Revenue Management of Consumer Options for Tournaments

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Abstract

Sporting event managers typically only offer advance tickets which guarantee a seat at a future sporting event in return for an upfront payment. Some event managers and ticket resellers have started to offer call options under which a customer can pay a small amount now for the guaranteed option to attend a future sporting event by paying an additional amount later. We consider the case of tournament options where the event manager sells team-specific options for a tournament final, such as the Super Bowl, before the finalists are determined. These options guarantee a final game ticket to the bearer if her team advances to the finals.

We develop an approach by which an event manager can determine the revenue maximizing prices and amounts of advance tickets and options to sell for a tournament final. We show that, under certain conditions, offering options will increase expected revenue for the event. We establish bounds for the revenue improvement and show that introducing options can increase social welfare. We present a numerical application of our approach to the 2012 Super Bowl.

Subject Classifications: Revenue Management; Real Options; Sports

Area of Review: Revenue Management

1 Introduction

The World Cup final, the Super Bowl, and the final game of the NCAA Basketball Tournament in the United States (a.k.a. "March Madness") are among the most popular sporting events in the world. Typically, demand exceeds supply for the tickets for these events, even when the tickets cost hundreds of dollars. However, since these events are the final games of a tournament, the identities of the two teams who will be facing each other are typically not known until shortly before the event. For example, the identity of the two teams who faced each other in the 2010 World Cup final was determined only after the completion of the two semi-final games, five days prior to the final. Yet, tickets for the World Cup Final are offered for sale many months in advance. While there may be many fans who are eager to attend the final game no matter who plays, many fans would only be interested in attending if their favored team, say Germany, were playing in the final. These fans face a dilemma. If they purchase an advance ticket, and Germany does not advance to the final, then they have potentially wasted the price of the ticket, especially if there is no secondary market. On the other hand, tickets are likely to be sold out well before it is known who will be playing in the finals, so if fans wait, they may be unable to attend at all. In response to this dilemma, some sporting events have begun to offer "ticket options" in which a fan can pay a nonrefundable deposit up front for the right to purchase a seat later once the identity of the teams playing is known. Essentially, this is a call option by which the fan can limit her cost should her team not make the final while guaranteeing a seat if her team does make the final. In this paper, we address the revenue management problem faced by the event manager (or promoter) of a tournament final who has the opportunity to offer options for the final. We examine when it is most profitable to offer options to consumers and how the manager should set prices and availabilities for both the advance tickets and the options. We also address the social welfare implications of offering options.

Over the past five years, a number of events and third-parties have begun to offer call options for sporting event tickets. For example, the Rose Bowl is an annual post-season event in which two American college football teams are chosen to play against each other based on their records during the regular season. The identity of the teams playing is not known until a few weeks prior to the event, however, the Rose Bowl sells tickets many months in advance. In addition to general "advance tickets", the Rose Bowl also sells "Team Specific Reservations". As described on the Rose Bowl's web-site (http://teamreserve.tournamentofroses.com/markets/sports/college-fb/event/2011-rose-bowl):

> One Team Specific Ticket Reservation guarantees one face value ticket if your team makes it to the 2011 Rose Bowl. Face value cost is a charge over and above the price you pay for your Team Specific Ticket Reservation. If your team doesn't make it to the Game, there are no refunds for your purchased Team Specific Ticket Reservations, and tickets will not be provided.

Ticket options have become so popular that there is a software company, TTR that specializes in selling Internet platforms to teams and events that wish to offer options. In addition, at least one web site, www.OptionIT.com offers options for a variety of sporting events.

While options can be offered for any sporting event, in this paper we consider only the case of *tournament options*, which are sold for a future event in which the two opponents who will face each other are *ex ante* unknown. We assume that there are potential customers – "fans" – whose utility of attending the game is dependent upon whether or not their favored team is playing. In this case, the tournament option enables a fan to hedge against the possibility that her favored team is not selected to play in the game of interest – e.g. the World Cup final.

While we derive results that are applicable to more general tournament structures, we pay particular attention to *dyadic* tournaments. In a dyadic tournament, the remaining teams can always be partitioned into two sets, $\mathcal{T}_1$ and $\mathcal{T}_2$ so that the final will feature a team from $\mathcal{T}_1$ facing a team in $\mathcal{T}_2$. The most common example of a dyadic tournament is a single-elimination tournament in which 2^n teams play each other at each stage with 1/2 of the teams being eliminated until the last remaining two teams play in the final. Another example of a dyadic tournament occurs when the winners of two different leagues are chosen to play each other. As an example, prior to 1988, the Rose Bowl featured the Pac-10 Conference champion against the Big-10 Conference champion. There are also dyadic tournaments such as those used by the Super Bowl and the World Cup that combine round-robin and single-elimination structures.

1.1 Main Contributions

In this paper we look at the revenue management of consumer options for sporting events. We study the potential revenue improvements of offering options, relative to only offering advance tickets.

We propose a demand model where consumers are segmented by their preferred teams. We do not enforce any a priori segmentation across products. Instead, we postulate a neoclassical, risk-neutral, choice model where consumers maximize their expected surplus. We allow fans to choose which product to purchase based on (i) prices, (ii) product availability, (iii) their intrinsic willingness-to-pay, and (iv) their rational expectations about the likelihood of the different outcomes. Thus, in our model, the demands for products are not independent, and a price-sensitive consumer choice model naturally arises.

In order to capture fans' sensitivity to the teams playing in the final game, we introduce a parameter termed *love-of-the-game* that measures the value to a fan of attending a game in which their favorite team does not play. The higher the value of this parameter, the more utility that fans derive from a game in which their favorite team is not playing. This parameter turns out to be critical in our model, and strongly influences the profitability of introducing options. Estimation of the fans' willingness-to-pay and their sensitivity to the teams playing in the final could be estimated, for example, with an empirical study similar to the one of Sainam et al. (2009) who estimated the willingness to pay of consumers for advance tickets and options under various probabilities of their favorite team playing in a final.

We address the joint problem of pricing and capacity allocation. We assume the event manager announces ticket prices at the beginning, and these remain fixed throughout the sales horizon. However, as demand realizes, the manager can control ticket sales by dynamically managing the availability of products. The sequential nature of these decisions suggests a two-stage optimization problem: set prices in the first stage, and allocate capacity given the fixed prices in the second stage. the capacity allocation problem in the second stage resembles a network revenue management problem under a discrete choice model. This continuous time stochastic problem is intractable.

Different methods have been proposed in the literature to solve the network capacity allocation problem. For example, Zhang and Adelman (2006) proposed an approximate dynamic programming approach in which the value function is approximated with an affine function of the state vector. Another popular approach, which we adopt here, considers a deterministic approximation of the capacity allocation problem, in which random variables are replaced by their means and products are allowed to be sold in fractional amounts (Gallego et al., 2004). The deterministic approximation results in a linear program. Unfortunately, the resulting LP grows exponentially with the number of products (or teams in our case). One of our contributions is an approximation that only grows quadratically with the number of teams. This allows us to efficiently solve instances of moderate size jointly on prices and capacity allocation. Additionally, we give precise bounds for the performance of the deterministic approximation and show that this approximation is asymptotically optimal for the stochastic problem.

To provide some insight we analyze the symmetric problem, i.e., the case in which all teams have the same probability of reaching the final and the fans of all teams share the same valuations and "love-of-the game". These simplifying assumptions allow us to characterize the conditions under which offering options is beneficial to the event manager. Though not entirely realistic, this analysis provides simple rules of thumb that can be applied to the general case. Specifically, offering options benefits the event manager the most when seats are scarce and the love-of-the-game parameter is low. Whe seats are abundant, the event manager can do better selling advance tickets only and no options. The value to the event manager of offering options decreases as the love-of-the-game parameter increases. That is, as fans become more averse to seeing other teams play, options become more attractive to them, and the event manager can take advantage of this by offering options. We also show that, under some mild assumptions, the introduction of options increases the consumer's surplus. This should not be surprising because options allow fans to hedge against the risk of watching a team that it is not of their preference. Lastly, we consider two possible extensions to our model. The first extension explores the idea of *full-information pricing* where the event manager prices the tickets after the finalists are determined. The other extension investigates how can the event manager can avoid arbitrage in the presence of a secondary market.

1.2 Literature Review

This paper addresses a particular case of the classic revenue management problem of pricing and managing constrained capacity to maximize expected revenue in the face of uncertain demand. Overviews of revenue management can be found in Talluri and van Ryzin (2004) and Phillips (2005). While the revenue management literature is vast, there has been relatively little research on its applications to sporting events. Barlow (2000) discusses the application of revenue management to Birmingham FC, an English Premier League soccer team. Chapter 5 of Phillips (2005) discusses some pricing approaches used by baseball teams and Phillips et al. (2006) describe a software system for revenue management applicable to sporting events. Duran et al. (2011) and Drake et al. (2008) consider the optimal time to switch from offering bundles (e.g. season tickets) to individual tickets for sports and entertainment industries. None of these works address the use of options.

Research specifically on the use of options for sports events is very scarce. The first attempt to analyze such options was by Sainam et al. (2009). The authors devise a simple analytical model to evaluate the benefits of offering options to sports event organizers. They show that organizers can potentially increase their profits by offering options to consumers in addition to advance tickets. They also conduct a small numerical study to support their theoretical findings. However, they do not address the problem of pricing options or determining the number to sell.

In the absence of discounting, a consumer call option for a future service is equivalent to a partially refundable ticket. Gallego and Sahin (2010) show how such partially refundable tickets can increase revenue relative to either fully refundable or non-refundable tickets and that they can be used to allocate the surplus between consumers and capacity providers. They show that offering an option wherein an initial payment gives the option of purchasing a service for an additional payment at a later date can provide additional revenue for sellers. Gallego and Stefanescu (2012) discuss this as one of several "service engineering" approaches that sellers can use to increase profitability. The same result holds for a consumer call option in the case when the identity of the teams is known *ex ante*. Our work extends their work by incorporating the correlation structure on *ex post* customer utilities imposed by the structure of the tournament.

1.3 Overview

§2 describes how we model a tournament and consumer demand for tickets. §3 introduces the pricing problems for advance tickets and options. §3.1 analyzes the problem of pricing advance tickets when no options are offered. The options pricing problem is defined in §3.2. The deterministic approximation for the capacity allocation problem, together with the efficient formulation is introduced in §3.3. In §3.4 we discuss implementation issues and practical considerations.

In §4 we give some important theoretical results for a symmetric tournament. §5.1 explains how offering options to consumers affects their surplus. The idea of full-information pricing is examined in §5.2, and §5.3 analyzes how to inhibit a third party from taking advantage of prices to obtain a risk-free profit. Results of numerical experiments are given in §6. §7 concludes with some final remarks.

2 Model

We consider a tournament with $N (N \geq 3)$ teams, where there is uncertainty about the finalists. The final is held in a venue with a capacity of C seats, which we assume of uniform quality. In the case where the seats have heterogeneous quality, the stadium can be partitioned in sections, and then each section can be considered independently. Alternatively, one could consider all sections simultaneously using a nested revenue management model with upgrades á la Gallego and Stefanescu (2009). In this model the event manager may upgrade customers to higher quality seats. Even though we do not pursue this direction in here, we note that upgrades help balance demand and supply by shifting excess capacity of high grade products to low grade products with excess demand.

We address the problem of pricing and management of tickets and options for the final game. The event manager offers $N+1$ different products for the event: *advance tickets*, denoted by A and *options* for each team i, denoted by O^i. Advance tickets require a payment of p_a in advance and guarantee a seat at the final game. An option O^i for team i is purchased at a price p_o^i, and confers the buyer a right to exercise and purchase the underlying ticket at a strike price p_e^i.

The event manager faces the problem of pricing the products, and determining the number of products of each type to offer so as to maximize her expected revenue. The event manager is a monopolist and can influence demand by varying the price. Sales are allowed during a finite horizon T that ends when the tournament starts. A common practice in sporting events is that prices are announced in advance, and the promoter commits to those prices throughout the horizon. We adhere to that static pricing practice in our model. However, the event manager does not commit in advance to allocate a fixed number of seats for each product. Thus, she can dynamically react to demand by changing the set of products offered at each point in time.

For ease of exposition, the event manager is assumed to be risk-neutral, and performs no

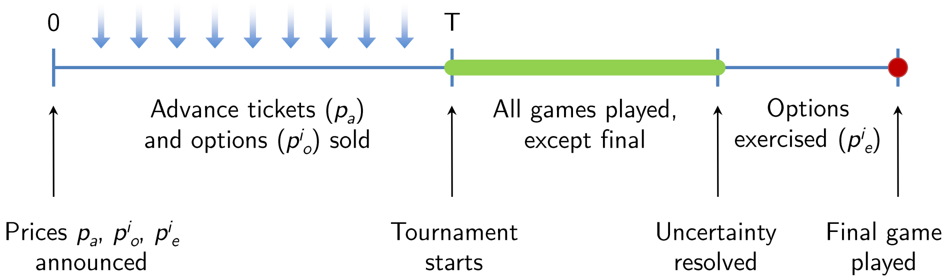

Figure 1: Sales horizon and actions involved in each period.

discounting. These assumptions can be easily relaxed and accommodated in our model. All costs incurred by the event manager are assumed to be sunk, so that there is no marginal cost for additional tickets sold. From the event manager's point of view seats are perishable, that is, unsold seats have no value after the tournament starts since they cannot be sold anymore. The event manager maximizes her expected revenue. Additionally, no overbooking is allowed.

The timing of the events is as follows. First, the event manager announces the advance ticket's price p_a, and the options' premium and strike price (p^i_o, p^i_e) for each team i. Then, the box office opens, and advance tickets and options are sold at those prices. When the tournament starts the sales horizon concludes and no more tickets are sold. Afterwards, the tournament is played out, and the two teams playing in the final are revealed. At this point the holders of options for the finalists decide whether to exercise their rights and redeem a seat at the corresponding strike price. Finally, the championship game is played and the fans attend the event. Figure 1 illustrates the timing of the events.

$\mathcal{T}$ denotes the set of possible combinations of teams that might advance to the finals. For example, in the case where any combination of teams may play in the final game, we have $\mathcal{T} = \{\{i, j\} : 1 \leq i < j \leq N\}$. In the case of a dyadic tournament such as a single-elimination tournament, teams can be divided into two groups, denoted by $\mathcal{T}_1 = \{1, \ldots, N/2\}$ and $\mathcal{T}_2 = \{N/2 + 1, \ldots, N\}$, in such a way that one team from each group advances to the final game. In this case the space of future outcomes is $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$.

A probability distribution $\{q^s\}_{s\in\mathcal{T}}$ for every possible set of outcome is assumed to be common knowledge and invariant throughout the sales horizon. Tournament participants' characteristics such as past performance and injury status are generally common knowledge. In turn, this information may be used to calculate the betting odds which are also available to the public, from where one can obtain $\{q^s\}_{s\in\mathcal{T}}$. For our second assumption, since the box office closes before the tournament starts, we only require the probabilities to be invariant before the games start. Finally, note that given $\{q^s\}_{s\in\mathcal{T}}$ the actual probability of team i advancing to the final is $q^i = \sum_{s\in\mathcal{T}:i\in s} q^s$ and that $\sum_{i=1,\ldots,N} q^i = 2$.

