Framework Agreements in Procurement:
An Auction Model and Design Recommendations

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Abstract
Framework agreements (FAs) are procurement mechanisms commonly used by buying agencies around the world to satisfy demand that arises over a certain time horizon. This paper is one of the first in the literature that provides a formal understanding of FAs, with a particular focus on the cost uncertainty faced by bidders over the FA time horizon. We introduce a novel model that generalizes standard auction models to include this salient feature of FAs; we analyze this model theoretically and numerically. First, we show that FAs are subject to a sort of winner’s curse that induces higher expected buying prices relative to running first-price auctions as needs arise. Then, our results provide concrete design recommendations that alleviate this issue and decrease buying prices in FAs, highlighting the importance of (i) monitoring the price charged at the open market by the FA winner and using it to bound the buying price; (ii) investing in implementing price indexes for the random part of suppliers’ costs; and (iii) allowing suppliers the flexibility to reduce their prices to compete with the open market throughout the selling period. These prescriptions are already being used by the Chilean government procurement agency that buys US$2 billion worth of contracts every year using FAs.

Keywords: procurement, supply chain management, auction theory, mechanism design

1 Introduction

1.1 Background and motivation
Governments around the world spend billions of dollars every year buying a wide range of products and services from private firms. While standard auctions are often used to allocate contracts in these procurement processes, recent years have seen a tremendous increase in the adoption of an alternative class of mechanisms in various public procurement settings: the so-called framework agreements (FAs), also called indefinite-delivery/indefinite-quantity (IDIQ) contracts in the United

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States. FAs award tens of billions of dollars worth of contracts annually around the globe and constitute a steadily increasing fraction of governments’ procurement processes. For example, FAs awarded €85 billion in 2010 in the European Union only, accounting for 17% of the total value of all contracts awarded, and their use has increased in the EU at an average rate of 18% since 2006.¹

Broadly speaking, FAs are anticipated arrangements for the delivery of goods and services over a certain period of time.² There are important characteristics of FAs that intrigue practitioners and this paper is one of the first in the literature that provides a more formal understanding of them. More specifically, in this work we introduce a novel auction model to study one salient feature of FAs and use it to propose concrete design recommendations. This work is the result of a collaboration with the Chilean government procurement agency Dirección ChileCompra (ChileCompra for short) that buys around 10 billion dollars worth of products and services every year, of which 2 billion are bought using FAs.

**What are FAs?** Consider a government that is interested in buying computers for its public agencies (e.g., school, hospitals, ministries) along a time horizon of two/three years. During this time horizon, many demand requests can be expected from various government agencies. On one hand, running an auction whenever such a request arises is too expensive administratively, especially considering that these requests may be frequent and of a small volume (e.g., few laptop computers for a rural school). On the other hand, letting each agency run its own procurement process does not exploit the central government’s bargaining power and buying know-how. FAs are meant to strike a balance between a decentralized procurement process and the central government’s bargaining power. Suppliers, on their end, are motivated to participate in FAs due to the large demand that is typically associated with them.

A typical FA is composed of two stages: in the first (the **auction** stage), an auction-type mechanism, typically first-price, takes place to select one supplier as the **FA winner** for a given product or service. The FA winner is required to sell over the time horizon of the FA at the bid price determined at this first stage. In the second (the **buying** stage), the various government agencies may buy the product from the FA winner as needs arise. It is common that government agencies have the obligation to buy from the FA winner unless they can provide evidence of a more convenient procurement option in the open market (that we also refer to as the outside or the spot market).


²The European Parliament defined FAs as “an agreement between one or more contracting authorities and one or more economic operators, the purpose of which is to establish the terms governing contracts to be awarded during a given period, in particular with regard to price and, where appropriate, the quantity envisaged”. (The Directive 2004/18/EC of the European Parliament and of the council of March 31, 2004.)
market). Overall, an FA can be viewed as a *government call option* to buy at a predetermined price over the time horizon.\(^3\)

In this paper we focus on one distinctive feature that FAs exhibit relative to running a standard first-price auction whenever a need arises. When participating in the auction stage, bidders face significant cost uncertainty; while in FAs the price of a product or service is locked at the beginning of the time horizon, the suppliers’ costs may change over that period. Anecdotal and empirical evidence suggests that providers “charge” for this uncertainty through higher bids. Guillermo Burr, head of the Research Department at ChileCompra, says “we wanted to enhance our understanding of why in some categories standard auctions resulted in lower prices relative to FAs. Further, we wanted to understand how to alleviate this problem and reduce ChileCompra’s buying prices.”

1.2 Main contributions

Despite their practical importance, there is little academic research on FAs. In the current paper we develop a model for FAs that considers the *cost uncertainty* faced by suppliers and the resulting bidding incentives. Our paper contributes to the literature on procurement mechanisms in the interface between operations and economics, and at the same time has concrete practical implications.

The contributions of the paper can be categorized along the following three dimensions.

**Modeling and bidding incentives.** We introduce an auction model for FAs that generalizes standard auction models to incorporate the cost uncertainties faced by suppliers in FAs, and characterizes their bidding incentives. Specifically, in our model all suppliers face a common cost that is unknown at the auction stage and gets realized when providing the good (for instance, the cost of gas in supplying a transportation service). At the buying state the procurement agency has the option to buy from the FA at the agreed price given by the auction, or if it is more convenient, she can buy from the open market. The spot market price has a similar structure to the bidders’ costs; in particular, it also incorporates the common cost component, as an outside provider may also need to pay this cost to provide the good.

Given this structure, we identify that FAs are subject to a sort of *winner’s curse* (Krishna 2002), because the events under which the FA winner sells the product are positively correlated with high cost realizations. Intuitively, the FA option is exercised when the locked-in price is attractive relative to the spot market price; this coincides with large cost realizations for suppliers. The *FA curse* formalizes the practitioners’ intuition that FAs may result in larger prices relative to running standard first-price procurement auctions as needs arise.

\(^3\)Additional details on FAs can be found in Albano and Sparro (2010).
Analysis. We study the Bayes Nash equilibrium (BNE) of the game of incomplete information between sellers induced by the FA auction. We begin by analyzing the integral equations or the ordinary differential equations that characterize the equilibria of the different models; unfortunately, these do not have closed-form solutions. Rather, we compare the expected buying prices among the different mechanisms we consider (different variants of FAs, as well as first-price auctions) using an envelope theorem approach. By itself, the latter analysis has some novelties relative to standard mechanism design, as the outside option given by the spot market price is endogenous and depends on suppliers’ private information. We complement our theoretical results by supporting numerical experiments that demonstrate the robustness of our findings.

Design prescriptions. We show that monitoring the price offered by the FA winner in the outside market and forcing the FA winner to match it whenever it is lower than her winning bid, significantly reduces expected buying prices. Hence, governments may capture significant value in monitoring prices charged by FA winners in the open market. Monitoring FAs are not always used by procurement agencies, however, because in practice the costs associated to monitoring prices in the outside market may be large, specially considering that thousands of products are bought using FAs.

As an alternative, we show that using a perfect price index (PPI) in which the auctioneer perfectly observes the realization of the common cost and indexes the bid of the FA winner to its changes lowers the expected buying prices in many settings of practical interest. Hence, if possible, governments should make an effort to invest in finding and implementing price indexes for the random common part of suppliers’ costs.

The previous results suggest that using PPIs is useful; such price indexes are available in practice when the common cost is a commodity such as gas. However, for many of the goods and services procured with FAs (e.g., computers, office equipment) such indexes do not exist or are hard to build. Thus motivated, we study a variation of the standard FA, the flexible FA (FLE). Here, if the FA winner has a bid larger than the spot market price offered by a different supplier, the FA winner is allowed to match it and sell the product. We show that the flexible FA achieves expected buying prices that are often comparable with the PPI FA and with running first-price auctions as needs arise. The flexible FA is very practical as it does not require monitoring costs nor PPIs.

These prescriptions are already being applied by ChileCompra to improve the design of their FAs. Mr. Burr says, “These results have provided important insights regarding the design of our FAs. They have encourage our FA department to make larger efforts to build adequate price indexes. They have also motivated a larger effort in policing spot market prices from FA winners,
by allowing buyers to report low spot market prices when they observe them. They have also showed us the types of pricing flexibility that we should encourage.”

1.3 Related literature

As previously mentioned, our work is related to classic work in common value auctions and the associated ‘winner’s curse.’ More specifically, due to the cost structure, our FA model is similar to auction models with both private and common values; for more details see, e.g., Goeree and Offerman (2003). In addition, because the spot market price plays a similar role to a random reserve price, our analysis share some similarities with Elyakime et al. (1994).

Our work relates to a growing stream of work in operations that studies supply chains and procurement processes under different forms of uncertainty. In particular, various studies have focused on demand uncertainty that is faced by buyers and/or suppliers: Chen (2007) and Duenyas et al. (2013) study optimal procurement mechanisms in a newsvendor-like setting where a buyer facing uncertain demand determines both the quantity and purchasing price through interactions with suppliers. Li and Scheller-Wolf (2011) consider a buyer facing uncertain demand and suppliers that need to invest in capacity before the demand uncertainty is resolved, and study whether the buyer should offer a pull or push contract. On related work, Zhang (2010) also studies a procurement mechanism in a supply chain setting, but includes supplier delivery performance and price-sensitive market demand.

