

1 Interest Rate Theory and the Pricing of Bonds

To understand bonds we need to understand the concept of interest rate compounding and the discounting of cash flows.

1.1 Discounting Cash Flows

Let i be the nominal interest rate per year and let m be the number of interest rate periods per year. Then the effective interest rate per interest period is i/m . For example, if $i = 18\%$ and $m = 12$ we have monthly compounding and the interest rate per month is $i/m = 1.5\%$. We are interested in determining how a deposit of \$1 accumulates over time. After one interest period the deposit grows to $(1 + i/m)$, after two interest periods the deposit increases to $(1 + i/m)^2$. In general, after n interest periods the deposit grows to $(1 + i/m)^n$. For example, if $i = 18\%$ and $m = 12$ what is the value of the deposit after $N = 2$ years? There are $n = Nm = 24$ interest periods in two years so the deposit grows to $(1.015)^{24} = 1.4295$. In general, the deposit after N years is worth

$$\begin{aligned} (1 + i)^N & \quad \text{if } m = 1, \\ \left(1 + \frac{i}{m}\right)^{mN} & \quad \text{if } m \in \{2, \dots\}, \\ e^{Ni} & \quad \text{if } m = \infty \end{aligned}$$

A related question is to ask how much money do we need to deposit today so that it grows to \$1 after n interest periods. The answer is obviously $(1 + i/m)^{-n}$. In the above example, if we deposit $0.6995 = 1/1.4295$ at 18% compounded monthly for 24 months we get one dollar. We say that $(1 + i/m)^{-n}$ is the present value at nominal interest rate i of one dollar n interest periods from now given that there are m interest periods per year.

1.2 Bonds

A **bond** is a note issued by a corporation or government entity for the purpose of obtaining needed capital for financing projects. The repayment conditions are specified at the time the bonds are issued. These conditions include the bond **face** value, the bond **interest rate**, the bond **interest repayment period**, and the bond **maturity date**.

In addition to U.S. government bonds, bonds are usually classified as **mortgage**, **debenture**, or **municipal** bonds. *Mortgage* bonds are backed by mortgages or specified assets. There are several types of mortgage bonds including first and second mortgage bonds. In the event of foreclosure first-mortgage bonds take precedence during liquidation.

Debenture bonds are not backed by any form of collateral. The reputation of the company is important for attracting investors to this type of bond. They can be convertible, non-convertible, subordinate or junk bonds. Convertible bonds can be converted into common stock at a fixed rate as long as the bonds are outstanding. Subordinate debentures represents debt that ranks behind other debt in the event of liquidation or reorganization. Junk bonds are debenture bonds rated BBB or lower by Standard and Poor's bond rating.

Finally, *municipal* bonds are income tax free bonds.

The bond face values is usually a multiple of \$1000. The face value:

- (i) represents the sum paid at maturity date,
- (ii) determines the interest payment.

Let

- F : bond face value,
- r : bond interest rate,
- m : number of payments per period.

N : years to maturity.

$T = Nm$ is the number of interest periods over the life of the bond. The bond coupon at interest period $t = 1, 2, \dots, T - 1$ is $C(t) = Fr/m$. At interest period T the coupon includes the face value, so $C(T) = F(1 + r/m)$. Let P be the current price of the bond. A bond is said to sell at a **discount** if $P < F$ or at a **premium** if $P > F$. If $P = F$ we say that the bond sells at **par**.

1.2.1 Current-Yield

The current-yield is simply the annual coupon payment dividend divided by the current price. Using the above notation the current yield is Fr/P . This is of very limited use. For example, zero-coupon bonds (bonds sold at a deep discount with a single payment at maturity) have a current-yield of zero.

1.2.2 Yield-to-Maturity

Let x be such that

$$P = \sum_{t=1}^T C(t)/(1+x)^t.$$

The value of x can be found by trial and error. Given x we can define the bond equivalent yield $y = mx$ (nominal) or the bond effective yield $y_E = (1+x)^m - 1$.

Remarks: x in the above calculation is the interest rate per interest period that makes the present value of the bond cash flow equal to its current value. This value is also known as the internal rate of return (per interest period) of the bond cash flow. We remark, however, that x is the rate of return (per interest period) of the original investment P only if all cash flows are reinvested at the same rate (x per interest period) until the end of the horizon. There are additional problems with yield-to-maturity as a measure of the desirability of a bond. In particular, the yield-to-maturity on a portfolio is not the weighted average of the yields on the bonds that comprise it. Finally, the yield-to-maturity is not generally the expected return on the bond if the bond is sold before the maturity.