A critical assumption of our model is that tickets and options are not transferrable. This can be enforced, for instance, by demanding some proof of identification at the entry gate. Non-transferability precludes the existence of a secondary market for tickets, that is, tickets cannot be resold and they can only be purchased from the event manager. This assumption, although somewhat restrictive, simplifies the analysis.

Now, let us examine the consumers' choice behavior. In the model, consumers are risk-neutral and maximize their expected surplus. The market is segmented with respect to team preference, with N disjoint segments corresponding to each team. We refer to consumers within segment i as *fans* of team i. In our model, demand is stochastic and price sensitive, with customers arriving according to independent Poisson processes with homogeneous intensity Λ^i for segment i. Time-dependent arrival intensities can be handled by partitioning the sales horizon into intervals where the arrival rate is constant (Liu and van Ryzin, 2008).

A fan of team i has two sources of utility, (i) attending a final game with his favorite team playing, and (ii) attending the event with any other team playing. A fan is characterized by his private valuation for attending his preferred team's game, denoted by V. Valuations are random and drawn independently from a distribution with c.d.f. $F_v^i(\cdot)$. We assume that $F_v^i(\cdot)$ is time-homogenous, and fans do not update their valuations with time. As a result, expected utilities of the possible alternatives remain constant, and the fans do not switch decisions, so there are no cancellations or no-shows. When his preferred team is not playing, a fan obtains only a fraction $\ell^i \in [0,1]$ of his original valuation, and his total value for attending the event is $\ell^i V$. When $\ell^i = 1$, fans' utility of attending the game is independent of which teams are playing. When ℓ^i is close to zero, fans have a strong preference towards their team, and are willing to attend the game only if their team is playing. We refer to ℓ^i as the "love-of-the-game". This parameter turns out to be critical in our model, and determines the profitability of introducing options to a great extent. All ℓ, $F_v^i(\cdot)$, and λ are common knowledge.

At the moment of purchase, a fan of team i has three choices, (i) buy an advance ticket, (ii)

Decision	Pays	Value	Ex. Utility
n: don't buy	0	0	0
a: buy A	p_a	V w.p. q^i $\ell^i V$ w.p. $1-q^i$	$(q^i+(1-q^i)\ell^i)V-p_a$
o: buy O^i	$p_o^i+p_e^i$ w.p. q^i p_o^i w.p. $1-q^i$	V w.p. q^i 0 w.p. $1-q^i$	$q^iV-(p_o^i+q^ip_e^i)$

Table 1: Expenditures, values and expected utilities related to each decision.

buy an option for his preferred team, or (iii) not purchase anything. The first choice requires the payment of the advance ticket price p_a. Then, with probability q^i, the fan expects to get a value of V from seeing his team in the final and with probability $1-q^i$ he expects to get a value of $\ell^i V$. Hence, the fan's expected utility for product A given a valuation of V, denoted by $U_a^i(V)$, is $U_a^i(V)=(q^i+(1-q^i)\ell^i)V-p_a$. The second choice, buying the option O^i, requires the payment of the premium price p_o^i at the moment of purchase. Since valuations are not updated over time, once a fan buys an option, he will always exercise if his team makes the final. Hence, with probability q^i his preferred team advances to the final, and he exercises by paying the strike price p_e^i and extracts a value V in return. The expected utility for product O^i given a valuation of V, denoted by $U_o^i(V)$, is $U_o^i(V)=q^iV-(p_o^i+q^ip_e^i)$. Finally, the utility of no purchase is $U_n=0$. Table 1 summarizes the expenditures, values and expected utilities related to each decision.

A fan makes the choice that maximizes his expected utility. The actual decision, however, depends on the availability of advance tickets and options at the moment of arrival to the box office. For instance, when the first-best choice is not available, the consumer pursues his second-best choice, and if this is also not available, he buys nothing.

We now address the problem of characterizing the demand rate of every product subject to a given set of offered products. We partition the space of valuations for each market segment into five disjoint sets as shown in Table 2. I_{xyz}^i denotes the set of valuations for which decision x is the first-best choice, y is the second-best choice, and z is the least preferred for segment i. For example, I_{aon}^i corresponds to the set of valuations where an advance ticket is the most highly preferred product, an option is the second most highly preferred product and buying nothing is the least preferred choice for segment i. The linearity of expected utilities implies that these sets are intervals of $\mathbb{R}_+$. Figure 2 illustrates the expected utility for the three choices versus

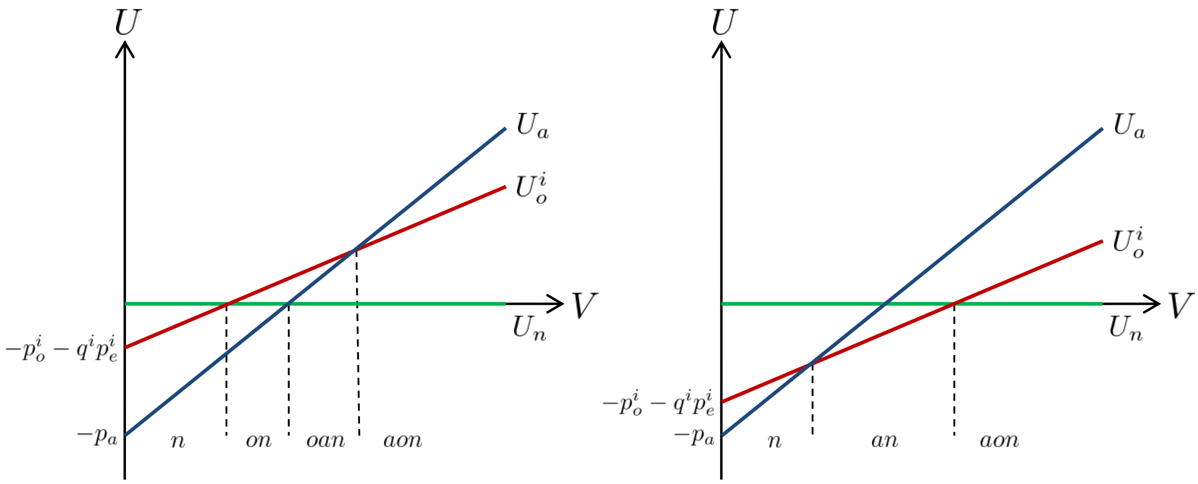

Figure 2: Graphs showing expected surplus for the three choices. The horizontal axis is divided in segments matching each decision. For instance, if V falls in the segment *oan* the fan would buy an option, and else she would buy an advance ticket.

the realized value of V for the particular market segment i, and the corresponding valuation intervals. Observe that depending on prices and problem parameters, this graph can take on two forms, either $I_n^i \cup I_{on}^i \cup I_{oan}^i \cup I_{aon}^i = \mathbb{R}_+$ (the first graph) or $I_n^i \cup I_{an}^i \cup I_{aon}^i = \mathbb{R}_+$ (the second graph).

Decision Priorities	Valuation Sets (I_{xyz})	Probability (π_{xyz})
n	$\{V : U_a(V) \leq 0, U_o(V) \leq 0\}$	$F_v(\min(c, b))$
on	$\{V : U_o(V) \geq 0 \geq U_a(V)\}$	$(F_v(c) - F_v(b))^+$
an	$\{V : U_a(V) \geq 0 \geq U_o(V)\}$	$(F_v(b) - F_v(c))^+$
oan	$\{V : U_a(V) \geq U_o(V) \geq 0\}$	$(F_v(a) - F_v(c))^+$
aon	$\{V : U_o(V) \geq U_a(V) \geq 0\}$	$1 - F_v(\max(a, b))$

Table 2: Decision priorities and corresponding valuation sets. For simplicity we drop the superscript indicating the team. The intersection points are given by $a = \frac{p_a - (p_o + qp_e)}{(1-q)\ell}$, $b = \frac{1}{q}(p_o + qp_e)$, and $c = \frac{p_a}{q + (1-q)\ell}$. Additionally, $(x)^+ = \max\{x, 0\}$.

Using the distribution of valuations in the population, the event manager can compute the probability that the private valuation of an arriving customer of team i belongs to a particular interval. In the following, we denote by $\pi_{xyz}^i = \mathbb{P}\{V \in I_{xyz}^i\}$ the probability that x is the first-best choice for a consumer of team i, y is his second-best, and z the least preferred choice. These probabilities can be conveniently expressed in terms of the intersection points of the utility functions of the different choices, as given by $a^i = \inf\{V : U_a^i(V) \geq U_o^i(V)\}$, $b^i = \inf\{V : U_o^i(V) \geq 0\}$, and $c^i = \inf\{V : U_a^i(V) \geq 0\}$. Table 2 shows the valuation sets and corresponding probabilities for each possible choice ordering.

Now we turn to the problem of determining the demand rate for each product when the event manager offers only a subset $S \subseteq \mathcal{S} \equiv \{A, O^1, \dots, O^N\}$ of the available products. Under our model the instantaneous arrival rate of fans of team i purchasing advance tickets when offering $S \subseteq \mathcal{S}$, denoted by $\lambda_a^i(S)$, is

$$\lambda_a^i(S) = \Lambda^i \mathbf{1}_{\{A \in S\}}(\pi_{an}^i + \pi_{aon}^i + \mathbf{1}_{\{O^i \notin S\}}\pi_{oan}^i). \tag{1}$$

The arrival rate for advance ticket purchases is composed of three terms. The first term accounts for fans that are only willing to buy those tickets. The second term accounts for fans that are willing to buy the advance tickets, but when they are no longer available will buy the options as a second choice. Finally, the third term considers fans that prefer options as their first choice, but may end up buying advance tickets when they are not available. The aggregate arrival rate for advance ticket purchases when offering subset S is $\lambda_a(S) = \sum_{i=1}^N \lambda_a^i(S)$.

Similarly, the arrival rate of fans of team i buying options when offering S, denoted by $\lambda_o^i(S)$, is

$$\lambda_o^i(S) = \Lambda^i \mathbf{1}_{\{O^i \in S\}}(\pi_{on}^i + \pi_{oan}^i + \mathbf{1}_{\{A \notin S\}}\pi_{aon}^i). \tag{2}$$

3 Pricing Problem

3.1 Advance Ticket Pricing Problem

We first consider the problem where no options are offered, and only advance tickets are sold. In this case the organizer's problem is to find the price that maximizes the expected revenue under the constraint that at most C tickets can be sold. The maximum expected profit R_a^* for the organizer is

$$R_a^* \equiv \max_{p_a} \mathbb{E}\left[p_a \min\{C, D_a(p_a)\}\right], \tag{3}$$

where $D_a(p_a)$ is the demand for advance tickets under price p_a. $D_a(p_a)$ is a Poisson random variable with mean $T\lambda_a(p_a)$ where $\lambda_a(p_a)$ denotes the arrival intensity of advance ticket purchase requests for all teams' fans under price p_a. In turn, using (1) the arrival intensity is

$$\lambda_a(p_a) \equiv \sum_{i=1}^{N} \Lambda^i \mathbb{P}\{U_a^i(V) \geq 0\} = \sum_{i=1}^{N} \Lambda^i \bar{F}_v \left(\frac{p_a}{q^i + (1-q^i)\ell^i} \right) \tag{4}$$

The exact solution to problem (3) can be given in terms of the elasticity of demand with respect to price as in Bitran and Caldentey (2003). However, we do not follow that path here. Instead, we approximate the exact solution using the deterministic version of the model (or the certainty equivalent policy) where random variables are replaced by their means, and discrete quantities are assumed to be continuous. The resulting solution, which is often easier to compute, is asymptotically optimal to the original problem.

The maximum revenue under the deterministic approximation, denoted R_a^D, is

$$R_a^D = \max_{p_a} \{p_a \min\{C, T\lambda_a(p_a)\}\}. \tag{5}$$

The *run-out rate*, given by $\lambda_a^0 = C/T$, is defined to be the rate of advance ticket sales at which the organizer sells all of her seats uniformly over the time horizon T. The corresponding *run-out price*, denoted by p_a^0, is the price which enables the organizer to achieve the run-out sales rate and is obtained from $\lambda_a(p_a^0) = C/T$. Let λ_a^* be the least maximizer of the revenue rate function $\lambda_a p_a(\lambda_a)$, then from Gallego and van Ryzin (1994) we know that

$$R_a^D = T \min(p_a^* \lambda_a^*, p_a^0 \lambda_a^0).$$

Hence, if the capacity of the stadium is large ($C > \lambda_a^* T$), the organizer ignores the problem of running out of seats and prices at the level that maximizes the revenue rate. In this case the organizer ends with $C - \lambda_a^* T$ unsold seats. If the seats are scarce ($C < \lambda_a^* T$), the organizer can afford to price higher, and it indeed prices at the highest level that still enables it to sell all the items. Notice that in the final game of a tournament it is likely that the number of seats will be scarce, so the second situation is more likely to prevail. As advance ticket prices increase, fans become more sensitive to the finalists so options should be more attractive when seats are scarce. We will later show that this turns out to be the case.

Consider the asymptotic optimality of the deterministic approximation using an inequality obtained from Gallego and van Ryzin (1994)

$$1 \geq \frac{R_a^*}{R_a^D} \geq 1 - \frac{1}{2\sqrt{\min(C, \lambda_a^* T)}}. \tag{6}$$

From inequality (6) we see that the fluid model approximation is asymptotically optimal in two limiting cases: (i) the capacity of the stadium is large ($C \gg 1$) and there is plenty of time to sell them ($C < \lambda_a^* T$); or (ii) there is the potential for a large number of sales at the revenue maximizing price ($\lambda_a^* T \gg 1$), and there are enough seats to satisfy this potential demand ($C > \lambda_a^* T$). Thus, we see that if the volume of expected sales is large, the approximation performs quite well.

3.2 Advance Ticket and Options Pricing Problem

We now consider the combined problem of pricing and managing both advance tickets and options. Recall that prices are determined in advance, disclosed at the beginning, and remain constant during the sales horizon. However, the number of seats allocated to each product are not disclosed in advance, and may be used by the organizer to adjust her strategy as sales occur. The organizer can control the number of tickets and options sold to dynamically react to the demand by playing with the availability of the products.

The sequential nature of the decisions involved suggests a partition of the problem into a two-stage optimization problem. The decision variables are prices in the first stage and product availabilities in the second stage. In the first stage, the organizer looks for the set of prices $p = (p_a, p_o, p_e)$ that maximizes the optimal value of the second-stage problem, which is the maximum expected revenue that can be extracted under fixed prices p. This partition is well-defined because prices are determined before the demand is realized, and are independent of the actual realization of the demand. The optimal value of the first-stage problem, denoted by R^*, is

$$R^* \equiv \max_{p \geq 0} R^*(p)$$

where $R^*(p)$ denotes the optimal value of the second-stage problem.

