

# 1 An Introduction to State Pricing

## 1.1 Arbitrage and State Prices

Assume that there are a finite number of possible states that describe the possible outcomes of a specific investment situation. At the initial time it is only known that one of these states will occur. At the end of the period, one specific state will be revealed. Sometimes states describe certain physical phenomena. For example, we might define two weather states for tomorrow: sunny and rainy. We do not know today which of these will occur, but tomorrow this uncertainty will be resolved. States define uncertainty in a very basic manner. It is not even necessary to introduce probabilities of the states, although this will be done later. Indeed, one of the main points is that a great deal can be said without reference to probabilities.

We assume a finite set of  $S$  of states, one of which is revealed as true. There are  $N$  securities. Let  $D$  be an  $N \times S$  matrix where  $D_{ij}$  is the payoff of security  $i$  in state  $j$  at the end of the period. Let  $D_i = (D_{ij})_{j=1}^S$  be the payoff vector of security  $i = 1, \dots, N$ . Let  $p \in \mathbb{R}^N$  be the vector of current security prices. A portfolio  $x \in \mathbb{R}^N$  has current market value  $p'x \in \mathbb{R}$  and future payoff  $D'x \in \mathbb{R}^S$ .

A special form of security is one that has a payoff in only one state. Indeed, we can define  $S$  elementary securities with payoff vectors  $e_j$ ,  $j = 1, \dots, S$ . Thus, elementary security  $j$  has a payoff of 1 in state  $j$  and a payoff of 0 in all other states. If such a security exists we denote its price by  $\psi_j$ . Necessarily  $\psi_j > 0$  since otherwise we have an arbitrage opportunity.

When a complete set of elementary securities exists we can define the *state-price vector*  $\psi = (\psi_j)_{j=1}^S \in \mathbb{R}_{++}^S$ . We can then use the state-price vector and the law of one price to find the prices of other securities via

$$p = D\psi.$$

### Example

Suppose there are only two states: 1 and 2. A security that pays \$1 if state 1 occurs has price  $\psi_1 = 0.5$ , while a security that pays \$1 if state 2 occurs has price  $\psi_2 = 0.3$ . In addition, to these elementary securities there is a stock that pays \$2 in state 1 and \$0.5 in state 2, a bond that pays \$1 in both states, and a call option with strike price \$1.5 on the stock which pays \$0.5 on state 1 and zero on state 2. Let

$$D = \begin{pmatrix} 2 & 0.5 \\ 1 & 1 \\ 0.5 & 0 \end{pmatrix}$$

and let

$$\psi = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

Then the price of the securities represented in  $D$  are

$$p = D\psi = \begin{pmatrix} 1.15 \\ 0.8 \\ 0.25 \end{pmatrix}.$$

If the elementary securities do not exist, it may be possible to construct them artificially by combining securities that do exist. That is, by forming portfolios whose payoff  $D'x$  replicates the elementary securities. We say that the markets are *complete* if  $\text{span}(D) = \{D'x : x \in \mathbb{R}^N\} = \mathbb{R}^S$ , and are otherwise *incomplete*. If the markets are complete then it is possible to replicate elementary securities. This can be always be done if  $D$  has full column rank.  $D$  has full column rank whenever there are at least  $S$  independent securities.

### Example:

Suppose that there are three securities and two states with price vector

$$p = \begin{pmatrix} 1.15 \\ 0.8 \\ 0.25 \end{pmatrix}$$

and payoff matrix

$$D = \begin{pmatrix} 2 & 0.5 \\ 1 & 1 \\ 0.5 & 0 \end{pmatrix}.$$

Find the state-price vector. Since the first two securities have independent payoffs we can find the state-price vector  $\psi$  by first finding a portfolio  $x_j$  such that  $D'x_j = e_j$   $j = 1, 2$ . By the law of one price the price of portfolio  $x_j$  will be  $\psi_j = p'x_j$ . For example, portfolio  $x_1 = (0, 0, 2)$  has payoff 1 in state 1 and payoff 0 in state 2. The price of this portfolio is  $\psi_1 = p'x_1 = 0.5$ . On the other hand, portfolio  $x_2 = (0, 1, -2)$  had payoff 0 in state 1 and payoff 1 in state 2. The price of this portfolio is  $\psi_2 = p'x_2 = 0.3$ .

**Example:**

Suppose that there are two securities and three states with price vector

$$p = \begin{pmatrix} 1.15 \\ 0.8 \end{pmatrix}$$

and payoff matrix

$$D = \begin{pmatrix} 2 & 0.5 & 0.5 \\ 1 & 1 & 1 \end{pmatrix}.$$

In this case there are more states than securities and the state-price vector is not unique. Indeed,

$$\psi = \begin{pmatrix} 0.5 \\ 0.3 - z \\ z \end{pmatrix}$$

is a state-price vector for all  $0 < z < 0.3$ . Here the market is incomplete. We can only find the correct prices for securities that have the same payoff in states 2 and 3. To be able to determine  $z$  we need a third security that is independent of 1 and 2. Suppose that there is a third security with payoff

$$D_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

and price \$1.15. Then it must be that  $z = 0.15$  so the state-price vector is now uniquely defined as

$$\psi = \begin{pmatrix} 0.5 \\ 0.15 \\ 0.15 \end{pmatrix}$$

An *arbitrage* is a portfolio  $x$  with  $p'x \leq 0$  and  $D'x > 0$  (type A arbitrage), or  $p'x < 0$  and  $D'x \geq 0$  (type B arbitrage). (Here  $x > 0$  mean  $x \geq 0$  and  $x_i > 0$  for at least some  $i$ .)

