

# 1 Active Portfolio Management

This lecture is based on the book Active Portfolio Management by Richard C. Grinold and Ronald N. Kahn.

Active portfolio management starts with estimates of expected excess returns that do not agree with the consensus returns. We will argue that a straight maximization of a sensible utility function may lead to active portfolios that have beta values that are significantly different from 1. This exposes active portfolio managers to large business risk. This will lead us to consider beta-neutral portfolios, i.e., portfolios with beta equal to one. Among these portfolios we will see that which maximizes the *information ratio*, i.e., the ratio of active return to active risk.

## 1.1 Consensus returns

The consensus expected returns are given by  $q$ . The variance-covariance matrix by  $V$ . The benchmark portfolio is

$$x_q = \frac{V^{-1}q}{e'V^{-1}q}.$$

## 1.2 Sensible Utility Function

Consider the utility function

$$u(x) = q'x - \frac{1}{\tau}x'Vx, \quad (1)$$

where  $\tau$  is the risk tolerance. We know from an earlier assignment that  $u(x)$  is maximized at

$$x_e + \tau z$$

where  $x_e$  is the minimum variance portfolio and

$$z = \frac{e'V^{-1}q}{2}(x_q - x_e)$$

is a swap vector.

We want to find the value of  $\tau$ , say  $\tau^*$ , that would induce an investor to select the tangency portfolio  $x_q$ . We need

$$x_e + \tau^* z = x_q,$$

or equivalently,

$$\tau^* z = x_q - x_e.$$

By inspection, we see that

$$\tau^* = \frac{2}{e'V^{-1}q}.$$

We have argued before that  $e'V^{-1}q = q_q/\sigma_q^2$ , so this allow us to write

$$\tau^* = 2 \frac{\sigma_q^2}{q_q}.$$

An investor with risk tolerance  $\tau^*$  and utility function (1) would select the tangency portfolio over all risky portfolios in the efficient frontier.

### 1.3 Forecast Returns Different From Consensus Returns

Suppose that your estimate of excess expected returns is given by the vector  $g \neq q$ . Then the tangency portfolio based on  $g$  is

$$x_g = V^{-1}g/e'V^{-1}g.$$

If you have risk tolerance  $\tau^*$ , and you believe the excess returns are given by  $g$  you would invest in a combination of the risk-free rate and portfolio  $x_g$ . If you invest a fraction  $a > 0$  in portfolio  $x_g$  and a fraction  $(1 - a)$  in the risk-free rate, the excess expected return will be given by  $ag_g = ag'x_g$ , and the variance by  $a^2\sigma_g^2 = a^2x_g'Vx_g$ . The optimal choice of  $a$  maximizes

$$ag_g - \frac{1}{\tau^*}a^2\sigma_g^2.$$

Solving, we obtain

$$a^* = \frac{\tau^*}{2} \frac{g_g}{\sigma_g^2}.$$

### 1.4 Beta of Active Portfolio

The optimal portfolio invests a fraction  $a^*$  in  $x_g$  and a portion  $(1 - a^*)$  in the risk-free rate. What is the beta of the resulting portfolio. Let  $x_p = a^*x_g$ , and let  $\beta = \frac{Vx_q}{\sigma_q^2}$ . The beta of the resulting portfolio is

$$\beta_p = a^*\beta'x_g = a^*\beta_g.$$

We will now argue that

$$\beta_p = \frac{g_q}{q_q}.$$

By a CAPM like argument, the excess return of any portfolio  $x_p$  is related to the excess return of portfolio  $x_g$  by

$$g_p \equiv g'x_p = \frac{\text{Cov}(R_p, R_g)}{\sigma_g^2} g'x_g = \frac{\text{Cov}(R_p, R_g)}{\sigma_g^2} g_g.$$

In particular,

$$g_q = g'x_q = \frac{\text{Cov}(R_q, R_g)}{\sigma_g^2} g_g.$$

By multiplying and dividing the right hand side by  $\sigma_q^2$  we obtain

$$g_q = \beta_g \frac{\sigma_q^2}{\sigma_g^2} g_g.$$

Therefore,

$$\begin{aligned} \beta_p &= a^*\beta_g \\ &= a^* \frac{g_q}{g_g} \frac{\sigma_g^2}{\sigma_q^2} \\ &= \frac{\tau^*}{2} \frac{g_g}{\sigma_g^2} \frac{g_q}{g_g} \frac{\sigma_g^2}{\sigma_q^2} \\ &= \frac{\sigma_q^2}{q_q} \frac{g_g}{\sigma_g^2} \frac{g_q}{g_g} \frac{\sigma_g^2}{\sigma_q^2} \\ &= \frac{g_q}{q_q}. \end{aligned}$$

Thus, the beta of the optimal portfolio is the ratio of the excess return of the tangency portfolio under the active manager and the consensus forecast.