The second-stage problem takes prices as given, and optimizes the expected revenue by controlling the subset of products that is offered at each point in time. Notice that the second-stage decision variable is a control policy over the offer sets, which is determined as the demand realizes. We refer to this second-stage problem as the *Capacity Allocation Problem*. Next, we turn to the problem of determining the optimal value of the second-stage problem under fixed prices p.

Once prices are fixed, the organizer attempts to maximize her revenue by implementing adaptive non-anticipating policies that offer some subset $S \subseteq \mathcal{S} \equiv \{A, O^1, \ldots, O^N\}$ of the available products at each point in time. A control policy μ maps states of the system to control actions, i.e. the set of offered products. We denote by $S_\mu(t)$ the subset of products offered under policy μ at time t. Let $X_a(S_\mu(t))$ be the total number of advance tickets sold up to time t. Under our assumptions, $X_a(S_\mu(t))$ is a non-homogeneous Poisson process with arrival intensity $\lambda_a(S_\mu(t))$ as defined in (1). The organizer can affect the arrival intensity of purchase requests by controlling the offer set $S_\mu(t)$. An advance ticket is sold at time t if $\mathrm{d}X_a(S_\mu(t)) = 1$. Similarly, let $X_o^i(S_\mu(t))$ be the number of options sold for team i up to time t, and $\mathrm{d}X_o^i(S_\mu(t)) = 1$ when an option is sold at time t. Again, $X_o^i(S_\mu(t))$ is a non-homogeneous Poisson process with arrival intensity $\lambda_o^i(S_\mu(t))$ as defined in (2). With some abuse of notation, we define $X_a = X_a(S_\mu(T))$ and $X_o^i = X_o^i(S_\mu(T))$ to be the total number of advance tickets and options sold, respectively.

The organizer seeks to maximize its expected revenue, which is given by

$$\mathbb{E}\left[X_a p_a + \textstyle\sum_{i=1}^N X_o^i (p_o^i + q^i p_e^i)\right].$$

The first term accounts for the revenue from advance ticket sales and the second term accounts for the revenue from options under the assumption that *all options are exercised*, which was previously discussed in §2. Notice that because prices remain constant during the time horizon, the revenue depends on the total number of tickets sold. Moreover, as a result of the linearity of expectation, the revenue depends only on the expected number of tickets sold.

The second-stage or Capacity Allocation Problem can be formalized as the following stochastic control problem which is similar to the one given in Liu and van Ryzin (2008):

$$
\begin{aligned}
R^*(p) = \max_{\mu \in \mathcal{M}} \mathbb{E} & \left[p_a X_a + \sum_{i=1}^{N} (p_o^i + q^i p_e^i) X_o^i \right] \\
\text{s.t. } X_a &= \int_0^T \mathrm{d}X_a(S_\mu(t)), \\
X_o^i &= \int_0^T \mathrm{d}X_o^i(S_\mu(t)), \qquad \forall i = 1, \dots, N, \\
X_a + X_o^i + X_o^j &\le C, \qquad \text{(a.s.) } \forall \{i,j\} \in \mathcal{T},
\end{aligned} \tag{7}
$$

where $\mathcal{M}$ is the set of all adaptive non-anticipating policies, and $R^*(p)$ is the expected revenue under the optimal policy μ^*.

Next, we state the Hamilton-Jacobi-Bellman (HJB) equation for the second-stage problem. Let the value function $V(t, X_a, X_o)$ be the maximum expected revenue that can be extracted when t time units are remaining, and X_a number of advance tickets and X_o number of options have been sold. Our goal is to find $R^*(p) = V(T, 0, 0)$ and the HJB equation for the second-stage problem can be written:

$$
\begin{aligned}
\frac{\partial V(t, X_a, X_o)}{\partial t} = \max_{S \subseteq \mathcal{S}} \Bigg\{ & \lambda_a(S) \left(p_a + V(t, X_a + 1, X_o)\right) \\
& + \sum_{i \in S_o} \lambda_o^i(S) \left(p_o^i + q^i p_e^i + V(t, X_a, X_o + e_i)\right) \\
& + \lambda_n(S) V(t, X_a, X_o) \Bigg\},
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
V(0, X_a, X_o) &= 0 \qquad \text{for all } X_a, X_o, \\
V(t, X_a, X_o) &= -\infty \qquad \text{if } X_a + X_o^i + X_o^j > C \text{ for some outcome } \{i,j\} \in \mathcal{T}.
\end{aligned}
$$

Here, $\lambda_n(S) = \Lambda - \lambda_a(S) - \sum_{i=1}^N \lambda_o^i(S)$ is the arrival rates of fans who decide not to purchase either advance tickets or options when S is on offer and $\Lambda = \sum_{i=1}^N \Lambda^i$ is the total arrival rate.

Unfortunately, the resulting HJB equation is a partial differential equation that is in most cases very difficult to solve. The next section gives a tractable and provably good deterministic approximation of (7).

3.3 Choice-based Deterministic Linear Programming Model

In this section we follow Gallego et al. (2004), and solve a deterministic approximation of (7) in which random variables are replaced by their means and quantities are assumed to be continuous. We denote by $r_a = p_a$ the expected revenue from selling an advance ticket, and by $r_o^i = p_o^i + q^i p_e^i$ the expected revenue from selling an option of team i. Under this approximation, when a subset of products S is offered, advanced tickets (resp. options for team i) are purchased at a rate of $\lambda_a(S)$ (resp. $\lambda_o^i(S)$). Since r_a (resp. r_o^i) is the expected revenue from the sale of an advance ticket (resp. option for team i), the rate of revenue generated from the sales of advance tickets is $r_a\lambda_a(S)$ (resp. $r_o^i\lambda_o^i(S)$ for options of team i). Additionally, because demand is deterministic and the choice probabilities are time homogeneous, we only care about the total amount of time each subset of products is offered and not the order in which they are offered. Thus, we only need to consider the amount of time each subset S is offered, denoted by $t(S)$, as the decision variables. Under this notation, the number of advance tickets sold is $\sum_{S\subseteq\mathcal{S}} t(S)\lambda_a(S)$, while the number of options sold for team i is $\sum_{S\subseteq\mathcal{S}} t(S)\lambda_o^i(S)$. Finally, the total revenue of the organizer is $\sum_{S\subseteq\mathcal{S}} r(S)t(S)$, where $r(S) = r^T\lambda(S)$ is the revenue rate when subset S is offered, and $r = \left(r_a, r_o^1, \ldots, r_o^N\right)$ is the vector of expected revenues.

Thus, we obtain the following choice-based deterministic LP model (CDLP):

$$R^{CDLP}(p) \equiv \max_{t(S)} \sum_{S\subseteq\mathcal{S}} r(S)t(S) \tag{8}$$

$$\text{s.t.} \sum_{S\subseteq\mathcal{S}} t(S) = T,$$

$$\sum_{S\subseteq\mathcal{S}} t(S)\left(\lambda_a(S) + \lambda_o^i(S) + \lambda_o^j(S)\right) \le C, \qquad \forall\{i,j\}\in\mathcal{T} \tag{9}$$

$$t(S) \ge 0 \qquad \forall S\subseteq\mathcal{S}$$

where $R^{CDLP}(p)$ denotes the maximum revenue of the CDLP under prices p. Notice that, no matter which teams advance to the final, the maximum number of options sold is $C - X_a$. Let $\mathcal{S}^* = \{S\subseteq\mathcal{S} : t^*(S) > 0\}$ be the sets offered in the optimal solution of the CDLP. Even though the number of offer sets is exponentially large, in a dyadic tournament there are at most $N^2/4+1$ basic variables (one for each possible constraint). Thus, there exists an optimal solution to the

CDLP in which the number of sets offered is at most $N^2/4+1$ (see, e.g., Bertsimas and Tsitsiklis (1997)). In the next section, we will show that this number is at most $2N+1$.

As with most deterministic approximations, it is the case that the optimal value of the CLDP provides an upper bound to the optimal value of the stochastic program (7) (see, e.g., Liu and van Ryzin (2008)). In the next result, we show that for every fixed price the revenue difference between the CDLP and the stochastic problem is of order $O(\sqrt{T})$. In order to show this bound we use an argument similar, yet slightly simpler, to that of Gallego et al. (2004). First, we construct a theoretical *offer time* (OT) policy from the optimal solution of the CDLP. In such a policy (i) each set is offered for the time prescribed by the deterministic solution in an arbitrary order, and (ii) the number of products sold in each set is limited to the expected demand. We then show that the expected time each set is offered in the OT policy is close to the one of the deterministic solution, and then conclude that the performance of such a policy is close to the deterministic upper bound.

Theorem 1. *Fix prices $p \geq 0$. Let $r_{\max}$ be the maximum revenue rate and $\lambda_{\min}$ be the minimum arrival rate of the solution. Suppose further that $\lambda_{\min}^{-1} \leq (N+1)T$. Then, we have that the revenue loss of the stochastic control problem with respect to the CDLP is bounded by*

$$0 \leq R^{CDLP}(p) - R^*(p) \leq 2r_{\max}|\mathcal{S}^*|\sqrt{(N+1)\lambda_{\min}^{-1}T}.$$

The proof of this result is in the appendix. As a corollary, we also get that the CDLP is asymptotically optimal. To see this, let $R^*_\theta(p)$ be the optimal objective of a scaled stochastic problem in which capacity is set to θC and time horizon to θT for some $\theta \geq 1$. Similarly, let $R^{CDLP}_\theta(p)$ be the optimal objective of a scaled CDLP. Notice that the CDLP is insensitive to the scaling, that is, $\frac{1}{\theta}R^{CDLP}_\theta(p) = R^{CDLP}(p)$. Then from Theorem 1 one gets that the CDLP is asymptotically optimal for the second-stage problem, or equivalently $\frac{1}{\theta}R^*_\theta(p)$ converges to $R^{CDLP}(p)$ as $\theta \to \infty$ for all $p \geq 0$. Moreover, it is not hard to see that the asymptotic optimality of the CDLP carries over to the first-stage problem. Solving the CDLP instead of the stochastic control in the second-stage is asymptotically optimal for the first-stage problem. Another interesting consequence of the previous results is that the OT policy, in spite of its simplicity, is asymptotically optimal for the stochastic control problem.

3.3.1 Efficient Formulation

Since the linear program in (8) has one primal variable for each offer subset, it has 2^{N+1} primal variables in total. For instance, if the tournament has 32 teams the program would have more than 8 billion primal variables! Fortunately, by exploiting the structure of our choice model it is possible to derive an alternative formulation with a linear number of variables and constraints.

Recall that consumers are partitioned into N different market segments, each associated with a different team. To any given segment $i = 1, \ldots, N$ two different products are potentially offered: (i) advance tickets (A) and (ii) options for the associated team (O^i). We denote by $\mathcal{S}^i = \{A, O^i\}$ the set of products available for market segment i. Demands across segments are independent, and different segments are only linked through the capacity constraints. Since each segment has two products, only four offer sets need to be considered. Thus, for each market segment we only need the following decision variables: (i) the time both advance tickets and options are offered, denoted by $t^i\left(\{A, O^i\}\right)$, (ii) the time only advance tickets are offered, denoted by $t^i\left(\{A\}\right)$, (iii) the time only options are offered, denoted by $t^i\left(\{O^i\}\right)$, and (iv) the time no product is offered, denoted by $t^i\left(\emptyset\right)$. Given $t(S)\ \forall S \subseteq \mathcal{S}$, the value of the new decision variables can be computed as follows:

$$
\begin{aligned}
t^i\left(\{A, O^i\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \in S, O^i \in S} t(S), & t^i\left(\{A\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \in S, O^i \notin S} t(S) \\
t^i\left(\{O^i\}\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \notin S, O^i \in S} t(S), & t^i\left(\emptyset\right) &\equiv \sum_{S \subseteq \mathcal{S}: A \notin S, O^i \notin S} t(S)
\end{aligned}
\qquad (10)
$$

Observe that for each segment offer times should sum up to length of the horizon, that is $\sum_{S \subseteq \mathcal{S}^i} t^i(S) = T$. An important observation is that by requiring $\sum_{S \subseteq \mathcal{S}^i \setminus \emptyset} t^i(S) \leq T$ we do not need to keep track of the time in which no product is offered for each segment. Additionally, in order for the offer sets to be consistent across market segments, the total time that advance tickets are offered should be equal for all segments, i.e., for some $T_a \geq 0$ it should be the case that $t^i\left(\{A, O^i\}\right) + t^i\left(\{A\}\right) = T_a$ for all $i = 1, \ldots, N$ where T_a denotes the total time advance tickets are offered throughout the sales horizon.

After applying the aforementioned changes, we obtain the following market-based deterministic LP (MBLP)

$$R^{MBLP}(p) \equiv \max_{t^i(S),T_a} \sum_{i=1}^{N} \sum_{S\subseteq\mathcal{S}^i} r^i(S)t^i(S) \tag{11}$$

$$\text{s.t.} \sum_{S\subseteq\mathcal{S}^i} t^i(S) \leq T \qquad \forall i = 1,\dots,N \tag{12}$$

$$t^i\left(\{A, O^i\}\right) + t^i(\{A\}) = T_a \qquad \forall i = 1,\dots,N \tag{13}$$

$$\sum_{k=1}^{N}\sum_{S\subseteq\mathcal{S}^k} t^k(S)\lambda_a^k(S) + \sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda_o^i(S) + \sum_{S\subseteq\mathcal{S}^j} t^j(S)\lambda_o^j(S) \leq C \qquad \forall\{i,j\}\in\mathcal{T} \tag{14}$$

$$T_a \geq 0, t^i(S) \geq 0 \qquad \forall S \subseteq \mathcal{S}^i, i = 1,\dots,N,$$

where $r^i(S) = p_a\lambda_a^i(S) + r_o^i\lambda_o^i(S)$ is the revenue rate from market segment i when subset $S \subseteq \mathcal{S}^i$ is offered. Notice that the new optimization problem has $3N+1$ variables, which is much less than the original CDLP, and $O(N^2)$ constraints.

Proposition 1. *The MBLP is equivalent to the CDLP, i.e.* $R^{MBLP}(p) = R^{CDLP}(p)$ *for all prices* $p \geq 0$.

The proof of this proposition can be found in the appendix. An interesting consequence of the proof is that the optimal policy has a nested structure. Because demands across segments are independent, one can sequentially order the offer sets containing advance tickets such that each set is a subset of the previous one. The same holds for offer sets that do not include advance tickets. Additionally, the number of active sets in the optimal solution is at most $2N+1$.

3.4 Implementation and Practical Considerations

As we previously discussed, the optimal value of the capacity allocation problem $R^*(p)$ is hard to compute. Hence, in order to tackle our problem, we replace the objective of the first-stage problem with the upper-bound provided by the deterministic approximation $R^{MBLP}(p)$. This new problem provides an upper bound to the truly optimal objective value R^*. However, in view of the asymptotic optimality of the deterministic approximation, and the large scale of the problem in terms of stadium's capacity, our policy is expected to perform reasonably well. In

Section 6 we show that this is indeed the case through some numerical experiments.