**Theorem 1** *There is no arbitrage if and only if there is a state-price vector.*

Proof

Suppose  $\psi \in \mathfrak{R}_{++}^S$  exists,  $p'x < 0$  and  $D'x \geq 0$ . Then  $0 > p'x = \psi'D'x \geq 0$  is a contradiction. On the other hand, if  $p'x \leq 0$  and  $D'x > 0$  we have  $0 \geq p'x = \psi'D'x > 0$  which again is a contradiction. Thus, if  $\psi \in \mathfrak{R}_{++}^S$  exists there is no arbitrage.

Now suppose there is no arbitrage. This means that there exist no portfolio  $x$  such that  $(-p'x, D'x) > 0$ . Let  $A = [D, -p]$  be an  $N \times (S+1)$  matrix. By Steimke's lemma either  $Dy - pz = 0$  with  $y \in \mathfrak{R}_{++}^S$  and  $z > 0$  or there exists a  $x \in \mathfrak{R}^N$  such that  $(-p'x, D'x) > 0$ . Since the second inequality is precluded by the no arbitrage condition we see that  $\psi = y/z \in \mathfrak{R}_{++}^S$  is a state-price vector. □

If there is no arbitrage we are guaranteed the existence (but not the uniqueness) of a state-price vector  $\psi \in \mathfrak{R}_{++}^S$ . The problem of finding a vector  $\psi \in \mathfrak{R}_{++}^S$  such that  $D\psi = p$  is related to that of finding a strictly positive solution to the problem  $Ax = 0$  which can be solved by minimizing  $e'x$  subject to  $Ax = 0$  and  $x \geq e$ . Notice that if  $x^*$  is the output of the linear program then  $x = \alpha x^*$  is a positive solution to  $Ax = 0$  for all  $\alpha > 0$ . Identifying  $(D, -p)$  with  $A$  we can find  $\psi$  via the linear program:

$$\min e'y + z$$

subject to

$$Dy - pz = 0$$

$$y \geq e$$

and

$$z \geq 1.$$

If this program has a solution then  $\psi = \frac{1}{z}y$ . Remark: If  $D$  has full column rank then there is a unique state-price vector.

Question: Given  $D$  and  $p$  how can we find out if there is an arbitrage opportunity? Answer: If the solution to the linear program does not exist then the alternative of Steimke's lemma must hold and there is arbitrage. To find the arbitrage minimize  $p'x$  subject to  $D'x \geq 0$ . If the answer to this problem is  $p'x = 0$  there is no type A arbitrage. In this case we search for type B arbitrage by solving the linear program: maximize  $e'D'x$  subject to  $p'x = 0$  and  $D'x \geq 0$ . If  $e'D'x > 0$  there is type B arbitrage.

## 2 Risk-Neutral Probabilities

Suppose that there are positive state prices  $\psi_j, j = 1, \dots, S$ . As stated before we have

$$p = D\psi. \tag{1}$$

Consider a security with  $D_{ij} = 1$  for all  $j = 1, \dots, S$ . You can think of this security as a bond paying one unit regardless of the state of the system. Since this bond is risk-free, it should earn the risk-free rate  $R_f$  so its current price should be  $1/r$  where  $r = 1 + R_f$ .<sup>1</sup>

On the other hand, by (1) we know that the price of this security is also equal to  $\sum_{j=1}^S \psi_j$ . Consequently,

$$\frac{1}{r} = \sum_{j=1}^S \psi_j.$$

Let  $q_j = r\psi_j, j = 1, \dots, S$ , and let  $q = (q_j)_{j=1}^S$ .<sup>2</sup> Notice that  $\sum_{j=1}^S q_j = 1$ . This allow us to think of  $q$  as a vector of (artificial) probabilities. We can now write

$$p = \frac{1}{r}Dq$$

resulting in

$$p_i = \frac{1}{r}\hat{E}[D_i] = \frac{1}{r}\sum_{j=1}^S D_{ij}q_j, \tag{2}$$

so the current price  $p_i$  of security  $i$  is the present value of the expected payoff of security  $i$ , where we interpret here  $D_i$  is a random variable taking values  $D_{ij}$  with *probability*  $q_j, j = 1, \dots, S$ . The pricing formula (2) corresponds to that of an individual that is risk-neutral. For this reason we

<sup>1</sup>Notice that here  $r$  is a scalar denoting the total return of a risk-free investment. It is not the vector of expected returns used in portfolio theory.

<sup>2</sup>Notice that this is not the vector of excess returns used in portfolio theory.

refer to  $q_j$ ,  $j = 1, \dots, S$  as the risk-neutral probabilities. We use notation  $\hat{E}$  to denote expectation with respect to the the risk-neutral probabilities  $q_j$  for  $j = 1, \dots, S$  to distinguish this from the expectation  $E$  with respect to the real probability measure.

The risk-neutral probabilities can be found, as above, from positive state prices by multiplying them by  $r$ . Alternatively, the risk-neutral probabilities can be found directly if there are at least  $S$  independent securities with known prices and no arbitrage possibility by solving the system of equations

$$p_i = \frac{1}{r} \sum_{j=1}^S D_{ij} q_j \quad i = 1, \dots, S$$

for the  $S$  unknown  $q_j$ 's.