Although investing fraction  $a^*$  on  $x_g$  is optimal for an investor with risk tolerance  $\tau^*$ ,  $\beta_p$  may be significantly different from  $\beta_q = 1$  exposing active portfolio managers to significant residual risk relative to the benchmark  $x_q$ . In terms of total risk/return analysis, small differences between  $q$  and  $g$  may lead to very high levels of residual risk relative to the benchmark. Active managers loath taking the risk of significantly under-performing the benchmark because they may get fired.

## 1.5 Active Excess Return and Active Risk with Beta Equal to One

In this section we will restrict ourselves to portfolios with beta equal to one, and will concentrate on the tradeoff between active excess return and active risk.

The consensus excess returns are given by

$$q = \beta q_q$$

where  $\beta$  is as above and  $q_q = q'x_q$ .

In general, we can write our estimate of excess returns as

$$g = \beta(q_q + \Delta q_q) + \alpha,$$

where  $\Delta q_q$  is a scalar and  $\alpha$  is a vector. Interpret  $\Delta q_q$  as our disagreement of the excess return of the benchmark, and  $\alpha$  as the residual returns. In this section we will assume that we agree that the excess return of the benchmark is equal to  $q_q$ , so  $\Delta q_q = 0$  and so

$$g = \beta q_q + \alpha = q + \alpha.$$

In addition, assume that  $g$  is such that  $g_q = g'x_q = q_q$ . This last assumption forces

$$\alpha_q = \alpha'x_q = 0.$$

Thus, the benchmark portfolio is assumed to be alpha-neutral.

Here the vector  $\alpha$  denotes the abnormal returns. Under the CAPM  $\alpha = 0$ , thus our estimates differ from the consensus returns by the presence of alpha. Looking forward, alpha is a forecast of residual return. Looking backward alpha is the average of realized residual return. For any portfolio  $x_p$  we can write

$$\begin{aligned} g'x_p &= \beta'x_p q_q + \alpha'x_p \\ &= \beta_p q_q + \alpha_p. \end{aligned}$$

Suppose that we invest in a certain portfolio  $x_p$ . The active holding in risky assets is given by  $x_a \equiv x_p - x_q$ .

Then the *active* return is given by

$$\begin{aligned} g'x_a &= \beta'(x_p - x_q)q_q + \alpha'(x_p - x_q) \\ &= (\beta_p - 1)q_q + \alpha_p - \alpha_q \\ &= \beta_a q_q + \alpha_a \end{aligned}$$

where  $\beta_a \equiv \beta_p - 1$ , and  $\alpha_a \equiv \alpha_p - \alpha_q = \alpha_p$ .

The *active* risk is given by

$$\begin{aligned} \psi_p^2 &\equiv x_a' V x_a \\ &= \sigma_p^2 - 2\sigma_{pq} + \sigma_q^2 \\ &= \sigma_p^2 - 2\beta_p \sigma_q^2 + \sigma_q^2 \\ &= \sigma_p^2 - \beta_p^2 \sigma_q^2 + \sigma_q^2 (1 - \beta_p)^2. \end{aligned}$$

Here we have used the fact that  $\sigma_{pq} = \beta_p \sigma_q^2$ .

If we restrict ourselves to portfolios with  $\beta_a = 0$ , or equivalently  $\beta_p = 1$ , then the active expected return is given by

$$q'x_a = \alpha_a = \alpha'x_a,$$

while the active risk is given by

$$\psi_p^2 = \sigma_p^2 - \sigma_q^2.$$

Notice that we can also write the active risk as

$$\begin{aligned}\psi_p^2 &= x_a' V x_a \\ &= x_a' [\beta \beta' \sigma_q^2 + T] x_a \\ &= \beta_a^2 \sigma_q^2 + \omega_p^2,\end{aligned}$$

where  $\omega_p^2$  is the variance of the residual return. For portfolios with  $\beta_a = 0$  we have

$$\psi_p^2 = \sigma_p^2 - \sigma_q^2 = \omega_p^2.$$

Therefore, for portfolios with  $\beta_p = 1$  the active risk is equal to the residual risk. Notice that all such portfolios have variance  $\sigma_p^2 = \sigma_q^2 + \omega_p^2 \geq \sigma_q^2$ , so the benchmark is the minimum variance portfolio with beta equal to one.