Now, the first-stage problem amounts to optimizing the non-linear function $R^{MBLP}(p)$ over the polyhedron of prices. Because the objective is not necessarily convex as a function of price, multiple different starting points need to be taken. Given our efficient method to evaluate the approximate objective value of the capacity allocation problem we are able to solve real problems of moderate size, despite the non-convexity of the objective.

After the optimal solution is computed, a remaining issue is how tickets should be effectively sold. Clearly, the event manager should announce the optimal prices p^* at the beginning, and commit to that price throughout the sales horizon. However, one important issue is the capacity allocation of the tickets, and constructing a good dynamic control policy from the output of the approximation. The optimal solution of the deterministic approximation prescribes only how long each subset should be offered, but does not specify how to implement the actual policy. One straightforward approach is to offer each subset S for the amount of time given by $t(S)$, which we referred as the *offer time* policy. As pointed out by Liu and van Ryzin (2008), this approach has a few problems. First, the order in which the sets are offered is not specified, and the resulting policy is static and does not react to changes in demand.

Various heuristics have been proposed to address the first problem. Liu and van Ryzin (2008) proposed a decomposition approach in which the dual optimal solutions of the deterministic problem are used to decompose the network dynamic program into a collection of leg-level DPs which can be solved exactly. These are then used to construct a control policy. **?** improved upon this idea by considering an alternative dynamic programming decomposition method that performs the allocations by solving an auxiliary optimization problem. Alternatively, Zhang and Adelman (2006) employ an approximate dynamic programming scheme in which the value function is approximated with affine functions of the state vector. This allows them to obtain dynamic bid-prices that are later used to construct control policies.

Inspired by our efficient formulation, we propose a simple *sales limit* policy. The policy offers all tickets from the beginning, and limits the number of each product sold to the expected value given by the deterministic approximation. That is, tickets are sold either until the end of horizon or the limit is reached, whichever happens first. The limits are given by X_a for the advance tickets, and X_o^i for the i^{th} team options. Such booking limit policies are not guaranteed to be optimal in the general network revenue management problem, but nevertheless performs surprisingly well in our setting. Two attractive features of this policy are its ease of implementation, and the fact that it concurs with the current sales practice.

To address the second problem, the static nature of the control policy, one could attempt to periodically resolve the deterministic approximation. Recently, Jasin and Kumar (2010) showed that carefully chosen periodic resolving schemes together with probabilistic allocation controls can achieve a bounded revenue loss w.r.t. to the optimal online policy (static control policies are guaranteed to achieve a revenue loss that grows as the squared root of the size of the problem). We do not pursue this direction in here, but note that one could attempt to periodically resolve the MBLP to improve the performance.

4 The Symmetric Case

In this section we consider the symmetric problem, i.e., the case in which all teams have the same probability of advancing to the final and arrival rates. Also, we assume that valuations are i.i.d. across teams and that the love-of-the-game is constant throughout the population. These assumptions, albeit not entirely realistic, allows us to theoretically characterize the benefits of introducing options. As we shall later see, the conditions under which options are beneficial frequently carry over to the most general case. In the remainder of this section we first identify several conditions under which offering options is beneficial to the organizer. Second, we provide an asymptotic analysis for the case where the number of teams is large.

The following analysis will be based on the deterministic approximation of the problem and not the actual stochastic performance. Due to the asymptotic optimality of the deterministic approximation and the large scale of the problem, it should be expected that these results carry over to the fully stochastic setting. In Section 6 we show, through some numerical experiments, that this is indeed the case.

4.1 Advance Ticket Pricing Problem with Symmetric Teams.

In a symmetric problem with N teams, each team has the same probability $q = \frac{2}{N}$ of advancing to the final game. The arrival rate of fans of each team is $\lambda = \frac{\Lambda}{N}$, where Λ denotes the aggregate arrival rate. Using (4), the total arrival intensity under price p_a is now $\lambda_a(p_a) = \Lambda \bar{F}_v \left(\frac{p_a}{q+(1-q)\ell} \right)$.

We assume that the c.d.f. of V is continuous and strictly increasing. Thus, there is a one-to-one correspondence between prices and arrival rates, and the function $\lambda_a(p_a)$ has an inverse $p_a(\lambda_a)$ given by

$$p_a(\lambda_a) = (q + (1-q)\ell)\, \bar{F_v}^{-1}\left(\frac{\lambda_a}{\Lambda}\right).$$

The one-to-one correspondence between prices and arrival intensity allows us to recast the problem with the arrival intensity as the decision variable; the promoter determines a target sales intensity λ_a and the market determines the price $p_a(\lambda_a)$ based on this quantity. Under this change of variables, the deterministic approximation of the advance ticket pricing problem (5) becomes

$$\begin{aligned} R_a^D = \max_{\lambda_a \geq 0} \quad & T\,(q + (1-q)\ell)\, v(\lambda_a) \\ \text{s.t.} \quad & T\lambda_a \leq C, \lambda_a \leq \Lambda, \end{aligned} \tag{15}$$

where we have written the objective in terms of the *value rate* $v(\lambda_a) = \lambda_a \bar{F_v}^{-1}\left(\frac{\lambda_a}{\Lambda}\right)$. We denote the objective function of (15) by $R_a^D(\lambda_a)$. Note that the advance ticket pricing problem (15) is equivalent to the problem of a monopolistic seller pricing a zero-cost product with limited capacity.

In the following, we assume that value rate is *regular* and differentiable. Regularity implies that v is continuous, bounded, concave, satisfies $\lim_{\lambda_a \to 0} v(\lambda_a) = 0$, and has a least maximizer λ_a^*. These assumptions are common in the RM literature (see, e.g., Gallego and van Ryzin (1994)). A consequence of regularity is that the objective of program (15) is concave. Additionally, because the objective is continuous and the feasible set is compact, by Weierstrass' Theorem one concludes that program (15) achieves a maximum (Luenberger, 1969).

Let $h(x) = f_v(x)/\bar{F}_v(x)$ be the *failure rate* of the random variable V. The random variable V is said to have an *increasing failure rate* (IFR) if $h(x)$ is non-decreasing. A sufficient condition for the concavity of the value rate is that valuations have IFR.

Lemma 1. *If the valuation random variable has IFR, then the value rate is strictly concave.*

The proof of this lemma is in the appendix. Lariviere (2006) and van den Berg (2006) give weaker sufficient conditions for the uniqueness of the optimal solution and the concavity of the objective. All the results that follow in this section hold under the weaker assumption that the valuation random variable has an *increasing generalized failure rate* (IGFR), that is, it suffices for generalized failure rate $xh(x)$ to be non-decreasing. IFR implies IGFR but the reverse does not hold.

4.2 Advance Ticket and Options Pricing Problem with Symmetric Teams.

Next, we turn to the deterministic approximation of the advance ticket and options pricing problem. We look for symmetric solutions in which we charge the same expected price $r_o = p_o + qp_e$ for options for all teams, and hence sell the same number of options for all teams. At this point it should be noted that it is optimal to price products so that we never run out of tickets before the end of the sales horizon. Otherwise, we would leave some unsatisfied demand that could be captured by raising prices, resulting in increased revenue. As a consequence, we do not need to control the availability of the products.

From Proposition 2 the aggregate arrival intensity under prices p_a and r_o can be computed as

$$\begin{aligned} \lambda_a(p_a, r_o) &= \Lambda \bar{F}_v\left(\frac{p_a - r_o}{(1-q)\ell}\right), \\ \lambda_o^{\Sigma}(p_a, r_o) &= \Lambda\left[\bar{F}_v\left(\frac{r_o}{q}\right) - \bar{F}_v\left(\frac{p_a - r_o}{(1-q)\ell}\right)\right], \end{aligned}$$

where we denote by λ_o^{Σ} the aggregate arrival intensity of all consumers buying options. Again, we work with arrival intensities as decision variables since there is a one-to-one correspondence between prices and arrival rates. The inverse functions are given by

$$\begin{aligned} r_o(\lambda_a, \lambda_o^{\Sigma}) &= q\bar{F}_v^{-1}\left(\frac{\lambda_a + \lambda_o^{\Sigma}}{\Lambda}\right), \\ p_a(\lambda_a, \lambda_o^{\Sigma}) &= q\bar{F}_v^{-1}\left(\frac{\lambda_a + \lambda_o^{\Sigma}}{\Lambda}\right) + (1-q)\ell\bar{F}_v^{-1}\left(\frac{\lambda_a}{\Lambda}\right). \end{aligned}$$

Let us now look at the advance ticket and options pricing problem. After the change of variables, the deterministic approximation of the advance ticket and options pricing problem

(15) becomes

$$R_o^D = \max_{\lambda_a \geq 0, \lambda_o^\Sigma \geq 0} T(1-q)\ell v(\lambda_a) + Tqv(\lambda_a + \lambda_o^\Sigma) \tag{16}$$

$$\text{s.t. } T\lambda_a + T\frac{2}{N}\lambda_o^\Sigma \leq C, \tag{17}$$

$$\lambda_a + \lambda_o^\Sigma \leq \Lambda,$$

where we have written the objective in terms of the value rate similar to the advance ticket pricing problem. Since the objective function is the linear combination of concave functions, it is also concave (see Boyd and Vandenberghe (2004)). Hence, (16) and (17) specify a concave maximization problem with linear inequality constraints. Again, the Weierstrass' Theorem applies, and guarantees the existence of an optimal solution.

We are now in a position to characterize some conditions under which options are beneficial to the organizer.

Proposition 2. *In the symmetric case, when the seats are scarce ($C < \lambda_a^* T$) and fans strictly prefer their own team ($\ell < 1$), introducing options increases the revenue of the organizer ($R_o^D > R_a^D$). However, when the capacity of the stadium is large ($C \geq \lambda_a^* T$) or fans are indifferent among teams ($\ell = 1$) options do not increase the revenue ($R_a^D = R_o^D$).*

The proof can be found in the appendix.

In this result we showed that options are beneficial to the event manager only when the demand is high with respect to the stadium's capacity and fans strictly prefer their own team over any other. Proposition 2, however, provides no information on how the benefit from options is affected by ℓ. Next, we perform some comparative statics w.r.t. to the love-of-the-game. As ℓ is increased, fans become less sensitive to the teams playing at the final. As a result, options start to lose their attractiveness, and demand for advance tickets increases. Hence, one should expect the benefit from introducing options to decrease as ℓ is increased. In the next result we show that this is indeed the case.

Proposition 3. *When seats are scarce ($C < \lambda_a^* T$) and fans strictly prefer their own team over any other ($\ell < 1$), both the absolute and relative benefit of introducing options decreases as ℓ is* *increased.*

The proof can be found in the appendix.

Next, we study the asymptotic behavior of different pricing schemes as the number of teams grows to infinity and seats are scarce. (If seats are not scarce, Proposition 2 shows that offering options does not increase revenue). We show that when fans obtain a positive surplus from attending a game without their own team ($\ell > 0$), revenue under options pricing converges to the revenue under advance selling as N grows to infinity. Furthermore, the convergence rate is $O\left(\frac{1}{N}\right)$. The intuition behind this result is that, as the number of teams grows, fans are aware that the probability of their own team reaching the final event decreases. So, in order to keep options attractive for consumers, the organizer needs to set lower prices, and thus revenues generated by options subside. Because fans also obtain a positive surplus from attending a game without their own team, more consumers choose to buy advance tickets as the number of teams grows.

Proposition 4. *When seats are scarce ($C < \lambda_a^* T$) and fans obtain a positive surplus from attending a game without their own team ($\ell > 0$), the revenue under options pricing converges to the revenue under advance selling as N grows to infinity. Moreover, the convergence rate is given by*

$$1 \leq \frac{R_o^D}{R_a^D} \leq 1 + \frac{2}{N\ell}\frac{v(\lambda_a^*)}{v(\lambda_a^0)}.$$

The proof can be found in the appendix.

In the case where fans obtain zero surplus from attending a game without their own team ($\ell = 0$), the previous discussion no longer holds. Now, options and advance tickets are equivalent to customers, and they are only interested in one outcome: their own team advancing to the final game. Because the probability of that outcome converges to zero, the number of tickets sold converges to zero as well. This observation, combined with the existence of the *null price* (or $\lim_{\lambda_a \to 0} v(\lambda_a) = 0$), causes the organizer's revenue to diminish to zero in all pricing schemes as the number of teams increases. Surprisingly, even though the revenues when only advance tickets are offered and, when both advance ticket and options are offered converge to zero, they do so at different rates. The rationale is that when the organizer offers only options each team has up to $C/2$ tickets available. Hence, the capacity of the stadium is extended, and for a suitable large N the organizer may price according to the revenue maximizer rate λ_a^*.

Proposition 5. *When seats are scarce ($C < \lambda_a^* T$) and fans obtain zero surplus from attending a game without their own team ($\ell = 0$), the revenue obtained when both advance tickets and options are offered strictly dominates the case when only advance tickets are offered. Moreover, their ratio is given by*

$$\begin{aligned}\frac{R_o^D}{R_a^D} &= \frac{v\left(\min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\}\right)}{v(\lambda_a^0)} \\ &= \frac{v(\lambda_a^*)}{v(\lambda_a^0)} > 1 \quad \textit{when } N \geq 2\left\lceil \frac{\lambda_a^*}{\lambda_a^0} \right\rceil.\end{aligned}$$

Proof. If $\ell = 0$ options and advance tickets are equivalent to customers, and customers choose the product with the lowest price. Thus, we only need to consider the case where the organizer sells only options the whole time horizon. The options pricing problem is now

$$\begin{aligned} R_o^D = \max_{\lambda_o^\Sigma \geq 0} \quad & T\frac{2}{N} v(\lambda_o^\Sigma) \\ \text{s.t. } & T\lambda_o^\Sigma \leq \frac{N}{2} C, \quad \lambda_o^\Sigma \leq \Lambda. \end{aligned}$$

This problem is similar to the advance selling problem (15) except that capacity is scaled by $\frac{N}{2}$. Scarcity implies that $C < \lambda^* T$, and thus the optimal solution is $\lambda_o^{(N)} = \min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\}$. Then, the optimal value is $R_o^D = T\frac{2}{N} v(\min\left\{\lambda_a^*, \lambda_a^0 \frac{N}{2}\right\})$. Finally, observe that for $N \geq 2\left\lceil \frac{\lambda_a^*}{\lambda_a^0} \right\rceil$ the organizer may price according to the revenue maximizer rate λ_a^* and $R_o^D = T\frac{2}{N} v(\lambda_a^*)$. □

5 Extensions

In this section we consider a number of extensions of the model. First, we investigate the effect of options on social efficiency. Then, we discuss full-information pricing, that is, the case in which the event manager prices tickets after the uncertainty is resolved. The last extension explores how can the event manager price tickets avoid the possibility of arbitrage.

