A General Attraction Model and an Efficient Formulation for the Network Revenue Management Problem
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Abstract

This paper addresses two concerns with the state of the art in network revenue management with dependent demands. The first concern is that the basic attraction model (BAM), of which the multinomial model (MNL) is a special case, may overestimate recapture in certain cases. The second concern is that the choice based deterministic linear program (CBLP), currently in use to derive heuristics for the stochastic network revenue management (SNRM) problem, has an exponential number of variables. We introduce a generalized attraction model (GAM) that has both the BAM and the independent demand model (IDM) as special cases. We also provide an axiomatic justification for the GAM and an E-M to estimate its parameters. As a choice model, the GAM should be of interest to those seeking a model that is not as optimistic as the BAM nor as pessimistic as the IDM in estimating recaptured demand. Our second contribution is a new formulation called the Sales Based Linear Program (SBLP) for the GAM. This formulation avoids the exponential number of variables in the CBLP and is essentially the same size as the formulation for the IDM. The SBLP should be of interest to revenue managers (even if their preferred choice model is the BAM) as it dramatically reduces the number of variables. Together these two contributions move forward the state of the art for network revenue management and allow for a wide range of effects including: partial demand dependencies, multiple time periods, inventory sharing across cabins, and competitive effects. In addition, the formulation yields new insights into the assortment problem that arises when capacities are infinite.

1 Introduction

One of the leading areas of research in revenue management (RM) has been incorporating demand dependencies into forecasting and optimization models. Developing effective models for suppliers to estimate how consumer demand is redirected as the set of available products changes is critical in determining the revenue maximizing set of products and prices to offer for sale. In the airline industry, several terms are used to describe different types of demand redirection. First-choice demand denotes the 'natural' demand for a product and price given a wide range of competing choices. However, when a product-price alternative is unavailable, its first-choice demand is redirected to other available alternatives (including the 'no-purchase' one). From a supplier's perspective, this turned away demand for a product-price can result in two possible outcomes: 'spill' or 'recapture'. Spill refers to redirected demand that is lost to a competitor (or the no-purchase alternative). Recapture refers to redirected demand that results in the sale of a different product-price on the same supplier. The work presented in this paper provides a mechanism to incorporate these effects into the RM optimization process through use of customer-choice models.

An early motivation for studying demand dependencies came from the airline industry which was concerned with incorporating a special type of recapture known as 'upsell demand'; upsell is recapture from closed discount fare classes into open, higher valued ones on the same flight (also called 'same-flight upsell'). More specifically, there was a perceived need to model the probability that a customer would buy up to a higher fare class if his preferred fare was not available. The inability to account for upsell demand makes the independent demand model (IDM) too pessimistic, resulting in too much inventory offered at lower fares and in a phenomenon known as 'revenue spiral down'; see Cooper et al. [7]. To our knowledge, the first authors to account for upsell potential in the context of a single resource were Brumelle et al. [6] who worked out implicit formulas for the two fare class problem. More explicit formulas were proposed by Belobaba and Weatherford [3] who made fare adjustments while keeping the assumption of independent demands. Gallego et al. [10] developed improved heuristics that combined fare adjustments with dependent demand structure for multiple fares. Their work and alternative approaches by Belobaba [2] suggest that considering same-flight upsell has significant revenue impact, leading to improvements ranging from 3-6%.

For network models, in addition to upsell, it is also important to consider 'cross-flight recapture' that occurs when a customer's preferred choice is unavailable, and he selects to purchase a fare on a different flight instead of buying up to a higher fare in the same flight. Authors that have tried to estimate upsell and recapture effects include Andersson [1], Ja et al. [15], and Ratliff et al. [22]. In markets in which there are multiple flight departures, recapture rates typically range between 15-55% [15]. Other authors have dealt with the topic of demand estimation from historical sales and availability data. The most recent methods use maximum likelihood estimation techniques to estimate primary demands and market flight alternative selection probabilities (including both same-flight upsell and cross-flight recapture); for details, see Vulcano et al. [31] and Meterelliyoz [21].

Most of the optimization work in network revenue management problem with dependent demands is based on formulating and solving large scale linear programs that are then used as the basis to develop heuristics for the stochastic network revenue management (SNRM) problem; e.g. Bront, Mendez-Diaz and Vulcano [5], Fiig et al. [8], Gallego et al. [12], Kunnumkal and Topaloglu [17], [18], Liu and van Ryzin [20], and Zhang and Adelman [34]. Our paper addresses two significant concerns with the current state of the art. The first concern is with the limitations of the multinomial choice model (MNL) which is universally used to handle demand dependencies. The MNL is a special case of the basic attraction model (BAM) that states that the demand for a product is the ratio of the direct attractiveness of the product divided by the sum of the direct attractiveness of the available products (including a no purchase alternative). This means that closing a high demand product will result in spill, and a significant portion of that spill may be recaptured by other available products. In practice, however, we have seen situations where the BAM overestimates the relative attractiveness of alternative products thus leading to underestimation of spill to the competitor and no-purchase alternatives. In a sense, one could also say that the BAM tends to be too optimistic about recapture. Our goal in demand modeling is to develop a more general attraction model (GAM) that is less pessimistic than the IDM but not as optimistic as the BAM. Indeed, under the GAM the choice probabilities depend both on the products offered and the products not offered; this approach provides more flexibility in terms of handling product-level spill. Moreover, the GAM includes both the IDM and the BAM as special cases. The GAM is justified in this paper by slightly modifying one of the axioms that gives rise to the BAM. It can also be justified by studying the BAM under competition allowing for different mean utilities for the same product at different locations but assuming that customers have the same idiosyncratic noise associated with their valuations at different locations.

The second concern addressed in this paper is that the linear programming formulation for the BAM, known as the Choice Based Linear Program (CBLP), has an exponential number of variables [12]. This is because the model strives to find the amount of time each subset of products is offered, and each subset requires a variable. The number of products itself is usually very large; for example, in airlines, it is the number of origin-destination fares (ODFs), and the number of subsets is two raised to the number of ODFs for each market segment. This forces the use of column generation and makes resolving frequently or solving for randomly generated demands impractical. Our goal here is to reformulate the Choice Based Linear Program (CBLP) to avoid using an exponential number of variables. For this purpose we propose an alternative formulation called the sales based linear program (SBLP), which we show is equivalent to the CBLP not only under the BAM but also under the GAM. Due to a unique combination of demand balancing and scaling constraints, the new formulation has a linear number of variables and constraints. The SBLP allows for direct solution without the need for large-scale column generation techniques. This has the benefit of enabling more efficient heuristics for the SNRM problem and providing interesting insights into the assortment problem (a special case that arises when capacities are infinite).

The remainder of this paper is organized as follows. In §2 we present the GAM as well as a justification of the model from first principles and then present an E-M to estimate its parameters. In §3 we present the stochastic, choice based, network revenue management problem. The choice based linear program is reviewed in section §4. The new, sales based linear program is presented in §5 as well as examples that illustrate the use of the formulation. The infinite capacity case gives rise to the assortment problem and is studied in §6. An upgrade model is presented in §7, sales of flexible products in §8, and our conclusions and directions for future research are in §9.

2 Generalized Attraction Model

There is considerable empirical evidence that the basic attraction model (BAM)[1]:

$$\pi_j(S) = \frac{v_j}{v_0 + v(S)}, \tag{1}$$

where $v(S) = \sum_{j\in S} v_j$ may be too optimistic in estimating recapture probabilities when the customer's first choice is not part of the offer set S. Here $\pi_j(S)$ is the probability that a customer will select $j \in S$ when the offer set is S, and in addition to S the customer can select the no-purchase alternative, denoted by 0. The v_j values are the direct attractiveness of the different choices. One could more formally write $\pi_j(S_+)$, where $S_+ = S \cup \{0\}$, instead of $\pi_j(S)$ since the customer is really selecting from S_+. However, we will refrain from this formalism and follow the convention of writing $\pi_j(S)$ to mean $\pi_j(S_+)$. Since the model with attractiveness αv results in the same probabilities, it is customary to normalize the model by setting $\alpha = 1/v_0$, resulting after rescaling in $v_0 = 1$.

The BAM assumes that even if a customer prefers $j \notin S_+$, he must select among $k \in S_+$. This ignores the possibility that the customer may look for products $j \notin S_+$ elsewhere or at a later time. As an example, suppose that a customer prefers a certain wine, and the store does not have it. The customer may then either buy one of the wines in the store, go home without purchasing, or drive to another store and look for the wine he wants. The BAM precludes the last possibility; it implicitly assumes that the search cost for an alternative source of product $j \notin S_+$ is infinity. Gallego, Li and Ratliff [10] addressed this concern in part through use of ad-hoc adjustments to the null alternative attractiveness. Although this approach was effective in varying recapture rates, the v_0 corrections affected all the alternatives in the choice set, and no specific guidance was provided as to the size of the adjustments required. This motivated us to find an improved approach that allows for alternative-specific dependencies rather than aggregate ad-hoc adjustments.

As an example, suppose that the consideration set is $N = \{1, 2\}$ and that the direct attractiveness values are $v_0 = v_1 = v_2 = 1$, so $\pi_k(\{1, 2\}) = 1/3$ for $k = 0, 1, 2$. Under the BAM, eliminating choice 2 results in $\pi_k(\{1\}) = 50\%$ for $k = 0, 1$. Suppose, however, that product 2 is available across town and that the customer's attraction for product 2 from the alternative source is $w_2 = 0.5 \in [0, v_2]$. Then his choice set, when product 2 is not offered, is in reality $S = \{1, 2'\}$ with $2'$ representing product 2 in the alternative location with switching attractiveness w_2. From the point of view of the store the probabilities are now:

$$\pi_0(\{1\}) = \frac{1.5}{2.5} = 60\%, \quad \pi_1(\{1\}) = \frac{1}{2.5} = 40\%.$$

This formulation may also help mitigate the optimism of the traditional attraction model in inter-temporal models where a choice $j \notin S_+$ may become available at a later time. In this case w_2 is the switching attractiveness of choice 2 discounted by time and the risk that it may not be available in the future.

[1]The BAM arises, for example, from the multinomial model where the direct attractiveness $v_j = e^{(\rho u_j)}$ for utilities u_j where $\rho > 0$ is a parameter that is inversely related to the variance of the Gumbel distribution.

This idea can be used to revisit the red-bus, blue-bus paradox, see Ben-Akiva and Lerman [4], where a person has a choice between driving a car and taking either a red or blue bus (it is implicitly assumed that both buses have ample capacity and depart at the same time). The paradox emerges because the probability of selecting to drive the car decreases when the second bus is added. Let v_1 represent the direct attractiveness of driving a car and let v_2, w_2 represent the direct and switching attractiveness of the two buses. If $w_2 = v_2$ then the probabilities are unchanged if one of the buses is removed (as one intuitively expects). The model with $w_2 = 0$ represents the case where there is no second bus.

To formally define the GAM we assume that in addition to the direct attraction values $v_k, k \in N = \{1, \ldots, n\}$, there are switching attraction values $w_k \in [0, v_k], k \in N$ such that for any subset $S \subset N$

$$\pi_j(S) = \frac{v_j}{v_0 + w(S') + v(S)} \quad j \in S, \tag{2}$$

where $S' = \{j \in N : j \notin S\}$ and for any $R \subset N$, $w(R) = \sum_{j \in R} w_j$. For the GAM, $\pi_0(S) = \frac{v_0 + w(S')}{v_0 + w(S') + v(S)}$ is no-purchase probability. The case $w_k = 0, k \in N$ recovers the BAM, while the case $w_k = v_k, k \in N$ recovers the independent demand model (IDM). As for the BAM it is possible to normalize the parameters so that $v_0 = 1$. We will later describe how to estimate the parameters of the GAM from data. The parsimonious formulation $w_j = \theta v_j \ \ \forall j \in N$ for $\theta \in [0, 1]$ can serve to test the competitiveness of the market, e.g., $H_0 : \theta = 0$ or $H_0 : \theta = 1$ against obvious alternatives to determine whether one is better off deviating from either the BAM or the IDM.

There is an alternative, perhaps simpler, way of presenting the GAM by using the following transformation: $\tilde{v}_0 = v_0 + w(N)$ and $\tilde{v}_k = v_k - w_k, k \in N$. For $S \subset N$, let $\tilde{v}(S) = \sum_{j \in S} \tilde{v}_j$. With this notation the GAM becomes:

$$\pi_j(S) = \frac{v_j}{\tilde{v}_0 + \tilde{v}(S)} \quad \forall j \in S \quad \text{and} \quad \pi_0(S) = 1 - \pi_S(S). \tag{3}$$

where $\pi_S(S) = \sum_{j \in S} \pi_j(S)$. For $S \subset T$ we will use the notation $\pi_{S_+}(T) = \sum_{j \in S_+} \pi_j(T)$.