In addition, Schummer and Vohra (2003) study the mechanism design problem of a buyer that can procure purchase options from capacity constrained sellers to satisfy an unknown future demand. The focus of their work is to study how options can be used to hedge against random demand when suppliers have a cost associated to reserving capacity. Instead, in our case FAs can be viewed as government call options to lock-in a price when suppliers’ costs fluctuate. More broadly, while these operations papers relate to demand uncertainty, the focus of our work is on studying the impact of bidders’ cost uncertainty. In that respect, our work also relates to the one of Elmaghraby and Oh (2013) that study, in a very different setting, how to structure two sequential auctions in the presence of learning-by-doing, and whether the buyer is better-off by limiting competition and contracting with a single supplier in the hope of extracting a better future price. These questions are at some level related to designing the structure of competition with the spot market price in the buying stage of FAs (e.g., whether to allow the flexibility to match it or not). In another related paper, Tunca and Zenios (2006) provide conditions under which procurement auctions are preferred over relational contracts, as similarly to FAs they identify the most cost efficient supplier.
As previously mentioned, the literature that directly studies FAs is limited. Subsequent to the first version of this paper, Saban and Weintraub (2015) studied another distinctive feature of FAs, namely how to optimize the trade-off between product variety and price competition. Their auction and mechanism design analysis considers an FA model with multiple imperfect substitute products, but ignores suppliers’ cost uncertainty. In this sense, the two papers are complementary as they study different aspects of FAs. In this paper we abstract away from the main features studied in Saban and Weintraub (2015) and other complexities that arise in FAs to focus on the cost uncertainty faced by suppliers. However, we believe our design prescriptions are valid in these more general settings as well. Finally, we note that we have presented preliminary results of this work in a conference paper that appeared in a practitioners’ outlet (Anonymous 2012). However, that paper studies a simpler model and does not contain the theoretical results that support the managerial insights and design prescriptions of the current work.

Structure of paper. The rest of the paper is structured as follows. In §2 we present an auction-based model for FAs. In §3 we prove the existence of symmetric BNE and develop BNE bidding function in FAs for important cases of practical interest, quantifying the impact of the “FA curse” on the bidding behavior. In §4 we compare the expected buying prices among the various mechanisms using an envelope theorem approach. In §5 we introduce the flexible FA, and study its performance. In §6 we provide numerical analysis that complements the theoretical results of the previous sections. In §7 we conclude with key design prescriptions. Selected proofs appear in Appendix A. Additional proofs and auxiliary results are deferred to Appendices B, C, D, and E, which appear in an online companion.

2 The Framework Agreement Model

We model a framework agreement (FA) as a game of incomplete information between suppliers, similarly to the classical modeling approach in auction theory (see, e.g., Milgrom (2004) and Krishna (2002)). Consider a buyer (or auctioneer) that is interested in procuring a product/service to satisfy demand over an horizon of \( T \) time periods. We assume that demand quantities are a sequence of independent and identically distributed (i.i.d.) random variables that are independent of all other random quantities subsequently defined. To simplify the notation and without loss of generality we normalize the expected demand at each time step to be equal to one. To simplify notation we also assume that agents do not discount future payments; our analysis can be easily extended to

\footnote{Albano and Sparro (2008) studies a similar trade-off to Saban and Weintraub (2015) but in a simpler model of complete information.}
accommodate discounting.

**Suppliers and cost structure.** We denote by $\mathcal{M}$ the set of $M$ risk neutral potential suppliers that could provide the good or service being procured. The set of potential suppliers is assumed to be fixed throughout the FA horizon. The (constant) marginal cost faced by bidder $i \in \mathcal{M}$ when providing the good at time period $t \in \{1, \ldots, T\}$ is given by the sum of two components: $c_i + X_t$. The first component, $c_i$, is bidder $i$’s private cost, known only to himself at $t = 0$. The private costs $\{c_i : i \in \mathcal{M}\}$ are assumed to be i.i.d. random variables, each with distribution function $F$, continuous density function $f$, and finite, non-negative support $[c, \bar{c}]$. The second cost component, $X_t$, is common to all bidders and its realization for all $t \geq 1$ is unknown at $t = 0$. The marginal distribution function of $X_t$ is denoted by $F_{X_t}$, and has a continuous density function denoted by $f_{X_t}$, and a finite, non-negative support $[\underline{x}, \bar{x}]$, for all $t \in \{1, \ldots, T\}$. The stochastic process $\mathcal{X} = \{X_t : t \geq 1\}$ is independent of the private costs $\{c_i : i \in \mathcal{M}\}$.

The private cost $c_i$ represents idiosyncratic characteristics of supplier $i$, such as his managerial ability, logistics and production costs, or technology, that for the most part do not change over time. On the other hand, the cost component $X_t$ is common, and we interpret it as being related to the price of inputs that all suppliers require in order to provide the product/service. The process $\mathcal{X}$ is random because these prices may change over the time horizon. To illustrate this cost structure, consider for example the provision of a transportation service at period $t \geq 1$. Costs associated with the logistics of the firm and its transportation network are private and assumed to be fixed over time; these costs are represented by $c_i$. On the other hand, the costs of some inputs such as gas are common to all firms and subject to random fluctuations between $t = 0$ and time period $t$; these costs are represented by $X_t$.

**Auction stage.** At time $t = 0$, we assume that a subset $\mathcal{N} \subseteq \mathcal{M}$ of $N$ suppliers participate in the auction stage. We do not model participation decisions at the entry stage explicitly; for simplicity, we take the participants of the auction as exogenously given. For example, it may be that among the $M$ suppliers, the $N$ that participate in the auction stage are the ones with the lowest private cost realizations; in such case the FA attracts the most cost efficient providers. Alternatively, it may be that a random sample of $N$ suppliers among the $M$ potential suppliers participates in the auction stage, or that simply $\mathcal{N} = \mathcal{M}$. We remain agnostic about this entry process, unless otherwise stated. At the auction stage suppliers simultaneously submit sealed bids and the lowest

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5This may be a particularly reasonable assumption for relatively mature products or services, such as transportation services or office supplies. While in reality, a fraction of the private costs may also be subject to uncertainty between $t = 0$ and some period $t$, we abstract away from this effect, focusing on the impact of the common cost uncertainty.
bid wins; we refer to the supplier with the lowest bid as the FA winner.

**Buying stage.** The FA winner is committed to sell the good at every period \( t \geq 1 \) of the time horizon if required to do so. More specifically, at any time period \( t \geq 1 \), after the realization of the common cost \( X_t \), the buyer has the option to buy from the FA winner at a price equals to the winning bid. As an alternative, she can also buy from a spot or open market representing an outside option for the procuring organization. We assume that at every \( t \geq 1 \) the buyer buys from the outside option if and only if the realized spot market price is lower than the winning bid.

**Market structure and outside option.** The price which is used by the procuring agency to compare the winning bid against is driven by the prices that are charged in the spot market. We assume this market consists of the aforementioned set \( \mathcal{M} \) of potential suppliers, and that the terms of the FA mechanism do not affect prices in the open market.\(^6\)

Our main objective is to understand bidding incentives and provide design recommendations at the auction stage. For this purpose, we introduce a spot market pricing model that has some richness, but that abstracts away from strategic considerations in the spot market to preserve tractability in the analysis of the auction stage. Specifically, we assume that the spot market price of supplier \( i \in \mathcal{M} \) at time \( t \geq 1 \) is given by \( \alpha (c_i + X_t + Z_{i,t}) \), where \( \alpha \geq 1 \) is a constant, and \( Z_{i,t} \) is an additive markup charged by firm \( i \) at time \( t \). We assume that for each \( i \in \mathcal{M} \) the markups \( \{Z_{i,t} : t \geq 1\} \) are unknown at time period \( t = 0 \) and that \( Z_{i,t} \) gets realized at time \( t \). While we do not model the price formation process explicitly, we view it as a reduced-form of the outcome of competition among the set of potential suppliers in the open market, which can be affected by changing market conditions. The parameter \( \alpha \) captures a common multiplicative markup. We believe this structure provides a reasonable compromise between realism and tractability, specially considering that our main focus is the auction outcome. Further, as we discuss below, the analysis is challenging even with this spot market specification.

To simplify the analysis we assume that for all \( t \in \{1, \ldots, T\} \), the random vector \( \mathbf{Z}_t = (Z_{1,t}, \ldots, Z_{M,t}) \) is independent from the common cost \( X_t \), and private cost vector \( \mathbf{c} \). (We use boldfaces to denote vectors throughout the paper.) However, our model does not rule out time correlation on the joint process \( \{(\mathbf{Z}_t, X_t), t = 1, 2, \ldots, T\} \). For each \( t \in \{1, \ldots, T\} \), we allow correlations among the random variables \( \{Z_{i,t}, i \in \mathcal{M}\} \), but assume they share the same marginal

\(^6\)This assumption is reasonable for most products bought by ChileCompra based on the observation that the volume of sales transacted through FAs is typically significantly smaller than the overall volume transacted in the spot market. While there are exceptions (for example, in Chile the majority of the volume associated with certain health services such as dialysis is transacted through FAs), in most of FA categories (e.g., electronic devices, furniture, food, office supplies), the volume transacted through FAs is indeed much smaller than the overall national transacted volume.
distribution function $F_{Z_t}$ with continuous density function denoted by $f_{Z_t}$, and finite, non-negative support $[z, \bar{z}]$.

Given suppliers’ prices at the open market, the outside option that the FA winner price is compared against by the buyer is determined by the process by which a supplier is sampled from the open market. We give special attention to the following cases of interest, that are summarized in Figure 1. On one extreme is the monitored FA, in which the outside option is always determined

![Figure 1: Cases of interest for spot market price.](image)

by the price charged in the open market by the FA winner. Monitoring the spot market price of the FA winner is an approach taken by buying agencies that play a pro-active role not only in the initial tendering stage at $t = 0$, but also in providing outside options at the buying stage. For example, the Korean procurement agency has embraced this approach and is continuously monitoring FA winners’ prices in the open market (Kang 2013). The motivation to monitor the spot market price of the FA winner and use it as the outside option is that if both bids and outside prices are ordered according to costs (i.e, more efficient suppliers are more competitive in the auction and in the spot market), then the FA winner may exhibit the lowest spot market price among all FA participants. Further, if the most cost efficient suppliers participate in the FA, then the FA winner exhibits the lowest spot market price among all suppliers.

Due to the high costs associated to monitoring prices of the many good being bought with FAs, ChileCompra (and other agencies) have not pro-actively monitored spot market prices of FA winners. In this case, there is not a focal supplier selected as the outside option. An alternative, perhaps more realistic model for this situation may be the naive FA, in which the spot market price is randomly sampled from the set of potential suppliers. We view this model as representing a situation in which the agency buying through the FA (e.g., school, hospital, etc.) select an outside option based on idiosyncratic reasons, such as geographical proximity.