5.1 Social Efficiency

How do the introduction of options affect customers' surplus? Options allow fans to hedge against the risk of watching a team that it is not of their preference. As a consequence, a larger number of seats will be taken by fans of the teams that are playing in the final. So, intuitively we expect the introduction of options to increase the total surplus of the fans. In this section we show how to compute the total consumer surplus of an allocation. We will see that the surplus can conveniently be expressed in terms of the intersections of the utility lines as defined in Proposition 2 and the integrated tail of the valuations, which is defined as $\bar{G}_v^i(x) = \mathbb{E}\left[(V-x)^+\right] = \int_x^\infty \bar{F}_v^i(v)\mathrm{d}v$. In the next section, we will characterize some conditions under which options are beneficial for the consumers in the symmetric case. We will see that these conditions coincide with the ones we developed previously in Section 4 regarding the revenue improvement resulting from offering options.

First, consider the *surplus rate*, which is the instantaneous rate at which surplus is generated, for team i's fans who purchase advance tickets. We distinguish whether options are offered simultaneously or not. In the case when options are not offered, fans purchase advance tickets when their utility $U_a^i(V)$ is non-negative. The arrival rate of such consumers is $\lambda_a^i(\{A\}) = \Lambda^i \mathbb{P}\{U_a^i(V) \geq 0\}$, and their expected utility conditioned on them buying A is $\mathbb{E}\left[U_a^i(V)|U_a^i(V) \geq 0\right]$. Hence, the surplus rate is given by

$$s_a^i(\{A\}) = \Lambda^i \mathbb{E}\left[U_a^i(V)\mathbf{1}\{U_a^i(V) \geq 0\}\right] = \Lambda^i(q^i + (1-q^i)\ell^i)\bar{G}_v(c^i). \tag{18}$$

When advance tickets are offered simultaneously with options, we need to take into account that fans purchase advance tickets when both $U_a^i(V) \geq 0$, and $U_a^i(V) \geq U_o^i(V)$. Now, the surplus rate is given by

$$\begin{aligned} s_a^i(\{A, O^i\}) &= \Lambda^i \mathbb{E}\left[U_a^i(V)\mathbf{1}\{U_a^i(V) \geq 0, U_a^i(V) \geq U_o^i(V)\}\right] \\ &= \Lambda^i(q^i + (1-q^i)\ell^i)\left(\bar{G}_v(\max\{a^i, c^i\}) + (a^i - c^i)^+ \bar{F}_v(a^i)\right). \end{aligned}$$

The surplus rate for team's i fans purchasing options can be obtained in a similar way. When advance tickets are not offered, fans purchase options when their utility $U_o^i(V)$ is non-negative, and the surplus rate is given by

$$s_o^i(\{O^i\}) = \Lambda^i \mathbb{E}\left[U_o^i(V)\mathbf{1}\{U_o^i(V) \geq 0\}\right] = \Lambda^i q^i \bar{G}_v(b^i).$$

The last case remaining is when options are offered simultaneously with advance tickets. Here, we need to take into account that fans purchase options when both $U_o^i(V) \geq 0$, and $U_o^i(V) \geq U_a^i(V)$. The surplus rate is given by

$$\begin{aligned} s_o^i(\{A, O^i\}) &= \Lambda^i \mathbb{E}\left[U_o^i(V)\mathbf{1}\{U_o^i(V) \geq 0, U_o^i(V) \geq U_a^i(V)\}\right] \\ &= \Lambda^i q^i \left(\bar{G}_v(\max\{a^i, b^i\}) - \bar{G}_v(a^i) + (a^i - b^i)^+ \bar{F}_v(a^i)\right). \end{aligned}$$

As in the case of the revenue rates, we can compute the aggregate surplus rate when offering a subset S as $s(S) = \sum_{i=1}^N s_a^i(S) + s_o^i(S)$. Notice that the first summand is nonzero only if $A \in S$, while the second summand is nonzero if $O^i \in S$. Thus, given an allocation $\{t(S)\}_{S \subseteq \mathcal{S}}$, the total expected surplus is $\sum_{S \subseteq \mathcal{S}} s(S)t(S)$.

Symmetric Case. In the following we will show that, under some assumptions, the introduction of options increases the customers' surplus. Consequently, options can benefit both the promoter and the consumers. We proceed as in the case of the revenue. First, we obtain the surplus of the advance ticket pricing problem in terms of intensities as decision variables. Second, we move on to the advance ticket and options pricing problem, and give a simple expression for the consumer surplus. We conclude by identifying a sufficient assumption and proving our result.

From equation (18) the total surplus of consumers that will buy an advance ticket when the arrival intensity is λ_a, denoted by $S_a^D(\lambda_a)$, is

$$\begin{aligned} S_a^D(\lambda_a) &= T\Lambda(q + (1-q)\ell)\bar{G}_v\left(\frac{p_a(\lambda_a)}{q + (1-q)\ell}\right) \\ &= T(q + (1-q)\ell)s(\lambda_a) \end{aligned}$$

where the surplus rate is defined as $s(\lambda_a) = \Lambda\bar{G}_v\left(\bar{F}_v^{-1}\left(\frac{\lambda_a}{\Lambda}\right)\right)$. Notice that the surplus rate is defined on $[0, \Lambda]$. Additionally, it is increasing, continuous, differentiable, non-negative, and bounded. The monotonicity stems from the fact that $\bar{F}_v^{-1}$ is decreasing and $\bar{G}_v$ is non-increasing.

Moreover, it satisfies $\lim_{\lambda_a \to 0} s(\lambda_a) = 0$, and $\lim_{\lambda_a \to \Lambda} = \Lambda \mathbb{E} V$. In contrast to the revenue rate, the maximum is reached when the intensity is set to Λ, or equivalently the price set to zero. Not surprisingly, total consumer surplus is maximized when the tickets are given away for free.

In the advance ticket and options pricing problem, two sources contribute to the total consumer surplus. The first source is consumers who choose advance tickets over options. The second source is consumers who chose options over advance tickets. Some algebra shows that the total consumer surplus in terms of the arrival intensities, denoted by $S_o^D(\lambda_a, \lambda_o^\Sigma)$, is

$$S_o^D(\lambda_a, \lambda_o^\Sigma) = T(1-q)\ell s(\lambda_a) + Tqs(\lambda_a + \lambda_o^\Sigma).$$

Observe that the formula for consumer surplus is similar to the organizer's revenue with the exception that the value rate is replaced by the surplus rate.

Next, we study how introducing options impacts the consumer surplus. Recall that, from Proposition 2, options are beneficial to the organizer only if capacity is scarce and fans strictly prefer their own team over any other. Hence, we only need to consider the consumer surplus under those assumptions, else the organizer has no incentive to sell options. The following proposition shows that if the surplus rate is convex, options do increase consumer surplus.

Proposition 6. *Suppose that the surplus rate is convex. When seats are scarce ($C < \lambda_a^* T$) and fans strictly prefer their own team over any other ($\ell < 1$) introducing options increases the consumer surplus ($S_o^D > S_a^D$).*

Proof. First, let $(\lambda_a, \lambda_o^\Sigma)$ be the optimal solution to the options pricing problem. Since seats are scarce, the capacity constraint (17) is binding in the optimal solution. Then $\lambda_a^0 = C/T = \lambda_a + q\lambda_o^\Sigma = (1-q)\lambda_a + q(\lambda_a + \lambda_o^\Sigma)$, where we have written λ_a^0 as a convex combination of λ_a and $\lambda_a + \lambda_o^\Sigma$. Consider the convex combination of the same points, denoted by $\hat{\lambda}_a$, in which we multiply the first weight by ℓ and re-normalize. Hence, $\hat{\lambda}_a$ is given by

$$\hat{\lambda}_a = \frac{(1-q)\ell}{q + (1-q)\ell}\lambda_a + \frac{q}{q + (1-q)\ell}(\lambda_a + \lambda_o^\Sigma).$$

Notice that $\hat{\lambda}_a > \lambda_a^0$. This follows from $\lambda_o^\Sigma > 0$ implying that the second point is strictly greater than the first, and the weight of this larger point being larger in $\hat{\lambda}_a$ than in λ_a^0.

Finally, we have that

$$\begin{aligned} S_o^D = S_o^D(\lambda_a, \lambda_o^\Sigma) &= T(1-q)\ell s(\lambda_a) + Tqs(\lambda_a + \lambda_o^\Sigma) \\ &\geq T(q + (1-q)\ell)s(\hat{\lambda}_a) > T(q+(1-q)\ell)s(\lambda_a^0) = S_a^D(\lambda_a^0) = S_a^D, \end{aligned}$$

where the first inequality follows from the convexity of the surplus rate, the second inequality from the fact that the surplus rate is increasing and $\hat{\lambda}_a > \lambda_a^0$, and the last equality from λ_a^0 being the optimal solution to the advance selling problem when seats are scarce. Thus, the introduction of options increases the consumer surplus. As a side note, any feasible solution to the options pricing problem in which the capacity constraint (17) is binding verifies that $S_o^D(\lambda_a, \lambda_o^\Sigma) \geq S_o^D$. □

Fortunately, the surplus rate turns out to be convex for many distributions. Lemma 2 gives one sufficient condition, namely the surplus rate is convex when the valuation random variable has IFR. Thus, options are beneficial for consumers whose valuations have IFR.

Lemma 2. *If the valuation random variable has (strict) IFR, then the surplus rate is (strictly) convex.*

Proof. The derivative of the surplus rate w.r.t. λ_a is

$$\frac{ds}{d\lambda_a}(\lambda_a) = \frac{\lambda_a}{\Lambda} \frac{1}{f_v\left(\bar{F_v}^{-1}(\lambda_a/\Lambda)\right)}.$$

Composing the derivative with $\lambda_a(c) = \Lambda \bar{F}_v(c)$ we get

$$\left(\frac{ds}{d\lambda_a} \circ \lambda_a\right)(c) = \frac{ds}{d\lambda_a}\left(\Lambda \bar{F}_v(c)\right) = \frac{\bar{F}_v(c)}{f(c)} = \frac{1}{h(c)}.$$

IFR implies that the composite function is non-increasing in c. Because $\lambda_a(c)$ is decreasing, we conclude that original derivative is non-decreasing and s is convex. The proof for the strict case follows similarly. □

5.2 Full information pricing

In this section we consider the hypothetical case where the event manager sets prices after the finalists are determined. This is referred to as the *full information pricing problem* (Sainam et al., 2009). In practice, the event manager is averse to the notion of pricing tickets after the identity of teams is revealed for at least two reasons: (i) deferring the sale of tickets increases the organizer's exposure to risk, (ii) the period of time left from the point when the uncertainty is revealed until the final game is usually too short. Nevertheless, some insight can be gained from benchmarking options against such a pricing scheme.

In full information pricing, the event manager needs to decide a price p_f^s to maximize the revenue in each possible outcome $s \in \mathcal{T}$. In turn, in outcome s the event manager faces an advance ticket pricing problem as described in Section 3.1 with a demand intensity given by $\lambda_f^s(p_f^s) = \sum_{i \in s} \Lambda^i \bar{F}_v(p_f^s) + \sum_{i \notin s} \Lambda^i \bar{F}_v(p_f^s/\ell^i)$. The first term of the arrival intensity accounts for fans of the teams that reach the final, and the second term for the fans whose teams do not reach the final. Assuming that the sales horizon remains the same, the maximum expected revenue over all outcomes, denoted by R_f^* is given by

$$R_f^* \equiv \sum_{s \in \mathcal{T}} q^s \max_{p_f^s} \ \mathbb{E}[p_f^s \min\{C, D_f^s(p_f^s)\}],$$

where $D_f^s(p_f^s)$ is a Poisson random variable with mean $T\lambda_f^s(p_f^s)$.

Symmetric Case. A natural question is how does the revenue under full information pricing compare to the revenue under advance tickets and options. In the symmetric case, we can show that options always perform at least as well as full information pricing. This follows from setting both the advance ticket price, and the exercise price of options equal to the full information price. That way customers with high valuations that want to buy a ticket independent of the outcome are captured with advance tickets, and options help capture the customers with lower valuations that will buy a ticket only when their preferred team plays.

Proposition 7. *In the symmetric case, the maximum expected revenue under options is greater or equal to that of full information pricing, i.e.,* $R_o^D \geq R_f^D$.

Proof. Since all teams are equivalent, then every outcome has a uniform probability of $q^s =$

$(2/N)^2$, and, under full information pricing, we offer tickets at price p_f independent of the outcome. Additionally, the expected revenue over all outcomes is equal to the revenue of any given outcome. The deterministic approximation problem is given by

$$\begin{aligned} R_f^D = \max_{p_f} \quad & T\lambda_f(p_f)p_f \\ \text{s.t. } & T\lambda_f(p_f) \leq C, p_f \geq 0, \end{aligned} \tag{19}$$

where the arrival rate is given by $\lambda_f(p_f) = 2\Lambda\bar{F}_v(p_f)/N + (N-2)\Lambda\bar{F}_v(p_f/\ell)/N$.

Let p_f be a solution to problem (19). We show that by taking $p_a = p_f$ and $r_o = qp_f$ we obtain a solution to the options pricing problem (16) that achieves the same revenue. First, under those prices, the demand intensities for advance tickets and options are $\lambda_a(p_f, qp_f) = \Lambda\bar{F}_v(p_f/\ell)$, and $\lambda_o^\Sigma(p_f, qp_f) = \Lambda\bar{F}_v(p_f) - \Lambda\bar{F}_v(p_f/\ell)$ respectively. Second, using the fact that $T\lambda_f(p_f) \leq C$, it is easy to see that the new solution satisfies the capacity constraint (17). Lastly, the revenue is given by

$$\begin{aligned} R_o^D(p_f, qp_f) &= T\lambda_a(p_f, qp_f)p_f + T\lambda_o^\Sigma(p_f, qp_f)qp_f = Tp_f\left(\lambda_a(p_f, qp_f) + 2/N\lambda_o^\Sigma(p_f, qp_f)\right) \\ &= T\lambda_f(p_f)p_f = R_f^D. \end{aligned}$$

□

Thus, selling advance ticket and options under this pricing policy performs as good as full information pricing in the symmetric case. However, this result does not generalize. In the case in which teams are not symmetrical, one can construct counterexamples in which offering options is dominated by full information pricing.

5.3 No-Arbitrage Pricing

We want to exclude the possibility of a third party, *the arbitrageur*, from taking advantage of differences in prices to obtain a risk-free profit. For instance, an arbitrageur may simultaneously offer options to fans and buy advance tickets to fulfill the obligations, or offer options for some teams while buying options from others.

In the following, we denote by $\theta = (\theta_a, \theta_o^1, \ldots, \theta_o^N) \in \mathbb{R}^{|\mathcal{S}|}$ a portfolio that assigns weight

θ_i to product i. By convention, a positive value for θ_i indicates that the arbitrageur is buying product i from the organizer, while when θ_i is negative she is selling product i in the market. Using this notation, today's *market value* of the portfolio is given by

$$p^T\theta = \theta_a p_a + \sum_{i=1}^{N} \theta_o^i p_o^i.$$

Uncertainty is represented by the finite set $\mathcal{T}$ of *states*, one of which will be revealed as true. When state $\{i,j\} \in \mathcal{T}$ realizes, the *payoff* of the portfolio is $-\theta_o^i p_e^i - \theta_o^j p_e^j$. These can be written more compactly in matrix notation as $R\theta$, where $R \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{S}|}$ is the matrix of future payoffs. Notice that exploiting the structure of the problem we can write the payoff matrix as $R = \begin{pmatrix} \mathbf{0} & -\Lambda^T \mathrm{diag}(p_e) \end{pmatrix}$, where $\Lambda \in \mathbb{R}^{N\times|\mathcal{T}|}$ is such that $(\Lambda)_{is} = 1$ if in state s team i advances to the final and 0 otherwise.