We now revisit the Luce Axioms, see [19], that give birth to the BAM and then suggest a generalization of one of the axioms to show that the GAM satisfies the resulting set of axioms. The original Luce axioms are:

- Axiom 1: If $\pi_i(\{i, j\}) \in (0, 1)$ for all $i, j \in T$, then for any $R \subset S_+$, $S \subset T$
$$\pi_R(T) = \pi_R(S)\pi_{S_+}(T).$$
- Axiom 2: If $\pi_i(\{i, j\}) = 0$ for some $i, j \in T$, then for any $S \subset T$ such that $i \in S$
$$\pi_S(T) = \pi_{S-\{i\}}(T - \{i\}).$$

The most celebrated consequence is that $\pi_i(S)$ satisfies the Luce Axioms if and only if $\pi_i(S)$ follows the basic attraction model (1). Axiom 1 holds trivially for $R = \emptyset$ since $\pi_\emptyset(S) = 0$ for all $S \subset N$. For $\emptyset \neq R \subset S$ the formula can be written as

$$\frac{\pi_R(T)}{\pi_R(S)} = \pi_{S_+}(T) = 1 - \pi_{T-S}(T),$$

so the right hand side is independent of R, and it is the complement of the probability of selecting $T - S$ when T is offered. We want to generalize Axiom 1 so that the ratio remains independent of R for any non-empty $R \subset S \subset T$ but in such a way that we have more control over how much of the relative demand is lost by adding alternatives $T - S$ to S. Indeed, empirical evidence suggests that the BAM allows too much of this demand to be lost, and the flip side of the coin is that too much of the demand $\pi_{T-S}(T)$ will find its way to $R \subset S$ when $T - S$ is removed from T. This suggests the following generalization of Axiom 1.

- Axiom 1': If $\pi_i(\{i, j\}) \in (0, 1)$ for all $i, j \in T$, then for any $\emptyset \neq R \subset S \subset T$

$$\frac{\pi_R(T)}{\pi_R(S)} = 1 - \pi_{T-S}(T) + \sum_{j \in T-S} \theta_j \pi_j(T)$$

  for some set of values $\theta_j \in [0, 1], j \in N$.

Notice that the special case $\theta_j = 0$ for all j recovers Axiom 1 and therefore results in the BAM. On the other hand, the case $\theta_j = 1$ for all j makes the right hand side equal to one, so $\pi_R(S)$ is independent of S for all S containing R, corresponding to the IDM. We will call the set of Axioms 1' and 2 the Generalized Luce Axioms (GLA). The next result states that a choice model satisfies the GLA if and only if it is a GAM.

Theorem 1 *A choice model $\pi_i(S)$ satisfies the GLA if and only if it is of the GAM form.*

A proof of this theorem may be found in the Appendix. The GAM can also be justified from the multinomial logit under competition. A detailed justification of this construction is provided in Appendix.

2.1 Limitation and Heuristic Uses of the GAM

One limitation of the GAM as we propose it is that, in practice, the utility associated with the products at different locations may depend on a set of covariates that may be customer or product specific. This can be dealt with either by assuming a latent class model on the population of customers or by assuming a customer specific random set of coefficients to capture the impact of the covariates. Another limitation is that we are implicitly modeling customers who visit the store first. This can be fixed by dividing the customers among market segments and having another GAM model for customers who visit the competitor's store first. The issues also arise when using the BAM. In fact, often a single BAM is used to try to capture the choice probabilities of all customers whether they visit the store first or not. This stretches the limits of the BAM and of the GAM, as this should really be handled by adding market segments. However the GAM can cope much better with this situation as the following example illustrates.

Example 1: Suppose there are two stores (denoted l), and each market store attracts half of the population. We will assume that there are two products (denoted k) and that the GAM parameters v_{lk} and w_{lk} for the two stores are as given in Table 1:

v_{10}	v_{11} w_{11}	v_{12} w_{12}	v_{20}	v_{21} w_{21}	v_{22} w_{22}
1	1 0.9	1 0.8	1	0.9 0.8	0.9 0.7

Table 1: Attraction Values (v, w)

With this type of information we can use the GAM to compute choice probabilities $\pi_j(S, R)$ that a customer will purchase product $j \in S$ from Store 1 when Store 1 offers set S, and Store 2 offers set R.

Suppose Store 1 has observed selection probabilities for different offer sets S conditional on Store 2 offering $R = \{1, 2\}$ as provided in Table 2:

S	0	1	2
$\emptyset$	100%	0%	0%
$\{1\}$	82.1%	17.9%	0%
$\{2\}$	82.8%	0.0%	17.2%
$\{1, 2\}$	66.7%	16.7%	16.7%

Table 2: Observed $\pi_j(S, R)$ for $R = \{1, 2\}$

To estimate these choice probabilities using a single market GAM we seek to find $v_k, w_k, k = 1, 2$ with $v_0 = 1$ to minimize the sum of squared errors of the predicted probabilities. This results in the GAM model: $v_0 = 1$, $v_1 = v_2 = 0.25$, $w_1 = 0.20$ and $w_2 = 0.15$. This model perfectly recovers the probabilities in Table 2. When limited to the BAM, the largest error is 1.97% in estimating $\pi_0(\{1, 2\})$.

A similar situation arises when demand comes from a mixture of the BAM and the IDM. In this case, the two extreme models (BAM and IDM) fit the overall data very poorly, but the GAM can do a reasonable job of approximating the choice probabilities.

2.2 GAM Estimation and Examples

It is possible to estimate the parameters of the GAM by refining and extending the Expectation Maximization method developed by Vulcano, van Ryzin and Ratilff (VvRR) [31] for the BAM. The details of the estimation procedure are given in the Appendix. Here we present three examples to illustrate how the procedure works when $w = 0$, $w = \theta v$ and $w \leq v$. These examples are based on Example 1 in VvRR. The examples involve five products and 15 periods where the set $N = \{1, 2, 3, 4, 5\}$ was offered in four periods, set $\{2, 3, 4, 5\}$ was offered in two periods, while sets $\{3, 4, 5\}, \{4, 5\}$ and $\{5\}$ were each offered in three periods. Notice that product 5 is always offered making it impossible to estimate w_5. On the other hand, estimates of w_5 would not be needed unless the firm planned to offer other products in the absence of product 5.

The parameters of Example 1 of VvRR are $\lambda = 50$, $v_0 = 1$, and $v = (1.0, 0.7, 0.40, 0.20, 0.05)$, resulting in share $s = v(N)/(1 + v(N)) = .71$. To construct a GAM-version of the original VvRR problem, we assume the true parameter for the parsimonious GAM is $\theta = 0.2$, and the true parameters for the non-parsimonious GAM are $w = (0.25, 0.35, 0.05, 0.05)$. In their paper, VvRR present the estimation results for a specific realization of sales $z_t, t = 1, \ldots, T = 15$ corresponding to the 15 offer sets $S_t, t = 1, \ldots, 15$. We instead generate 500 simulated instances of sales, each over 15 periods, and then compute our estimates for each of the 500 simulations. We report the mean and standard deviation of λ, v and either θ or w over the 500 simulations. We feel these summary statistics are more indicative of the ability of the method to estimate the parameters. We report results when s is assumed to be known, as in [31], and some results for the case when s is unknown[2].

Estimates of the model parameters tend to be quite accurate when s is known and the estimation of λ and v become more accurate when $w \neq 0$. The results for s unknown tend to be accurate when $w = 0$ but severely biased when $w \neq 0$. The bias is positive for v and w and negative for λ. Since it is possible to have biased estimates of the parameters but accurate estimates of demands, we measure how close the estimated demands

$$\hat{\lambda}\hat{\pi}_i(S) = \hat{\lambda}\frac{\hat{v}_i}{1 + \hat{v}(S) + \hat{w}(S')} \quad i \in S, \quad S \subset N$$

are from the true expected demands $\lambda\pi_i(S)$, $i \in S$, $S \subset N$ and also how close the estimated demands $\hat{\lambda}\hat{p}_i(S_t)$ $i \in S_t, t = 1, \ldots, T$ are from sales $z_{ti}, i \in S_t, t = 1, \ldots, T$. These measures are reported, respectively, as the Model Mean Squared Error (M-MSE) and the Data Mean Squared Error (D-MSE). Surprisingly the results with unknown s, although strongly biased in terms of the model parameters, often produce M-MSEs and D-MSEs that are similar to the corresponding values when s is known. Nonetheless, due to the strong bias, we suggest the fitting a GAM with unknown s is as an area for future research.

Example 2 (BAM) In the original VvRR example, the true value of w is 0. Assuming s is known, our estimates of λ and v based on 500 simulations had means $\hat{\lambda} = 50.265$ and $\hat{v} = (0.9875, 0.6969, 04003, 0.1996, 0.0491)$ with respective bias 0.53% and $(-1.25\%, 0.44\%, -0.07\%, -0.20\%, 1.80\%)$. The standard deviation among the 500 estimates was 3.2862 for λ and $(0.1001, 0.0765, 0.0576, 0.02921, 0.0114)$ for v. The D-MSE and the M-MSE were respectively 253.42 and 47.57. We also tried our method when s is unknown and obtained $\bar{\lambda} = 51.2589$, $\bar{v} = (1.2117, 0.8183, 0.4485, 0.2190, 0.0535)$ with respective bias 2.518% and $(21.77\%, 16.90\%, 12.13\%, 45.50\%, 7.00\%)$. The standard deviation among the 500 estimates was 11.4116 for λ and $(0.7581, 0.4674, 0.2216, 0.0871, 0.0215)$ for v. The mean share was $\bar{s} = 0.7015$ and the standard deviation of the share was 0.0951. Finally, the D-MSE and the M-MSE were respectively 247.50 and 58.44

Example 3 (Parsimonious GAM) For this example v is the same as above but $w = \theta v$ with $\theta = 0.20$. Assuming s is known, our estimates of λ, v and θ based on 500 simulations had mean $\hat{\lambda} = 50.1248$, $\hat{v} = (0.9988, 0.7022, 0.3973, 0.2010, 0.0507)$ and $\hat{\theta} = 0.2384$ with respective bias 0.25%, $(-0.12\%, 0.31\%, -0.68\%, 0.50\%, 1.45\%)$, and 19.2%. The standard deviations among the 500 estimates was 4.3897 for λ, $(0.1110, 0.0786, 0.0624, 0.0361, 0.0142)$ for v and 0.2167 for

[2]The authors are grateful to Anran Li for coding the EM procedure and conducting the experiments.

θ. Finally, the D-MSE and the M-MSE were respectively 226.2 and 50.37. We also tried our method when s is unknown and obtained severely biased estimates of the parameters so they are not reported. However, the D-MSE and M-MSE with s unknown were respectively 228.54 and 53.84.

Example 4 (GAM) We tried the same example but with $w = (0.25, 0.35, 0.15, 0.05, 0.05)$. Assuming s is known, our estimates of λ, v and w based on 500 simulations had mean $\hat{\lambda} = 50.1883$, $\hat{v} = (0.9935, 0.7001, 0.4036, 0.2023, 0.0506)$ and $\hat{w} = (0.3213, 0.2881, 0.1858, 0.1064, 0.0101)$, with respective bias .38%, $(-0.65\%, 0.01\%, 0.90\%, 1.15\%, 1.20\%)$ and $(28.50\%, -17.69\%, 23.86\%, 112.76\%, -79.77\%)$. The standard deviation among the 500 estimates was 3.9776 for λ, $(0.1025, 0.0793, 0.0558, 0.0349, 0.0142)$ for v and $(0.2819, 0.2518, 0.1796, 0.1021, 0.0028)$ for w. Finally, the D-MSE and the M-MSE were respectively 204.71 and 48.56. We also tried our method when s is unknown and obtained hugely biased results so they are not reported. However, the D-MSE and M-MSE with s unknown were 203.70 and 54.33 with a surprisingly good fit to expected demands.

The results from these experiments suggest that the EM method devised for the GAM improves the estimates of λ and v when s is known and $w \neq 0$. The results for s unknown result in parameter estimates that are severely biased when $w \neq 0$. However, the fit to the data and to the model is almost as good as the case when s is known. This can be best appreciated by reporting the largest absolute relative error in estimating demands $\lambda\pi_i(S)$ $i \in S, S \subset N$ for a randomly generated sales instance of each of the three examples. For the GAM examples, the largest relative error was only 2% higher for s unknown than for s known.