1.2.3 Spot Rates

Zero-coupon yields have $r = 0$, so $C(t) = 0$ for $t = 1, \dots, T - 1$ with $C(T) = F$. If the bond matures in T interest periods and there are m interest period per year, we find the bond equivalent yield by first solving

$$P = F/(1+x)^T$$

and then setting $S_{0,T} = mx$ as the bond equivalent yield or annualized yield rate. Since $x = (F/P)^{1/T} - 1$ we see that

$$S_{0,T} = m[(F/P)^{1/T} - 1].$$

In most cases $m = 2$ indicating that semi-annual compounding. In this case

$$S_{0,T} = 2[(F/P)^{1/T} - 1].$$

For example, if a bond that pays $F = 1000$ three years from now is currently selling at $P = 725.25$ and $m = 2$, then $T = 6$ and

$$S_{0,6} = 2[(1000/725.25)^{1/6} - 1] = 11\%.$$

As a second example, if a bond that pays $F = 1000$ two and a half years from now is currently selling at $P = 783.53$ and $m = 2$, then $T = 5$ and

$$S_{0,5} = 2[(1000/783.53)^{1/5} - 1] = 10\%.$$

1.2.4 Forward Rates

Forward rates are interest rates on bonds where the date the commitment is made and the date the money is loaned are different. Let m be the number of interest periods per year. Then f_{st} is the nominal annual rate of a loan that starts at the end of interest period s and ends at the end of interest period t . If P is the amount to be lent at the end of interest period s and F is the amount to be repaid at the end of interest period t then f_{st} is the solution to

$$\left(1 + \frac{f_{st}}{m}\right)^{t-s} = \frac{F}{P}.$$

Clearly,

$$f_{st} = m[(F/P)^{1/(t-s)} - 1].$$

In particular, if $m = 2$ then

$$f_{st} = 2[(F/P)^{1/(t-s)} - 1].$$

For example, if $m = 2$, $s = 1$, $t = 3$, $P = 924.56$ and $F = 1000$ then

$$f_{13} = 2[(1000/924.56)^{1/2} - 1] = 8\%.$$

1.2.5 Relationship between Spot Rates and Forward Rates

Consider an investor wishing to hold money for t interest periods. The investor can buy and hold a t period zero-coupon bond. The ending value per \$1 invested would be

$$\left(1 + \frac{S_{0,t}}{m}\right)^t.$$

Alternatively, the investor could buy a $s < t$ period zero-coupon bond and at the same time agree to invest the proceeds at the forward rate for $t - s$ periods. A dollar invested under this scheme would grow to

$$\left(1 + \frac{S_{0,s}}{m}\right)^s \left(1 + \frac{f_{s,t}}{m}\right)^{t-s}.$$

In the absence of arbitrage both investments would grow to the same amount resulting in

$$\left(1 + \frac{f_{s,t}}{m}\right)^{t-s} = \frac{\left(1 + \frac{S_{0,t}}{m}\right)^t}{\left(1 + \frac{S_{0,s}}{m}\right)^s}.$$

Solving for f_{st} we obtain

$$f_{st} = m \left(\frac{\left(1 + \frac{S_{0,t}}{m}\right)^{t/(t-s)}}{\left(1 + \frac{S_{0,s}}{m}\right)^{s/(t-s)}} - 1 \right).$$

In particular, if $m = 2$ we obtain

$$f_{st} = 2 \left(\frac{\left(1 + \frac{S_{0,t}}{2}\right)^{t/(t-s)}}{\left(1 + \frac{S_{0,s}}{2}\right)^{s/(t-s)}} - 1 \right).$$

1.2.6 Short Rates

Short rates are forward rates spanning a single period. The short rate at time t is consequently $r_t = f_{t,t+1}$. From the above we see that

$$\left(1 + \frac{S_{0,t+1}}{m}\right)^{t+1} = \left(1 + \frac{S_{0,t}}{m}\right)^t \left(1 + \frac{r_t}{m}\right).$$

From this we can infer that

$$\left(1 + \frac{S_{0,t+1}}{m}\right)^{t+1} = \prod_{s=0}^t \left(1 + \frac{r_s}{m}\right)$$

and similarly

$$\left(1 + \frac{f_{s,t}}{m}\right)^{t-s} = \prod_{u=s}^{t-1} \left(1 + \frac{r_u}{m}\right).$$

1.3 Bond Prices and Spot Rates

Given m and the spot rates S_{0t} we know that one dollar invested at time zero grows to $\left(1 + \frac{S_{0,t}}{m}\right)^t$ by the end of interest period t . Thus, an investment of

$$d_t = \frac{1}{\left(1 + \frac{S_{0,t}}{m}\right)^t}$$

now grows to one dollar by the end of interest period t .