Finally, in order to fulfill future obligations, the portfolio needs to satisfy $\theta_a + \theta_o^i + \theta_o^j \geq 0$ whenever state $(i,j) \in \mathcal{T}$ realizes. For instance, if the arbitrageur sells one option for team i and another for team j, then she needs to hold at least two advance tickets for the case that both teams advance in the final. Similarly, we write the obligation restriction in matrix notation as $A\theta \geq 0$, where $A \in \mathbb{R}^{|\mathcal{T}|\times|\mathcal{S}|}$ is the obligation matrix. Again, we may exploit the structure of the problem, and write the obligation matrix as $A = \begin{pmatrix} \mathbf{1} & \Lambda^T \end{pmatrix}$.

Textbook arbitrage requires no capital and entails no risk. Thus, an arbitrage opportunity is a transaction that involves no negative cash flow future state and a positive cash flow today.

Definition 1. *An arbitrage opportunity is a portfolio $\theta \in \mathbb{R}^{|\mathcal{S}|}$ with $A\theta \geq 0$ such that $p^T\theta \leq 0$ and $R\theta \geq 0$ with at least one strict inequality.*

The following theorem characterizes the set of arbitrage-free prices. The requirement that y is strictly positive for all outcomes is due to our definition of arbitrage. If we adopted a *strong arbitrage* definition as in LeRoy and Werner (2000), then we would only require y to be non-negative.

Theorem 2. *Prices constitute an arbitrage-free market if and only if there exists $z, y \in \mathbb{R}^{|\mathcal{T}|}$*

such that $z \geq 0$, $y > 0$, *and*

$$\sum_{s\in\mathcal{T}} z_s = p_a, \sum_{s\in\mathcal{T}:i\in s} z_s = p_o^i + p_e^i \sum_{s\in\mathcal{T}:i\in s} y_s, \quad \forall i = 1,\ldots,N.$$

Proof. We want to show that there is no portfolio θ with $A\theta \geq 0$, $\begin{pmatrix} -p^T \\ R \end{pmatrix} \theta \geq 0$, and $\begin{pmatrix} -p^T \\ R \end{pmatrix} \theta \neq 0$. Equivalently, from Tucker's Theorem of the Alternative Mangasarian (1987) there exists $z, y \in \mathbb{R}^{|\mathcal{T}|}$ such that $R^T y + A^T z = p$, $y > 0$, and $z \geq 0$. The result follows by exploiting the fact that $A^T = \begin{pmatrix} \mathbf{1}^T \\ \Lambda \end{pmatrix}$, and $R^T = \begin{pmatrix} \mathbf{0} \\ -\mathrm{diag}(p_e)\Lambda \end{pmatrix}$. □

Notice that by normalizing to 1, we can interpret z as a probability distribution over the set of outcomes. This suggests that the result can be further simplified by aggregating outcomes, and considering the probabilities of each team advancing to the final. Indeed, we may rewrite the arbitrage conditions in terms of $z^i = \sum_{s\in\mathcal{T}:i\in s} z_s$, and $y^i = \sum_{s\in\mathcal{T}:i\in s} y_s$. It is clear that every distribution over outcomes induces a distribution over teams, but the converse does not necessarily hold. Lemma 3 identifies the set of attainable distributions over teams for dyadic tournament.

Lemma 3. *Consider a single elimination tournament, and let the cone* $\mathcal{C} = \left\{\alpha \in \mathbb{R}^N \mid \alpha = \Lambda y, y \geq 0\right\}$. *Then,* $\alpha \in \mathcal{C}$ *if and only if* $\sum_{i\in\mathcal{T}_1} \alpha^i = \sum_{i\in\mathcal{T}_2} \alpha^i$, *and* $\alpha \geq 0$.

The proof is in the appendix. For example in the case of a single elimination tournament we require that the teams in both branches sum up to the same value, that is, $\sum_{i\in\mathcal{T}_1} y^i = \sum_{i\in\mathcal{T}_2} y^i$ and $\sum_{i\in\mathcal{T}_1} z^i = \sum_{i\in\mathcal{T}_2} z^i$. Hence, after eliminating z^i from the system, we find that there is no arbitrage if there exists that a strictly positive $y \in \mathbb{R}^N$ such that

$$\begin{aligned} p_a &= \sum_{i\in\mathcal{T}_1} p_o^i + y^i p_e^i = \sum_{i\in\mathcal{T}_2} p_o^i + y^i p_e^i, \\ \sum_{i\in\mathcal{T}_1} y^i &= \sum_{i\in\mathcal{T}_2} y^i. \end{aligned}$$

Thus, one may incorporate the previous set of constraints in the first-stage problem to exclude arbitrage opportunities.

6 Numerical Examples

In this section we describe the results of several numerical experiments conducted to evaluate the improvements from offering options. Our numerical example is based on Superbowl XLVI which will take place on February 12th, 2012. For the sake of computational simplicity we assume that pricing decisions are made at the Conference Championship level where only four teams are left and we also assume that the teams that will play in the Superbowl will be the winners of New Orleans Saints vs. Minnesota Vikings, and Indianapolis Colts vs. New York Jets games. Let us note that those teams were the divisional round winners in the 2009 season. The probabilities of each team advancing to the Superbowl were obtained from the betting odds of a major online betting company (Vegas.com). The probabilities are given by $q = (.6, .4, .65, .35)$ for (Saints, Vikings, Colts, Jets). We estimated arrival rates arrival rates proportional to the population of each team's hometown, $\lambda = (0.1271, 0.0477, 0.0675, 0.7576)$.

6.1 The Benefits of Introducing Options

In this first experiment we measure the benefits of introducing options in terms of both the event manager's revenue and the consumers' surplus as well as the sensitivity of the results to changes in the demand model.

To measure the impact in our model of the distribution of valuations V, we conducted the experiment with two different distributions, namely, uniform and truncated normal. In order to obtain comparable results the distributions were chosen with equal means and variances, where the mean was \$2000. Also, we checked the sensitivity of our results against different love-of-the-game parameters and *load factors*. The load factor is defined by $l_f = (T\Lambda)/C$, and measures the total demand relative to the size of the stadium: the lower the load factor, the scarcer tickets are. Five different values were used for ℓ – the love-of-the-game paramater: $(0.001, 0.1, 0.2, 0.5$ and $0.9)$ and two different values for l_f (1 and 3). Changes in the load factor were implemented by changing the length of the time horizon T.

Table 3 reports the revenues, as given by the deterministic approximations, for advance tickets and options for the different demand scenarios. The third column shows the relative improvement in the event manager's revenue generated by the introduction of options. Relative improvements in the revenue are plotted in Figure 3. Table 4 reports the consumer surplus when

Uniform Valuations

	$l_f = 3$			$l_f = 1$		
ℓ	Advance	Options	Improv.	Advance	Options	Improv.
0.01	$71.92	$80.73	12.24%	$26.97	$28.31	5.60%
0.1	$83.84	$87.88	4.83%	$31.44	$31.44	0.00%
0.2	$95.68	$98.30	2.74%			
0.5	$130.41	$130.87	0.35%			
0.9	$175.53	$175.53	0.00%			

Truncated Normal Valuations

	$l_f = 3$			$l_f = 1$		
ℓ	Advance	Options	Improv.	Advance	Options	Improv.
0.01	$72.01	$81.95	13.79%	$28.88	$30.50	4.97%
0.1	$83.70	$88.91	6.23%	$33.74	$33.74	0.00%
0.2	$95.29	$99.03	3.93%			
0.5	$130.06	$130.06	0.00%			
0.9	$173.69	$173.69	0.00%			

Table 3: Revenues(in millions) from the deterministic approximation for different parameters

offering advance tickets and options under the different demand scenarios.

From Table 3 we see that offering options is most beneficial when ℓ is low. As ℓ is increased, the potential benefit the organizer can get from offering options decreases. This result is intuitive since options target fans who care about the teams playing in the finals. As ℓ is increased, the utility that the fans get from watching other teams increases, so fans will be less sensitive to the finalists and be more willing to buy advance tickets. Consequently, options become less attractive for the fans and the organizer does not benefit as much from offering them.

Table 3 also confirms that options are most beneficial when capacity is scarce. This coincides with the results obtained about symmetric tournaments in §4. It is seen from the table above that once the load factor is decreased, which is equivalent to making capacity abundant, options do not result in any revenue improvement. In order to keep options attractive the organizer has to set its sales price lower than the advance ticket price. Thus, the organizer has to sell multiple options to get the same revenue it does from selling a single advance ticket. When the capacity is abundant, the total number of advance tickets or options that the organizer can sell is less than the stadium's capacity. So, the organizer will prefer selling only advance tickets over options.

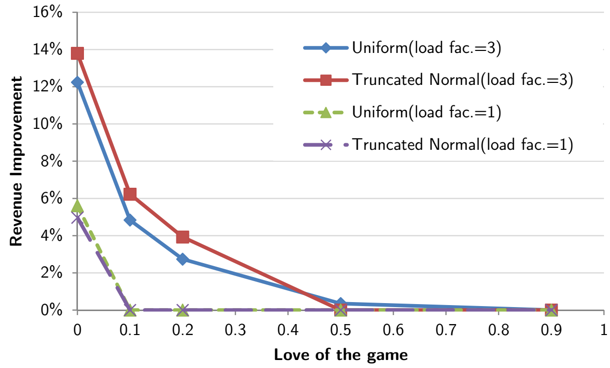

Figure 3: Relative improvement in the event manager's revenue generated by the introduction of options as given by the deterministic approximations.

Uniform Valuations

	$l_f = 3$			$l_f = 1$		
ℓ	Advance	Options	Improv.	Advance	Options	Improv.
0.01	\$26.26	\$27.50	4.73%	\$16.25	\$14.16	-12.87%
0.1	\$27.20	\$34.92	28.37%	\$17.80	\$17.80	0.00%
0.2	\$28.52	\$35.09	23.02%			
0.5	\$34.10	\$34.73	1.85%			
0.9	\$43.93	\$43.93	0.00%			

Table 4: Social surpluses(in millions) for different parameters

Lastly, we see that our results do not appear to be very sensitive to the shape of the distribution since the results are similar for uniformly and normally distributed valuations.

6.2 Simulation of Control Policies

In this second round of experiments we simulate the performance of different control policies for the capacity allocation problem. In each scenario we tested the performance of two different control policies: (i) offer time (OT) control, and (ii) sales limit (SL) control. Both policies were described in Section 3.4. In the OT control each subset S is offered for the amount of time given by the optimal solution $t(S)$. Since the deterministic approximation does not prescribe any particular ordering for the subsets, an order is chosen at random at the beginning of the horizon. In the SL control all tickets are offered from the beginning, and the number of each product sold is limited to the expected value given by the MBLP. Additionally, we simulate the performance of advance ticket pricing problem.

Uniform Valuations

	$l_f = 3$			$l_f = 1$		
ℓ	Advance	Options OT(Gap)	Options BL(Gap)	Advance	Options OT(Gap)	Options SL(Gap)
0.01	\$71.80	\$80.62 (0.13%)	\$80.55 (0.22%)	\$26.97	\$28.31 (0.00%)	\$28.20 (0.37%)
0.1	\$83.70	\$87.74 (0.16%)	\$87.73 (0.18%)	\$31.44	\$31.44 (0.00%)	\$31.38 (0.18%)
0.2	\$95.53	\$98.13 (0.17%)	\$98.13 (0.18%)			
0.5	\$130.22	\$130.67 (0.15%)	\$130.66 (0.16%)			
0.9	\$175.25	\$175.25 (0.16%)	\$175.26 (0.15%)			

Truncated Normal Valuations

	$l_f = 3$			$l_f = 1$		
ℓ	Advance	Options OT(Gap)	Options SL(Gap)	Advance	Options OT(Gap)	Options SL(Gap)
0.01	\$71.89	\$81.82 (0.15%)	\$81.76 (0.23%)	\$28.89	\$30.51 (0.01%)	\$30.40 (0.33%)
0.1	\$83.55	\$88.75 (0.18%)	\$88.78 (0.15%)	\$33.74	\$33.74 (0.00%)	\$33.68 (0.18%)
0.2	\$95.14	\$98.86 (0.18%)	\$98.86 (0.18%)			
0.5	\$129.10	\$129.85 (0.16%)	\$130.25 (-0.15%)			
0.9	\$173.42	\$173.42 (0.15%)	\$173.43 (0.15%)			

Table 5: Simulated revenues (in millions) of different control policies. Gaps between the simulation and the deterministic upper bound are also reported.

Table 5 reports the average revenue of the different policies over 100 different sample paths for the different demand scenarios. In the advance ticket pricing problem, simulation results are compared to the upper bound provided by the deterministic approximation (5). In the options pricing problem, the simulated revenues of the OL and SL policies are compared to the upper bound provided by the MBLP. In the simulation results, the Options OT column corresponds to the offer time policy and the Options SL column corresponds to the sales limit policy.

6.3 Discussion

From Table 5 we see that all gaps between the simulation and the deterministic upper bound are below .5%. Additionally, the deterministic upper bound performs very well for both distributions and is robust to the different values of ℓ and l_f.

In practice, enforcing the offer times obtained from the MBLP can be difficult. The difficulty arises, for example, from the need to coordinate different sales booths, explain the policy to the salespeople, etc. On the other hand, the sales limit policy provides a simple solution: the event manager only needs to allocate a fixed number of tickets to each product, and sell the products

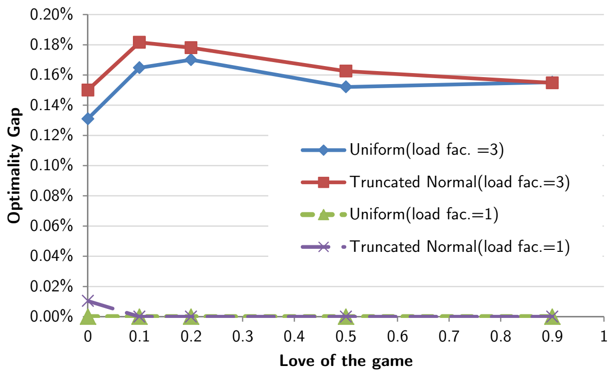

Figure 4: Gaps between deterministic approximation and simulation for options OT policy.

until the end of the horizon or the limits are reached. Table 5 shows that the difference between the sales limit policy and the offer time policy is not very significant. In some cases, the sales limit policy performs even better than the offer time policy. Thus, event managers can offer options in a straightforward way: after solving the MBLP they just need to enforce the sales limits obtained from the deterministic approximation.