The BAM has been universally used over the last few years in revenue management (RM) as an alternative to the pessimistic IDM that ignores upsell and recapture. Our experience, however, reveals that the BAM can suffer from being overly optimistic. As the GAM allows greater flexibility and has both the BAM and the IDM as special cases, we feel that it is a more appropriate model for network revenue management. In sections 3, 4 and 5, applications of the GAM to the network revenue management problem will be discussed in detail.

3 Stochastic Revenue Management over a Network

We will assume that there are L customer market segments and that customers in a particular market segment are interested in a specific set of itineraries and fares[3] from a set of origins to one or more destinations. For example, a market segment could be morning flights from New York to San Francisco that are less than \$500, or all flights to the Mexican Caribbean. Notice that there may be multiple fares with different restrictions associated with each itinerary. At any time the set of origin-destination-fares (ODFs) offered for sale to market segment l is a subset $S_l \subset N_l, l \in \mathcal{L} = \{1, \ldots, L\}$ where N_l is the consideration set for market segment $l \in \mathcal{L}$ with associated fares $p_{lk}, k \in N_l$. If the consideration sets are non-overlapping over the market segments then there is no need to add further restrictions to the offered sets. This is also true when consideration sets overlap if the capacity provider is free to independently decide what

[3] The airfare value used in the RM optimization may be adjusted to account for other associated revenues (e.g. baggage fees) or costs (e.g. meals or in-flight wireless access charges).

fares to offer in each market. As an example, different fares are offered for round-trips from the US to Asia depending on whether the customer buys the fare in the US or in Asia. If the provider cannot make independent decisions by market segment, then he needs to post a single offer set S so that the projection subsets $S_l = S \cap N_l$ for all l are offered for sale. Customers in market segment l select from S_l and the no-purchase alternative. We will abuse notation slightly and write $\pi_{lk}(S_l)$ to denote the probability of selecting $k \in S_{l+}$ from market segment l. For convenience we define $\pi_{lk}(S_l) = 0$ if $k \notin S_{l+}$.

The state of the system will be denoted by (t, x) where t is the time-to-go and x is an m-dimensional vector representing remaining inventories. The initial state is (T, c). For ease of exposition we will assume that customers arrive to market segment l as a Poisson process with time homogeneous rate λ_l. We will denote by $V(t, x)$ the maximum expected revenues that can be obtained from state (t, x). The Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\frac{\partial V(t,x)}{\partial t} = \sum_{l \in \mathcal{L}} \lambda_l \max_{S_l \subset N_l} \sum_{k \in S_l} \pi_{lk}(S_l) \left(p_{lk} - \Delta_{lk} V(t,x)\right) \tag{4}$$

where $\Delta_{lk} V(t, x) = V(t, x) - V(t, x - A_{lk})$, where A_{lk} is the vector of resources consumed by $k \in N_l$. The boundary conditions are $V(0, x) = 0$ and $V(t, 0) = 0$. Talluri and van Ryzin [28] developed the first stochastic, choice-based, dependent demand model for the single resource case in discrete time. The first choice-based, network formulation is due to Gallego, Iyengar, Phillips and Dubey [12].

4 The Choice Based Linear Program

An upper bound $\bar{V}(T, c)$ on the value function $V(T, c)$ can be obtained by solving a deterministic linear program. The justification for the bound can be obtained by assuming a perfect foresight model, arguing that the objective is concave on the realized demands and then using Jensen's inequality. Approximate dynamic programming with affine functions can also be used to justify the same bound.

We now introduce additional notation to write the linear program with value function $\bar{V}(T, c) \geq V(T, c)$. Let $\pi_l(S_l)$ be the $n_l = |N_l|$-dimensional vector with components $\pi_{lk}(S_l), k \in N_l$. Clearly $\pi_l(S_l)$ has zeros for all $k \notin S_l$. The vector $\pi_l(S_l)$ gives us the probability of sale for each fare in N_l per customer when we offer set S_l. Let $r_l(S_l) = p_l' \pi_l(S_l) = \sum_{k=1}^{n_l} p_{lk} \pi_{lk}(S_l)$ be the revenue rate per customer associated with offering set $S_l \subset N_l$ where $p_l = (p_{lk})_{k \in N_l}$ is the fare vector for segment $l \in \mathcal{L}$. Notice that $\pi_l(\emptyset) = 0 \in \Re^{n_l}$ and $r_l(\emptyset) = 0 \in \Re$. Let $A_l \pi_l(S_l)$ be the consumption rate associated with offering set $S_l \subset N_l$ where A_l is an identity matrix with columns A_{lj} associated with ODF $j \in N_l$.

The choice based deterministic linear program (CBLP) for this model can be written as

$$
\begin{array}{llll}
\bar{V}(T,c) = \max & \sum_{l\in\mathcal{L}} \lambda_l T \sum_{S_l\subset N_l} r_l(S_l)\alpha_l(S_l) & & \\
\text{subject to} & \sum_{l\in\mathcal{L}} \lambda_l T \sum_{S_l\subset N_l} A_l\pi_l(S_l)\alpha_l(S_l) & \leq c & \forall \\
& \sum_{S_l\subset N_l} \alpha_l(S_l) & = 1 & l\in\mathcal{L} \\
& \alpha_l(S_l) & \geq 0 & S_l\in N_l \quad l\in\mathcal{L}.
\end{array}
\tag{5}
$$

The decision variables are $\alpha_l(S_l), S_l \subset N_l, l \in \mathcal{L}$ which represent the proportion of time that set $S_l \subset N_l$ is used. The case of non-homogeneous arrival rates can be handled by replacing $\lambda_l T$ by $\int_0^T \lambda_{ls} ds$. There are $\sum_{l\in\mathcal{L}} 2^{n_l}$ decision variables. The CBLP is due to Gallego, et al. [12]. They showed that $V(T,c) \leq \bar{V}(T,c)$ and proposed a column generation algorithm for a class of attraction models. This formulation has been used and analyzed by other researchers including Bront, Mendez-Diaz and Vulcano [5], Kunnumkal and Topaloglu [17], [18], Liu and van Ryzin [20] and Zhang and Adelman [34].

5 The Sales Based Linear Program for the GAM

In this section we will present a linear program that is equivalent to the CBLP but is much smaller in size (essentially the same size as the well known IDM) and easier to solve as it avoids the need for column generation. The new formulation is called the sales based linear program (SBLP) because the decision variables are sales quantities. The number of variables in the formulation is $\sum_{l\in\mathcal{L}}(n_l+1)$, where $x_{lk}, k = 0, 1 \ldots, n_l, l \in \mathcal{L}$ represents the sales of product k in market segment l. Let $D_l = \lambda_l T, l \in \mathcal{L}$ (or $\int_0^T \lambda_{ls} ds$ if the arrival rates are time varying). With this notation the formulation of the SBLP is given by:

$$
\text{R(T,c)} = \max \quad \sum_{l\in\mathcal{L}} \sum_{k\in N_l} p_{lk} x_{lk} \tag{6}
$$

subject to

$$
\begin{array}{lrcll}
\textit{capacity}: & \sum_{l\in\mathcal{L}} \sum_{k\in N_l} A_{lk} x_{lk} & \leq & c & l\in\mathcal{L} \\
\textit{balance}: & \frac{\tilde{v}_{l0}}{v_{l0}} x_{l0} + \sum_{k\in N_l} \frac{\tilde{v}_{lk}}{v_{lk}} x_{lk} & = & D_l & l\in\mathcal{L} \\
\textit{scale}: & \frac{x_{lk}}{v_{lk}} - \frac{x_{l0}}{v_{l0}} & \leq & 0 & k\in N_l \quad l\in\mathcal{L} \\
\textit{non-negativity:} & x_{lk} & \geq & 0 & k\in N_l \quad l\in\mathcal{L}
\end{array}
$$

Theorem 2 *If for each market segment $\pi_{lj}(S_l)$ is of the form (2) then $\bar{V}(T,c) = R(T,c)$.*

We demonstrate the equivalence of the CBLP and SBLP formulations by showing that the SBLP is a relaxation of the CBLP and then proving that the dual of the SBLP is a relaxation of the dual of the CBLP. Full details of this proof may be found in the Appendix.

The intuition behind the SBLP formulation is as follows. Our decision variables represent the amount of inventory to allocate for sales across ODFs for different market segments. The objective is to maximize revenues. Three types of constraints are required. The capacity constraints limit the sum of the allocations. The balance constraints ensure that the sale allocations are in certain hyperplanes, so when some service classes are closed for sale, the demand for the remaining open service classes (including the null alternative) must be modified to offset the amount of the closed demands. The scale constraints operate on the principle that demands are rescaled depending on the availability of other items in the choice set.

The decision variables x_{l0} play an important role in the SBLP. Since the null alternative has positive attractiveness and is always available, the dependent demand for any service class is upper bounded based on its size relative to the null alternative. The scale constraints imply that $x_{lk} \leq v_{lk}x_{l0}/v_{l0}$. Therefore, the scale constraints favor a large value of x_{l0}. On the other hand, the balance constraint can be written as $\sum_{k \in N} \tilde{v}_{lk}x_{lk}/v_{lk} \leq D_l - \tilde{v}_{l0}x_{l0}/v_{l0}$, favoring a small value of x_{l0}. The LP solves for this tradeoff taking into account the objective function and the capacity constraint. We will have more to say about x_{l0} in the absence of capacity constraints in Section 6. The combination of the balance and scale constraints ensures that GAM properties are represented and that the correct ratios are maintained between the sizes of each of the open alternatives.

The special case of the SBLP with $\tilde{v}_{lk} = v_{lk}$ for all $k \in N_l, l \in \mathcal{L}$ reduces to the BAM. For the BAM the demand balance constraint simplifies to $x_{l0} + \sum_{k \in N_l} x_{lk} = D_l$. For the GAM, the balance constraint can alternatively be written as $x_{l0} + \sum_{k \in N_l} x_{lk} + \sum_{k \in N_l} w_{lk}(x_{l0}/v_{l0} - x_{lk}/v_{lk}) = D_l$. Since the third term of this equation is non-negative on account of the scale constraints, it follows that for the GAM $x_{l0} + \sum_{k \in N_l} x_{lk} \leq D_l$. For the independent demand case $\tilde{v}_{lk} = 0$ for all $k \in N_l$ and $\tilde{v}_{l0} = v_{l0} + \sum_{k \in N_l} v_{lk}$, so the balance constraint reduces to $x_{l0} = (v_{l0}/\tilde{v}_{l0})D_l$, and the scale constraint becomes $x_{lk} \leq (v_{lk}/\tilde{v}_{l0})D_l, k \in N_l$, so clearly $x_{l0} + \sum_{k \in N_l} x_{lk} \leq D_l$.

A major advantage of the SBLP formulation is that it neatly avoids the usual non-linearities associated with calculating dependent demands under the general attraction model. The SBLP allows us to determine which flight service classes should be open versus closed for sale. Based on the specific pattern of open versus closed classes, the demands will vary (due to same-flight upsell and cross-flight recapture effects). The compact formulation of the SBLP makes practical the use of randomized linear programming (RLP) to obtain tighter upper bounds on revenues and robust bid-price controls by repeatedly solving the SBLP for simulated demands and averaging the objective function and the dual variables of the capacity constraints. See Talluri and van Ryzin [29] for more on the RLP for the independent demand model as well as Talluri [27] who proposes a concave programming method for general choice models.

Most modern RM optimization models for handling dependent demands require special transformations of fare inputs and/or pre-calculation of 'maximum demands' (conditional on closure of lower valued classes). Weatherford et al. [32] discusses several of these approaches (e.g. DAVN-MR and choice-based EMSR); however, the required fare and demand transformations complicate both the optimization and real-time control processing. An important practical benefit of the SBLP formulation is that it does not require any special transformations of the original inputs; the special structure of the demand, scale and balance constraints enforces automatic rescaling of the dependent demands and correctly models the associated revenues (inclusive of upsell and recapture effects) in the LP.

Another point worth noting is that the CBLP and SBLP formulations remain equivalent even if the market segments have products that overlap. The caveat is that both formulations assume in this case that each market segment will be managed independently. This works only if the capacity provider has the ability to offer a product for a market segment without offering it to other market segments. On the other hand, if a fare cannot be offered in one market without offering to other markets then both the CBLP and the SBLP formulations are upper bounds that can be tightened by adding additional constraints. This approach was used by Bront et al. [5] for the CBLP. They use column generation assuming that each market segment follows a BAM and show that the column generation step is NP-hard.