When considering the naive FA it is worthwhile to distinguish two different kinds of markets:
(i) a concentrated market, in which $M$ is a relatively small number and each bidder has a relatively large probability of being sampled as the outside option in the buying stage); and (ii) a diffused market, in which $M$ is very large and each bidder has a relatively small probability of being sampled as the outside option in the buying stage. The histogram in Figure 2 includes a distribution of the number of suppliers that compete in FAs in Chile (note that the histogram depicts numbers of FA participants, which are a lower bound for the number of potential suppliers). Indeed, many FA markets in Chile are composed by numerous suppliers, while many other markets are concentrated. In Chile, food and computers are representative examples of diffused markets, while examples of concentrated markets include airline tickets and dialysis services.

Figure 2: The histogram summarizes the number of competitors in 83 different product categories of FAs taken place between 2007 and 2011 in Chile. We note that in some of these FAs, in particular those with tens of suppliers, it may be the case that some of them only submit bids for a subset of products/services auctioned in the FA. Source: Dirección ChileCompra.

The above cases of interest will be analyzed in the following sections, where we will see that these distinctions impact both bidding incentives as well as expected payments for the procuring agency.

Bayes-Nash equilibrium. Following the standard auction framework, we use pure strategy Bayes-Nash equilibrium (BNE) as solution concept of the game of incomplete information between the $N$ suppliers that participate in the FA (see, e.g., Mas-Colell et al. (1995)). The bidding strategy of $i$ is a mapping $\beta_i : [\underline{c}, \bar{c}] \to A$, where $A$ is an interval of $\mathbb{R}_+$ unless otherwise stated. We denote by $\beta_{-i} = \{\beta_j, j \neq i\}$ as the vector of bidding strategies of $i$’s competitors.

Comment. We note that our model has abstracted away from various complexities that arise in real-world FAs, such as the existence of many products and having more than one winner at $t = 0$. Our objective is to introduce a model that studies one important and salient feature of FAs: the cost uncertainty faced by bidders over the FA time horizon.
3 Analysis of BNE Bidding Strategies

In §2 we distinguished between naive and monitored FAs, where the difference between the two lies in the mechanism used to construct the outside option with which the FA winning bid is compared in the buying stage. Both cases represent realistic practical settings. In the case of naive FAs we also distinguished between diffused and concentrated markets, where the difference between the two lies in the extent of competition in the open market. As we mentioned, all of these regimes are practically relevant.

In this section we characterize BNE bidding functions for the above cases of interest, and discuss their main elements to provide intuition on suppliers’ bidding incentives in FAs. For the sake of concreteness we focus on the case $T = 1$; in §4 we will return to consider the general case $T \geq 1$, when analyzing expected buying prices. In this section we study symmetric, continuous, and strictly increasing BNE strategies. Throughout this section we assume the existence of such equilibria, and at the end of the section we provide conditions under which such equilibria actually exist.

3.1 Naive FA

In the naive FA to construct the outside option in the buying stage, a supplier is randomly sampled from the $M$ potential suppliers in the spot market. We assume that this sampling is independent of all other random quantities previously defined. The spot market for supplier $i$ at time $t = 1$ is given by $\alpha(c_i + X + Z_i)$, where we ignored the time index because we are considering $T = 1$.

3.1.1 Diffused Markets

In a diffused market the number of suppliers that compete in the open market from which an outside option is drawn at the buying stage, $M$, is very large, while the number of firms participating in the FA, $N$, remains finite (and potentially small). Hence, the chance that a given FA participant is sampled at the spot market stage becomes negligible. Formally, we assume that each potential supplier $i \in \mathcal{M}$ is sampled at the spot market independently from any other random quantity with probability $q_{i,M}$, with $q_{i,M} \to 0$ as $M \to \infty$ for any $i \in \mathcal{M}$, and $\sum_{i=1}^{M} q_{i,M} = 1, \forall M$.

We denote by $c_{i(1:N),-i}$ the lowest order statistic among the costs of the suppliers that compete with $i$ in the auction stage. Given a strictly increasing competitors’ strategy $\beta$, the expected profit of seller $i$ with private cost $c_i$ and bid price $b_i$ in the naive FA with diffused markets is given by:
\[
\pi_i^{NDFA}(b_i, c_i, \beta) = \lim_{M \to \infty} E \left[ \mathbb{I}\{b_i \leq \beta(c_{i(1:N)}, -i)\} \cdot \left( \sum_{j=1}^{M} q_{j,M} \mathbb{I}\{b_i \leq \alpha(c_j + X + Z_j)\} \right) \cdot (b_i - c_i - X) \right]
= \bar{F}^{N-1}(\beta^{-1}(b_i)) \mathbb{E}\{b_i \leq \alpha(c_j + X + Z_j)\} \cdot (b_i - c_i - X),
\]

for any \(i \in \mathcal{N}\), where \(\bar{F}(x) = 1 - F(x)\), and \(\mathbb{I}\{\cdot\}\) denotes the indicator function. The first equation follows because the bidder sells the good if he defeats his competitors at the auction stage and also the randomly sampled spot market price in the buying stage. Further, because \(M \to \infty\), the bidder ignores the event in which himself is sampled in the spot market, and the cost of the sampled supplier in the spot market \((c_j)\) becomes independent from the costs of the FA participants. Note that at the time the bid is submitted at \(t = 0\), \(X\) and \(Z\) are random from the bidder’s perspective, and \(Z_j\)’s share the same marginal distributions.

To study the equilibrium strategies we could solve the ordinary differential equation (ODE) derived from the first-order condition associated to the maximization of the profit function above. However, differently to standard first price auctions, to the best of our knowledge, this ODE does not have a closed-form solution, because of the presence of the random spot market price. Instead, we derive the integral equation that describes the BNE strategy, by using the envelope theorem (see, e.g., chapter 4 in Milgrom 2004 for a treatment of this approach). This equation provides more intuition regarding the equilibrium relative to the ODE. We have the following result.

**Proposition 1. (BNE bids in the naive FA with diffused markets)** Suppose that \(T = 1\). Let \(\beta^{NDFA}(\cdot)\) be a strictly increasing, continuous, and symmetric BNE strategy profile for the naive FA in diffused markets. Then, \(\beta^{NDFA}(\cdot)\) satisfies the following integral equation for all \(c_i \in [\underline{c}, \bar{c}]\):

\[
\beta^{NDFA}(c_i) = c_i + \mathbb{E}\left[ X \mathbb{I}\{\beta^{NDFA}(c_i) \leq \alpha(c_j + X + Z_j)\} \right] \cdot \frac{\int_{c_i} f_{\bar{c}}^{E^{N-1}(\cdot)}(c) \cdot \mathbb{P}\{\beta^{NDFA}(c) \leq \alpha(c_j + X + Z_j)\} \, dc}{\mathbb{P}\{\beta^{NDFA}(c_i) \leq \alpha(c_j + X + Z_j)\}}.
\]

We refer to the first two components (A) of the right-hand-side of the equation above as the **implied cost**: the expected cost of the bidder, conditional on offering a better price than the spot market. We refer to the third term (B) as the **markup** the bidder charges on top of the implied cost for having a private cost lower than \(\bar{c}\). The last term is similar to the “information rent” term in a standard first price auction; however, in our model the bidder needs to defeat not only all the other bidders but also the spot market. Regarding term (A), one may obtain:

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\(^7\)Elyakime et al. (1994) provide a related discussion when analyzing a first-price auction with a random reserve and derive a somewhat similar integral equation to the one below.
for any realized private cost $c_i$. This expression captures an important feature of FAs we refer to as the FA curse for its similarity with the winner’s curse in common value auctions (Krishna 2002). The FA curse captures the fact that for the FA winner selling the good is in some sense ‘bad news’, because the selling event is positively correlated with the event in which his costs are high. This is driven by the dependence of both the spot market price and the supplier’s costs on $X$. Hence, in equilibrium bidders anticipate this effect and instead of charging the expected value of $X$, they charge the conditional expectation, which is larger.

To further emphasize the role of this positive correlation in the outcome of the FA, we next describe the different scenarios the FA winner faces depending on the outcome of $X$ (these scenarios are illustrated in Figure 3). For a given realization $c_j + z_j$, the spot market price is $\alpha X + \alpha (c_j + z_j)$.

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<th>$X$ Realization Scenario</th>
<th>Graphic Description</th>
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<td>Moderate $X$ realization</td>
<td>$c_i + X$</td>
<td>$i$ wins with positive margin</td>
</tr>
<tr>
<td></td>
<td>Market price</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_i$</td>
<td></td>
</tr>
<tr>
<td>High $X$ realization</td>
<td>$c_i + X$</td>
<td>$i$ wins with negative margin</td>
</tr>
<tr>
<td></td>
<td>Market price</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_i$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3: The “FA curse”. Conditional on a realization $c_j + z_j$, the market price is $\alpha X + \alpha (c_j + z_j)$. Then, the three different scenarios lead to different outcomes for the FA winner.

First, if the realization of $X$ is low enough, the spot market price is lower than the FA winning bid, and demand is satisfied in the spot market. Second, for a moderate realization of $X$, the FA winning bid may be lower than the spot market price, but larger than his realized costs $c_i + X$; hence, the FA winner provides the product and makes positive profits. Finally, if the realization of $X$ is large enough, the FA winner provides the product with negative profits because of the high realized costs.

The FA curse formalizes the observation by practitioners described in §1 that bid prices may be higher in a FA relative to a standard first-price auction run at the time the good is needed (see §1). This generalization of $E[X|X \geq a] \geq E[X]$ for some random variable $X$ and constant $a$ is directly obtained by observing that for any random variable $Y$ with cdf $H(\cdot)$ and support $[y, \bar{y}]$ (possibly $y = -\infty$ and/or $\bar{y} = \infty$), the function

$$B(a,b) = E[Y \mid a < Y < b] = \int_a^b t dH(t), \quad y \leq a < b \leq \bar{y},$$

is increasing in $a$ and $b$. (See Lemma D1 in Appendix D.)
Proposition 7 below). An important objective of the rest of the paper will be to derive mechanisms that alleviate the impact of the FA curse on buying prices by eliminating or reducing the common cost uncertainty.