7 Conclusion

In this paper, we analyzed consumer options that are contingent on a specific team reaching the tournament final. Offering options, in addition to advance tickets, allows an organizer to segment fans. The organizer targets fans with a higher willingness to pay, who are less sensitive to the outcome, with advance tickets, whereas options target fans who receive more value from attending when their favored team is in the final. Our results show that the organizer can potentially increase her profits by taking advantage of this segmentation and that offering options is beneficial for the fans as well.

In this work we specifically addressed the problem of pricing and capacity control of such options and advance tickets, under a stochastic and price-sensitive demand. The organizer faces the problem of pricing the tickets and options, and determining the number of tickets to offer so as to maximize her expected revenue. We propose solving the organizer's problem using a two-stage optimization problem. The first stage optimizes over the prices, while the second optimizes the expected revenue by controlling the subset of products that is offered at each point in time using a discrete choice revenue management model. The second-stage problem can be formulated as a stochastic control problem, and in most cases is very difficult to solve. Hence, we propose an efficient deterministic approximation, which is shown to be asymptotically optimal. Furthermore, the deterministic approximation worked extremely well in our numerical tests.

To develop some insight, we provide a theoretical characterization of the problem in the symmetric case, i.e., when all teams are equal in terms of arrival rate and other characteristics. Under some mild assumptions, we show that when the seats are scarce and fans strictly prefer their own team over any other, introducing options increases both the revenue of the organizer, and the surplus of the consumers. We show that the benefits of options decreases as the number of teams grow.

Investigating dynamic pricing of options is a promising topic for reseach. Dynamic pricing may provide higher revenues at the cost of substantially increased complexity. Two other natural extensions are relaxing the no-resale restriction and allowing secondary markets, and selling tickets after the tournament starts. Relaxing these two assumptions can affect the fans' decisions substantially, and deserve special attention. However, this may result in an intricate model since the fans may now delay their decisions of buying tickets and options. Lastly, the single quality seat restriction may be relaxed by dividing the stadium according to seat quality. This has the possibility of complicating the consumer choice part of the model considerably. Instead of having to choose from at most two products, fans now may face a wide array of choices.

Appendix: Proofs of Selected Propositions

Proof of Theorem 1

Proof. Fix prices p. The first bound follows from Proposition 1 in Liu and van Ryzin (2008), where they proved such result by using the optimal policy μ^* of the stochastic control problem to construct a candidate solution for the CDLP. In the candidate solution each set is offered for an amount of time $t_{\mu^*}(S)$, which is defined as the expected time set S is offered under policy μ^*. Or equivalently $t_{\mu^*}(S) = \mathbb{E}\left[\int_0^T \mathbf{1}\{S_{\mu^*}(t) = S\}\, dt\right]$. Such a solution is easily shown to be feasible for the CDLP and to attain, in that problem, the same objective value that in the original stochastic problem. Thus, one concludes that $R^*(p) \leq R^{CDLP}(p)$ since every solution of the CDLP is upper bounded by its optimal value.

In order to show the second bound we use an argument similar to that of Gallego et al. (2004). First, we construct a theoretical *offer time* (OT) policy from the optimal solution of the CDLP. In such a policy one offers each set for the time prescribed by the deterministic solution in an arbitrary order. Additionally, the number of products sold in each set is limited to the expected demand, and each set is offered until either the time runs out or any of the products runs out. We denote by $R^{OT}(p)$ to be the expected revenue of the offer time control. Clearly, it is the case that $R^{OT}(p) \leq R^*(p)$. We shall bound the difference between $R^{OT}(p)$ and the upper bound $R^{CDLP}(p)$.

We construct the OT policy as follows. Let $t^*(S)$ be the optimal solution of the CDLP. With some abuse of notation with refer to advance tickets as the zero option, i.e., $A \equiv O^0$, $X_a \equiv X_o^0$, $p_a \equiv r_o^0$, and $\lambda_a(S) \equiv \lambda_o^0(S)$. Under the OT policy set S is offered for a time $\tau^{OT}(S) \overset{d}{=} \min\{t^*(S), \min_{O^i \in S} \tau_o^i(S)\}$, where $\tau_o^i(S)$ is the first time we run out of the i^{th} team options in an alternate system in which products are sold independently of each other. More formally, we have that

$$\tau_o^i(S) = \inf\{t : X_o^i(S,t) \geq \lfloor \lambda_o^i(S) t^*(S) \rfloor\},$$

where $X_o^i(S,t)$ is the number of i^{th} team options sold by time t when offering set S in the alternate system, and $\lfloor x \rfloor$ is largest integer not greater than x. For the sake of simplicity we assume that the limits on the number of tickets sold are stricly positive, else they can be excluded from the offer set. Notice that $\tau_o^i(S)$ is an Erlang random variable with rate $\lambda_o^i(S)$ and shape parameter $\lfloor \lambda_o^i(S) t^*(S) \rfloor$.

Before proceeding we state some definitions. Let $r_{\max} = \max_{S \subseteq \mathcal{S}^*, O^i \in S}\{r_o^i(S)\}$ be the maximum revenue rate and $\lambda_{\min} = \min_{S \subseteq \mathcal{S}^*, O^i \in S}\{\lambda_o^i(S)\}$ be the minimum arrival rate.

We can lower bound the expected value of the random time $\tau^{OT}(S)$ using the bound for the minimum of random variables from Aven (1985) by

$$
\begin{aligned}
\mathbb{E}\left[\tau^{OT}(S)\right] &\geq \min\{t^*(S), \min_{O^i \in S} \mathbb{E}[\tau_o^i(S)]\} - \sqrt{\frac{|S|}{|S|+1} \sum_{O^i \in S} \operatorname{Var}\left[\tau_o^i(S)\right]} \\
&\geq t^*(S) - \max_{O^i \in S} \lambda_o^i(S)^{-1} - \sqrt{t^*(S) \sum_{O^i \in S} \lambda_o^i(S)^{-1}} \\
&\geq t^*(S) - \lambda_{\min}^{-1} - \sqrt{t^*(S)(N+1)\lambda_{\min}^{-1}},
\end{aligned}
$$

where the second inequality follows from the fact that $\mathbb{E}\left[\tau_o^i(S)\right] = \lfloor \lambda_o^i(S) t^*(S) \rfloor / \lambda_o^i(S) \geq t^*(S) - 1/\lambda_o^i(S)$, and $\operatorname{Var}\left[\tau_o^i(S)\right] = \lfloor \lambda_o^i(S) t^*(S) \rfloor / \lambda_o^i(S)^2 \leq t^*(S)/\lambda_o^i(S)$.

Next, we bound the expected revenue of the offer time policy. Using the fact that $\tau^{OT}(S)$ is a bounded stopping time together with the previous bound we obtain that

$$
\begin{aligned}
R^{OT}(p) = \mathbb{E}\left[\sum_{S \subseteq \mathcal{S}} \sum_{O^i \in S} r_o^i X_o^i(S, \tau^{OT}(S))\right] &= \sum_{S \subseteq \mathcal{S}} \sum_{O^i \in S} r_o^i \lambda_o^i(S) \mathbb{E}\left[\tau^{OT}(S)\right] \\
&\geq \sum_{S \subseteq \mathcal{S}} r(S) t^*(S) - \sum_{S \subseteq \mathcal{S}} r(S)(\lambda_{\min}^{-1} + \sqrt{t^*(S)(N+1)\lambda_{\min}^{-1}}) \\
&\geq R^{CDLP}(p) - r_{\max}\lambda_{\min}^{-1}|\mathcal{S}^*| - r_{\max}\sqrt{(N+1)\lambda_{\min}^{-1}} \sum_{S \subseteq \mathcal{S}} \sqrt{t^*(S)} \\
&\geq R^{CDLP}(p) - r_{\max}|\mathcal{S}^*| \left(\lambda_{\min}^{-1} + \sqrt{(N+1)\lambda_{\min}^{-1}}\sqrt{T}\right),
\end{aligned}
$$

where the last inequality follows from the fact that $\sum_{S \subseteq \mathcal{S}} \sqrt{t^*(S)} \leq |\mathcal{S}^*| \sqrt{\sum_{S \subseteq \mathcal{S}} t^*(S)}$. We conclude by noting that $R^{OT}(p) \leq R^*(p)$, and using the fact that $\lambda_{\min}^{-1} \leq (N+1)T$. $\square$

Proof of Proposition 1

Proof. We first show that $R^{CDLP}(p) \leq R^{MBLP}(p)$ by showing that any solution of the CDLP can be used to construct a feasible solution to the MBLP with the same objective value. Let

$\{t(S)\}_{S\subseteq\mathcal{S}}$ be a feasible solution to the CDLP. First, using the decision variables given by (10), the total number of advance tickets sold can be written as

$$
\begin{aligned}
X_a &= \sum_{S\subseteq\mathcal{S}} t(S)\lambda_a(S) = \sum_{S\subseteq\mathcal{S}} t(S) \sum_{i=1}^{N} \Lambda^i \mathbf{1}_{\{A\in S\}}(\pi^i_{an} + \pi^i_{aon} + \mathbf{1}_{\{O^i\notin S\}}\pi^i_{oan}) \\
&= \sum_{i=1}^{N} \Big(\sum_{S\ni A, S\ni O^i} t(S)\Big)\Lambda^i(\pi^i_{an} + \pi^i_{aon}) + \Big(\sum_{S\ni A, S\not\ni O^i} t(S)\Big)\Lambda^i(\pi^i_{an} + \pi^i_{aon} + \pi^i_{oan}) \\
&= \sum_{i=1}^{N} t^i\left(\{A, O^i\}\right)\lambda^i_a\left(\{A, O^i\}\right) + t^i\left(\{A\}\right)\lambda^i_a\left(\{A\}\right) = \sum_{i=1}^{N}\sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda^i_a(S),
\end{aligned} \tag{20}
$$

where the second equality follows from (1), the third from exchanging summations, and the fourth from (1) again. Similarly, the number of option sold in market segment i can be written as

$$
\begin{aligned}
X^i_o &= \sum_{S\subseteq\mathcal{S}} t(S)\lambda^i_o(S) = \sum_{S\subseteq\mathcal{S}} t(S)\Lambda^i \mathbf{1}_{\{O^i\in S\}}\left(\pi^i_{on} + \pi^i_{oan} + \mathbf{1}_{\{A\notin S\}}\pi^i_{aon}\right) \\
&= \Big(\sum_{S\ni A, S\ni O^i} t(S)\Big)\Lambda^i(\pi^i_{on} + \pi^i_{oan}) + \Big(\sum_{S\not\ni A, S\ni O^i} t(S)\Big)\Lambda^i(\pi^i_{on} + \pi^i_{oan} + \pi^i_{aon}) \\
&= t^i\left(\{A, O^i\}\right)\lambda^i_o\left(\{A, O^i\}\right) + t^i\left(\{A\}\right)\lambda^i_o\left(\{A\}\right) = \sum_{S\subseteq\mathcal{S}^i} t^i(S)\lambda^i_o(S),
\end{aligned} \tag{21}
$$

where the second equality follows from (2), the third from exchanging summations, and the fourth from (2) again. Thus, the capacity constraint (14) is verified.

Second, the non-negativity constrains and the time-horizon length constraints (12) follow trivially. Next, for the advance selling market consistency constraints (13) notice that for all $i = 1, \ldots, N$ we have that

$$
t^i\left(\{A, O^i\}\right) + t^i\left(\{A\}\right) = \sum_{S\subseteq\mathcal{S}: A\in S, O^i\in S} t(S) + \sum_{S\subseteq\mathcal{S}: A\in S, O^i\notin S} t(S) = \sum_{S\subseteq\mathcal{S}: A\in S} t(S) = T_a.
$$

Thus, advance tickets are offered the same amount of time in all markets.

Finally, the next string of equalities show that both solutions attain the same objective value

$$\begin{aligned}\sum_{S\subseteq\mathcal{S}} r(S)t(S) &= \sum_{S\subseteq\mathcal{S}} r^T\lambda(S)t(S) = \sum_{S\subseteq\mathcal{S}}\sum_{i=1}^{N}\left(r_a\lambda_a^i(S)+r_o^i\lambda_o^i(S)\right)t(S)\\ &= \sum_{i=1}^{N}\sum_{S\subseteq\mathcal{S}^i} r_a\lambda_a^i(S)t^i(S)+r_o^i\lambda_o^i(S)t^i(S) = \sum_{i=1}^{N}\sum_{S\subseteq\mathcal{S}^i} r^i(S)t^i(S),\end{aligned}$$

where the third equality follows from (20) and (21).

Next, we show that $R^{CDLP}(p) \geq R^{MBLP}(p)$ by showing that any solution of the MBLP can be used to construct a feasible solution to the CDLP with the same objective value. Let $\left\{t^i(S)\right\}_{S\subseteq\mathcal{S}^i,i=1,\dots,N}$ be a feasible solution to the MBLP. In the following, we give a simple algorithm to compute a feasible solution $\{t(S)\}_{S\subseteq\mathcal{S}}$ for the CDLP.

First, we deal with offer sets containing advance tickets, and compute $t(S)$ for all $S\in\mathcal{S}$ such that $A\in S$. Let $[i]_{i=1,\dots,N}$ be the permutation in which teams are sorted in increasing order with respect to $t^i\left(\{A,O^i\}\right)$, i.e. $t^{[i]}\left(\{A,O^{[i]}\}\right)\leq t^{[i+1]}\left(\{A,O^{[i+1]}\}\right)$. Consider the following offer sets

$$\begin{aligned}S^{[i]} &= \left\{A,O^{[i]},O^{[i+1]},\dots,O^{[N]}\right\} \quad \forall i=1,\dots,N\\ S^{[N+1]} &= \{A\}\end{aligned}$$

and associated times $t\left(S^{[i]}\right)=t^{[i]}\left(\{A,O^{[i]}\}\right)-t^{[i-1]}\left(\{A,O^{[i-1]}\}\right)$ for all $i=1,\dots,N+1$, with $t^{[0]}\left(\{A,O^{[0]}\}\right)=0$, and $t^{[N+1]}\left(\{A,O^{[N+1]}\}\right)=T_a$. Since teams are sorted with respect to $t^i\left(\{A,O^i\}\right)$, we have $t\left(S^{[i]}\right)\geq 0$. Notice that this construction is valid because the market consistency constraints (13) guarantee that advance tickets are offered the same amount of time in all markets. Figure 5 sketches a graphical representation of the algorithm.

Next, we look at the intuition behind this construction. Although the order is not important, consider a solution for the CDLP that offers the sets $S^{[i]}$ in sequential order; it starts with $S^{[1]}$, then $S^{[2]}$, and so forth until $S^{[N+1]}$. Hence, at first it offers all products, then team 1's options are removed, then team 2's options are removed, and so forth until the end when only advance tickets are offered. Hence, the optimal policy has a nested structure.