Risk-neutral probabilities also arise in the context of maximizing utility. Let  $p_i$  be the current price of security  $i$  and assume that the future price of security  $i$  takes value  $D_{ij}$  with probability  $\tilde{q}_j$   $j = 1, \dots, S$ . Given current wealth  $W_0$  the problem is to determine how many units, say  $X_i$ , of stock  $i$  to buy to maximize the expected utility of future wealth. The problem reduces to

$$\max EU\left(\sum_{i=1}^N D_i X_i\right)$$

subject to

$$\sum_{i=1}^N p_i X_i = W_0.$$

The first order conditions are

$$p_i = \frac{1}{\lambda} E[U'(W_1^*) D_i] \quad (3)$$

and

$$\sum_{i=1}^N p_i X_i^* = W_0 \quad (4)$$

where  $W_1^* = \sum_{i=1}^N D_i X_i^*$ . Let  $W_{1j}^* = \sum_{i=1}^N D_{ij} X_i^*$  be the future wealth in state  $j$ , then the first order condition (3) can be written as

$$p_i = \frac{1}{\lambda} \sum_{j=1}^S U'(W_{1j}^*) D_{ij} \tilde{q}_j$$

implying that

$$\psi_j = \frac{1}{\lambda} U'(W_{1j}^*) \tilde{q}_j.$$

If there is a risk-free investment with  $p_i = 1$  and  $D_{ij} = r = (1 + R_f)$  for all  $j$  then

$$\lambda = r E[U'(W_1^*)] = r \sum_{j=1}^S U'(W_{1j}^*) \tilde{q}_j$$

resulting in

$$\psi_j = \frac{1}{r} \frac{U'(W_{1j}^*) \tilde{q}_j}{\sum_{j=1}^S U'(W_{1j}^*) \tilde{q}_j}.$$

and in

$$q_j = r \psi_j = \frac{U'(W_{1j}^*) \tilde{q}_j}{\sum_{j=1}^S U'(W_{1j}^*) \tilde{q}_j}.$$

If  $U(W) = W$  then  $U'(W) = 1$  so

$$q_j = \frac{\tilde{q}_j}{\sum_{j=1}^S \tilde{q}_j} = \tilde{q}_j.$$

If  $U(W) = \ln(W)$  then  $U'(W) = 1/W$  so

$$q_j = r\psi_j = \frac{\frac{\tilde{q}_j}{W_{1j}^*}}{\sum_{j=1}^S \frac{\tilde{q}_j}{W_{1j}^*}}.$$

### 3 Option Pricing

Suppose that there are two states 1 (up) and 2 (down), and two securities 1 (stock) and (bond). The payoff matrix is

$$D = \begin{pmatrix} uS & dS \\ r & r \end{pmatrix}$$

and the price vector is

$$p = \begin{pmatrix} S \\ 1 \end{pmatrix}$$

where here  $S > 0$  is the current price of the stock.<sup>3</sup>

We assume that  $d < r < u$  since otherwise there is arbitrage. Here, as usual  $r = 1 + R_f$ . For example, if  $d > r$  then the stock is always better than the bond, or if  $u < r$  then the bond is always better than the stock. One could also argue that  $r < 1$  is necessary to avoid arbitrage. If  $r < 1$  we can short the bond, keep the money under the blanket and make a profit at the end of the period. This assumes, however, that there is no possibility of capital loss due to theft. If there is a possibility of loss then an additional state should be included to account for this possibility. Under extreme conditions people may be willing to make deposits that earn negative interest rates. In the United States there have been periods where short term *real* rates are negative although the nominal rates have always been positive. In 1998 Japan issued short term notes with a negative nominal interest rate.

We know that the state-price vector must satisfy  $D\psi = p$ . Since  $D$  has full rank we can invert to find

$$\psi = D^{-1}p = \frac{1}{r} \begin{pmatrix} \frac{r-d}{u-d} \\ \frac{u-r}{u-d} \end{pmatrix}.$$

Thus,

$$q = r\psi = \begin{pmatrix} \frac{r-d}{u-d} \\ \frac{u-r}{u-d} \end{pmatrix}$$

is the vector of risk-neutral probabilities. Now let  $K$  be a constant and consider a security that pays  $C_u = (uS - K)_+$  in the up-state and  $C_d = (dS - K)_+$  in the down-state. This is the payoff, at time 1, of a call option with strike price  $K$ . According to the law of one price the current price, say  $C_0$  given by

$$C_0 = \frac{1}{r} \hat{E}[C_1]$$

where  $C_1$  is the random variable with payoff  $C_u$  in the upstate and  $C_d$  in the down state. Clearly,

$$\begin{aligned} C_0 &= \frac{1}{r} [q_u C_u + q_d C_d] \\ &= \frac{1}{r} \left[ \frac{r-d}{u-d} C_u + \frac{u-r}{u-d} C_d \right]. \end{aligned}$$

<sup>3</sup>In this section  $S$  is the current price of the stock. Not the number of states.

For example, if  $r = 1.1$ ,  $S = 1$ ,  $u = 1.3$ ,  $d = .9$ ,  $K = 1$  then  $C_u = .3$ ,  $C_d = 0$ ,  $q_u = q_d = .5$ , so

$$C_0 = \frac{1}{1.1}[0.5(.3) + 0.5(0)] = \frac{.15}{1.1} = 0.136.$$

## 4 Multi-Period Binomial Model

Consider now a two period model where in each period the stock price can go either up by a factor  $u$  or down by a factor  $d$ . After two periods the price of the option is  $C_{uu} = (u^2S - K)_+$  if the stock went up in both periods, it is  $C_{ud} = (udS - K)_+$  if the stock went up in the first period and down in the second or if the stock went down in the first period and up in the second. Finally, the price is  $C_{dd} = (d^2S - K)_+$  if the stock goes down in both periods. We assume that in any period the stock goes up with probability  $q_u = (r - d)/(u - d)$ . Assuming that we do not exercise the option early (it is possible to show that this is in fact optimal) we can find the values of  $C_u$  and  $C_d$  at the end of period 1 using the single period calculations discussed earlier. In particular,