3.1.2 Concentrated Markets

In concentrated markets we assume the number of suppliers in the open market is small, and thus bidders do not neglect the possibility of being sampled from the open market in the buying stage. Assuming a strictly increasing competitors’ strategy \( \beta \), the expected profit of supplier \( i \) with private cost \( c_i \) and bid price \( b_i \) is given by:

\[
\pi_i(b_i, c_i, \beta) = \mathbb{E}\left[ \mathbb{I}\{b_i \leq \beta(c_{(1:N),-i})\} \left( \sum_{j=1,j \neq i}^{M} q_{j,M} \mathbb{I}\{b_i \leq \alpha(c_j + X + Z_j)\} \right) \cdot (b_i - c_i - X) + q_{i,M} \mathbb{I}\{b_i \leq \beta(c_{(1:N),-i})\} \left( \min\{b_i, \alpha(c_i + X + Z_i)\} - c_i - X \right) + q_{i,M} \mathbb{I}\{b_i > \beta(c_{(1:N),-i}) > \alpha(c_i + X + Z_i)\} \left( \alpha(c_i + X + Z_i) - c_i - X \right) \right].
\]

The first term captures the potential profit from winning the FA with a bid that is lower than the spot market price sampled from one of the competitors. The second term captures profits from the FA winner while also being sampled in the spot market; hence, revenues are the minimum between the bid submitted by \( i \) at \( t = 0 \) and the spot market price of \( i \) at \( t = 1 \). The third term captures potential profits from losing the FA, but being sampled from the spot market with a price lower than the winning bid.

Proposition 2. (BNE bids in the naive FA with concentrated markets) Suppose that \( T = 1 \). Let \( \beta^{NFA}(\cdot) \) be a strictly increasing, continuous, and symmetric BNE strategy profile for the naive FA. Then, \( \beta^{NFA}(\cdot) \) satisfies the following integral equation for all \( c_i \in [\underline{c}, \bar{c}] \):
\[ \beta^{NFA}(c_i) = c_i + E \left[ \frac{X I[c_i \leq c(1:N), -1]}{E I[c_i \leq c(1:N), -1]} \left( \sum_{j=1}^{M} q_j M I[\beta^{NFA}(c_i) \leq \alpha(c_j + X + Z)] \right) \right] \]

\[ - \int_{c_i}^{E \beta^{NFA}(c_i, c, \beta^{NFA})} dc \]

\[ q_{i,M} E \left[ ((\alpha - 1)(c_i + X) + \alpha Z_i) I[\beta^{NFA}(c(1:N), -1)] \right] \]

\[ q_{i,M} E \left[ ((\alpha - 1)(\bar{c} + X) + \alpha Z_i) I[\beta^{NFA}(c(1:N), -1)] \right] \]

The structure of the partial derivative in \((B)\) is specified in the proof of the proposition in Appendix B. Term \((A)\) captures the FA curse as before. Term \((B)\) is the information rent term considering that the FA winner may or may not be selected as the outside option in the spot market when the market is concentrated. Elements \((C1)\) and \((C2)\) captures the \emph{spot market opportunity effect}, representing the additional margin the FA winner may get if selected from the spot market, while adjusting for the respective margin the least efficient supplier realizes. Because of the spot market opportunity effect, the analysis of the concentrated market regime is significantly more challenging than that of the diffused market. Hence, some of our results will specialize for the latter regime, which as we already argued, is by itself relevant in practice.

The procurement agency cares about the \emph{expected buying price} that corresponds to the overall sourcing cost, which considers the possibility to buy from the FA winner \emph{or} from the spot market. In the naive FA the expected buying price is given by:

\[ E [P^{NFA}] = E \left[ \min \left\{ \beta^{NFA}(c_i;N), \sum_{j=1}^{M} q_j M \alpha(c_j + X + Z) \right\} \right]. \quad (3) \]

### 3.2 Monitored FA

In the monitored FA model, the buyer compares the price set in the auction stage with the one charged by the FA winner in the open market (monitored by the buyer), and sets the buying price to be the lowest of the two. We note that in the monitored FA demand will always be satisfied by the FA winner. Assuming a strictly increasing competitors’ strategy \(\beta\), the expected profit of supplier \(i\) with private cost \(c_i\) and bid price \(b_i\) is given by:
\[
\pi_i(b_i, c_i, \beta) = \mathbb{E}[I\{b_i \leq \beta(c_{(1:N)}, i)\} \cdot (\min\{b_i, \alpha(c_i + X + Z_i)\} - c_i - X)],
\]

where \(Z_i\) is the random markup that \(i\) charges in the spot market. The expression above considers that bidder \(i\) wins the FA if and only if he has the lowest bid, and that in that case, the transacted payment is the lowest between the bid submitted by \(i\) at \(t = 0\) and his spot market price at \(t = 1\).

**Proposition 3. (BNE bids in the monitored FA)** Suppose that \(T = 1\). Let \(\beta^{MFA}(\cdot)\) be a strictly increasing, continuous, and symmetric BNE strategy profile for the monitored FA. Then, \(\beta^{MFA}(\cdot)\) satisfies the following integral equation for all \(c_i \in [c, \bar{c}]\):

\[
\beta^{MFA}(c_i) = c_i + \mathbb{E}\left[\frac{\mathbb{I}\{\beta^{MFA}(c_i) \leq \alpha(c_i + X + Z_i)\}}{\mathbb{E}[\mathbb{I}\{\beta^{MFA}(c_i) \leq \alpha(c_i + X + Z_i)\}]}ight] F^{N-1}(c) \cdot \mathbb{E}\left[\mathbb{I}\{\beta^{MFA}(c_i) \leq \alpha(c_i + X + Z_i)\}\right] dc
\]

\[
+ \int_{c}^{\bar{c}} F^{N-1}(c) \cdot \mathbb{E}\left[\mathbb{I}\{\beta^{MFA}(c) \leq \alpha(c + X + Z_i)\}\right] dc - \frac{\mathbb{E}\left[\mathbb{I}\{\beta^{MFA}(c_i) > \alpha(c_i + X + Z_i)\}\right]}{\mathbb{E}[\mathbb{I}\{\beta^{MFA}(c_i) \leq \alpha(c_i + X + Z_i)\}]}. \tag{A}
\]

As in the previous cases, term (A) captures the FA curse. Element (B) is the ‘adjusted’ information rent that incorporates the impact of the FA winner’s bid competing against its own spot market price. Finally, element (C) captures the market opportunity effect: the added value of delivering the good from the open market. In the monitored FA this value is fully captured by the FA winner and therefore it makes equilibrium bids more aggressive.

The expected buying price in the monitored FA is given by:

\[
\mathbb{E}\left[p^{MFA}\right] = \mathbb{E}\left[\min\left\{\beta^{MFA}(c_{(i:N)}), \alpha(c_{(i:N)} + X + Z_{(i:N)})\right\}\right], \tag{4}
\]

where \(Z_{(i:N)}\) is the markup charged in the open market by the lowest cost supplier in the FA (the one that achieves \(c_{(i:N)}\)). Comparing equations (3) and (4), it is immediate that the expected value of the right-hand side of the latter is smaller than the expected value of the right-hand side of the former; this is a direct effect of monitoring the market price of the most efficient FA participant. However, comparing the left-hand sides is not trivial as the equations for the BNE bidding strategies in Propositions 2 and 3 do not admit closed-form solutions. Further, later we show with numerical experiments that equilibrium bids between monitored and naive FAs cannot be ordered in general. In §4 we propose a mechanism design approach that will more easily allow us to compare the expected buying prices between the different FAs and also with other benchmarks that we introduce below.
3.3 Existence of Equilibrium

We finish this section by proving the existence of symmetric BNE for the naive FA with diffused markets and the monitored FA. We note that these results do not follow by standard existence results for first price auctions, because of the presence of the random common cost component and its correlation with the random spot market price.

**Proposition 4.** Assume that the action space $\mathcal{A}$ is restricted to be finite. (i) The naive FA game in diffused market admits a symmetric BNE in increasing strategies. (ii) Assume $\alpha = 1$, the monitored FA game also admits a symmetric BNE in increasing strategies.

We prove this result by applying and specializing the result of Athey (2001) that establishes the existence of BNE for a large class of games of incomplete information in two steps. For the corresponding game with a finite action space, the so-called *single-crossing condition (SCC)* is shown to be sufficient for the existence of an increasing and symmetric BNE. We note that while Proposition 4 assumes a single period, extending it to multiple periods is direct. We also note that to establish the SCC we specialize the monitoring FA to the case $\alpha = 1$. For games with continuous and compact action spaces (which include the settings discussed in the current paper) Athey (2001) also establishes the existence of a symmetric BNE by taking a limit of a sequence of games with finite action space as the granularity of the action space increases. In Appendix D we extend Proposition 4 from a finite action set to a continuous and compact action space by showing that the regularity conditions required for the limiting argument are valid for naive FAs.

Now, we provide conditions under which BNE strategies are continuous and strictly increasing. We have the following result.

**Proposition 5.** Any symmetric and increasing BNE strategy must satisfy: (i) Under monitored FA with $\alpha = 1$, it is continuous and strictly increasing in $c \in [\underline{c}, \bar{c}]$; (ii) under naive FA with diffused market, it is strictly increasing in $c \in [\underline{c}, \bar{c}]$. Further, if $\arg\max_{b \in \mathcal{A}} \mathbb{E}[(b - c - X)\mathbb{1}\{b \leq \alpha(X + c_j + Z_j)\}]$ is unique for all $c \in [\underline{c}, \bar{c}]$, it must also be continuous in $c \in [\underline{c}, \bar{c}]$.

A sufficient condition for the argmax to be unique is that $\mathbb{E}[(b - c - X)\mathbb{1}\{b \leq \alpha(X + c_j + Z_j)\}]$ is strictly quasi-concave in $b$, for all $c$. As an example, one can show this is the case when the common cost $X$ has a uniform distribution and $\mathbb{E}[X] - x > z - \underline{c}$.