Under these conditions, the active portfolio manager tries to find the active position  $x_\alpha$  that maximizes the ratio of active return  $\alpha'x_\alpha$  to active risk  $\sqrt{x_\alpha' V x_\alpha}$ . This is known as the information ratio.

$$\frac{\alpha'x_\alpha}{\sqrt{x_\alpha' V x_\alpha}}. \quad (2)$$

Following the same logic used to find the portfolio with the maximum Sharpe ratio, we see that

$$x_\alpha = \frac{V^{-1}\alpha}{e'V^{-1}\alpha},$$

maximizes (2) provided  $e'V^{-1}\alpha > 0$ . Under this assumption, the maximum information ratio is given by

$$\text{IR} = \frac{\alpha_\alpha}{\sigma_\alpha} = \sqrt{\alpha'V^{-1}\alpha},$$

where  $\alpha_\alpha = \alpha'x_\alpha$ , and  $\sigma_\alpha^2 = x_\alpha' V x_\alpha$ . If  $e'V^{-1}\alpha < 0$ , then  $\alpha_\alpha = \alpha'x_\alpha < 0$ , and  $-x_\alpha$  maximizes the information ratio. To handle both cases at the same time we redefine

$$x_\alpha = \frac{V^{-1}\alpha}{\text{abs}(e'V^{-1}\alpha)},$$

but the reader should be aware that  $e'x_\alpha$  can be either 1 or -1.

It is easy to verify that  $\beta_\alpha = 0$ . Indeed,

$$\begin{aligned}\beta_\alpha &= \beta'x_\alpha \\ &= \frac{x_q' V}{\sigma_q^2} \frac{V^{-1}\alpha}{e'V^{-1}\alpha} \\ &= \frac{x_q' \alpha}{\sigma_q^2 (e'V^{-1}\alpha)} \\ &= 0.\end{aligned}$$

As a consequence, any portfolio of the form  $x_p = x_q + ax_\alpha$  will have  $\beta_p = 1$ , active return  $\alpha_p \equiv a\alpha'x_\alpha$  and active risk  $\omega_p^2 \equiv a^2 x_\alpha' V x_\alpha = a^2 \sigma_\alpha^2$ . Notice that the information ratio of portfolio  $x_p$  is the same as that of portfolio  $x_\alpha$ , so

$$\alpha_p = \text{IR } \omega_p.$$

This defines the active manager's opportunity set.

The objective of the active portfolio manager is to find a portfolio that maximizes the value added function

$$\alpha_p - \frac{1}{\tau_r} \omega_p^2,$$

where  $\tau_r$  is the tolerance for residual risk. Since  $\alpha_p = \text{IR } \omega_p$ , we can write the objective function as

$$\text{IR } \omega_p - \frac{1}{\tau_r} \omega_p^2. \quad (3)$$

The active risk level that maximizes the value added function is  $\omega_p^* = \frac{\tau_r}{2} \text{IR}$ , which results in value added equal to

$$\frac{1}{4} \text{IR}^2 \tau_r = \frac{\omega_p^* \text{IR}}{2}.$$

This formula states that the ability of the manager to add value increases as the square of the information ratio and with the manager's tolerance for risk. Therefore, a manager's information ratio determines his or her potential to add value. One further implication is that every investor seeks the strategy or manager with the highest information ratio. Investors will differ only in how aggressively they implement the strategy depending on their risk tolerance.

We have solved the problem in terms of the residual risk. To determine the value of  $a$  recall that  $\omega_p = a\sigma_\alpha$ , so the optimal choice is to invest a fraction

$$a^* = \frac{\tau_r \text{IR}}{2 \sigma_\alpha}$$

in the active portfolio  $x_\alpha$ . Notice that  $a^*$  is linear in the risk tolerance, so an investor with more tolerance for risk will be more aggressive in implementing this strategy.

### 1.5.1 Cash Position

The resulting portfolio is  $x_p = x_q + a^* x_\alpha$ . Notice that  $e'x_p = e'x_q + a^* e'x_\alpha = 1 + a^* e'x_\alpha = 1 \pm a^*$ , since by construction  $e'x_\alpha = \pm 1$ . Consequently, portfolio  $x_p$  has a cash position equal to  $-a^* e'x_\alpha$ , or equivalently  $-a^*$  if  $e'x_\alpha = 1$  or  $a^*$  if  $e'x_\alpha = -1$ . Thus, implementing this strategy may require borrowing or lending money.

## 1.6 Active Excess Return and Active Residual Risk with Arbitrary Beta

Under the CAPM, the excess returns are given by

$$R_i - R_f = \beta_i (R_m - R_f) + \epsilon_i.$$

This gives rise to a variance-covariance matrix of the form

$$V = \sigma_q^2 \beta \beta' + T.$$

If we allow portfolios with arbitrary beta, then the *residual* risk of a portfolio  $x_p$  is given by

$$\begin{aligned} \omega_p^2 &= x_p' T x_p \\ &= x_p' [V - \beta \sigma_q^2 \beta'] x_p \\ &= x_p' V x_p - (\beta' x_p)^2 \sigma_q^2 \\ &= \sigma_p^2 - \beta_p^2 \sigma_q^2. \end{aligned}$$

Our objective is now to find a portfolio  $x_p$  that

$$\alpha' x_p - \frac{1}{\tau_r} x_p' T x_p,$$

where  $\tau_r$  is the tolerance for residual risk.