$$C_u = \frac{1}{r} [q_u C_{uu} + q_d C_{ud}]$$

and

$$C_d = \frac{1}{r} [q_u C_{du} + q_d C_{dd}]$$

Finally, the price at time zero is given by

$$C_0 = \frac{1}{r} [q_u C_u + q_d C_d]$$

or equivalently,

$$C_0 = \frac{1}{r^2} [q_u^2 C_{uu} + 2q_u q_d C_{ud} + q_d^2 C_{dd}].$$

Notice that

$$C_0 = \frac{1}{r^2} \hat{E}[C_{u^X, d^{2-X}}]$$

where

$$C_{u^X, d^{2-X}} = (u^X d^{2-X} S - K)_+,$$

and  $X$  is a binomial random variable with  $n = 2$  and probability  $q_u$ .

### 4.1 $n$ period model

After  $n$  periods the price of the stock is

$$u^X d^{n-X} S$$

where  $X \in \{0, 1, \dots, n\}$  is the number of times the stock goes up during the first  $n$  periods and  $n - X$  is the number of times the stock goes down during the first  $n$  periods. At time  $n$  the call option has payoff

$$C_{u^X, d^{n-X}} = (u^X d^{n-X} S - K)_+.$$

The problem is to find the current price, say  $C_0$  of the call option. We will assume that there is a bond that earns total return  $r = 1 + R_f$  in each period. We conjecture that

$$C_0 = \frac{1}{r^n} \hat{E}[(u^X d^{n-X} S - K)_+] \quad (5)$$

where under the risk-neutral probability  $X$  is an  $(n, q_u)$  binomial random variable. This results in

$$C_0 = \frac{1}{r^n} \sum_{j=0}^n \binom{n}{j} q_u^j q_d^{n-j} C_{u^j, d^{n-j}}.$$

To verify this conjecture we proceed by induction. We know the result holds for 1 period. Assume the result works for  $n$  periods, we will show that it works for  $n + 1$  periods. After the first period the stock goes either up or down. Let  $C_u$  be the value of the call at the end of period 1 if the stock goes up, and let  $C_d$  be the value of the call at the end of period 1 if the stock goes down. Since there are  $n$  periods to go, and the price of the stock at the end of period one is  $uS$  in the up state and  $dS$  in the down state it follows that

$$C_u = \frac{1}{r^n} \hat{E}[(u^X d^{n-X} uS - K)_+],$$

and that

$$C_d = \frac{1}{r^n} \hat{E}[(u^X d^{n-X} dS - K)_+].$$

From this, we see that

$$\begin{aligned} C_0 &= \frac{1}{r} [q_u C_u + q_d C_d] \\ &= \frac{1}{r^{n+1}} \hat{E}[(u^{X+Y} d^{n+1-X-Y} S - K)_+], \end{aligned}$$

where  $Y$  is a Bernoulli random variable independent of  $X$  with parameter  $q_u$ , so  $Y$  takes value 1 with probability  $q_u$  and value 0 with probability  $q_d = 1 - q_u$ . Since  $X$  is a  $(n, q_u)$  binomial and it is independent of  $Y$ , it follows that  $X + Y$  is an  $(n + 1, q_u)$  binomial completing the proof that (5) gives the right form for the value of the call option.

In general, we can price the option at time  $n$  after  $j$  up periods  $C_{u^j, d^{n-j}}$  from the prices at time  $n + 1$  via

$$C_{u^j, d^{n-j}} = \frac{1}{r} (q_u C_{u^{j+1} d^{n-j}} + q_d C_{u^j d^{n+1-j}}).$$

We now work from (5) to obtain an equivalent expression that is easier to compute. Let  $a = \min\{j \geq 0 : u^j d^{n-j} S \geq K\}$  and assume that  $a \leq n$ . You can think of  $a$  as the smallest number of times the stock needs to go up so that the call ends in the money. Then  $(u^j d^{n-j} S - K)_+ = u^j d^{n-j} S - K$  for all  $j \geq a$  and  $(u^j d^{n-j} S - K)_+ = 0$  otherwise. Then

$$\begin{aligned} C_0 &= \frac{1}{r^n} \sum_{j=a}^n \binom{n}{j} q_u^j q_d^{n-j} u^j d^{n-j} S - \frac{1}{r^n} K \sum_{j=a}^n \binom{n}{j} q_u^j q_d^{n-j} \\ &= S \sum_{j=a}^n \binom{n}{j} (q_u u/r)^j (q_d d/r)^{n-j} - \frac{1}{r^n} K \sum_{j=a}^n \binom{n}{j} q_u^j q_d^{n-j}. \end{aligned}$$

Let  $\tilde{X}$  be a binomial random variable with parameters  $n$  and  $q_u u/r$ . It is easy to check that  $q_u u + q_d d = r$ . Then

$$C_0 = S \Pr(\tilde{X} \geq a) - \frac{1}{r^n} K \Pr(X \geq a).$$

If  $n$  is large then  $\tilde{X}$  is approximately normal with mean  $nq_u u/r$  and variance  $n(q_u u/r)(q_d d/r)$ , while  $X$  is approximately normal with mean  $nq_u$  and variance  $nq_u(1 - q_u)$  so

$$\begin{aligned} C_0 &\simeq S \Pr\left(Z \geq \frac{a - nq_u u/r}{\sqrt{n(q_u u/r)(q_d d/r)}}\right) - \frac{1}{r^n} K \Pr\left(Z \geq \frac{a - nq_u}{\sqrt{nq_u q_d}}\right) \\ &= S \Pr\left(Z \leq \frac{nq_u u/r - a}{\sqrt{n(q_u u/r)(q_d d/r)}}\right) - \frac{1}{r^n} K \Pr\left(Z \leq \frac{nq_u - a}{\sqrt{nq_u q_d}}\right) \end{aligned}$$