Suppose a coupon paying bond has coupon payments $C_i(t)$ at the end of interest periods $t = 1, \dots, T$. By the law of one-price, the current price P_i of such a bond should be equal to

$$\sum_{t=1}^T C_i(t) d_t.$$

This would be the case if the bonds were default-free, there were no bid-ask spreads, and all the bonds traded at the same time since otherwise we would have an arbitrage opportunity. The formula fails to hold in practice to the extent of nonsynchronous trading, bid-ask spreads, differences in bond characteristics, or because prices are out of equilibrium.

One may postulate a slightly different formula for bond prices

$$P_i = \sum_{t=1}^T C_i(t) d_t + \epsilon_i.$$

Since the P_i 's and $C_i(t)$'s are known, the d_t 's can be estimated via linear regression under the constraint that the d_t 's are non-increasing. The problem with this approach is that a large number of bonds pay interest on different dates, and the spot rates must be interpolated for use on these dates. An alternative is to estimate a continuous discount function. For example a functional form, say $D(t) = a_0 + a_1 t + a_2 t^2$ may be postulated and the coefficients can be estimated via regression.

2 The Determinants of Bond Prices

The yields to maturity of bonds differ for a number of reasons including:

1. The length of time before the bonds matures
2. The risk of not receiving coupon and principal payments
3. The tax status of the cash flows

4. The existence of provisions that allow the corporation or government to redeem the debt before maturity
5. The amount of the coupon

2.1 Term to Maturity and Term Structure Theory

The relationship between yield and time is usually called the term structure. More precisely the theory of the term structure of interest rates deals with why pure discount bonds of different maturities have different yields to maturity. In analyzing the effect of maturity on yield, all other influences (such as risk, tax features, and redemption features) are held constant. Most of the analysis is done using government bonds without early redemption features. This theory will be developed in detail in Financial Engineering II.

2.2 Default Risk

Unlike government bonds, for corporate bonds there is a risk that the coupon or principal payment will not be met. It is necessary to make a distinction between promised and expected return. For example, a bond could promise a return of 12%, but its expected return could be only 10%. In addition, since there is risk associated with these bonds, investors should require that the expected return be greater than the return on a similar bond that is default free. We refer to the difference between the promised and the expected return as the default-premium. The difference between the expected return and the return on a default-free instrument is the risk premium. The total promised return should therefore have both a risk-premium and a default-premium. Three large investment services estimate the likelihood of default for most corporate bonds: Moody's, Standard and Poor's, and Fitch.

2.3 Tax Effects

Municipal bonds are not subject to federal taxation and usually are not subject to tax in the state where they are issued.

2.4 Option Features of Bonds

To be discussed later when dealing with options.

3 The Management of Bond Portfolios

3.1 Duration and Convexity

Suppose the yield curve is flat, i.e., $S_{0t} = y$ for all $t > 0$. Moreover, assume that $m = 1$ or if $m > 1$ that we are working with the effective interest rate per interest period $x = y/m$. In this case the discount factor is given by $d_t = (1 + y)^{-t}$, so the present value of the coupon payment $C(t)$ at time t is given by

$$PV(t) = \frac{C(t)}{(1 + y)^t}.$$

Assume that $PV(t) > 0$ for all t . Now,

$$\frac{dPV(t)}{dy} = -\frac{tC(t)}{(1 + y)^{t+1}} = -\frac{t}{1 + y}PV(t) < 0,$$

indicating that an increase in the yield will result in a decrease in the present value of the cash flow $C(t)$. This tells us that up to a first-order approximation the decrease is proportional to t meaning that coupons far into the future are more sensitive to changes in interest rates.

Notice also that

$$\frac{d^2 PV(t)}{dy^2} = \frac{t(t+1)C(t)}{(1+y)^{t+2}} = \frac{t(t+1)}{(1+y)^2} PV(t) > 0$$

indicating that present value of a cash flow is decreasing convex function of the yield.

Now let $PV = \sum_{t=1}^T PV(t)$ be the present value of the cash flow $C(t), t = 1, \dots, T$. We see that

$$\frac{dPV}{dy} = \sum_{t=1}^T \frac{d}{dy} PV(t) = -\frac{1}{1+y} \sum_{t=1}^T tPV(t).$$

Dividing by PV we obtain

$$\begin{aligned} \frac{1}{PV} \frac{dPV}{dy} &= -\frac{1}{1+y} \frac{1}{PV} \sum_{t=1}^T tPV(t) \\ &= -\frac{1}{1+y} D, \end{aligned}$$

where

$$D = \sum_{t=1}^T t \frac{PV(t)}{PV}.$$

Notice that D is the weighed average of the cash flow times. Clearly $1 \leq D \leq T$. Zero-coupon bonds have durations equal to their maturity.