We remark that the CBLP and therefore the SBLP can be extended to multi-stage problems since the inventory dynamics are linear. This allows for different choice models of the GAM family to be used in different periods. The different GAMs may reflect time heterogeneity and the products that planners expect competitors to be offering. Thus, the probabilities may be of the form $\pi_j(S, R^i)$ for $t \in T_i$ if R^i is the set of products offered by competitors over the time interval T_i.

5.1 Feasibility, Sub-optimality and Numerical Examples

In this section we present two results that relate the solutions to the extreme cases $w = 0$ and $w = v$ to the case $0 \leq w \leq v$. The proofs of the propositions are in the Appendix.

The reader may wonder whether the optimistic solution that assumes $w_j = 0$ for all $j \in N$ is feasible when $w_j \in (0, v_j)$ $j \in N$. The following proposition confirms and generalizes this idea.

Proposition 1 *Suppose that $\alpha_l(S_l), S_l \subset N_l, l \in \mathcal{L}$ is a solution to the CBLP for the GAM for fixed $v_{lk}, k \in N_{l+}, l \in \mathcal{L}$ and $w_{lk} \in [0, v_{lk}], k \in N_l, l \in \mathcal{L}$. Then $\alpha_l(S_l), S_l \subset N_l, l \in \mathcal{L}$ is a feasible, but suboptimal, solution for all GAM models with $w'_{lk} \in (w_{lk}, v_{lk}], k \in N_l, l \in \mathcal{L}$.*

An easy consequence of this proposition is that if $\alpha_l(S_l), l \in \mathcal{L}$ is an optimal solution for the CBLP under the BAM, then it is also a feasible, but suboptimal, solution for all GAMs with $w_k \in (0, v_k]$.

As mentioned in the introduction, the IDM underestimates demand recapture. As such optimal solutions for the SBLP under the IDM are feasible but suboptimal solutions for the GAM. The following proposition generalizes this notion.

Proposition 2 *Suppose that $x_{lk}, k \in N_l, l \in \mathcal{L}$ is an optimal solution to the SBLP for the GAM for fixed $v_{lk}, k \in N_{l+}$ and $w_{lk} \in [0, v_{lk}], k \in N_l, l \in \mathcal{L}$. Then $x_{lk}, k \in N_{l+}, l \in \mathcal{L}$ is a feasible solution for all GAM models with $w'_{lk} \in [0, w_{lk}], k \in N_l, l \in \mathcal{L}$ with the same objective function value.*

An immediate corollary is that if $x_{lk}, k \in N_l, l \in \mathcal{L}$ is an optimal solution to the SBLP under the independent demand model then it is also a feasible solution for the GAM with $w_k \in [0, v_k]$ with the same revenue as the IDM.

Example 5: The following example problem was originally reported in Liu and van Ryzin [20]. There are three flights AB, BC and AC. All customers depart from A with destinations to either B or C. The relevant itineraries are therefore AB, ABC and AC. For each of these itineraries there are two fares (high and low) and three market segments. Market segment 1 are customers who travel to B. Market segment 2 are customers who travel to C but only at a high fare. Market segment 3 are customers who travel to C but only at the low fare. The six fares as well as $v_{lj}, j \in N_l$ and v_{l0} for $l = 1, 2, 3$ are given in Table 3. Assume that market sizes are $\lambda_1 = 6, \lambda_2 = 9, \lambda_3 = 15$ and that capacity $c = (10, 5, 5)$ for flights AB, BC and AC.

	1	2	3	4	5	6	0
Products	AC_H	ABC_H	AB_H	AC_L	ABC_L	AB_L	$Null$
Fares	\$1,200	\$800	\$600	\$800	\$500	\$300	
(AB, v_{1j})			5			8	2
(AC_{high}, v_{2j})	10	5					5
(AC_{low}, v_{3j})				5	10		10

Table 3: Market Segment and Attraction Values v

In this example we will consider the parsimonious model $w_{lj} = \theta v_{lj}$ for all $j \in N_l, l \in \{1, 2, 3\}$ for some global parameter $\theta \in [0, 1]$. We are interested in seeing how robust are the solutions to the two extreme cases $\theta = 0$ and $\theta = 1$ to situations where $\theta \in (0, 1)$.

The extreme case $\theta = 0$ corresponds to the BAM. The solution to the CBLP for the BAM is given by $\alpha(\{1, 2, 3\}) = 60\%$, $\alpha(\{1, 2, 3, 5\}) = 23.3\%$ and $\alpha(\{1, 2, 3, 4, 5\}) = 16.7\%$ with $\alpha(S) = 0$ for all other subsets of products. Equivalently, the solution spends 18, 7 and 5 units of time, respectively, offering sets $\{1, 2, 3\}$, $\{1, 2, 3, 5\}$ and $\{1, 2, 3, 4, 5\}$. The solution results in \$11,546.43 in revenues.[4] By Proposition 1, the solution $\alpha(\{1, 2, 3\}) = 60\%$, $\alpha(\{1, 2, 3, 5\}) = 23.3\%$, $\alpha(\{1, 2, 3, 4, 5\}) = 16.7\%$ is feasible for all $\theta \in (0, 1]$ with a lower objective function value. The corresponding SBLP solution is shown in Table 4 and provides the same revenue outcome as the CBLP.

The case $\theta = 1$ corresponds to the IDM. The solution to the SBLP for the IDM and the corresponding revenue is given in Table 5. By Proposition 2, the solution to the IDM is a feasible solution for all $\theta \in [0, 1)$ with the same objective value function.

Figure 1 shows how the two extreme solutions (CBLP for $\theta = 0$) and (SBLP for $\theta = 1$) perform compared to the optimal solution for the GAM for $\theta \in (0, 1)$. A couple of observations are in order. First, the BAM solution does well only for small values of θ, say $\theta \leq 20\%$ and performs poorly for large values of θ. This result is intuitive because the modified problem with θ treats the competitor alternatives much differently than the (uncorrected) BAM assumes; in

[4]This result is a slight improvement on the one reported in Liu and van Ryzin [20] for which we calculate a profit of \$11,546.20 based on an allocation of 16.35, 2.48, 10.3 and 0.87 units of time, respectively, on the sets $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 3, 4, 5\}$.

	1	2	3	4	5	6	0	
Products	AC_H	ABC_H	AB_H	AC_L	ABC_L	AB_L	x_{l0}	Profit
(AB, x_{1j})			4.29				1.71	$2,571.43
(AC_{high}, x_{2j})	4.50	2.25					2.25	$7,200.00
(AC_{low}, x_{3j})				0.50	2.75		11.75	$1,775.00
Total	4.50	2.25	4.29	0.50	2.75	0.0	15.71	$11,546.43

Table 4: SBLP Allocation Solution for BAM with ($\theta = 0$)

	1	2	3	4	5	6	0	
Products	AC_H	ABC_H	AB_H	AC_L	ABC_L	AB_L	x_{l0}	Profit
(AB, x_{1j})			2.00			3.00	0.80	$2,100.00
(AC_{high}, x_{2j})	4.50	2.25					2.25	$7,200.00
(AC_{low}, x_{3j})				0.50	2.75		6.00	$1,775.00
Total	4.50	2.25	2.00	0.50	2.75	3.00	9.05	$11,075.00

Table 5: SBLP Allocation Solution for IDM with ($\theta = 1$)

practice, ad-hoc increases to the size of the null alternative would be required in the BAM to avoid overstating expected recapture rates. Left uncorrected (which is what these results are based on), the BAM solution gives up about 10% of the optimal profits at the extreme value of $\theta = 1$. Second, the solution for the IDM gives up significant profits only for small values of θ. In particular, the IDM solution loses over 4% of the optimal profits for $\theta = 0$ but less than 0.25% of profits for $\theta \geq 40\%$. Finally, the optimal value function for the GAM is only marginally better than the maximum of the (GAM-estimated) value functions for the BAM and the IDM. Indeed, a heuristic that uses the BAM for $\theta \leq 20\%$ and the IDM for $\theta > 20\%$ works almost as well as the GAM. Please note that these results apply to the special case where all items in the choice set have had their attractiveness downgraded by the same relative amount. The more typical application of the GAM in practice is that at most a few of the items in the choiceset would be modified (because they are especially competitive relative to the rest of the set), so the revenue differences between the BAM and GAM results wouldn't be quite as large. Nonetheless, the example does highlight the asymmetry of revenue performance associated with overestimating versus underestimating the attractiveness of competitor offerings. A related finding was reported by Hartmans [14] of Scandinavian Airlines; his simulation studies showed clear positive revenue potential of including sellup effects, but assuming strong sellup in cases where it doesn't really exist led to revenue loss.

Example 6: We continue using the running example in Liu and van Ryzin but we now assume that demand is governed by the specific GAM presented in Table 6. The model reflects positive w values for lower fares. The optimal revenue under this model is $11,225.00, and the optimal allocations are given in Table 7. We know from Proposition 1 that the CBLP solution for the BAM is feasible for this GAM. The resulting revenue is $11,185.86. The solution to the IDM for the SBLP is also feasible for the GAM and results in revenue $11,075.00. It is clear that in this case the BAM performs better than the IDM solution, and both are slightly worse than the GAM.

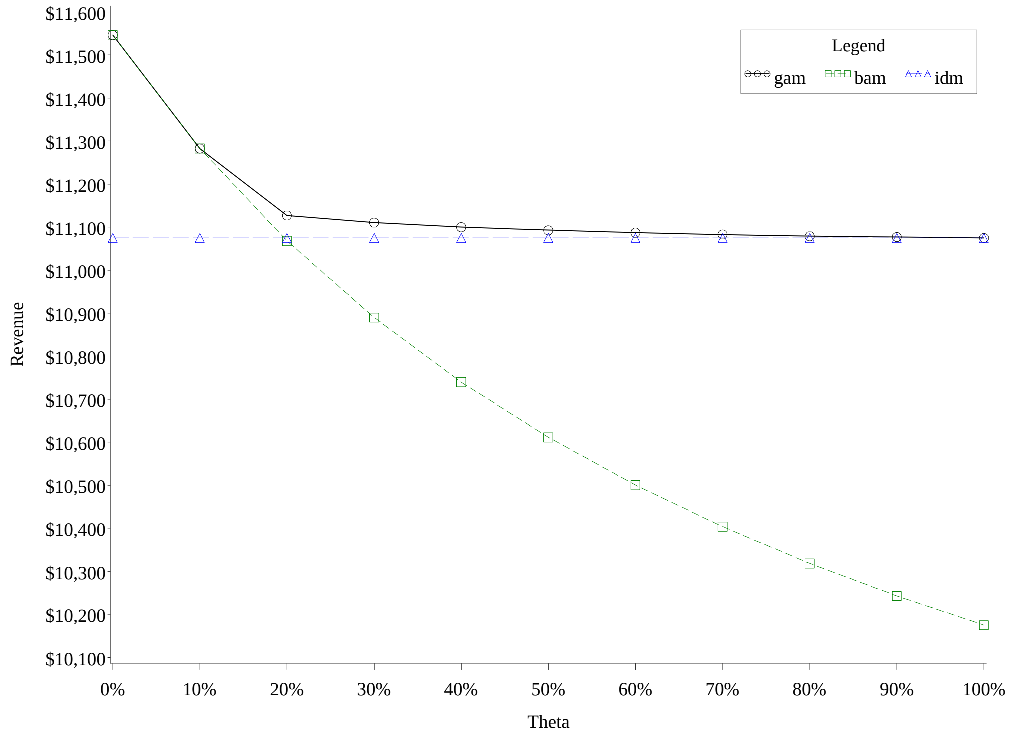

Figure 1: GAM, BAM and IDM Revenue for Different θ Values

5.2 Non-Linear Objective Functions

Recently, Topaloglu, Birbil, Frenk, and Noyan [30] extended the results in this paper to deal with non-linear objective functions for the BAM. They do this by showing how to construct a primal solution to the CBLP from the SBLP. Here we present how to construct a primal solution to the CBLP for the GAM to allow also for non-linear objective functions[5]: Suppose that x_{lk} is an optimal solution to the SBLP. For each market segment, sort the items in decreasing order of x_{lk}/v_{lk}, so that

$$x_{l0}/v_{l0} \geq x_{l1}/v_{l1} \geq \ldots \geq x_{ln_l}/v_{ln_l}$$

where n_l is the cardinality of N_l. Then form the sets $S_{l0} = \emptyset$ and $S_{lk} = \{l1, \ldots, lk\}$ for $k = 1, \ldots, n_l$. For $k = 0, 1, \ldots, n_l$ set

$$\alpha_l(S_{lk}) = \left(\frac{x_{lk}}{v_{lk}} - \frac{x_{l,k+1}}{v_{l,k+1}} \right) \frac{\tilde{v}(S_{lk})}{D_l},$$

where for convenience we define $x_{l,n_l+1}/v_{l,n_l+1} = 0$. Set $\alpha_l(S_l) = 0$ for all $S_l \subset N_l$ not in the collection $S_{l0}, S_{l1}, \ldots, S_{ln_l}$. It is then easy to verify that $\alpha_l(S_l) \geq 0$, that $\sum_{S_l \subset N_l} \alpha_l(S_l) = 1$ and that $\sum_{S_l \subset N_l} \alpha_l(S_l)\pi_{lk}(S_l) = x_{lk}$. Applying this to the CBLP we can immediately see that all constraints are satisfied and the objective function coincides with that of the SBLP. Notice that for each market segment the set of offered sets $S_{lk}, k = 0, 1 \ldots, n_l$ are nested

[5] The authors are indebted to Huseyin Topalogulu for suggestions this result.