Finally, a similar argument holds for offer sets not containing advance tickets. $\square$

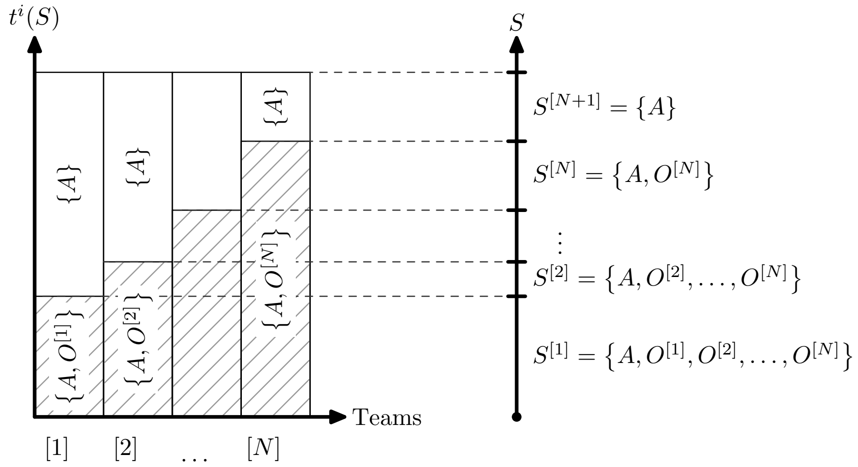

Figure 5: Computing a feasible solution for the CDLP (showed on the right) from a feasible solution from the MBLP (on the left) in the case of offer sets containing advance tickets.

Proof of Lemma 1

Proof. The derivative of the value rate w.r.t. λ_a is

$$\frac{dv}{d\lambda_a}(\lambda_a) = \bar{F_v}^{-1}(\lambda_a/\Lambda) - \frac{\lambda_a}{\Lambda} \frac{1}{f_v\left(\bar{F_v}^{-1}(\lambda_a/\Lambda)\right)}.$$

Composing the derivative with $\lambda_a(c) = \Lambda \bar{F}_v(c)$ we get

$$\left(\frac{dv}{d\lambda_a} \circ \lambda_a\right)(c) = \frac{dv}{d\lambda_a}\left(\Lambda \bar{F}_v(c)\right) = c - \frac{\bar{F}_v(c)}{f(c)} = c - \frac{1}{h(c)}.$$

IFR implies that the composite function is increasing in c. Since $\lambda_a(c)$ is decreasing, we conclude that the original derivative is decreasing and v is strictly concave. □

Proof of Proposition 2

Proof. First, we look at the case where the seats are scarce ($C < \lambda_a^* T$). In the advance ticket pricing problem (15) the organizer can afford to price higher, and prices at the run-out rate $\lambda_a^0 = C/T$, i.e., the intensity at which all seats are sold over the time horizon. Note that λ_a^0 is a constrained global optimum of the advance selling problem, and $v'(\lambda_a^0) > 0$. Starting from $(\lambda_a^0, 0)$ in the options pricing problem, we will study the impact of increasing the options' intensity on the revenue.

Clearly, $(\lambda_a^0, 0)$ is a feasible solution of (16). Since capacity is binding, to compensate for an increase in λ_o^Σ the organizer needs to decrease the intensity of advance tickets. Thus, from (17) we obtain that $\frac{d\lambda_a}{d\lambda_o^\Sigma} = -\frac{2}{N} = -q$. The total derivative of the objective w.r.t. λ_o^Σ is then

$$\frac{dR_o^D}{d\lambda_o^\Sigma} = \frac{\partial R_o^D}{\partial \lambda_o^\Sigma} + \frac{\partial R_o^D}{\partial \lambda_a}\frac{d\lambda_a}{d\lambda_o^\Sigma}. \tag{22}$$

Evaluating (22) at $(\lambda_a^0, 0)$ we obtain

$$\begin{aligned}\frac{dR_o^D}{d\lambda_o^\Sigma}(\lambda_a^0, 0) &= Tqv'(\lambda_a^0) - Tq\left((1-q)\ell + q\right)v'(\lambda_a^0) \\ &= Tq(1-q)(1-\ell)v'(\lambda_a^0) > 0.\end{aligned}$$

This implies that the current solution can be improved by introducing options.

Second, we consider the case where the capacity of the stadium is large ($C \geq \lambda_a^* T$). In the advance ticket pricing problem (15) the organizer ignores the problem of running out of seats and prices according to the revenue maximizing rate λ_a^*. Note that λ_a^* is an unconstrained global optimum, and thus $v'(\lambda_a^*) = 0$. We will show that $(\lambda_a^*, 0)$ is an optimal solution to the options pricing problem.

Clearly, $(\lambda_a^*, 0)$ is a feasible solution of (16). The gradient of the objective is

$$\begin{aligned}\frac{\partial R_o^D}{\partial \lambda_a} &= T(1-q)\ell v'(\lambda_a) + Tqv'(\lambda_a + \lambda_o^\Sigma), \\ \frac{\partial R_o^D}{\partial \lambda_o^\Sigma} &= Tqv'(\lambda_a + \lambda_o^\Sigma).\end{aligned}$$

Using the fact that $v'(\lambda_a^*) = 0$, we obtain that the gradient is zero at $(\lambda_a^*, 0)$. Hence, this solution is an unconstrained local optimum. Finally, the concavity of the program implies that any local optimum is a global optimum. Thus, both problems attain the same objective value, and $R_o^D = R_a^D$.

Third, we consider the case where fans are indifferent among teams ($\ell = 1$). Note that the objective functions of the advance selling and options pricing problems become $R_a^D(\lambda_a) = Tv(\lambda_a)$, and $R_o^D(\lambda_a, \lambda_o^\Sigma) = T(1-q)v(\lambda_a) + Tqv(\lambda_a + \lambda_o^\Sigma)$, respectively. Let $(\lambda_a, \lambda_o^\Sigma)$ be any feasible solution to (16). We will show $\tilde{\lambda}_a = \lambda_a + q\lambda_o^\Sigma$ is a feasible solution for (15) with greater revenue.

Clearly, $\tilde{\lambda}_a$ is feasible. Regarding revenues

$$\begin{aligned} R_a^D(\tilde{\lambda}_a) &= Tv\left(\lambda_a + q\lambda_o^{\Sigma}\right) \\ &= Tv\left((1-q)\lambda_a + q(\lambda_a + \lambda_o^{\Sigma})\right) \\ &\geq T(1-q)v(\lambda_a) + Tqv(\lambda_a + \lambda_o^{\Sigma}) = R_o^D(\lambda_a, \lambda_o^{\Sigma}), \end{aligned}$$

where the inequality follows from concavity of v. Thus, $\tilde{\lambda}_a$ always dominates the original solution.

□

Proof of Proposition 3

Proof. Let $R_a^D(\ell)$ and $R_o^D(\ell)$ be the optimal values of (15) and (16) as a function of ℓ, respectively. First, we show that the absolute benefit of introducing options decreases as ℓ increases. We proceed by showing that the difference $R_o^D(\ell) - R_a^D(\ell)$ is decreasing in ℓ. Notice that the objective function of both problems is convex as a function of ℓ. By the Maximum Theorem the functions $R_a^D(\ell)$ and $R_o^D(\ell)$ are convex, and differentiable almost everywhere. We proceed by calculating the total derivatives of $R_a^D(\ell)$ and $R_o^D(\ell)$ with respect to ℓ.

For the advance ticket pricing problem, we have that the optimal solution of (15) is λ_a^0 because $C < \lambda_a^* T$. Then, any change in ℓ does not affect the optimal solution and the derivative of $R_a^D(\ell)$ with respect to ℓ is given by

$$\frac{dR_a^D(\ell)}{d\ell} = T(1-q)v(\lambda_a^0). \tag{23}$$

For the options pricing problem, we have from the Envelope Theorem that the derivative of $R_o^D(\ell)$ with respect to ℓ is

$$\frac{dR_o^D(\ell)}{d\ell} = T(1-q)v(\lambda_a(\ell)), \tag{24}$$

where $\lambda_a(\ell)$ denotes the optimal arrival intensity for advance tickets in 16 for fixed ℓ.

A trivial consequence of the capacity constraint (17) is that $\lambda_a(\ell) \leq \lambda_a^0$. Additionally, because seats are scarce we have $\lambda_a^0 < \lambda_a^*$. Finally, because the value rate is increasing in $[0,\lambda_a^*)$ we conclude that $\frac{dR_a^D(\ell)}{d\ell} \geq \frac{dR_o^D(\ell)}{d\ell}$, and the difference is decreasing in ℓ.

For the relative benefit, we first write the ratio of revenues as $\frac{R_o^D(\ell)}{R_a^D(\ell)} = 1 + \frac{R_o^D(\ell) - R_a^D(\ell)}{R_a^D(\ell)}$. From (23) it is clear that $R_a^D(\ell)$ is increasing in ℓ, and the result follows. $\square$

Proof of Proposition 4

Proof. Observe that since capacity is scarce, the optimal solution of the advance ticket pricing problem (15) is the run-out rate $\lambda_a^0 = C/T$, and it is independent of the number of teams. Let $\{(\lambda_a^{(N)}, \lambda_o^{(N)})\}_N$ be a sequence of optimal solutions to the advance ticket and options pricing problem (16) indexed by the number of teams. Scarcity of seats together with concavity guarantee that the capacity constraint (17) is binding at the optimal solution. Since intensities are bounded from above by Λ, this guarantees that $\lim_{N\to\infty} \lambda_a^{(N)} = \lambda_a^0$. As a side note, it is not necessarily the case that $\lambda_o^{(N)}$ converges to zero as N goes to infinity.

Second, we show that the following inequality holds

$$\lambda_a^{(N)} \leq \lambda_a^0 \leq \lambda_a^{(N)} + \lambda_o^{(N)} \leq \lambda_a^*. \tag{25}$$

The first inequality is a trivial consequence of the capacity constraint (17). For the second inequality observe that the capacity constraint (17) is binding, and thus $\lambda_a^0 = \lambda_a^{(N)} + \frac{2}{N}\lambda_o^{(N)} \leq \lambda_a^{(N)} + \lambda_o^{(N)}$. For the third inequality suppose that $\lambda_a^{(N)} + \lambda_o^{(N)} > \lambda_a^*$ for some N, and consider an alternate solution in which the options' intensity is decreased to $\tilde{\lambda}_o^{(N)} = \lambda_a^* - \lambda_a^{(N)}$. Clearly, $\tilde{\lambda}_o^{(N)} \geq 0$, and the new solution satisfies the capacity constraint and the third inequality. Moreover,

$$\begin{aligned} R_o^D\left(\lambda_a^{(N)}, \lambda_o^{(N)}\right) &= T(1 - \frac{2}{N})\ell v(\lambda_a^{(N)}) + T\frac{2}{N} v(\lambda_a^{(N)} + \lambda_o^{(N)}) \\ &\leq T(1 - \frac{2}{N})\ell v(\lambda_a^{(N)}) + T\frac{2}{N} v(\lambda_a^*) = R_o^D\left(\lambda_a^{(N)}, \tilde{\lambda}_o^{(N)}\right), \end{aligned}$$

where the first inequality follows since λ_a^* is the least maximizer of v. Thus, the new solution is also optimal. This shows that if the third inequality does not hold for any N, we can construct a solution $\left(\lambda_a^{(N)}, \tilde{\lambda}_o^{(N)}\right)$ for which it holds. So, without loss of generality, we can conclude that the third inequality holds.

So, the ratio of optimal revenues can be written as

$$\begin{aligned}
\frac{R_o^D\left(\lambda_a^{(N)},\lambda_o^{(N)}\right)}{R_a^D(\lambda_a^0)} &= \frac{T(1-\frac{2}{N})\ell v(\lambda_a^{(N)}) + T\frac{2}{N}v(\lambda_a^{(N)}+\lambda_o^{(N)})}{T\left[(1-\frac{2}{N})\ell + \frac{2}{N}\right]v(\lambda_a^0)} \\
&= \frac{N\ell-2\ell}{N\ell+2(1-\ell)}\frac{v(\lambda_a^{(N)})}{v(\lambda_a^0)} + \frac{2}{N\ell+2(1-\ell)}\frac{v(\lambda_a^{(N)}+\lambda_o^{(N)})}{v(\lambda_a^0)} \\
&\le \frac{v(\lambda_a^{(N)})}{v(\lambda_a^0)} + \frac{2}{N\ell}\frac{v(\lambda_a^{(N)}+\lambda_o^{(N)})}{v(\lambda_a^0)} \le 1 + \frac{2}{N\ell}\frac{v(\lambda_a^*)}{v(\lambda_a^0)},
\end{aligned}$$

where the second equation is obtained by algebraic manipulation, the first inequality follows from bounding the leading factor of the first term by 1 and the leading factor of the second term by $\frac{2}{N\ell}$, and the second inequality follows from 25 together with the fact that $v(.)$ is non-decreasing in $[0, \lambda_a^*]$. □

Proof of Lemma 3

Proof. For the only if, take any $\alpha \in \mathcal{C}$ and observe that

$$\sum_{i\in\mathcal{T}_1}\alpha^i = \sum_{i\in\mathcal{T}_1}\sum_{s\in\mathcal{T}}(\Lambda)_{is}y_s = \sum_{i\in\mathcal{T}_1}\sum_{j\in\mathcal{T}_2}y_{(i,j)} = \sum_{j\in\mathcal{T}_2}\sum_{i\in\mathcal{T}_1}y_{(i,j)} = \sum_{j\in\mathcal{T}_2}\sum_{s\in\mathcal{T}}(\Lambda)_{js}y_s = \sum_{j\in\mathcal{T}_2}\alpha^j.$$

For the if part we proceed by contradiction. If $\alpha = 0$ the result is trivial, so we assume that $\alpha \neq 0$. Since the cone $\mathcal{C}$ is closed and convex and $\alpha \notin \mathcal{C}$, by the Strictly Separating Hyperplane Theorem there exists an hyperplane that strictly separates them Boyd and Vandenberghe (2004). Alternatively, there is a vector $q \in \mathbb{R}^N$ such that $q^T\alpha < q^T\Lambda y$ for all $y \geq 0$.

Pick any $i' \in \mathcal{T}_1$, and set y such that $y_{(i,j)} = \alpha_j$ if $i = i'$ and 0 otherwise. Evaluating the right hand side at y we get

$$q^T\Lambda y = \sum_{i\in\mathcal{T}_1}\sum_{j\in\mathcal{T}_2}(q_i+q_j)y_{(i,j)} = \sum_{j\in\mathcal{T}_2}(q_i'+q_j)\alpha_j = q_i'\sum_{j\in\mathcal{T}_2}\alpha_j + \sum_{j\in\mathcal{T}_2}q_j\alpha_j = \sum_{i\in\mathcal{T}_1}q_i'\alpha_i + \sum_{j\in\mathcal{T}_2}q_j\alpha_j,$$

where the last equality follows from the hypothesis. Hence, $\sum_{i\in\mathcal{T}_1\setminus i'}(q^i - q^{i'})\alpha_i < 0$ from the separating hyperplane theorem, and we conclude that $q^{i''} < q^{i'}$ for some $i'' \in \mathcal{T}_1 \setminus i'$ since $\alpha \geq 0$. Repeating the argument with i'' we obtain that $q^{i'''} < q^{i''} < q^{i'}$ for some $i''' \in \mathcal{T}_1 \setminus i''$. By repeatedly applying the same argument we reach a contradiction. □

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