The first order conditions are

$$\frac{2}{\tau_r}(V - \sigma_q^2 \beta \beta') x_p = \alpha,$$

or equivalently,

$$V x_p = \frac{\tau_r}{2} \alpha + \beta_p \sigma_q^2 \beta.$$

Notice that

$$\begin{aligned} \alpha &= \frac{\alpha' x_\alpha}{x_\alpha' V x_\alpha} V x_\alpha \\ &= \frac{\alpha_\alpha}{\sigma_\alpha^2} V x_\alpha \\ &= \frac{\text{IR}}{\sigma_\alpha} V x_\alpha, \end{aligned}$$

while

$$\sigma_q^2 \beta = V x_q.$$

Thus,

$$\begin{aligned} V x_p &= \frac{\tau_r}{2} \frac{\text{IR}}{\sigma_\alpha} V x_\alpha + V \beta_p x_q \\ &= V[a^* x_\alpha + \beta_p x_q]. \end{aligned}$$

Consequently,

$$x_p = a^* x_\alpha + \beta_p x_q.$$

Notice also that  $x_p$  has beta equal to  $\beta_p$ , and that  $x_p$  has active cash position equal to  $1 - a^* - \beta_p$  assuming  $e' x_\alpha = 1$ .

## 1.7 Active Management with Zero Cash Position

Here we will consider portfolios that are fully invested in risky securities  $e' x = 1$ , so they have no cash position. Our objective is to find the minimum variance fully invested, beta-neutral portfolio with expected excess return  $\hat{b}$ . Suppose  $x$  is a beta-neutral portfolio,  $\beta' x = 1$ . Then the excess expected return is

$$g' x = \beta' x q_q + \alpha' x = q_q + \alpha' x_a = q_q + \alpha_a,$$

where  $x_a = x_p - x_q$ . Recall  $\alpha' x_q = 0$  since the benchmark portfolio is  $\alpha$  neutral. Thus our objective  $g' x = \hat{b}$  is equivalent to  $\alpha' x_a = b \equiv \hat{b} - q_q$ . The requirement that the portfolio has no cash position is equivalent to  $e' x_a = 0$ . Let us now consider the variance of portfolio  $x = x_q + x_a$ . Clearly

$$\begin{aligned} x' V x &= \sigma_q^2 + 2x_q' V x_a + x_a' V x_a \\ &= \sigma_q^2 + 2\sigma_q^2 \beta' x_a + x_a' V x_a \\ &= \sigma_q^2 + x_a' V x_a, \end{aligned}$$

where we have used the fact that  $1 = \beta' x = \beta' x_q + \beta' x_a = 1 + \beta' x_a$  implies  $\beta' x_a = 0$ .

Consequently, the problem reduces to

$$\min x_a' V x_a$$

subject to

$$\begin{aligned} \alpha' x_a &= b, \\ \beta' x_a &= 0, \end{aligned}$$

and

$$e'x_a = 0.$$

It is an exercise to show that there exists constants  $\lambda, \mu$ , and  $\gamma$  such that

$$x_a = \lambda x_\alpha + \mu x_e + \gamma x_q,$$

satisfies the constraints. Let  $\alpha_\alpha = \alpha'x_\alpha$ ,  $\alpha_e = \alpha'x_e$ , and recall that  $\alpha_q = \alpha'x_q$ . The constraints are given by

$$\lambda\alpha_\alpha + \mu\alpha_e + \gamma\alpha_q = b,$$

$$\lambda\beta_\alpha + \mu\beta_e + \gamma\beta_q = 0,$$

$$\lambda e'x_\alpha + \mu e'x_e + \gamma e'x_q = 0.$$

To solve we use the following facts:  $\alpha_q = 0$ ,  $\beta_\alpha = 0$ ,  $\beta_q = 1$ , and  $e'x_\alpha = e'x_e = e'x_q = 1$ . Solving, we obtain the swap vector

$$x_a = \frac{b}{(1 - \beta_e)\alpha_\alpha - \alpha_e}[(1 - \beta_e)x_\alpha - x_e + \beta_e x_q].$$

The optimal portfolio is then

$$x_p = x_q + x_a.$$

The student can verify that  $e'x_a = 0$  and  $\beta'x_a = 0$ .