Let

$$d_1 = \frac{nq_u u/r - a}{\sqrt{n(q_u u/r)(q_d d/r)}}$$

and

$$d_2 = \frac{nq_u - a}{\sqrt{nq_u q_d}}.$$

Then

$$C_0 \simeq S_o \Pr(Z \leq d_1) - K_0 \Pr(Z \leq d_2)$$

where  $S_o = S$  is the initial price of the stock and  $K_0 = K/r^n$  is the present value of the strike price.

The method for calculating values of European put options is analogous to that for call options. The main difference is that the terminal values for the option are different. But once they are specified, the recursive procedure works in a similar way.

## 4.2 American Put Option

For an American put, early exercise may be optimal. This is easily accounted for in the recursive process as follows: At each node, first calculate the value of the put using the discounted risk-neutral formula; then calculate the value that would be obtained by immediate exercise of the put; finally select the larger of these two values as the value of the put at that node.

Let us see how this works. Suppose that we want to compute the price of the put at time  $n$  and that there have been  $j$  up periods. Let  $P_{u^j, d^{n-j}}$  denote the unknown price. Since the stock can either go up or down the price at time  $n+1$  is  $P_{u^{j+1}, d^{n-j}}$  if the stock goes up and  $P_{u^j, d^{n+1-j}}$  if the stock goes down. Accordingly, if we do not exercise the option the price will be

$$\frac{1}{r} [q_u P_{u^{j+1}, d^{n-j}} + q_d P_{u^j, d^{n+1-j}}].$$

If we do exercise the option the price will be

$$(K - u^j d^{n-j} S)_+.$$

Thus the recursion is

$$P_{u^j, d^{n-j}} = \max\{(K - u^j d^{n-j} S)_+, \frac{1}{r} [q_u P_{u^{j+1}, d^{n-j}} + q_d P_{u^j, d^{n+1-j}}]\}.$$

Example: Suppose that  $S = 62$ ,  $K = 60$ ,  $u = 1.060$ ,  $d = 0.943$ ,  $r = 1.008$ ,  $q_u = 0.554$ ,  $q_d = 0.446$ , and  $n = 5$ . At time  $n = 5$  the value of the put is  $p_{u^j, d^{5-j}} = (60 - u^j d^{5-j} 62)_+$  where  $j$  is the number of ups in the first five periods. This allows us to start the recursion above. We obtain  $P_0 = \$1.61$ .

## 4.3 How to Select the Parameters of the Binomial Model

Assume that the length of each period is  $\Delta t$ , and that the initial price of the stock is  $S_o$ . Then the stock at time  $n\Delta t$  is

$$S_n = S_o u^X d^{n-X}$$

where  $X$  is binomial  $(n, \theta)$ . Here  $\theta$  is the probability of the up state (not necessarily the risk neutral probability).

Stock prices are often model as

$$S_t = S_o e^{Y_t}$$

where  $Y_t$  is normal with mean  $\nu t$  and variance  $\sigma^2 t$ . Viewed as a stochastic process,  $Y_t$  is called Brownian motion (BM), and  $S_t$  geometric Brownian motion (GBM). Stock prices are modeled as GBM to keep stock prices always positive, and to make the returns proportional to the current stock price.

Notice that  $\ln(S_t/S_o) = Y_t$  and that

$$E \left[ \frac{1}{t} \ln(S_t/S_o) \right] = E \left[ \frac{1}{t} Y_t \right] = \nu$$

and

$$\text{Var} \left[ \frac{1}{\sqrt{t}} \ln(S_t/S_o) \right] = \text{Var} \left[ \frac{1}{\sqrt{t}} Y_t \right] = \sigma^2.$$

To have the discrete time model mimic the continuous time model we would like to have

$$E \left[ \frac{1}{n\Delta t} \ln(S_n/S_o) \right] = \nu$$

and

$$\text{Var} \left[ \frac{1}{\sqrt{n\Delta t}} \ln(S_n/S_o) \right] = \sigma^2$$

in the limit as  $\Delta t \rightarrow 0$ .

Now

$$\frac{1}{n\Delta t} \ln(S_n/S_o) = \frac{1}{n\Delta t} [n \ln(d) + X \ln(u/d)]$$

so

$$E \left[ \frac{1}{n\Delta t} \ln(S_n/S_o) \right] = \frac{1}{\Delta t} [\ln(d) + \theta \ln(u/d)].$$

On the other hand,

$$\frac{1}{\sqrt{n\Delta t}} \ln(S_n/S_o) = \frac{1}{\sqrt{n\Delta t}} [n \ln(d) + X \ln(u/d)]$$

so

$$\text{Var} \left[ \frac{1}{\sqrt{n\Delta t}} \ln(S_n/S_o) \right] = \frac{1}{\Delta t} [\theta(1 - \theta)] (\ln(u/d))^2.$$