Finally, we note that similar results to those alluded to in this section can be proved for the ‘flexible FA’ that will be introduced in §5. For brevity, we will omit the proofs of the latter results.
4 Mechanism Design Approach

The bidding analysis in §3 was aimed at developing intuition regarding the various elements that affect bidding behavior of suppliers in FAs, and in particular, highlighting the FA curse. In this section, we adopt a mechanism design approach to derive the expected buying price for various designs of FAs that were discussed in the previous sections, and that will more easily allow a comparison between them. In this section we come back to the original multi-period setting.

The analysis in this section has some novelties relative to existing mechanism design literature (e.g., Myerson 1981) as we now explain. Recall that the spot market price of supplier \(i\) at time \(t\) is given by \(\alpha(c_i + X_t + Z_{i,t})\). The sampled spot market price is a function of these market prices; in the monitored FA it is the market price of the FA winner, and in the naive FA it is a random sample of the market price among all potential suppliers. Hence, the spot market price is a function of the actual cost realizations \(c\), and the realizations of \(X_t\) and \(Z_t\). Therefore, the allocation rules defined below, which determine whether a good is bought to the FA winner at the winning auction bid price, implicitly depend on the actual private cost realizations of suppliers through the spot market prices. In other words, the outside option for the auctioneer is not exogenous but rather depends on the actual cost realizations, which differs from the classical mechanism design setup. More specifically, this dependence yields additional terms when applying the envelope theorem to express the payments as a function of the allocation rule; despite this challenge, we will derive useful expressions for expected buying prices.

4.1 Preliminaries

We consider the following class of first-price FA mechanisms that include the monitored and naive FAs as special cases. At time period \(t = 0\), the auctioneer receives bids from the FA participants and decides the FA winner. At time period \(t = 1, 2, \ldots, T\), the auctioneer observes the realized spot market price and decides whether to buy from the FA winner (at the submitted bid at \(t = 0\)) or to buy from the spot market at period \(t\). Formally, we define the allocation function \(r_{i,t} : \mathcal{A}^N \times [\bar{p}_t, \tilde{p}_t] \to [0, 1]\), where \(r_{i,t}(b_i, b_{-i}, p_t)\) is the probability bidder \(i\) sells the good through the FA if he submits a bid \(b_i\), his competitors submit bids \(b_{-i}\), and the realization of the spot market price at period \(t\) is given by \(p_t\). The interval \([\bar{p}_t, \tilde{p}_t]\) is the range of feasible spot market prices. We denote \(r_t = (r_{1,t}, \ldots, r_{N,t})\) and \(r = (r_1, \ldots, r_T)\) as the vectors of the allocation probabilities. In what follows we only consider allocation rules that satisfy \(\sum_{i=1}^N r_{i,t}(b, p) \leq 1\), for all \(b, p, t\).

Similarly, define the payment function \(m_{i,t} : \mathcal{A}^N \times [\bar{p}_t, \tilde{p}_t] \to \mathbb{R}\), where \(m_{i,t}(b_i, b_{-i}, p_t)\) is the
expected FA payment bidder $i$ receives for given bids $(b_i, b_{-i})$ and a given realization of the period $t$’s spot market price, $p_t$. We let $m_t = (m_{1,t}, \ldots, m_{N,t})$ and $m = (m_1, \ldots, m_T)$. Because we consider first-price FA mechanisms, it may also be convenient to write $m_{i,t}(b, p) = b_i r_{i,t}(b, p)$, where $b_i$ is the bid submitted by bidder $i$ at $t = 0$. An FA mechanism is given by the functions $w = (r, m)$ and we denote by $\mathcal{W}$ the set of all such mechanisms.

Let $A_j$ be the event in which bidder $j$ is selected in the spot market; when sampling randomly under the naive FA; $A_j = 1$ with probability $q_{j,M}$. Then, the spot market price is given by

$$p(c, X_t, Z_t) = \sum_{j=1}^{M} A_j p_j(c_j, X_t, Z_{j,t}),$$

where $p_j(c_j, X_t, Z_{j,t})$ is the price charged by firm $j$ at the spot market. For the monitored FA, the allocation function is:

$$r_{i,t}(b_i, b_{-i}, p) = \mathbb{I}\{b_i \leq b_j, \forall j \neq i\} \mathbb{I}\{b_i \leq p_i(c_i, X_t, Z_{i,t})\},$$

and for the naive FA, the allocation function is:

$$r_{i,t}(b_i, b_{-i}, p) = \mathbb{I}\{b_i \leq b_j, \forall j \neq i\} \mathbb{I}\{b_i \leq p(c, X_t, Z_t)\}.$$

While to simplify the exposition we ignore ties, we note that all the results in this section hold even if ties are allowed. In both cases above the FA winner is determined by the lowest submitted bid: in the monitored FA the winning bid is compared against the FA winner’s own spot market price, and in the naive FA it is compared against a randomly sampled spot market price. In both cases, the allocation function is equal to one if the good is allocated through the FA at the winning bid. However, FA winner’s profits also consider the event of supplying the good through the spot market. For example, in the monitored FA, the allocation function is equal to one if the FA winner sells the good at his winning bid $b_i$. The FA winner profit function also considers the event of selling the good at his own spot market price when the latter is lower than $b_i$.

We benchmark the expected payments under the monitored FA and the naive FA against the expected payment in a procurement mechanism where a first price auction (FPA) is held at every time period $t$ after $X_t$ gets realized, and then the winner (the supplier with the lowest bid) immediately delivers the good. In these FPAs, bidders do not face cost uncertainty. In such a mechanism, the profit with bid $b_i$, cost $c_i$, and realization $x_t$ is given by:

$$\pi_{i,t}(b_i, c_i, \beta) = \mathbb{P}\{b_i \leq \beta(c_j), \forall j \neq i\} \cdot (b_i - c_i - x_t).$$

We denote the payment in the mechanism in which a FPA is run at every time period by $P^{FPA}$. In a BNE, the expected payment for the FPA at period $t$ is given by:
\[
\mathbb{E}[P^{FPA}] = \mathbb{E}\left[\beta^{FPA}_N(c_{1:N})\right] = \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{1:N})/f(c_{1:N})\right],
\]  

where \(c_{(1:N)}\) is the lowest order statistic among the private costs of the \(N\) bidders. The function \(\beta^{FPA}_N(\cdot)\) is the BNE strategy of the FPA with \(N\) bidders. The second equation follows by standard mechanism design arguments based on the envelope theorem (Milgrom 2004). In the next subsection, we compare the payments in the naive FA with those of FPA to quantify the impact of the FA curse.

As the discussed FA mechanisms have an outside option, an alternative benchmark one may consider is a first price auction with a random reserve price that is set to be the spot market price (randomly sampled like in the diffused market naive FA); in Appendix C we show that expected payments under the two FPAs (with and without the outside option) are asymptotically equivalent as the number of bidders grows large. Hence, the comparison result shown in the following subsection also holds for FPA with an outside option that is given by the spot market price.

4.2 The Naive FA

We next provide a lower bound for expected payments in the naive FA that only depends on the allocation rule and will allow for easier comparisons with other mechanisms.

Proposition 6. (Bound on the expected buying price in naive FA) Let \(\beta(\cdot)\) be a strictly increasing symmetric BNE strategy profile induced by a naive FA mechanism \(w = (r, m)\). Then, the expected buying price for the auctioneer can be lower bounded by the sum over \(t\) of the following expression, \(\mathbb{E}\left[P^{NFA}\right] \geq \sum_{t=1}^{T} P_{t}^{NFA}\), where

\[
P_{t}^{NFA} = \sum_{i=1}^{N} \pi_{i,t}(\beta(\bar{c}), \bar{c}, \beta) - \sum_{i=1}^{N} (\alpha - 1)q_{t,M}(\bar{c} - \mathbb{E}[c_i]) + \mathbb{E}\left[\tilde{q}_0(c, X_t)\right] + \mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\beta(c), p_t)(\bar{v}_i(c_i) + X_t - \tilde{q}_0(c, X_t))\right],
\]

where \(\tilde{q}_0(c, X_t) = \sum_{i=1}^{M} A_i(c_i + X_t)\) and \(\bar{v}_i(c_i) = c_i + (1 + (\alpha - 1)A_i)\). Note that \(\bar{v}(c_i)\) is a modified virtual cost. Next, we compare the expected payments in the naive FA with those obtained in a procurement mechanism where a first price auction (FPA) is held, and then the winner immediately delivers the good. To derive the result we consider an asymptotic regime in which the number of participant firms in the FA, \(N\), grows to infinity. This dramatically simplifies the analysis because it allow us to get a better handle on the FA equilibrium bids and markups, which for every finite \(N\) are complex objects without closed-form expressions. Note that even in the case \(N \to \infty\) we obtain meaningful results, as expected payments can be
strictly ordered. In §6 we provide numerical results for small values of \( N \) that support the validity of the asymptotic results.

**Proposition 7. (Impact of the FA curse)** Assume \( \min_{c \in [c, \bar{c}]} f(c) > 0 \).

1. **[Concentrated Markets]** If \( \alpha = 1 \), then
   \[
   \lim_{N \to \infty} E[P^{NFA} - P^{FPA}] \geq 0.
   \]

2. **[Diffused Markets]** If \( \bar{z} + x < \mathbb{E}[X_t], \forall t, \) and the spot market is diffused, then:
   \[
   \lim_{N \to \infty} E[P^{NDA} - P^{FPA}] > 0.
   \]

The proposition formalizes the FA curse. In concentrated markets with \( \alpha = 1 \), the expected payments of the naive FA are larger than those of the FPA. Further, we can show that this inequality is strict in diffused markets (for any \( \alpha \geq 1 \)). Note that governments may still prefer to use FAs because running auctions at every period \( t \) has administrative costs associated to setting up the auctions, receiving bids, processing them, and so forth, that scale with the total number of periods \( T \). In contrast, when running an FA the administrative cost associated to running an auction is only payed once at \( t = 0 \).

### 4.3 The Monitored FA

We next demonstrate that one may improve the performance of FAs by monitoring the spot market price of the FA winner. We first derive an expression for the expected payments in monitored FAs.

