is also efficient. Can we recover the Security Pricing Line? Let us try write down a “Capital Market Line” that goes through (σ_m, \bar{R}_m) with slope λ_m equal to the slope of the efficient frontier at (σ_m, \bar{R}_m) . That is,

$$\bar{R} = \bar{R}_m + \lambda_m(\sigma - \sigma_m)$$

Let \bar{R}_z denote the intercept of this line at $\sigma = 0$. Later we will identify \bar{R}_z as the expected return of zero-beta portfolios. Then

$$\lambda_m = \frac{\bar{R}_m - \bar{R}_z}{\sigma_m}.$$

so the “CML” is given by

$$\bar{R} = \bar{R}_z + \lambda_m \sigma.$$

Since R_m and has maximum Sharpe ratio relative to \bar{R}_z , the market portfolio is the tangent portfolio that we would obtain if the risk-free rate were \bar{R}_z . By an earlier exercise

$$\bar{R}_i = \bar{R}_z + \beta_i(\bar{R}_m - \bar{R}_z).$$

where $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$. For portfolios we have

$$\bar{R}_p = \bar{R}_z + \beta_p(\bar{R}_m - \bar{R}_z).$$

In particular, for R_z we have

$$\bar{R}_z = \bar{R}_z + \beta_z(\bar{R}_m - \bar{R}_z)$$

which implies $\beta_z = 0$.

Let us briefly discuss how to find the minimum variance zero-beta portfolio. From the two-fund theorem, efficient portfolios can be written as $(1 - \alpha)x_e + \alpha x_r$ for some $\alpha \geq 0$. In particular, there exists and $\alpha_m \geq 0$ such that $x_m = (1 - \alpha_m)x_e + \alpha_m x_r$ is the market portfolio. To avoid trivialities, we will assume that $x_r \neq x_e$ which implies that $\alpha_m > 0$. On the other hand, all portfolios of the form

$$(1 - \gamma)x_e + \gamma x_r$$

with γ a real number are minimum variance portfolios. In fact, they are efficient if $\gamma \geq 0$, and inefficient (but minimum variance) if $\gamma < 0$.

To find the minimum variance zero-beta portfolio we need to find the value of γ that makes portfolio $x_\gamma = (1 - \gamma)x_e + \gamma x_r$ uncorrelated with x_m . Clearly,

$$\begin{aligned} x_\gamma' V x_m &= (1 - \gamma)(1 - \alpha_m)\sigma_e^2 + [(1 - \gamma)\alpha_m + \gamma(1 - \alpha_m)]\sigma_{em} + \alpha_m \gamma \sigma_r^2 \\ &= [1 - \gamma\alpha_m]\sigma_e^2 + \alpha_m \gamma \sigma_r^2 \\ &= \sigma_e^2 + \gamma\alpha(\sigma_r^2 - \sigma_e^2), \end{aligned}$$

where the first equality is justified since $\sigma_e^2 = \sigma_{er}$. Setting the last expression to zero and solving for γ results in

$$\gamma = -\frac{\sigma_e^2}{\alpha_m(\sigma_r^2 - \sigma_e^2)} < 0.$$

From the sign of γ we immediately see that the zero beta portfolio is inefficient (but minimum variance), and therefore has mean return that below that of the minimum variance portfolio, and variance that is at least as large as σ_e^2 .

It not difficult to find the mean and the variance of the zero-beta portfolio. Notice that all efficient portfolios can be formed by combining the zero-beta portfolio and the market portfolio, although not all such combinations are efficient.

1.11.1 Riskless Lending but no Borrowing

What if people can lend, but not borrow at R_f ? Then it must be the case that $R_f \leq \bar{R}_z$ because otherwise the market portfolio would be dominated by the tangency portfolio. If $R_f < \bar{R}_z$ then the tangent portfolio T is to the left of the market portfolio M . The efficient frontier goes through $R_f - T - M$ and then follows the efficient frontier of risky securities to the right of M . What can we say about the security market line?

For risky securities it is still

$$\bar{R}_i = \bar{R}_z + \beta_i(\bar{R}_m - \bar{R}_z),$$

and for risky portfolios it is still

$$\bar{R}_p = \bar{R}_z + \beta_p(\bar{R}_m - \bar{R}_z),$$

What about efficient portfolios in $R_f - T$? These portfolios are a combination of an investment in the risk-free rate and the tangency portfolio. Consider a portfolio with $1 - a \geq 0$ invested in R_f and $a \geq 0$ invested in R_T . We obtain

$$\bar{R}_a = R_f + a(\bar{R}_T - R_f).$$

Since

$$\beta_a = a\beta_T,$$

we can write

$$\bar{R}_a = R_f + \beta_a \frac{\bar{R}_T - R_f}{\beta_T}.$$

More generally, for portfolios in $R_f - T$ with $\beta < \beta_T$ we have

$$\bar{R} = R_f + \beta \frac{\bar{R}_T - R_f}{\beta_T}.$$

How does this return compare with that of a risky portfolio with the same β ? The return of such a portfolio would be

$$\bar{R}_z + \beta(\bar{R}_m - \bar{R}_z).$$

It is easy to see that

$$\begin{aligned} \bar{R} &= R_f + \beta \frac{\bar{R}_T - R_f}{\beta_T} \\ &= R_f + \beta \frac{\bar{R}_z - R_f + \beta_T(\bar{R}_T - \bar{R}_z)}{\beta_T} \\ &= \left(1 - \frac{\beta}{\beta_T}\right) R_f + \frac{\beta}{\beta_T} \bar{R}_z + \beta(\bar{R}_T - \bar{R}_z) \\ &\leq \bar{R}_z + \beta(\bar{R}_T - \bar{R}_z) \end{aligned}$$

This gives rise to a puzzle: The expected return of efficient portfolio in $R_f - T$ is lower than the expected return of risky portfolios (not involving R_f) with the same beta. The conclusion is that Beta is not the appropriate measure of risk when $0 < \beta < \beta_T$.