	1	2	3	4	5	6	0
Products	AC_H	ABC_H	AB_H	AC_L	ABC_L	AB_L	$Null$
Fares	$1,200	$800	$600	$800	$500	$300	
(AB, v_{1j}, w_{1j})			(5, 0)			(8, 1)	2
$(AC_{high}, v_{2j}, w_{2j})$	(10, 2)	(5, 1)					5
$(AC_{low}, v_{3j}, w_{3j})$				(5, 1)	(10, 0)		10

Table 6: Market Segment and Attraction Values (v, w)

	1	2	3	4	5	6	0	
Products	AC_H	ABC_H	AB_H	AC_L	ABC_L	AB_L	x_{l0}	Profit
Market 1 (AB)			3.75				1.50	$2,250.00
Market 2 (AC$_{high}$)	4.50	2.25					2.25	$7,200.00
Market 3 (AC$_{low}$)				0.50	2.75		10.25	$1,775.00
Total	4.50	2.25	2.01	0.50	2.75	2.99	9.84	$11,225.00

Table 7: SBLP Allocation Solution with GAM $(w > 0)$

$S_{l0} \subset S_{l1} \subset \ldots \subset S_{ln_l}$ and that set S_{lk} is offered if and only if $x_{lk}/v_{lk} > x_{l,k+1}/v_{l,k+1}$. In many situations the sequence looks like:

$$x_{l0}/v_{l0} = x_{l1}/v_{l1} = \ldots = x_{lk}/v_{lk} > x_{l,k+1}/v_{l,k+1} > 0 = x_{l,k+2}/v_{l,k+2} = \ldots = x_{ln_l}/v_{ln_l}.$$

In this case only the set S_{lk} and $S_{l,k+1}$ are offered for positive amounts of time. In this case fare $k+1$ is the marginal fare for market segment l. Unlike the BAM and the IDM the sorting order is not necessarily in decreasing order of fares. This is because fares that face more competition (w_{lj} close to v_{lj}) are more likely to be offered even if the corresponding fare p_{lj} is low. We will have more to say about sort ordering when we discuss the Assortment Problem in the next section.

6 The Assortment Problem

As a corollary to Theorem 2, the infinite capacity formulations for the SBLP and CBLP are equivalent. With infinite capacity the problem separates by market segment so it is enough to study a single market segment. To facilitate this we will drop the subscript l both in the formulation and in the analysis. The resulting problem can be interpreted as the that of finding the assortment of products $S \subset N$ and the corresponding sales $x_k, k \in N$ to maximize profits from a market segment of size D that makes decision based on a GAM choice model. Under this interpretation the p_k values are gross unit profits, net of unit costs.

$$
\begin{array}{llll}
\text{R(T)} = \max & \sum_{k\in N} p_k x_k & & (7)\\
\text{subject to} & & & \\
\textit{balance :} & \frac{\tilde{v}_0}{v_0}x_0 + \sum_{k\in N}\frac{\tilde{v}_k}{v_k}x_k = D & & \\
\textit{scale :} & \frac{x_k}{v_k} - \frac{x_0}{v_0} \leq 0 & k \in N & \\
\textit{non-negativity:} & x_k \geq 0 & k \in N_+ &
\end{array}
$$

The problem is similar to a bin-packing problem with constraints linking the x_k's to x_0. An analysis of the problem reveals that the allocations should be in descending order of the ratio of p_k to $\tilde{v}_k/v_k$. This ratio can be expressed as $p_k v_k/\tilde{v}_k$. For the BAM the allocation is in decreasing order of price. Including lower priced products improves sales, but it cannibalizes demands and dilutes revenue from higher priced products. As such, the allocation of demand to products needs to balance the potential for more sales against demand cannibalization. The sorting of products for the GAM depends not only on price but on the ratio $v_k/\tilde{v}_k$, so a low priced product with a high competitive factor $\theta_k = w_k/v_k$ may have higher ranking than a higher priced product with a low competitive factor. To explain how to obtain an efficient allocation of demand over the different products we will assume that the products are sorted in decreasing order of $p_k v_k/\tilde{v}_k$. Let $X_j = \{1, \ldots, j\}$ and consider the allocation $x_k = v_k x_0/v_0$ for all $k \in X_j$ and $x_k = 0$ for $k \notin X_j$. Solving the demand constraint for x_0 we obtain

$$x_0 = \frac{v_0}{\tilde{v}_0 + \tilde{v}(X_j)} D, \tag{8}$$

$$x_k = \pi_k(X_j) D \quad k \in X_j, \tag{9}$$

and revenue

$$R_j = \sum_{k\in S_j} p_k \pi_k(X_j) D. \tag{10}$$

Now consider the problem of finding the revenue R_{j+1} for set $X_{j+1} = X_j \cup \{j+1\}$. It turns out that R_{j+1} can be written as a convex combination of R_j and $p_{j+1} D v_{j+1}/\tilde{v}_{j+1}$. This implies that $R_{j+1} > R_j$ if and only if $R_j < p_{j+1} D v_{j+1}/\tilde{v}_{j+1}$. Thus the consecutive set X_j with highest profit has index

$$j = \max\{j \geq 1 : p_j \geq \sum_{k\in X_j} p_k \pi_k(X_j)(1 - \theta_j)\}. \tag{11}$$

The following proposition whose proof can be found in the Appendix establishes the optimality[6] of X_j.

Theorem 3 *The assortment X_j with j given by equation (11) with x_k, $k \in X_j$ given by (9) is optimal.*

[6]H. Topaloglu at Cornell independently found the same sorting for the GAM. Personal communication.

Notice that j in equation (11) is independent of D. This means that products $1, \ldots, j$ are offered for all demand levels $D > 0$ and products $j+1, \ldots, n$ are closed regardless of the value of D. Products $j+1, \ldots, n$ are inefficient in the sense they do not increase sales enough to compensate for the demand they cannibalize. Table 4 illustrates precisely this fact as product 5 priced at \$105 is inefficient because $p_5/(1-\theta_5) < R_4$. Notice that in this example it was better to keep product 1 at \$100 fare with a competitive factor $\theta_1 = 0.9$ than the higher product 5 at \$105 with competitive factor $\theta_5 = 0$. The intuition behind this is that closing product 5 is more likely to result in recapture than closing fare 1 (whose demand will most likely be lost to competition or to the no purchase alternative).

k	p_k	v_k	θ_k	x_k	$p_k x_k$
0		1		0.0233	
1	\$100	15	0.9	0.3488	\$34.88
2	\$110	6	0.8	0.1395	\$15.35
3	\$115	9	0.7	0.2093	\$24.07
4	\$120	12	0.6	0.2791	\$33.49
5	\$105	9	0	0.0000	\$0.00
Total					\$107.79

Table 8: Example Assortment Problem

If the problem consists of multiple market segments and each market segment is allowed to post its own available product set X_{lj} then the problem reduces to sorting the products for market segment l in decreasing order of $p_{lk}/(1-\theta_{lk})$, where $\theta_{lk} = w_{lk}/v_{lk}$, and then computing $j(l)$ using (11) for each l. The resulting sets $X_{lj(l)}, l \in \mathcal{L}$ are optimal. If, however, a single assortment set S is to be offered to all market segments then the problem is NP-hard, see Rusmevichientong et al. [23], even if there are only two market segments and the choice model is the BAM. However, the problem for the BAM is somewhat easier to analyze since the preference ordering for all the market segments is $p_1 > p_2 > \ldots > p_n$ whereas the preference ordering for the GAM may be different for different market segments. Assuming the order $p_1 > p_2 > \ldots > p_n$, [23] provide a two market segment example where it is optimal to offer $X_1 = \{1\}$ to market segment one, to offer $X_2 = \{1, 2\}$ to market segment two and to offer $X = \{1, 3\}$ to a composite of 50% of each of the two market segments. Given the NP-hardness of the assortment problem with overlapping sets, the research activity has concentrated on bounds, heuristics and performance warranties. We refer the reader to the working papers by Rusmevichientong, Shen, and Shmoys [24] and Rusmevichientong and Topaloglu [25]. If prices are also decision variables, with choices reflecting price sensitivities, then the problem involves finding the offer sets and prices for each market segment. We refer the reader to Schön [26] who allows for price discrimination, e.g., to offer different prices and offer sets to different market segments. The problem of offering a single offer set and a single vector of prices for all the market segments is again NP-hard.

7 Extension to Multiple Cabins and Planned Upgrades

In many revenue management applications temporal mismatches between supply and demand are addressed, at least in the short run, by serving the customer with an alternative product; Gallego and Stefanescu [11]. When the mismatch between supply and demand results in excess demand for a lower quality product and excess supply for a higher quality product, customers may be upgraded to allow for more sales without altering the fare structure. In airline applications the upgrade can be to a higher valued cabin within the same flight or to a more convenient non-stop flight. In cruises, the upgrade can be to a cabin in a higher deck, to a large cabin or to a cabin with a view. In hotels the upgrade can be to a deluxe room and in car rentals to a larger car.

To improve utilization and correct cases of excess capacity relative to demand, the SBLP formulation can be extended to accommodate planned upgrades. We will first provide a formulation where upgrades are given at no extra cost to the consumer. Later we will discuss how providers can plan for upgrades by charging nominal fees for desirable upgrades, or, alternatively, they can plan for downgrades by reimbursing customers for the inconvenience of serving them with another product.

To handle the network revenue management problem with planned upgrades it is convenient to first solve the problem without upgrades and then to look at the dual variables of the capacity constraints. It may be, for example, that a certain flight or a certain cabin class is severely constrained whereas some alternative, nominally superior, services are unconstrained. As an example, the coach class on a certain flight may be severely constrained while the business class has slack capacity. As a second example, a leg in a connecting flight may be severely constrained while a non-stop flight may be unconstrained. In car rentals, compact cars may be severely constrained while full size cars are not. In these cases selective upgrades can be given to alleviate the capacity constraint and improve revenues and profits.

As an example, the coach cabin on a certain flight may be severely constrained while the business cabin may have slack capacity. As a second example, a leg on a connecting flight may be severely constrained while a non-stop flight may be unconstrained. In car rentals, compact cars may be severly constrained while full size cars are not. In these cases, selective upgrades can be given to alleviate the product-specific capacity constraints and improve revenues and profits.

7.1 Free Upgrades

Suppose a customer from market segment l selects product $k \in N_l$ that is capacity constrained. This means that the customer preferred product k at price p_{lk} over other, higher-priced products offered in N_l, say j, at price p_{lj}. This preference may be due to the combined effect of the desirability of the product and the disutility of the prices. Moreover, the selection may vary from customer to customer due to their idiosyncratic preferences. For example, in the context of the MNL, decisions are based on the product attributes, the product prices and the draw of independent Gumbel random variables. However, if the provider were to offer a nominally superior product $j \in N_l$ at the lower price p_{lk} instead of p_{lj} then the customer may revise his choice. We will define V_{lk} as the set of sufficiently desirable upgrades at price p_{lk}. Let ρ_{lkj} denote the conditional probability that a customer who originally selected $k \in N_l$ will select a superior product j when offered at p_{lk}. This probability can be computed from the choice model; see [11]. The maximum number of customers who actually accept the upgrade is a binomial random variable with parameters x_{lk} and ρ_{lkj}. For planning purposes one can put upper bounds on the number of upgrades from k to j that are conservatively below the expected number $x_{lk} \cdot \rho_{lkj}$ of people who would agree to take the upgrade. Thus, our decision variable x_{lk} is comprised of customers accommodated in product k or upgrades from product k to superior ones. In most cases the actual number of upgrades needed to significantly improve profits (compared to the solution without upgrades) is relatively small. Moreover, a customer may be offered an alternative upgrade, say j' if he rejects j, so the bounds discussed here are rarely a problem in terms of finding people willing to take the upgrades.