The above gives rise to two equations:

$$\ln(d) + \theta \ln(u/d) = \nu \Delta t$$

$$\theta(1 - \theta)(\ln(u/d))^2 = \sigma^2 \Delta t$$

in three unknowns  $u$ ,  $d$  and  $\theta$ . There is one degree of freedom. One way to use this degree of freedom is to set  $d = 1/u$ . This reduces the system to

$$(2\theta - 1) \ln u = \nu \Delta t$$

and

$$4\theta(1 - \theta)(\ln u)^2 = \sigma^2 \Delta t$$

Solving for  $\theta$  and  $u$  we obtain

$$\theta = \frac{1}{2} + \frac{\frac{1}{2}}{\sqrt{\sigma^2/(\nu^2 \Delta t) + 1}},$$

and

$$\ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}.$$

Consequently,

$$\ln d = -\sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}.$$

For small  $\Delta t$  we can approximate the above quantities by

$$\theta = \frac{1}{2} + \frac{1}{2} \frac{\nu}{\sigma} \sqrt{\Delta t}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}.$$

An advantage of the choice  $d = u^{-1}$  is that our formulas for  $u$  and  $d$  are independent of  $\nu$ .

Example: Suppose that  $\sigma = .20$  and that  $r = e^{\alpha\Delta t}$  where  $\alpha = .05$ . Now for  $\Delta t = 1/52$  we have  $u = 1.028123206$ ,  $d = 0.972646074$ ,  $r = 1.000962001$ . Suppose  $n = 52$ ,  $S = 100$ , and  $K = 105.1271096$ . In this case  $q_u = 0.510407186$  and  $a = 27$ . Let

$$d_1 = \frac{nq_u u/r - a}{\sqrt{n(q_u u/r)(qd/r)}} = 0.072576765,$$

and

$$d_2 = \frac{nq_u - a}{\sqrt{nq_u qd}} = -0.127283096.$$

We have

$$Pr(Z \leq d_1) = 0.528928611$$

and

$$Pr(Z \leq d_2) = 0.449358131.$$

Thus

$$C_0 \simeq 100(0.528928611) - 105.1271096(0.449358131)/(1.000962001)^{52} = 7.957051365.$$

#### 4.4 Mean and Variance of Log-Normal Random Variable

Suppose

$$S_t = S_0 e^{Y_t}$$

where  $Y_t$  is a normal random variable with mean  $\nu t$  and variance  $\sigma^2 t$ . Notice that  $\frac{1}{t}Y_t$  is normal with mean  $\nu$  and variance  $\sigma^2/t$ . Thus as  $t$  increases  $Y_t/t$  converges to  $\nu$ . Since

$$Y_t = \nu t + \sigma\sqrt{t}Z$$

where  $Z$  is a standard normal random variable, we can write

$$S_t = S_0 e^{\nu t} e^{\sigma\sqrt{t}Z}.$$

This makes the task of computing  $E[S_t]$  and  $\text{Var}[S_t]$  a bit easier. Notice that

$$E[S_t] = S_0 e^{\nu t} E[e^{\sigma\sqrt{t}Z}].$$

and

$$\text{Var}[S_t] = S_0^2 e^{2\nu t} \text{Var}[e^{\sigma\sqrt{t}Z}],$$

so our problem reduces to that of computing the mean and variance of  $e^{aZ}$  where  $a = \sigma\sqrt{t}$ . Now

$$\begin{aligned} E[e^{aZ}] &= \int_{-\infty}^{\infty} e^{az} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz \\ &= e^{a^2/2} \int_{-\infty}^{\infty} e^{az} \frac{1}{\sqrt{2\pi}} e^{-0.5(z-a)^2} dz \\ &= e^{a^2/2}. \end{aligned}$$

Thus

$$E[S_t] = S_0 e^{\nu t} e^{0.5\sigma^2 t} = S_0 e^{\nu t + 0.5\sigma^2 t} = S_0 e^{\mu t},$$

where

$$\mu = \nu + \frac{\sigma^2}{2}.$$

Now

$$\begin{aligned}\text{Var}[e^{aZ}] &= E[e^{2aZ}] - E[e^{aZ}]^2 \\ &= e^{2a^2} - e^{a^2} \\ &= e^{a^2}[e^{a^2} - 1].\end{aligned}$$

From this we can write

$$\begin{aligned}\text{Var}[S_t] &= S_o^2 e^{2\nu t} e^{\sigma^2 t} [e^{\sigma^2 t} - 1] \\ &= S_o^2 e^{2\mu t} [e^{\sigma^2 t} - 1].\end{aligned}$$

Notice that

$$\begin{aligned}\sqrt{\text{Var}[S_t]} &= E[S_t] \sqrt{e^{\sigma^2 t} - 1} \\ &\simeq E[S_t] \sigma \sqrt{t}.\end{aligned}$$

## 5 Black Scholes Formula

We now know that

$$E[S_t] = S_o e^{\mu t}$$

where  $\mu = \nu + 0.5\sigma^2$ . We also know that if there are state prices and the risk free rate is  $e^{rt}$  then

$$S_o = e^{-rt} \hat{E}[S_t] \tag{6}$$

where  $\hat{E}$  denotes expectation with respect to the risk-neutral probability density. Our problem is to change the parameters of the log-normal distribution so that the risk-neutral formula (6) holds. Suppose the parameters were  $a$  and  $\sigma$ . Then

$$S_o = e^{-rt} \hat{E}[S_t] = e^{-rt} S_o e^{(a+0.5\sigma^2)t}$$

and

$$a = r - 0.5\sigma^2$$

results in the desired formula. Now let  $\hat{Y}_t$  be a normal random variable with mean  $at$  and variance  $\sigma^2 t$ , and let

$$S_t = S_o e^{\hat{Y}_t}.$$

It is now possible to value derivative securities based on the risk-neutral density. For example, a call option with strike price  $K$  has value

$$C_0 = e^{-rt} \hat{E}[(S_t - K)_+].$$

Then

$$\begin{aligned}C_0 &= e^{-rt} \hat{E}[(S_o e^{\hat{Y}_t} - K)_+] \\ &= e^{-rt} \int_{-\infty}^{\infty} [S_o e^{rt} e^{-0.5\sigma^2 t} e^{\sigma\sqrt{t}z} - K]_+ \phi(z) dz.\end{aligned}$$