To get some insight into this situation, consider portfolios A and A' both with the same $\beta \in (0, \beta_T)$. Portfolio A is in the efficient frontier $R_f - T$ while portfolio A' consists exclusively of risky securities. According to the SML, portfolio A has lower expected return than portfolio A'. However, it is possible to find portfolio A'' with the same expected return as A' (and higher β) but with lower total risk. Obviously, A'' is more desirable than A' so risk-averse investors would want to be in the line joining $(0, R_f)$ and (β_T, \bar{R}_T) .

What can we conclude from all this: If we draw the SML of the standard CAPM we see that it underestimates the expected returns of risky portfolios with $\beta < 1$ and overestimates the expected returns of securities with $\beta > 1$.

1.12 Other Forms of the CAPM

In addition to the zero-beta model, researchers have tried to relax other assumptions of the standard CAPM. In particular, efforts have been made to obtain equilibrium pricing equations under taxation, under heterogeneous expectations, and for multi-period models. All of these efforts lead to more complicated pricing formulas, some of which are similar to the standard CAPM. This makes it clear that the security market line under the standard CAPM is at best an approximation of the expected returns of portfolios under more realistic conditions, e.g., taxes, heterogeneous expectations, multi-period investment horizons, etcetera.

2 Style Analysis

The objective of style analysis is to ex-ray a portfolio in terms of the asset classes it invests over time. Once this is done, we can isolate what component of the return is due to investment style (asset allocation), and what component is due to selection (active portfolio). Style analysis was developed by William Sharpe at Stanford University.

The return of security i is modeled as

$$R_i = \sum_{k=1}^K b_{ik} I_k + \epsilon_i$$

where

- I_k is the value of factor k
- In style analysis, the factors represent returns of asset classes.
- b_{ik} is the sensitivity of R_i to factor I_k
- Force $\sum_k b_{ik} = 1$
- Require $b_{ik} \geq 0$.

Thus, style analysis is a constrained regression of the returns of a security on the returns of several asset classes.

Under this model the return of asset i is represented as the return on a portfolio invested in K asset classes plus a residual component ϵ_i . The component of $\sum_{k=1}^K b_{ik} I_k$ of R_i is attributable to style, while the component ϵ_i is attributable to selection.

Desirable properties of asset classes:

- mutually exclusive

- collectively exhaustive
- have statistically different returns

An example of asset classes with these properties would be:

- Treasury bills (Salomon Brothers 90 day)
- Intermediate Government Bonds (Lehman Brothers I)
- Long Term Government Bonds (Lehman Brothers L T)
- Corporate Bonds (Lehman Brothers C)
- Mortgage Related Securities (Lehman Brothers MB)
- Large Cap Value Stocks (BARRA V)
- Large Cap Growth Stocks (BARRA G)
- Mid-Cap Stocks (Barra M)
- Small-Cap Stocks (Barra S)
- Non US Bonds
- European Stocks
- Japanese Stocks
- Emerging Market Stocks

In the list above, Barra V is comprised by half the stocks in the SP 500 with the highest Book-to-Price Ratio, while Barra G is comprised by half the stocks in the SP 500 with the lowest Book-to-Price Ratio. Barra M is the top 80% of non SP 500 in terms of market capitalization, while Barra S is the bottom 20% of non SP500 in terms of market cap.

The exposure of an investor to the different asset classes is a function of the amounts invested in various funds, the amount invested by these funds in different securities, and the exposure of these securities to asset classes.

There are two ways we can approach finding the exposure:

- Internal. Detailed analysis of securities.
- External. Used realized returns to infer typical exposure of funds to asset classes.

For portfolios, we have

$$R_p = \sum_{k=1}^K b_{pk} I_k + \epsilon_p,$$

where $b_{pk} = \sum_{i=1}^n b_{ik} x_i$ for $k = 1, \dots, K$, and $\epsilon_p = \sum_{i=1}^n \epsilon_i x_i$ where x_i is the weight of security i in portfolio p . The second approach suggest estimating the sensitivities of portfolios directly from realize returns. Style analysis is based on the second approach. Here is how it works. Given monthly data: $R_{ph}, I_{1h}, \dots, I_{Kh}$, $h = 1, \dots, H$ representing historical returns of a portfolio and the asset classes, we select b_{pk} , $k = 1, \dots, K$ to minimize

$$\sum_{h=1}^H (R_{ph} - \sum_{k=1}^K b_{pk} I_{kh})^2$$

subject to

$$b_{pk} \geq 0,$$

and,

$$\sum_{k=1}^K b_{pk} = 1.$$

This is a quadratic program. Once the program is solved we can evaluate the performance of active portfolio managers. A passive fund manager provides an investor with an investment style. Active fund manager provides both style and selection. To evaluate the selection component we look at ϵ_p . If we can reject the hypothesis that $\epsilon_p < 0$, we can be confident that the selection is adding value to our portfolio. Otherwise, we would be better off simply following the style component.

See excel file style-vanguard.xls for an example of style analysis.