To model the problem with upgrades we will assume that the capacity vector c has sufficient granularity to take into account different cabin classes. With this understanding the problem can be modeled by adding variables $z_{lk,mj}$ which are the number of upgrades from customers who book product $k \in N_l$ to an upgrade product $j \in N_m$. Notice that we are allowing here $m \neq l$ as it may be that product $j \in N_m$ is not in the initial consideration set of customers in market segment l. As an example, suppose that market segment l consists of people who want to fly from A to C and pay a low fare. The consideration set for this market may consist of a low fare for a non-stop flight AC or a low fare in a connecting flight $A - B - C$. For a customer who selected the connecting flight, the set of upgrades can consist of the low fare in the non-stop flight or a higher fare in the same flight (which was not part of his initial consideration set). We will denote by V_{lk} the set of upgrade products that can potentially be used to fulfill requests from customers who select $k \in N_l$. For convenience we include lk in the set V_{lk}. Then the formulation has the same objective function, demand balance and demand scaling constraints as the formulation without upgrades. What changes is how the capacity constraint is handled through the use of the variables $z_{lk,mj}, mj \in V_{lk}$. The new capacity constraint is given by

$$\sum_{l \in L} \sum_{k \in N_l} \sum_{mj \in V_{lk}} A_{mj} z_{lk,mj} \leq c$$

To complete the formulation it is necessary to include the linking constraints

$$x_{lk} = \sum_{mj \in V_{lk}} z_{lk,mj}$$

and to add upper bounds to the variables $z_{lk,mj}$

$$z_{lk,mj} \leq x_{lk} \cdot \rho_{lkj}$$

Note that we assume that the maximum number of upgrades allowed is less than the expected number of customers willing to use the free upgrade (otherwise we would need to tighten our bounds).

A special case of the above formulation that occurs commonly in practice is when upgrades are limited to higher-valued cabins on the same-flight. The formulation remains unchanged, but it is simpler to manage because the dimensionality of the z variables is reduced from $z_{lk,mj}$

to only $z_{lk,lj}$ (where j is an upgrade from k). Another belief is that, in practice, most of the upgrade benefit can be gained by simply managing to the LP allocations (x) and bid prices (flight leg dual values) without requiring explicit z variable bounds in the reservations system. Strictly speaking, the full benefit depends on the demand order of arrival, but the negative impact of not enforcing the z upper bounds is mitigated with frequent re-optimization.

7.2 Paid Upgrades

The previous formulation allows for the typical case of upgrades to higher-valued products, but it does not allow for upsells where the customer is charged for a desirable upgrade. The formulation also does not allow for downgrades. As an example, consider a provider who is forced to reject high fare demand because capacity was previously sold at a low fare. The provider could improve his profits if he could displace some of the low fare demand customers into alternative products. To avoid the ill-will associated with such a practice, the provider can offer price sensitive customers a compensation for the right to assign such customers to one among several prespecified alternatives. The combination of offering upgrades, upsells and to compensate customers for their flexibility can result in a more efficient allocation of capacity.

To do this we need to expand the definition of V_{lk} and have associated fees $f_{lk,mj}$ (in addition to the base price p_{lk}) that change hands if and when the customers agrees to a change to mj. If non-negative, f is the upgrade fee that customers agreed to pay for an upgrade to $mj \in V_{lk}$. If non-positive, f can be the pre-agreed fee paid to customers fulfilled with $mj \in V_{lk}$ (this may be used as a mechanism to buy flexibility from customers). The bounds on $z_{lk,mj}$ may then be a conservative estimate of the number of people willing to upgrade to mj or the number of people who agreed to be compensated when switched to mj. The constraints for this problem remain the same but the objective is now modified to include the sum of the $f_{lk,jm} z_{lk,jm}$ terms.

8 Extension to Flexible Products

The SBLP also allows for the sale of flexible products. Flexible products were introduced by Gallego and Phillips [9] for a single leg model and by [12] for a network model. A flexible product, such as a morning flight from New York to San Francisco, gives the capacity provider the option of fulfilling its obligation with any of a set of pre-specified products. Flexible products allow the provider to buy flexibility from flexible customers and use this to have more capacity available for customers willing to pay high fares. Flexible products can be either be sold at an upfront discount or customers can select an pay for a specific product and agree to be reallocated to an alternative in a pre-specified set with and compensated only if they are reallocated.

For ease of exposition we will formulate the case where flexible products are sold at a discount. We will also assume a single market segment for specific and flexible products. The extension to non-overlapping market segments is straightforward. Let $N = \{1, \ldots, n\}$ be the set of specific products. The firm also offers f flexible products $F = \{1, \ldots, f\}$. Each flexible product j consists of a set $F_j \subset N$ of pre-specified alternatives. Every customer purchasing flexible product j must agree to be assigned to any of the products in F_j. The capacity provider has the obligation to fulfill every sale of flexible product j with exactly one of the $f_j = |F_j|$ alternatives in the set F_j. The assignment occurs at the end of the horizon. We assume that the sale probability vectors $\pi(S)$ and $\rho(S)$ for any $S \subset N \cup F$ are such that $\pi(S)$ (of dimension n) depends only on $S \cap N$ and $\rho(S)$ (of dimension f) depends only on $S \cap F$. We also assume that each of the choice models (for specific and flexible products) follows a GAM. For an extension to the case where the demand for specific and flexible products can depend on all the products offered we refer the reader to [12]. We complete the specification of the problem by letting p and q be the vectors of fares for the flexible and specific products and by letting D_s and D_f represent aggregate arrival rates over the sales horizon for specific and flexible products, respectively.

We will present the formulation of the SBLP directly. Let B_j be the sub-matrix of A corresponding to the columns of the specific products in F_j and let $B = (B_1, \ldots, B_f)$. U be a matrix with f rows and $f[1,f] = \sum_{j=1}^{f} f_j$ columns consisting of zeros except for ones in the entries (i,k) such that $f[1,i-1]+1 \leq k \leq f[1,i]$ for $i = 1,\ldots,f$, where $f[1,i] = \sum_{j=1}^{i} f_j$ and sums over empty sets are zero. With this notation the SBLP is given by:

$$
\begin{array}{llcll}
\text{R(T,c)= max} & \sum_{k\in N} p_k x_k + \sum_{j\in F} q_j y_j & & & \quad (12)\\
\text{subject to} & & & & \\
\textit{capacity:} & Ax + Bz & \leq & c & \\
 & y - Uz & = & 0 & \\
\textit{balance}(s): & \frac{\tilde{v}_0}{v_0} x_0 + \sum_{k\in N} \frac{\tilde{v}_k}{v_k} x_k & = & D_s & \\
\textit{scale}(s): & \frac{x_k}{v_k} - \frac{x_0}{v_0} & \leq & 0 & k \in N \\
\textit{balance}(f): & \frac{\tilde{\nu}_0}{\nu_0} y_0 + \sum_{j\in F} \frac{\tilde{\nu}_j}{\nu_j} y_j & = & D_f & \\
\textit{scale}(f): & \frac{y_j}{\nu_j} - \frac{y_0}{\nu_0} & \leq & 0 & j \in F \\
\textit{non-negativity:} & x_k & \geq & 0 & k \in N_+ \\
\textit{non-negativity:} & y_j & \geq & 0 & j \in F_+
\end{array}
$$

The vector y denotes the sales of flexible products. The vector z reallocates sales of flexible products to specific products. Bid-price heuristics for flexible products are described in detail in [12].

9 Conclusions and Future Research

The general attraction model formulation is a new variation of the basic attraction model. We have presented an E-M algorithm to estimate the parameters that is an extension and refinement of [31]. Work is currently underway to develop techniques to estimate the parameters in situations where the arrival rates λ_{lt} are driven by a set of covariates such as day-of-the-week, seasonality indices, and other factors that may affect demand. One known problem in applying the BAM to typical airline problems is that the IIA property can be violated in cases where there are large differences in the departure times of alternate flight services; in such cases, the recapture rates between these widely spread departures would be overestimated. This problem can be mitigated by segmenting customers by their preferred departure time. However, more elegant solutions may exist to handle this problem, and an area of future research is to incorporate features of the GAM into other commonly used types of customer choice models (e.g. nested logit or generalized extreme value).

One limitation of our model is that it provides only an upper bound when there are overlapping market segments, and only a single offer set can be posted at any given time. However, the SBLP formulation is more amenable to adding cuts and to finding polynomial time approximation algorithms to handle commonality constraints. We are currently working on these extensions for both the assortment problem and the case of finite capacities. It is also possible to extend the results in this paper to other objective functions given the equivalence of the solutions space for the CBLP and the SBLP[7]. The SBLP provides the basis for many heuristics for the stochastic network revenue management problem. Since the size of the problem is essentially the same as that of the independent demand model, applying the randomized linear program (RLP) of Talluri and van Ryzin [29] becomes viable for the GAM model. Other heuristics such as those proposed by Gallego et al. [10], by Jasin and Kumar [16], by Kunnumkal and Topaloglu [18], and by Liu and van Ryzin [20] are aided by the ease of solving the SBLP and enriched by the GAM.

We also showed that the SBLP and GAM can be successfully applied to the assortment problem that arises when the capacities are set at infinity; the assortment problem is important in its own right both in the context of perishable and non-perishable products. The new insight is that, under the GAM, it is not necessarily true that the optimal assortment consists of the highest priced products.

A Appendix

A.1 MNL Under Competition Leads to GAM

Consider two retailers, each of which can provide any subset of products in $N = \{1, \ldots, n\}$ at given prices $p_{ki}, i \in N, k = 1, 2$ where k indexes the retailer. Assume that customers derive net utility $u_{ki} + \epsilon_i$ from buying product i from retailer k. We will assume that the $\epsilon_i, i \in N$ are independent, standard Gumbel random variables with parameter ϕ. Notice that a single idiosyncratic random variable ϵ_i is selected for each product independently of the retailer.

Suppose retailer k selects set $S_k \subset N, k = 1, 2$. We are interested in finding the probabilities $\pi_{ki}(S_1, S_2)$ of selecting $i \in S_1 \cup S_2$ for $k = 1, 2$. Let $N_k = \{i : u_{ki} > u_{3-k,1}\}$ for $k = 1, 2$. Thus

[7] H. Topalogulu has used this idea to model the multiclass overbooking problem under discrete choice models in a presentation at the 2011 Informs Revenue Management Conference at Columbia University.

retailer $k = 1, 2$ is dominant over set N_k. We will assume that $N_1 \cup N_2 = N$ so there are no ties. We will first analyze the case $S_k \subset N_k$, $k = 1, 2$. Let $v_{ki} = \exp(\phi u_{ki})$ for all $i \in N$ and $k = 1, 2$. Then

$$\pi_{ki}(S_1, S_2) = \frac{v_{ki}}{v_0 + \sum_{i \in S_1} v_{1i} + \sum_{i \in S_2} v_{2i}} \quad i \in S_k \;\; k = 1, 2,$$

and $\pi_{ki} = 0$ if $i \in S_{3-k}$. Retailer k may have reasons to use a subset S_k that is strictly smaller than N_k, perhaps because adding a product $i \in N_k - S_k \equiv N_k \cap S_k'$ may cannibalize demand from higher profit products in S_k. However, this allows Retailer $3 - k$ to add some products in $N_k - S_k$ to S_{3-k}. This suggest that we may need to allow for situations where $S_k \subset N$, $k = 1, 2$ with non-empty overlap $S_1 \cap S_2$ even if $N_1 \cap N_2 = \emptyset$. We will now partition $S_1 \cup S_2$ into disjoint sets where retailer's dominate. More precisely, let

$$S_k^D = \{i \in S_k : i \in N_k \quad \text{or} \quad i \notin S_{3-k}\} \quad k = 1, 2$$

be the subset of S_k where retailer k dominates. It is clear that the $S_k^D, k = 1, 2$ are disjoint and that $S_1^D \cup S_2^D = S_1 \cup S_2$.