**Proposition 8. (Expected buying price in monitored FA)** Let \( \beta(\cdot) \) be a strictly increasing symmetric BNE strategy profile induced by a monitored FA mechanism \( \mathbf{w} = (\mathbf{r}, \mathbf{m}) \). Then, the expected total buying price for the auctioneer from period 1 to period \( T \) is given by:

\[
\mathbb{E}[P^{MFA}] = \sum_{t=1}^{T} P_t^{MFA},
\]

where

\[
P_t^{MFA} = \mathbb{E} \left[ - (\alpha - 1) \sum_{i=1}^{N} [c_{(1:N),-i} - c_i]^+ + q_0(c, X_t) + \sum_{i=1}^{N} r_{i,t}(\beta(c), p(c, X_t, Z_t)) (v(c_i) + X_t - q_0(c, X_t)) \right],
\]

with \( q_0(c, x) = c_{(1:N)} + x \), \( v(c_i) = c_i + \alpha \frac{F(c_i)}{\tilde{F}(c_i)} \), and \( c_{(1:N),-i} \) is the lowest order statistic among the cost realizations of firm \( i \)'s competitors.

In Proposition 8 we derive an expression for the expected payment that depends only on the allocation rule, even though the allocation rule itself also depends on the actual realizations of costs.
through the spot market price. A key idea behind this derivation lies in exploiting the fact that
some of the additional terms appearing in the envelope expression cancel out. In what follows we
leverage this result in comparing the performance of a monitored FA and running FPAs as needs
arise.

**Proposition 9. (Comparing buying prices of monitored FA and FPA)** The difference
between the expected buying prices of a monitored FA and a FPA is given by:

$$ E[P_{MFA} - P_{FPA}] = -T(\alpha - 1) \sum_{i=1}^{N} E[c(1:N)_{-i} - c_i] + \sum_{t=1}^{T} E[(\alpha I_{MFA} - 1)F(c(1:N))/f(c(1:N))] , $$

where $I_{MFA} = I\{\beta_{MFA}(c(1:N)) < \alpha(c(1:N) + X_t + Z(1:N),t)\}$. As a consequence, if $\bar{x} = 0$ and $\bar{z} = 0$, then there exists $\alpha$, such that $E[P_{MFA}] \leq E[P_{FPA}]$, for all $\alpha \in [1, \bar{\alpha}]$.

Proposition 9 shows that because bidders compete against their own spot market prices (that
are random at $t = 0$), there are values of $\alpha$ for which the monitored FA achieves lower expected
prices than the first price auction. We next provide conditions under which the expected buying
price of the monitored FA is strictly smaller than that of the naive FA.

**Theorem 1. (Comparing buying prices of naive and monitored FAs)** Assume $\min_{c \in [\underline{c}, \bar{c}]} f(c) > 0$.

1. [Concentrated Markets] If $\alpha = 1$, then

$$ \lim_{N \to \infty} E[P_{NFA} - P_{MFA}] \geq 0. $$

2. [Diffused Markets] If $\bar{x} + \bar{z} < E[X_t]$, $\forall t$, and the spot market is diffused under naive FA, then:

$$ \lim_{N \to \infty} E[P_{NFA} - P_{MFA}] > 0. $$

The result shows that monitoring alleviates the FA curse. This may be somewhat intuitive:
the monitored FA should provide lower expected buying prices relative to the naive FA, because it
uses a better outside option (the FA winner spot market price as oppose to a randomly sampled
spot market price). However, the analysis is not direct, because the expected buying price also
depends on the equilibrium bids which are hard to characterize. By taking $N \to \infty$ we can simplify
the dependence of the expected buying prices on equilibrium bids and prove the result. In §6 we
provide numerical results showing that the monitored FA performs better than the naive FA even
when $N$ is small.

The main argument behind Proposition 7 and Theorem 1 (for diffused markets) is to show that
the per period asymptotic expected payment under the monitored FA and first-price auction are
no larger than $c + \mathbb{E}[X_t]$. This follows because as $N$ grows the lowest cost supplier achieves $c$ and competition dissipates all markups. In contrast, one can also show that the expected payment under the naive FA is strictly larger than this quantity due to the FA curse, even in the asymptotic regime of large $N$.

5 Other Practical FA Mechanisms

The results that we have established in the previous section suggest that monitoring spot prices of FA winners may improve the performance of FAs. Nevertheless, as discussed before such an approach involves some practical challenges (such as monitoring costs) and therefore may not be necessarily followed by procurement agencies (as is the case for ChileCompra). In this section we discuss alternative (and potentially more practical) approaches to improve the performance of FAs.

To facilitate the analysis, in this section we focus on diffused markets. The spot market price at each period $t$ is given by $p_t = \sum_{j=1}^{M} A_j \alpha (c_j + X_t + Z_{j,t})$. It is simple to show that $p_t$ converges in distribution to $X_t + Z_t$ as $M \rightarrow \infty$, where we abuse notation assuming that $Z_t$ has the same distribution to the sum of the random variables $c_j + Z_{j,t}$. We assume $\alpha = 1$ to simplify some of our arguments and notation, but the results can be extended for $\alpha > 1$. Note that by the diffused market assumption, $Z_t$ is independent of $c_j$, for all $j \in \mathcal{N}$, because the chances of an FA participant to be sampled from the spot market become negligible as $M \rightarrow \infty$.

Recall that naive FAs result in large expected buying prices because of the FA curse that is driven by the common cost uncertainty. In this section we study FA designs that alleviate the common cost uncertainty and therefore achieve lower expected buying prices relative to naive FAs. In particular, we show that the optimal mechanism eliminates the common cost uncertainty by indexing payments to the common cost. One limitation of using this type of mechanisms in practice, though, is that finding price indexes may be challenging for many goods and services. For example, while price indexes may be easy to construct for commodities such as gas, these indexes do not exist and may be hard to establish for many of the goods and services procured with FAs (e.g., computers, office equipment, services). Thus motivated, we introduce a more practical FA design, the flexible FA, that achieves similar expected buying prices to the optimal mechanism and the monitored FA, without requiring a price index nor monitoring spot market prices.

5.1 The Flexible FA

One proxy for the realization of the common cost (which in general, is not observed by the procurement agency) is the realization of the spot market price. Hence, a practical FA variant may use
the observed spot market price to partially remove common cost uncertainty. In this subsection we introduce the flexible FA, in which at every point in time the FA winner has the option to decrease his bid to match the observed spot market price (in case the latter is lower than the winning bid). When the FA winner lowers his price to match the spot market price he is guaranteed to supply the good on that period. The decision to lower the price is done independently each period. Note that the flexible FA is simple to implement; relative to the naive FA, it only requires giving the FA winner the added option of matching the spot market price at every time period. The allocation rule of the flexible FA for time period $t$ is given by:

$$r_{i,t}(b_i, b_{-i}, z_t, x_t, c_i) = \mathbb{1}\{b_i \leq b_j, \ j \neq i\} \cdot (\mathbb{1}\{b_i \leq z_t + x_t\} + \mathbb{1}\{b_i > z_t + x_t, c_i \leq z_t\}) .$$

In this allocation rule the FA winner sells through the FA either when his bid is lower than the spot market price or when he can afford matching the spot market price (the latter is the case if $c_i \leq z_t$). The payment function of the flexible FA for time period $t$ is given by:

$$m_{i,t}(b_i, b_{-i}, z_t, x_t, c_i) = \mathbb{1}\{b_i \leq b_j, \ j \neq i\} \cdot (b_i \mathbb{1}\{b_i \leq z_t + x_t\} + (z_t + x_t) \mathbb{1}\{b_i > z_t + x_t, c_i \leq z_t\}) .$$

Note that the allocation depends on the actual cost realizations. Despite this dependence, using similar arguments to those in analyzing the monitored FA, we establish the following characterization of expected payments in flexible FAs.

**Proposition 10. (Expected buying price in Flexible FAs)** The expected payment in flexible FAs is given by:

$$\mathbb{E}[P_{FLE}] = \sum_{t=1}^{T} P_{FLE}^t,$$

where

$$P_{FLE}^t = \mathbb{E}[X_t + Z_t] + \mathbb{E}\left[(v(c_{1:N}) - Z_t) \cdot (\mathbb{1}\{c_{1:N} \leq Z_t\} + \mathbb{1}\{c_{1:N} \leq Z_t + X_t, c_{1:N} > Z_t\})\right].$$

(6)

In the following subsection we will use this expression to compare the expected payments of the flexible FA with those of various related mechanisms.

### 5.2 The Optimality of the Flexible FA

To evaluate the flexible FA we benchmark its performance against three mechanisms: the monitored FA; the FPA, in which an auction is run at every time period and bidders do not face cost uncertainty; and a mechanism that is the optimal in a class of mechanisms that generalizes the class defined in §4. While the development of the latter is described in detail in Appendix E, in what follows we summarize the setup and the main results.

We define the allocation function $r_{i,t} : \mathcal{A}^N \times [\bar{x}, \bar{x}] \times [\bar{z}, \bar{z}] \rightarrow [0, 1]$, where $r_{i,t}(b_i, b_{-i}, x, z)$ is
the probability bidder $i$ sells the product in period $t$ if he submit a bid $b_i$, his competitors submit bids $b_{-i}$, the common cost realization is $x$ and the realization of $Z$ is equal to $z$. We let $r_t = (r_{1,t}, ..., r_{N,t})$ and $r = (r_1, ..., r_T)$. We only consider allocation rules that satisfy $\sum_{i=1}^{N} r_{i,t}(b, x, z) \leq 1$, for all $b, x, z, t$. Similarly, define the payment function $m_{i,t} : \mathcal{A}^N \times [\bar{x}, \bar{x}] \times [\bar{z}, \bar{z}] \to \mathbb{R}$, where $m_{i,t}(b_i, b_{-i}, x, z)$ is the expected payment bidder $i$ receives in period $t$ for given bids $(b_i, b_{-i})$ and given realization of common cost $x$ and realization $z$. We let $m_t = (m_{1,t}, ..., m_{N,t})$ and $m = (m_1, ..., m_T)$. An FA mechanism is given by the functions $w = (r, m)$ and we denote by $\mathcal{W}$ the set of all such mechanisms. This class is more general than the one defined in Section §4, because it allows dependence on $x$ and $z$ separately, beyond just $p = x + z$ (but note that it does not include the FLE FA because the allocation rule of the latter depends on the cost realization).