Now let  $z_1$  solve

$$S_o e^{rt} e^{-0.5\sigma^2 t} e^{\sigma\sqrt{t}z} = K.$$

Clearly

$$z_1 = \frac{\ln(K_0/S_0) + 0.5\sigma^2 t}{\sigma\sqrt{t}}$$

where  $K_0 = Ke^{-rt}$  is the present value of the strike price. Now we can write

$$\begin{aligned} C_0 &= S_0 \int_{z_1}^{\infty} e^{-0.5(z-\sigma\sqrt{t})^2} \frac{1}{\sqrt{2\pi}} dz \\ &- K_0 \int_{z_1}^{\infty} \phi(z) dz \\ &= S_0 \Phi(\sigma\sqrt{t} - z_1) - K_0 \Phi(-z_1) \\ &= S_0 \Phi(d_1) - K_0 \Phi(d_2) \end{aligned}$$

where

$$d_2 = -z_1$$

and

$$\begin{aligned} d_1 &= \sigma\sqrt{t} + d_2 \\ d_1 &= \frac{\ln(S_0/K_0) + 0.5\sigma^2 t}{\sigma\sqrt{t}} \end{aligned}$$

and

$$d_2 = \frac{\ln(S_0/K_0) - 0.5\sigma^2 t}{\sigma\sqrt{t}}.$$

Example: Suppose that  $\sigma = .20$  and that  $r = .05$ ,  $t = 1$ ,  $S_0 = 100$  and  $K = 105.1271096$ . Notice that  $K_0 = Ke^{-rt} = 100$  so

$$d_1 = \frac{\ln(1) + 0.02}{.2} = .10$$

and

$$d_2 = \frac{\ln(1) - 0.02}{.2} = -.10$$

Consequently,

$$C_0 = 100\Phi(.10) - 100\Phi(-.10) = 100(0.539827896 - 0.460172104) = 7.965579107.$$

What changes if the current time is  $t$  and the option expires at time  $T > t$ ? The new formula is

$$C_t = S_t \Phi(d_1) - K_t \Phi(d_2)$$

where  $S_t$  is the current price of the stock (at time  $t$ ),  $K_t = Ke^{-r(T-t)}$ ,

$$d_1 = \frac{\ln(S_t/K_t) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S_t/K_t) - 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

## 5.1 Sensitivity of $C_t$ to Parameters

We now investigate the sensitivity of

$$C_t = S_t\Phi(d_1) - K_t\Phi(d_2)$$

relative to  $S_t$ ,  $K_t$ ,  $r$ , and  $\sigma$ . Notice that

$$\frac{\partial C_t}{\partial S_t} = \Phi(d_1) + S_t\phi(d_1)\frac{\partial d_1}{\partial S_t} - K_t\phi(d_2)\frac{\partial d_2}{\partial S_t}.$$

While this expression appears complex, it turns out that

$$S_t\phi(d_1) = K_t\phi(d_2),$$

and that

$$\frac{\partial d_1}{\partial S_t} = \frac{\partial d_2}{\partial S_t}.$$

Together the last two equations imply that

$$\frac{\partial C_t}{\partial S_t} = \Phi(d_1) > 0.$$

The first thing to note is that  $\Phi(d_1) > 0$ , so the value of the call increases as the value of the stock increases. Some authors call this the  $\Delta$  of the call option.

Now,

$$\frac{\partial C_t}{\partial K_t} = S_t\phi(d_1)\frac{\partial d_1}{\partial K_t} - K_t\phi(d_2)\frac{\partial d_2}{\partial K_t} - \Phi(d_2).$$

The expression simplifies because  $\frac{\partial d_1}{\partial K_t} = \frac{\partial d_2}{\partial K_t}$ , resulting in

$$\frac{\partial C_t}{\partial K_t} = -\Phi(d_2).$$

Notice that this is the change relative to the *present value* of the strike price  $K$ . Since

$$\frac{\partial K_t}{\partial K} = e^{-r(T-t)},$$

we have, by the chain rule,

$$\frac{\partial C_t}{\partial K} = -e^{-r(T-t)}\Phi(d_2) < 0.$$

Consequently, the value of the call option decreases as  $K$  increases.

Using similar arguments we can show that

$$\begin{aligned} \frac{\partial C_t}{\partial r} &= -\frac{\partial K_t}{\partial r}\Phi(d_2) \\ &= (T-t)K_t\partial r\Phi(d_2) \\ &> 0, \end{aligned}$$

so the value of the call option increases as  $r$  increases.

Finally,

$$\begin{aligned} \frac{\partial C_t}{\partial \sigma} &= S_t\phi(d_1)\frac{\partial d_1}{\partial \sigma} - K_t\phi(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= S_t\phi(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right] \\ &= S_t\phi(d_1)\sqrt{T-t} \\ &> 0. \end{aligned}$$

To summarize,  $C_t$  increases as  $S_t$ ,  $r$ , and  $\sigma$  increase, and  $C_t$  decreases as  $K$  increases.

## 6 Bounds on $C_t$ and Put-Call Parity

### 6.1 Lower Bound

Claim:  $C_t \geq (S_t - K_t)_+$ .