Then

$$\pi_{ki}(S_1, S_2) = \frac{v_{ki}}{v_0 + \sum_{i \in S_1^D} v_{1i} + \sum_{i \in S_2^D} v_{2i}} \quad i \in S_k^D \;\; k = 1, 2.$$

We are particularly interested in seeing how the choice probabilities change for a given retailer, say 1 as he changes his set, say $S = S_1$, assuming the offer set of the other retailer, say $R = S_2$ stays put. More precisely, we are interested in $\pi_i(S, R) = \pi_{1i}(S, R)$ $i \in S$ for fixed R. To make things more concrete, let us assume that $R = N$ and let $\pi_i(S) = \pi_{1i}(S, N)$.

With $R = N$, then $\pi_i(S) = 0$ if $i \notin N_1$ suggesting that $S \subset N_1$. For $S \subset N_1$ we have

$$\pi_i(S) = \frac{v_{1i}}{v_0 + \sum_{j \in S} v_{1j} + \sum_{j \in N - S} v_{2j}} \quad i \in S \subset N_1.$$

Let $\tilde{v}_0 = v_0 + \sum_{j \in N} v_{2j}$, $\tilde{v}_j = v_{1j} - v_{2j}$ and $v_j = v_{1j}$ for all $j \in N_1$. Then for all $S \subset N_1$ we have

$$\pi_i(S) = \frac{v_i}{\tilde{v}_0 + \tilde{v}(S_l)}.$$

Clearly this has the GAM form. With some care, GAM parameters can be found for $\pi_i(S, R)$ for $R \subset N$ and $S \subset N_1 \cup (N_2 - R)$.

A.2 Estimation-Maximization Method for the GAM

The method presented here is an extension and a refinement of the EM proposed by Vulcano, van Ryzin and Ratliff (VvRR) [31]. We assume that demands in periods $t = 1, \ldots, T$ arise from a GAM with parameters v and w and arrival rates $\lambda_t = \lambda$ for $t = 1, \ldots, T$. The homegeneity of the arrival rates is not curcial. In fact the method descibed here can be extended to the case where the arrival rates are govened by co-variates such as the day of the weak, seasonality index, and indicator variables of demand drivers such as conferences, concerts or games. Our main purpose here is show that it is possible to estimate demands fairly accurately.

The data is given by the offer sets $S_t \subset N$ and the realized sales $z_t \subset \mathcal{Z}_+^n$ for $t = 1, \ldots, T$. VvRR assume that the market share $s = \pi_N(N)$, when all products are offered, is known. While we do not make this assumption here, we do present results with and without knowledge of s.

VvRR update estimates of v by estimating demands for each of the products in N when some of them are not offered. To see how this is done, consider a generic period where set S was offered and the realized sale vector is $Z = z$. Let X_j be the demand for product j if set N instead of S was offered. We will estimate X_j conditioned on $Z = z$. Consider first the case $j \notin S$. Since X_j is Poisson with parameter $\lambda\pi_j(N)$, substituting the MLE estimate $e'z_t/\pi_{S_t}(S_t)$ of λ we obtain the estimate $\hat{X}_j = \hat{\lambda}\pi_j(N) = \frac{\pi_j(N)}{\pi_S(S)}e'z$. We could use a similar estimate for products $j \in S$ but this would disregard the information z_j. For example, if $z_j = 0$ then we know that the realization of X_j was zero as customers had product j available for sale. We can think of X_j, conditioned on (S, z) as a binomial random variable with parameters z_j and probability of selection $\pi_j(N)/\pi_j(S)$. In other words, each of the z_j customers that seleted j when S was available, would have selected j anyway with probability $\pi_j(N)/\pi_j(S)$. Then $\hat{X}_j = \frac{\pi_j(N)}{\pi_j(S)}z_k$ is just the expectation of the binomial. Notice that $\hat{X}[1, n] = \sum_{i=1}^m \hat{X}_j = \frac{e'z}{\pi_S(S)}\pi_N(N) = \hat{\lambda}s$. We can find $\hat{X}_0$ by treating it as $j \notin S$ resulting in $\hat{X}_0 = \frac{\pi_0(N)}{\pi_S(S)}e'z = \hat{\lambda}(1 - s)$. For any j let $Z_j = X_j + Y_j$ where Y_j is the demand recaptured or ceded by j from or to other products. Our estimate of Y_j is given by $\hat{Y}_j = -\hat{X}_j$ if $j \notin S$, and by $\hat{Y}_j = z_j - \hat{X}_j$ for $j \in S$. We cannot obtain $\hat{Y}_0$ from the formula $z_0 - \hat{X}_0$ because we do not observe z_0. However, we know by conservation of flow, that $\hat{Y}[0, n] = 0$, which leads directly to $\hat{Y}_0 = \frac{\pi_0(N)\pi_{S'}(N)}{\pi_S(S)}e'z$ or equivalently to $\hat{Y}_0 = \pi_0(N)\sum_{j \notin S}\hat{X}_j = \hat{\lambda}(1 - s)\pi_{S'}(N)$. We summarize the results in the following proposition:

Proposition 3 *(VvRR extended to the GAM): If $j \in S$ then*

$$\hat{X}_j = \frac{\pi_j(N)}{\pi_j(S)}z_j.$$

If $j \notin S$ or $j = 0$ we have

$$\hat{X}_j = \frac{\pi_j(N)}{\pi_S(S)}e'z.$$

Moreover, $\hat{X}[1, n] = \sum_{j \in N}\hat{X}_j = \frac{\pi_N(N)}{\pi_S(S)}e'z$. If $j \in S$ then

$$\hat{Y}_j = \frac{v(S') - w(S')}{1 + v(N)}z_j$$

for $j \notin S$

$$\hat{Y}_j = -\hat{X}_j.$$

Finally, for $j = 0$,

$$\hat{Y}_0 = \frac{\pi_0(N)\pi_{S'}(N)}{\pi_S(S)}e'z.$$

VvRR use the formulas for the BAM to estimate the parameters $\lambda, v_1, \ldots, v_n$ under the assumption that the share $s = v(N)/(1+v(N))$ is known. From this they use an E-M method to iterate between estimating the $\hat{X}_j$s given a vector v and then optimizing the likelihood function over v given the observations. The MLE has a closed form solution so the updates are given by $\hat{v}_j = \hat{X}_j/\hat{X}_0$ for all $j \in N$, where $\hat{X}_0 = r\hat{X}[1,n]$ and $r = 1/s - 1$. VvRR use results in Wu [33] to show that the method converges. We now propose a refinement of the VvRR method.

Using data $(S_t, z_t), t = 1\ldots, T$, VvRR obtain estimates $\hat{X}_{tj}$ that are then averaged over the T observations for each j. In effect, this amounts to pooling estimators from the different sets by giving each period the same weight. When pooling estimators it is convenient to take into account the variance of the estimators and give more weight to sets with smaller variance. Indeed, if we have T different unbiased estimators of the same unknown parameter, each with variance σ_t^2, then giving weights proportional to the inverse of the variance minimizes the variance of the pooled estimator. Notice that the variance of $\hat{X}_{tj}$ is proportional to $1/\pi_j(S_t)$ if $j \in S_t$ and proportional to $1/\pi_{S_t}(S_t)$ otherwise. Let $a_{tj} = \pi_j(S_t)$ if $j \in S_t$ and $a_{tj} = \pi_{S_t}(S_t)$ if $j \notin S_t$. We can select the set the weights for product j as

$$\alpha_{tj} = \frac{a_{tj}}{\sum_{s=1}^{T} a_{sj}}.$$

Our simulations suggests that the adding that our weighting scheme reduces the variance of the estimators by about 4%.

To extend the procedure to the GAM we combine ideas of the EM method based on MLEs and least squares (LS). It is aslo possible to refine the estimates by using iterative re-weighted least squares (Green [13]) but our computational experience shows that there is little benefit from doing this. To initialize the estimation we use formulas for the expected sales under $S_t, t = 1, \ldots, T$ for arbitrary parameters λ, v and w. More precisely, we estimate $E[Z_{tj}]$ by $\lambda\pi_j(S_t)$ for $j \in S_t$ and $t \in \{1, \ldots, T\}$. We then minimize the sum of squared errors between estimated and observed sales (z_ts) subject to the constraints $0 \le w \le v$ and $\lambda \ge 0$. This gives us an initial estimate $\lambda^{(0)}, v^{(0)}, w^{(0)}$ and of the share $s^{(0)} = v^{(0)}(N)/(1 + v^{(0)}(N))$. Let $r^{(0)} = 1/s^{(0)} - 1$. If s is known then we add the constraint $v(N) = r = 1/s - 1$. Given estimates of $v^{(k)}$ and $w^{(k)}$ the procedure works as follows

E-step Estimate the first choice demands $\hat{X}_{tj}^{(k)}$ and then aggregate over time using the current estimate of the weights $\alpha_{tj}^{(k)}$ to produce $\hat{X}_j^{(k)} = \sum_{t=1}^{T} \hat{X}_{tj}^{(k)}\alpha_{tj}^{(k)}$ and $\hat{X}_0^{(k)} = r^{(k)}\hat{X}^{(k)}[1,n]$ and update the estimates of v by

$$v_j^{(k+1)} = \frac{\hat{X}_j^{(k)}}{X_0^{(k)}} \qquad j \in N.$$

M-step Feed $v^{(k+1)}$ together with arbitrary λ and w to estimate sales under $S_t, t = 1, \ldots, T$ and minimize the sum of squared errors between observed and expected sales over λ and w subject to $w \le v^{(k+1)}$ and non-negativity constraint to obtain updates $\lambda^{(k+1)}$, $w^{(k+1)}$ and $s^{(k+1)}$.

The proof of convergence is a combination of the arguments in Wu, as in VvRR, and the fact that the least squared errors problem is a convex minimization problem. The case when s is known works exactly as above, except that we do not need to update this parameter. The M-step can be done by solving the score equations:

$$\sum_{t\in A_i} \pi_i(S_t)[e'z_t + \lambda\pi_0(S_t)] - \sum_{t\in A_i} z_{ti} = 0 \quad i = 1,\dots,n,$$

$$\sum_{t\notin A_i} \hat{\pi}_i(S_t)[e'z_t + \lambda\pi_0(S_t)] - \lambda \sum_{t\notin A_i} \hat{\pi}_i(S_t) = 0 \quad i = 1,\dots,n.$$

and

$$\sum_{t=1}^{T} \pi_{S_t}(S_t) - \sum_{t=1}^{T} e'z_t/\lambda = 0,$$

where A_i is the set of periods where product i is offered and

$$\hat{\pi}_i(S_t) = \frac{v_i}{1 + v(S_t) + w(S_t')} \quad t \notin A_i.$$

This is a system of up to $2n + 1$ equations on $2n + 1$ unknowns. Attempting to solve the score equations directly or through re-weighted least squares does not materially improve the estimates relative to the least square procedure described above.

A.3 Proof of Theorem 1:

Proof: If the choice model $\pi_j(S), j \in S$ satisfies GAM then there exist constants $v_i, i \in N_+$, $\tilde{v}_i \in [0, v_i], i \in N$ and $\tilde{v}_0 = v_0 + \sum_{j\in N}(v_j - \tilde{v}_j)$ such that $\pi_R(S) = v(R)/(\tilde{v}_0 + \tilde{v}(S)$ for all $R \subset S$. For this choice model, the left hand side of Axiom 1' is $(\tilde{v}_0 + \tilde{v}(S)/(\tilde{v}_0 + \tilde{v}(T))$. Let $\theta_j = 1 - \tilde{v}_j/v_j$. Then the right hand side of Axiom 1' is given by $1 - \sum_{j\in T-S}(1-\theta_j)v_j/(\tilde{v}_0 + \tilde{v}(T)) = (\tilde{v}_0 + \tilde{v}(S)/(\tilde{v}_0 + \tilde{v}(T))$. This shows that the GAM satisfies Axiom 1'. If $\pi_i(\{i,j\}) = 0$ then $v_i = 0$ and consequently $\tilde{v}_i = 0$. From this it follows that $\pi_{S-\{i\}}(T - \{i\}) = \frac{v(S)-v_i}{\tilde{v}_0+\tilde{v}(T)-\tilde{v}_i} = \frac{v(S)}{\tilde{v}_0+\tilde{v}(T)} = \pi_S(T)$, so Axiom 2 holds.