In Proposition E2 (Appendix E) we characterize the optimal mechanism, showing that a “modified” second price auction, in which the winner is payed $b_{(2)} + x_t$ ($b_{(2)}$ being the second lowest bid), with an appropriately chosen random reserve price (that depends on $z_t$) is optimal. Note that in the optimal mechanism, payments (derived in Appendix E) are indexed to the realization of the common cost $x_t$ in every period, effectively creating a price index that eliminates the common cost uncertainty for the FA winner, and therefore, eliminating the FA curse. It is important to observe that the practicality of the price index FA lies on the availability of the relevant price indexes. Further, in §6 we discuss the impact of implementing a perfect price index FA with a first-price rule.

The following result shows that the expected buying price under the FLE FA becomes asymptotically optimal, where we denote $P^{OPT}$ as the buying price over the horizon under the optimal mechanism. Further, the expected buying prices of the FLE FA asymptotically coincides with that of the monitored FA, and the FPA.

**Theorem 2. (Asymptotic optimality of the flexible FA)** Assume that $\min_{c \in [\underbar{c}, \bar{c}]} f(c) > 0$ and $\alpha = 1$. Then,

$$\lim_{N \to \infty} \mathbb{E}[P^{FLE} - P^{OPT}] = 0,$$
$$\lim_{N \to \infty} \mathbb{E}[P^{FLE} - P^{MFA}] = 0,$$
$$\lim_{N \to \infty} \mathbb{E}[P^{FLE} - P^{FPA}] = 0.$$

Theorem 2 implies that a flexible FA, an applicable mechanism that is easy to implement as a simply adjustment of the naive FA, achieves low expected payments when the number of bidders is large, comparable to those obtained by the benchmarks above (OPT, MFA, and FPA). Moreover,
in the numerical results that will be described in the next section we show that expected payments under the flexible FA are typically close to the benchmarks even for small values of $N$. The result stands in contrast with Theorem 1 that shows that the expected payment for the naive FA exceeds that of the monitored FA and FPA even in the asymptotic regime of large $N$.

6 Numerical Experiments

In this section, we complement the analytical results obtained in the previous section with numerical results, showing that the former are quite robust. In particular, the most important conclusions derived from the analytical results that inform the design of FAs are that the flexible FA achieves lower expected payments relative to the naive FA, that prices indexes are helpful, and that the performances of the flexible FA and the monitored FA are comparable. We confirm that these conclusions are valid even when $N$ is relatively small.

To simplify the analysis we assume that $Z_{i,t} = \Delta$ is deterministic and $\alpha = 1$. Hence, the spot market price under the monitored FA for bidder $i$ in period $t$ is $c_i + X_t + \Delta$. The spot market price under the naive FA in diffused markets is $c_0 + X_t + \Delta$, where we also assume $c_0$ is constant for simplicity (we take $z_0 = c_0 + \Delta$). Further, we assume that $\{X_t : t \geq 1\}$ are i.i.d. random variables. Under these assumptions, it is enough to report the expected buying prices per period, so without loss of generality we present all our results for $T = 1$.

To enrich the analysis, we will also consider a first-price perfect price index (PPI) FA in which the winning bidder with bid $b^*$ receives a payment of $b^* + (x - \mathbb{E}[X])$. The PPI FA removes the common cost uncertainty; costs for the PPI FA winner are effectively given by $c + \mathbb{E}[X]$. It is not hard to show that under the assumptions of the present section, this PPI FA is payoff equivalent to the flexible FA. Hence, for large enough $N$, the PPI FA achieves lower expected buying prices relative to the naive FA. One can also show that such PPI FA with an appropriate reserve price is optimal among the class of mechanisms introduced in §5 (that is, under the assumptions of this section, the optimal mechanism can be implemented as a first price PPI FA). Furthermore, in the setting of the present section, one can also show that the flexible FA is payoff equivalent to running a FPA after $X$ has been realized. Hence, the results obtained below for the flexible FA also apply to the first-price PPI FA and for running a FPA after $X$ has been realized.

Next we describe the methodology and setup of our numerical experiments. Following that, we describe the main results and the key insights.
6.1 Methodology

We numerically study the different FA models: naive FA with diffused market, monitored FA, and flexible FA. To do so, in the following result we derive the ordinary differential equations (ODEs) together with their corresponding boundary conditions that characterize their symmetric BNE strategies.

Proposition 11. (ODEs associated with FA mechanisms) Let $z_0 = c_0 + \Delta$. Then,

(i) A symmetric and differentiable BNE strategy under naive FA in diffused market satisfies the following ODE:

$$\frac{d\beta^{\text{NDFA}}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ (\beta^{\text{NDFA}}(c) - c - X) \cdot \mathbb{I}\{\beta^{\text{NDFA}}(c) \leq z_0 + X\} \right],$$

for any $c \in [\underline{c}, z_0]$, with the boundary conditions $\beta^{\text{NDFA}}(z_0) = z_0 + \bar{x}$, and $\frac{d\beta^{\text{NDFA}}}{dc}(z_0) = 0$.

(ii) A symmetric and differentiable BNE strategy under monitored FA satisfies the following ODE:

$$\frac{d\beta^{\text{MFA}}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \min\{\beta^{\text{MFA}}(c) - c - X, \Delta\} \right],$$

for any $c \in [\underline{c}, \bar{c}]$, with the boundary condition $\beta^{\text{MFA}}(\bar{c}) = \bar{c} + \Delta + \bar{x}$, where $K = \mathbb{E}[X] - \Delta$ if $\mathbb{E}[X] \leq \Delta$, and otherwise $K \in [0, \bar{x}]$ is the unique solution to the equation:

$$(K + \Delta) \bar{F}_X(K) - \int_{K}^{\bar{x}} x f_X(x) dx + \Delta \cdot \bar{F}_X(K) = 0.$$  

(iii) A symmetric and differentiable BNE strategy under flexible FA satisfies the following ODE:

$$\frac{d\beta^{\text{FLE}}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ (b - c - X) \cdot \mathbb{I}\{b \leq z_0 + X\} + (z_0 - c) \cdot \mathbb{I}\{b > z_0 + X\}\right]_{b=\beta^{\text{FLE}}(c)},$$

for any $c \in [\underline{c}, z_0]$, with the boundary conditions $\beta^{\text{FLE}}(z_0) = z_0 + \bar{x}$, and $\frac{d\beta^{\text{FLE}}}{dc}(z_0) = 0$.

We note that if $X$ is uniformly distributed over the interval $[0, \bar{x}]$, the boundary condition under monitored FA takes the following closed form expression: $\beta^{\text{MFA}}(\bar{c}) = \bar{c} + \Delta + \bar{x} - \sqrt{2\Delta\bar{x}}$. We further note that while the FA winner under the naive FA or flexible FA is competing against an outside market with price $z_0 + X$, and as a result bidders with private cost higher than $z_0$ never wins, under the monitored FA the FA winner is competing against his/her own spot market price, and therefore the whole interval $[\underline{c}, \bar{c}]$ should be considered.
Similarly to asymmetric first-price auctions, our ODEs are not well-behaved at the boundary, because at the right-hand-side of these we obtain $0/0$. Hubbard and Paarsch (2011) and Fibich and Gavish (2011) provide useful summaries of the challenges involved in numerically solving ODEs to establish BNE strategies in asymmetric first-price auctions, as well as ways to overcome these technical challenges. To avoid the singularity at the boundary, we make the approximation $\beta(z_0 - \epsilon) = \beta(z_0) - \epsilon$ for a small value of $\epsilon = 10^{-5}$. We solve the ODEs using a Runge-Kutta method in Matlab to obtain the BNE bid functions for the various FA mechanisms.

Having calculated the equilibrium bid functions, we use simulation to determine the expected buying prices for various parametric instances of each FA mechanism. In particular, we randomly generate the private costs $c_i$ for all suppliers and use the BNE bid functions to obtain the winning bid $\beta(c_{(1)})$, where $c_{(1)}$ is the lowest realized private cost. Then, we randomly generate a common cost $X$. The realized cost of the winning bidder is $c_{(1)} + X$, and the realized spot market price is $c_0 + \Delta + X$ for naive FA with diffused market and flexible FA, and $c_{(1)} + \Delta + X$ for monitored FA. Given these quantities, the buying price is determined according to the rules of the different mechanisms. To simulate the expected buying price, we replicate the above procedure 50,000 times for each problem instance and each FA mechanism; relative errors in the results below are all less than 0.5% with 98% confidence intervals.

### 6.2 Setup and Results

We consider $N$ bidders, where bidder $i$ has a private cost $c_i \sim U[0, \frac{1}{2}]$. We assume a uniform distribution in order to simplify the ODEs; we fix this distribution while varying other parameters. We consider a common cost $X$ that is distributed according to a truncated normal distribution with mean $\mu_x$, standard deviation $\sigma_x$, and support $[0, 2\mu_x]$; we denote by $\sigma_{\text{max}} = \frac{2\mu_x}{\sqrt{12}}$, the value of the standard deviation under which the distribution of $X$ becomes uniform over $[0, 2\mu_x]$. We consider all possible combinations of a large range of parameter values that capture different settings: $N \in \{2, 4, 6, 8, 10\}$, $c_0 \in \{\frac{1}{8}, \frac{1}{6}, \frac{1}{4}\}$, $\Delta \in \{\frac{1}{16}, \frac{1}{12}, \frac{1}{8}\}$, $\mu_x \in \{1, \frac{3}{2}, 2\} \times (c_0 + \Delta)$, and $\sigma_x \in \{\sigma_{\text{max}}/2, 3\sigma_{\text{max}}/4\}$. In total, there are 270 model instances.

All numerical results can be obtained from the authors upon request. The plot in the left side of Figure 4 shows the BNE bid functions under the different mechanisms for one representative instance. The bid functions in the plot are ordered in the “typical order”, which repeated itself over most instances. The equilibrium bid under the naive FA is higher than the equilibrium bid.