Dividing $\frac{d^2 PV}{dy^2}$ by PV we obtain

$$\begin{aligned} \frac{1}{PV} \frac{d^2 PV}{dy^2} &= \frac{1}{(1+y)^2} \frac{1}{PV} \sum_{t=1}^T t(t+1)PV(t) \\ &= \frac{1}{(1+y)^2} 2C, \end{aligned}$$

where

$$C = \frac{1}{2} \frac{1}{PV} \sum_{t=1}^T t(t+1)PV(t).$$

Why is all this important? Suppose that there is a change in interest rate from y to $y + \Delta y$, then using Taylor's approximation we have

$$\Delta PV \simeq \Delta y \frac{dPV}{dy} + \frac{1}{2} \Delta y^2 \frac{d^2 PV}{dy^2}.$$

Dividing by PV we obtain

$$\begin{aligned} \frac{\Delta PV}{PV} &\simeq \Delta y \frac{1}{PV} \frac{dPV}{dy} + \frac{1}{2} \Delta y^2 \frac{1}{PV} \frac{d^2 PV}{dy^2} \\ &= -\frac{\Delta y}{1+y} D + \frac{\Delta y^2}{(1+y)^2} C \end{aligned}$$

Example: $C(1) = C(2) = 100, C(3) = 1100, y = 10\%$ then $PV(1) = 90.91, PV(2) = 82.64$ and $PV(3) = 826.45$ then $D = 2.7321$ and $C = 5.288$. Accordingly,

$$\frac{\Delta PV}{PV} \simeq -2.7321 \frac{\Delta y}{1+y} + 5.288 \frac{\Delta y^2}{(1+y)^2}.$$

Δy	$-D \frac{\Delta y}{1+y}$	$C \frac{\Delta y^2}{(1+y)^2}$	sum	true
0.01	=0.02487	0.000438	-0.02443	-0.02444
0.02	=0.04974	0.001751	-0.04799	-0.04804

3.2 Protecting Against Term Structure Change

Let $L(t)$ denote the liabilities in period t , $C_i(t)$ the cash flow from bond i in period t , P_i the price of bond i at time zero, N_i the number of bonds of type i to be purchased (decision variable), F_t short term investment in period t , r one period interest rate.

The problem is to purchase now at minimal cost bonds to cover all liabilities. The formulation is given by

$$\min \sum_{i=1}^I N_i P_i$$

subject to

$$\sum_{i=1}^I N_i C_i(t) + F_{t-1}(1+r) - F_t \geq L(t) \quad t = 1, \dots, T$$

$$N_i \geq 0 \quad i = 1, \dots, I \quad F_t \geq 0 \quad t = 1, \dots, T$$

where $F_{-1} \equiv 0$.

Let $X_i = \frac{N_i P_i}{\sum_{j=1}^I N_j P_j}$, and let

$$D_i = \frac{\sum_{t=1}^T t PV_i(t)}{P_i}.$$

Then the duration of the bond portfolio is

$$\begin{aligned} D &= \frac{\sum_t \sum_i N_i PV_i(t)}{\sum_j N_j P_j} \\ &= \frac{\sum_i N_i P_i D_i}{\sum_j N_j P_j} \\ &= \sum_{i=1}^I X_i D_i. \end{aligned}$$

Similarly, we can compute the duration of the liabilities

$$D_L = \frac{\sum_{t=1}^T t L(t) (1+y)^{-t}}{\sum_{t=1}^T L(t) (1+y)^{-t}}.$$

To inoculate the portfolio against shifts in interest rates we can add the constraint

$$\sum_i N_i P_i (D_i - D_L) = 0.$$

This will force the duration of the bond portfolio to initially match the duration of the liabilities.

3.3 Active Bond Portfolio Management

Bond Swaps:

Let $P_B(i)$ be the cost of buying bond i and $P_S(i)$ be the cash received from selling bond i . Let $C_i(t)$ be the cash flow of bond i at time t , let $N_B(i)$ be the number of bonds of type i purchased and $N_S(i)$ be the number of bonds of type i sold. With these definitions the objective is to maximize

$$\sum_i N_S(i)P_S(i) - \sum_j N_B(j)P_B(j)$$

subject to

$$\sum_i N_B(i)C_i(t) + F_{t-1}(1+r) \geq \sum_j N_S(j)C_j(t) + F_t \quad t = 1, \dots, T$$

$$N_B(i) \geq 0 \text{ for all } i, \quad N_S(j) \geq 0 \text{ for all } j, \quad F_t \geq 0 \quad t = 1, \dots, T$$

where F_t is the short-term investment in period t and r be the one-period interest rate.