Conversely, suppose a choice model satisfies the GLA. Then by selecting $R = \{i\} \subset S \subset T = N$ we see from Axiom 1' that

$$\pi_i(S) = \frac{\pi_i(N)}{1 - \sum_{j\notin S}(1-\theta_j)\pi_j(N)}.$$

Since $\pi_0(N) + \sum_{j\in N}\pi_j(N) = 1$ the denominator can be written as $\pi_0(N) + \sum_{j\in N}\theta_j\pi_j(N) + \sum_{j\in S}(1-\theta_j)\pi_j(N)$, resulting in

$$\pi_i(S) = \frac{\pi_i(N)}{\pi_0(N) + \sum_{j\in N}\theta_j\pi_j(N) + \sum_{j\in S}(1-\theta_j)\pi_j(N)}.$$

Letting $v_j = \pi_j(N)$ for all $j \in N_+$, $\tilde{v}_0 = v_0 + \sum_{j\in N}\theta_j v_j$ and $\tilde{v}_j = (1-\theta_j)v_j \in [0,1]$ for $j \in N$ the choice model can be written as

$$\pi_i(S) = \frac{v_i}{\tilde{v}_0 + \tilde{v}(S)},$$

so it satisfies the GAM.

∎

A.4 Proof of Theorem 2:

Proof: Assume a solution $\alpha_l(S_l), l \in \mathcal{L}$ for the CBLP is given and for $k \in N_l$ let $x_{lk} = D_l \sum_{S_l \subset N_l} \alpha_l(S_l)\pi_{lk}(S_l)$. The objective function and the capacity constraint in terms of x_{lk} are given respectively by $\sum_{l\in\mathcal{L}} \sum_{k=1}^{n_k} p_{lk}x_{lk}$ and $\sum_{l\in\mathcal{L}} \sum_{k=1}^{n_k} A_{lk}x_{lk} \leq c$. Let

$$\tilde{x}_{l0} = D_l \sum_{S_l \subset N_l} \alpha_l(S_l)\pi_{l0}(S_l).$$

Notice that $\tilde{x}_{l0} = x_{l0} + \sum_{k\in\mathcal{N}_l} y_{lk}$ where

$$x_{l0} = D_l \sum_{S_l \subset N_l} \alpha_l(S_l) \frac{v_{l0}}{v_0 + w_l(S'_l) + v_l(S_l)}$$

and

$$y_{lk} = D_l \sum_{S_l \subset N_l : k \notin S_l} \alpha_l(S_l) \frac{w_{lk}}{v_0 + w_l(S'_l) + v_l(S_l)} \quad k \in N_l.$$

We now argue that $\frac{x_{lk}}{v_{lk}}$ is bounded above by $\frac{x_{l0}}{v_{l0}}$. This follows because

$$\frac{x_{lk}}{v_{lk}} = D_l \sum_{S_l \subset N_l : k \in S_l} \alpha_l(S_l) \frac{1}{v_0 + w_l(S'_l) + v_l(S_l)}$$

while

$$\frac{x_{l0}}{v_{l0}} = D_l \sum_{S_l \subset N_l} \alpha_l(S_l) \frac{1}{v_0 + w_l(S'_l) + v_l(S_l)}$$

and the second equation sums over at least as many subsets.

Notice that if $w_{lk} > 0$ then

$$\frac{x_{lk}}{v_{lk}} + \frac{y_{lk}}{w_{lk}} = \frac{x_{l0}}{v_{l0}},$$

or equivalently

$$y_{lk} = w_{lk} \left[\frac{x_{l0}}{v_{l0}} - \frac{x_{lk}}{v_{lk}} \right] \geq 0 \tag{13}$$

since $\frac{x_{lk}}{v_{lk}} \leq \frac{x_{l0}}{v_{l0}}$. On the other hand, by definition, $y_{lk} = 0$ if $w_{lk} = 0$. Thus formula (13) holds for all $k \in N_l$. Substituting y_{lk} into

$$x_{l0} + \sum_{k\in N_l} y_{lk} + \sum_{k\in N_l} x_{lk} = D_l$$

and collecting terms we obtain

$$\left(\frac{v_{l0}+\sum_{k\in N_l} w_{lk}}{v_{l0}}\right) x_{l0} + \sum_{k\in N_l}\left(\frac{v_{lk}-w_{lk}}{v_{lk}}\right) x_{lk} = D_l \quad \forall \quad l \in \mathcal{L},$$

which is equivalent to the demand balance constraint since $\tilde{v}_{l0} = v_{l0} + \sum_{k\in N_l} w_{lk}$ and $\tilde{v}_{lk} = v_{lk} - w_{lk}$ for all $k \in N_l$.

This shows that any solution to the CBLP can be converted into a feasible solution to the SBLP for the GAM with the same objective value. As a result, $\bar{V}(T,c) \le R(T,c)$.

Let z and $\tilde{\beta}_l, l \in \mathcal{L}$ be an optimal solution for the dual of the CBLP. Let $\beta_l = \tilde{\beta}_l / D_l$. Then the feasibility constraint for the dual of the CBLP takes the form

$$\beta_l + \sum_{j\in S_l} (z' A_{lj} - p_{lj}) \pi_{lj}(S_l) \ge 0 \quad S_l \subset N_l, \quad l \in \mathcal{L}. \tag{14}$$

For each market segment we define the sets:

$$F_l = \{k \in N_l : p_{lk} - z' A_{lk} - \beta_l \frac{\tilde{v}_{lk}}{v_{lk} > 0}\}, \quad l \in \mathcal{L}$$

and the quantities:

$$\gamma_{lk} = (p_{lk} - z' A_{lk}) v_{lk} - \beta_l \tilde{v}_{lk} \ge 0 \qquad k \in F_l.$$

For $k \notin F_l$ we set $\gamma_{lk} = 0$.

We will now show that $z, \beta_l, l \in \mathcal{L}$ and γ_{lk} $k \in N_l, l \in \mathcal{L}$ is a feasible solution to the dual of the SBLP:

$$\begin{aligned} R(T,c) = \min \quad & \textstyle\sum_{l\in\mathcal{L}} \beta_l D_l + z'c & & & \\ \text{subject to} \quad & \beta_l \tfrac{\tilde{v}_{lk}}{v_{lk}} + z' A_{lk} + \tfrac{\gamma_{lk}}{v_{lk}} & \ge \; & p_{lk} & k \in N_l, \quad l \in \mathcal{L} \\ & \beta_l \tfrac{\tilde{v}_{l0}}{v_{l0}} - \textstyle\sum_{j\in N_l} \tfrac{\gamma_{lj}}{v_{l0}} & \ge \; & 0 & l \in \mathcal{L} \\ & z \ge 0, \quad \gamma_{lk} & \ge \; & 0 & k \in N_l, \quad l \in \mathcal{L} \end{aligned} \tag{15}$$

By construction

$$\beta_l \frac{\tilde{v}_{lk}}{v_{lk}} + z' A_{lk} + \frac{\gamma_{lk}}{v_{lk}} \ge p_{lk}$$

for all $k \in N_l, l \in \mathcal{L}$.

Our next task is to show that

$$\beta_l \frac{\tilde{v}_{l0}}{v_{l0}} - \sum_{j\in N_l} \frac{\gamma_{lj}}{v_{l0}} \ge 0$$

Since $v_{l0} > 0$ and $\gamma_{lj} = 0$ for $j \notin F_l$ it is enough to show that

$$\beta_l \tilde{v}_{l0} - \sum_{j\in F_l} \gamma_{lj} \ge 0$$

Substituting the definition of γ_{lj} and rearranging terms, the inequality is equivalent to

$$\beta_l[\tilde{v}_{l0} + \tilde{v}_l(F_l)] + \sum_{j \in F_l}(z' A_{lj} - p_{lj}) v_{lj} \geq 0.$$

Dividing by $\tilde{v}_{l0} + \tilde{v}_l(F_l)$ the inequality is equivalent to

$$\beta_l + \sum_{j \in F_l}(z' A_{lj} - p_{lj}) \pi_{lj}(F_l) \geq 0.$$

However, this inequality holds because of the feasibility of the dual of the CBLP (14) applied to F_l. Since any optimal solution to the dual of the CBLP has a corresponding solution to the dual of the SBLP with the same objective function we have $R(T, c) \leq \bar{V}(T, c)$ completing the proof.

■

A.5 Proof of Proposition 1:

Proof: To show feasibility it is enough to show that the capacity constraint is satisfied and to do this it is enough to show that the vectors $\pi_l(S_l)$ are decreasing in w_k. To facilitate the notation we will drop the market segment index l. Then

$$\pi_j(S) = \frac{v_i}{v_0 + w(S') + v(S)}$$

which is decreasing in w_k. As a result, if we compare the demands under model $w_k, k \in N$ and model $w'_k \in (w_k, v_k], k \in N$, we see that the demands under the latter model are bounded above by the demands under the former model. Consequently, strictly less capacity is consumed under the latter model. Moreover, revenues suffer from lower demands so the original solution is suboptimal.

■

A.6 Proof of Proposition 2:

Proof: Suppose that $x_{lk}, k \in N_l, l \in \mathcal{L}$ is an optimal solution to the SBLP under the GAM for fixed $v_k, k \in N_+$ and $w_k \in [0, v_k], k \in N$ and consider a GAM model with the same v_ks and $w'_k \in [0, w_k], k \in N$. To see that the x_{lk} remains feasible it is enough to verify the demand balance constraint for each market segment. To facilitate the exposition we will drop the market segment subscript l. The demand balance constraint can be written as

$$x_0 + \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} w_k\left(\frac{x_0}{v_0} - \frac{x_k}{v_k}\right) = D.$$

Now consider a GAM with $w'_k \in [0, w_k]$. Then the demand scaling constraints guarantee that

$$x_0 + \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} w'_k(\frac{x_0}{v_0} - \frac{x_k}{v_k}) \leq D.$$

This shows that $x_k, k \in N$ together with some $x'_0 \geq x_0$ form a feasible solution to the demand balance constraint without violating the demand scaling constraints.

■

A.7 Proof of Theorem 3:

Proof: We first verify that x_0 and $x_k, k \in X_j$ as given by equations (8) and (9) with j given by equation (11) is feasible for the SBLP assortment problem. Suppose the $x_k, k \in X_j$ are as in equation (9). Then the demand balance equation is

$$\frac{\tilde{v}_0}{v_0}x_0 + \sum_{k \in X_j} \frac{\tilde{v}_k}{v_0}x_0 = D.$$

Solving for x_0 we obtain (8). But now $\frac{x_k}{v_k} = \frac{x_0}{v_0} = \frac{1}{\tilde{v}_0 + \tilde{v}(X_j)}D$ for all $k \in X_j$ together with the non-negativity of x_0 and $x_k, k \in X_j$ shows that the solution is indeed feasible. This feasible solution has objective value R_j given by equation (10).

To prove that R_j is the maximum revenue for the assortment problem we will show that there is a feasible solution to the dual with the same objective value. Consider the dual with objective function $D\beta$ and constraint set $\tilde{v}_0\beta - \sum_k \gamma_k \geq 0$ and $\tilde{v}_k\beta + \gamma_k \geq p_k v_k, k \in N$. We claim that $\beta = R_j/D$ together with $\gamma_k = p_k v_k - \tilde{v}_k\beta$ for $k \in X_j$ and $\gamma_k = 0$ for $k \notin X_j$ is a feasible solution to the dual. The non-negative constraint $\gamma_k \geq 0$ is satisfied by construction for $k > j$. For $k \in X_j$ the constraint is equivalent to $p_k v_k \geq \tilde{v}_l\beta$, which in turn is equivalent to $p_k D v_k/\tilde{v}_k \geq R_j$ for all $k \in X_j$. Since the items are sorted in decreasing order of $p_k v_k/\tilde{v}_k$ it is enough to show that $p_j D v_j/\tilde{v}_j \geq R_j$. However, we know that R_j is a convex combination of R_{j-1} and $p_j D v_j/\tilde{v}_k$. Moreover, since $R_j \geq R_{j-1}$ it follows that $R_j \leq p_j D v_j/\tilde{v}_j$. It remains to show that $\tilde{v}_0\beta - \sum_k \gamma_k \geq 0$. We will do this by showing that $\beta = R_j/D$ is a solution to equation $\tilde{v}_0\beta = \sum_{k \in X_j} \gamma_k$ where $\gamma_k = p_k v_k - \tilde{v}_k\beta$ for $k \in X_j$. This is equivalent to $[\tilde{v}_0 + \tilde{v}(X_k)]\beta = \sum_{k \in X_j} p_k v_k$. We see that $\beta = \sum_{k \in X_j} p_k \pi_k(X_j) = R_j/D$ completing the proof.

■

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