Consider two portfolios. Portfolio A consists of one call and  $K_t$  invested at the risk-free rate. Portfolio B consists of the stock. The payoff of portfolio A at time  $T$  is

$$(S_T - K)_+ + K = \max(S_T, K),$$

while the payoff of portfolio B is

$$S_T.$$

Since  $\max(S_T, K) \geq S_T$ , the value of portfolio A must be at least as large as the value of portfolio B. Consequently,

$$C_t + K_t \geq S_t.$$

From this we obtain

$$C_t \geq S_t - K_t.$$

On the other hand, we know that  $C_t \geq 0$ , so

$$C_t \geq (S_t - K_t)_+ \geq (S_t - K)_+.$$

This tells the option is always worth at least as much as the value of exercising it immediately. Thus, if  $c_t$  is the cost of an American call option (which allows early exercise) then  $c_t = C_t$  since it is never optimal to exercise early.

Another way to see this is to recognize that

$$\begin{aligned} C_t &= e^{-r(T-t)} \hat{E}[(S_T - K)_+] \\ &= \hat{E}[(e^{-r(T-t)} S_T - K_t)_+] \\ &\geq (e^{-r(T-t)} \hat{E}(S_T) - K_t)_+ \\ &= (S_t - K_t)_+ \\ &\geq (S_t - K)_+, \end{aligned}$$

where the first inequality is an application of Jensen's inequality to convex functions, the third equality uses the fact that  $e^{-r(T-t)} \hat{E}(S_T) = S_t$ , and the last inequality uses the fact that  $K \geq K_t$ .

Example: Suppose that  $\sigma = .20$  and that  $r = .05$ ,  $t = 1$ ,  $S_o = 100$  and  $K = 105.1271096$ . Notice that  $K_0 = K e^{-rt} = 100$  so

$$C_0 \geq (S_o - K_0)_+ = 0,$$

so the lower bound is trivial in this case.

### 6.2 Upper Bound

Claim:

$$C_t \leq \frac{1}{2} \left[ S_t - K_t + \sqrt{(S_t - K_t)^2 + S_t^2 (e^{\sigma^2(T-t)} - 1)} \right].$$

We know that  $C_t = e^{-r(T-t)} \hat{E}[(S_T - K)_+]$ , so it is enough to upper bound  $\hat{E}[(S_T - K)_+]$ . We do this by writing

$$\begin{aligned} \hat{E}[(S_T - K)_+] &= \frac{1}{2} \hat{E}[(S_T - K) + |S_T - K|] \\ &= \frac{1}{2} \left[ \hat{E}(S_T - K) + \hat{E}|S_T - K| \right] \\ &\leq \frac{1}{2} \left[ \hat{E}S_T - K + \sqrt{\hat{V}\text{ar}[S_T] + (\hat{E}S_T - K)^2} \right] \end{aligned}$$

We know the mean and the variance of  $S_T$ . Writing  $K_t = Ke^{-r(T-t)}$  we obtain after some (rather painful) algebra, the bound

$$C_t \leq \frac{1}{2} \left[ S_t - K_t + \sqrt{(S_t - K_t)^2 + S_t^2(e^{\sigma^2(T-t)} - 1)} \right].$$

In particular, if  $K_t \leq S_t$  then

$$C_t \leq \frac{1}{2} S_t \sqrt{e^{\sigma^2(T-t)} - 1} \simeq \frac{1}{2} S_t \sigma \sqrt{T-t}.$$

The above suggests that following approximate lower bound on  $\sigma$ ,

$$\sigma \geq \frac{2}{\sqrt{T-t}} \frac{C_t}{S_t}.$$

Example: Suppose that  $\sigma = .20$  and that  $r = .05$ ,  $t = 0$ , and  $T = 1$ ,  $S_0 = 100$  and  $K = 105.1271096$ . Notice that  $K_0 = Ke^{-rt} = 100$  so

$$C_0 \leq \frac{1}{2} \sqrt{100^2 e^{0.2^2 - 1}} \simeq \frac{1}{2} 100(.20) = 10$$

Since  $C_0 = 7.97$  we obtain the bound

$$\sigma \geq \frac{2}{\sqrt{1}} \frac{7.97}{100} = .1594.$$

## 7 Pull Call Parity

Let  $P_t, C_t, S_t$  be the current price of a European put, a European call, and a non-dividend paying stock at time  $t$ . We assume that the options expire at time  $T \geq t$ . Let  $K_t = Ke^{-r(T-t)}$  be the present value of the strike price. Let us form two portfolios:

Portfolio A. One European call option plus  $K_t$  in cash. This portfolio has value  $C_t + K_t$  at time  $t$  and value  $(S_T - K)_+ + K = \max(S_T, K)$  at time  $T$ .

Portfolio B. One European put option plus one share. This portfolio has value  $P_t + S_t$  at time  $t$  and value  $(K - S_T)_+ + S_T = \max(S_T, K)$  at time  $T$ .

Since both portfolios have the same terminal payoff their current price must be the same in the absence of arbitrage. Consequently,

$$C_t + K_t = P_t + S_t.$$

Thus,

$$P_t = C_t + K_t - S_t.$$

Recall that

$$C_t = S_t \Phi(d_1) - K_t \Phi(d_2).$$

Consequently,

$$\begin{aligned} P_t &= K_t(1 - \Phi(d_2)) - S_t(1 - \Phi(d_1)) \\ &= K_t \Phi(-d_2) - S_t \Phi(-d_1), \end{aligned}$$

where

$$d_1 = \frac{\ln(S_t/K_t) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S_t/K_t) - 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$