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9Note that a first order approximation yields $\beta(z_0 - \epsilon) = \beta(z_0)$, when $\beta'(z_0) = 0$. In addition, since we are looking for BNE in strictly increasing strategies, and to avoid a flat curve at the boundary, we subtract $\epsilon$ to $\beta(z_0)$ above.
Figure 4: A representative problem instance. Parameter values are: \( \mu_x = 0.5, \sigma_x = \frac{\mu_x}{\sqrt{12}} = \frac{\sigma_{\text{max}}}{2}, \)

\( c_0 = 0.25, \) and \( \Delta = \frac{1}{12}. \) \( \text{(Left)} \) BNE bid function under the NDFA, MFA, and FLE mechanisms with \( N = 10 \) suppliers. \( \text{(Right)} \) Expected buyer’s payment for different mechanisms as a function of the number of participating suppliers.

under the flexible FA. There is no universal order between equilibrium bids under naive FA and monitored FA. The plot in the right hand side of Figure 4 depicts the buyer’s expected payment as a function of the number of bidders, illustrating that the established comparisons in Theorems 1 and 2 may also hold with a relatively small number of bidders. In the representative example that is showed in the figure, monitored FA dominates naive FA for any number of bidders, while flexible FA dominates naive FA for any \( N \geq 5, \) and approaches the monitored FA as the number of bidders increases.

Comparing the performances of the monitored FA and the naive FA under a diffused market, we find that when \( c_0 = 1/4, \) the buyer’s expected payment under the former is always smaller than the payment under the latter for all scenarios, and that on average the prices are 9% lower. Note that this value of \( c_0 \) represents a sampling of an average cost seller from the spot market. However, when the spot market is cheaper than the average seller (that is, \( c_0 < 1/4 \)), there is a small number of instances for which the expected payment under monitoring is larger than that of naive FA. However, it is still the case that in most instances the monitored FA performs better, especially when \( X \) is volatile (with a larger coefficient of variation), and when there is more competition (when the number of sellers is relatively large). For example, with \( c_0 = 1/6 \) and \( \sigma_x = 3\sigma_{\text{max}}/4, \)

the expected payments of the monitored FA are lower than those of the naive FA in 31 out of 36 parametric combinations of \( \Delta, \mu_x, \) and \( N \geq 4. \) These findings represent the following insight that is consistent with Theorem 1: monitored FAs typically perform better than naive FAs, except in

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some cases in which the number of FA participants is small and very efficient suppliers are sampled from the spot market in the naive FA.

Comparing the performance of the flexible FAs with those of the monitored and the naive FAs, we observe that flexible FAs often dominate naive FAs under sufficient competition. For example, in 88% of the tested instances (191 of 225) in which \( N \geq 4 \), flexible FAs outperformed naive FAs with a buying price that was on average 4% lower than the one obtained by naive FAs. In cases where in addition in a naive FA an “average cost” supplier was sampled (\( c_0 = 1/4 \)), the flexible FA dominated the naive FA in an even more consistent manner (in more than 95% of these instances the buying prices obtained by flexible FAs were lower). Note that a naive FA may perform better than a flexible FA when \( N \) is small, because in naive FAs bidders face the additional competition of the random spot market price (which in flexible FAs can always be matched), incentivizing suppliers to bid more aggressively. This effects get dissipated quickly as \( N \) grows.

Regarding the comparison between flexible and monitored FAs, we observe that the competition level that is required for payments under a flexible FA to approach those under a monitored FA depend on the cost efficiency of the outside option sampling, which is captured by the value of \( c_0 \). When \( c_0 \) has a low value, we found that the performance of flexible FAs is close to that of monitored FAs even under moderate competition (for example, considering the cases of \( c_0 \in \{1/6, 1/8\} \), when \( N \geq 8 \), monitored and flexible FAs have similar performances, except in 3 out of 72 instances). However, when average suppliers are sampled (\( c_0 = 1/4 \)), monitored FAs outperformed flexible FAs in most of the instances that were tested even for \( N = 10 \). In such cases flexible FAs seem to require more aggressive competition (larger \( N \)) in order to achieve a performance comparable to monitored FAs. (See the right hand side of Figure 4 for one such instance.)

7 Conclusions

Discussions and Design Recommendations. In this paper we introduced a novel auction model to analyze FAs, procurement mechanisms that are commonly used around the world. The increasing popularity of FAs can be attributed, among other things, to the following features: (i) they avoid the administrative costs of running repeated first-price auctions; and (ii) they tend to screen more cost-efficient suppliers relative to a setting in which each public organization buys from the spot market. Despite these advantages, we show that FAs are subject to a sort of winner’s curse that makes expected buying prices higher relative to running first-price auctions when needs arise. Based on our theoretical and numerical results we suggest the following prescriptions for a practical design of FAs that alleviate this concern and reduce buying prices:
1. Monitoring the price that is charged by the FA winner in the open market and using it to upper bound the buying price significantly reduces procurement costs. Procurement agencies should engage in this type of monitoring activities in cases where the associated administrative costs are not too high.

2. Building and implementing perfect price indexes for the common random parts of costs may reduce buying prices. The potential value of implementing price indexes suggests that procurement agencies should try to be creative in building such indexes even when there are not obvious ways of doing so, for example, when buying services or non-commodities products.

3. In the absence of a perfect price index or when monitoring costs are high, allowing FA winners to lower and match the spot market price (implementing flexible FAs) may be an effective way of reducing buying prices.

While some of these prescriptions are intuitive, they need to be taken with some caution because they are not always valid. For example, when the number of FA participants is small and the outside option given by the spot market is attractive, the monitored and flexible FAs may perform worse relative to naive FAs. This concern notwithstanding, the above prescriptions are likely to be relevant in many cases of practical interest. In fact, these prescriptions are currently being considered by the Chilean government to improve their procurement processes and we hope that in the future they will also be considered by other buying agencies.

**Future Work.** Our work is the first that provides a formal understanding on how the cost uncertainty suppliers face in FAs affects their bidding behavior and the outcomes in FAs. We focus on this aspect of FAs but several interesting directions may be worth exploring in the future. In particular, throughout the paper we assumed risk neutral bidders, and even in this case, the discussion regarding cost uncertainty was rich and relevant. An interesting extension of our model would be to consider risk averse bidders. This would introduce additional complications in the analysis; for example, under risk aversion revenue equivalence-style arguments may no longer hold. On the other hand, introducing risk averse bidders would allow a meaningful discussion on bidders’ use of financial instruments such as options to hedge the risk associated to the common cost. We conjecture that the use of options may play a similar role to the perfect price index in the case of risk neutral bidders to alleviate the extent of the FA curse. Overall, we hope that this paper together with the follow up work it may generate, will improve how FAs and buying processes are designed in practice.
References


A Selected Proofs

Proof of Proposition 1. Taking derivative with respect to $c_i$, one has:

$$\frac{\partial \pi_i^{NFA}(b_i, c_i, \beta)}{\partial c_i} = -F^{-1}_N(\beta^{-1}(b_i))\mathbb{P}\{b_i \leq \alpha(c_j + X + Z_j)\}$$

Applying the envelope theorem (Milgrom 2004), one has that:

$$\pi_i(\beta(c_i), c_i, \beta) = \pi_i(\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(\beta(c), c, \beta)}{\partial c} dc.$$ 

The rest of the proof follows by taking $\pi_i(\beta(\bar{c}), \bar{c}, \beta) = 0$ and equating the above with

$$\pi_i(\beta(c_i), c_i, \beta) = \bar{F}^{-1}(c_i)\mathbb{E}\{\beta(c_i) \leq \alpha(c_j + X + Z_j)\} \cdot (\beta(c_i) - c_i - X).$$

\[\square\]

Proof of Proposition 7. By Proposition 6, we have $\mathbb{E}[P^{NFA}] \geq \sum_{t=1}^{T} P_t^{NFA}$. By standard arguments based on the envelope theorem (Milgrom 2004), it can be easily shown that the expected payment under first-price auction is given by $\mathbb{E}[P^{FPA}] = \sum_{t=1}^{T} \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{(1:N)})/f(c_{(1:N)})\right]$. Let $P_t^{FPA} = \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{(1:N)})/f(c_{(1:N)})\right]$. Denote the event of buying from the FA winner under naive FA be $\mathbb{I}^{NFA} = \{\beta(c_{(1:N)}) < p(c, X_t, Z_t)\}$.

By Proposition 6, one has (using the notation $\mu_c = \mathbb{E}[c_i]$):

$$P_t^{NFA} = \sum_{i=1}^{N} \pi_{i,t}(\beta(\bar{c}), \bar{c}, \beta) - \sum_{i=1}^{N} (\alpha - 1)q_i,M(\bar{c} - \mu_c) + \mathbb{E}[\bar{q}_0(c, X_t)]$$

$$+ \mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\beta(c), p_i) (\bar{r}_i(c_i) + X_t - \bar{q}_0(c, X_t))\right]$$

$$(a) \quad \sum_{i=1}^{N} \mathbb{E}[A_i(1 - \Pi^{NFA})((\alpha - 1)(\bar{c} + X_t) + \alpha Z_{i,t})] - \sum_{i=1}^{N} (\alpha - 1)q_i,M(\bar{c} - \mu_c) + \mathbb{E}[\bar{q}_0(c, X_t)]$$

$$+ \mathbb{E}\left[\Pi^{NFA}(\bar{v}_{(1:N)}(c_{(1:N)})) + X_t - \bar{q}_0(c, X_t)\right]$$

$$(b) \quad \sum_{i=1}^{N} \mathbb{E}[A_i(1 - \Pi^{NFA})((\alpha - 1)(\bar{c} + X_t) + \alpha Z_{i,t})] - \sum_{i=1}^{N} (\alpha - 1)q_i,M(\bar{c} - \mu_c)$$

$$+ \mathbb{E}\left[\sum_{i=1}^{M} A_i(c_i + X_t) (1 - \Pi^{NFA}) + \mathbb{E}\left[\Pi^{NFA}\left(\hat{v}(c_{(1:N)}) + (\alpha - 1)A_{(1:N)} \frac{F(c_{(1:N)})}{f(c_{(1:N)})} + X_t\right)\right]\right],$$

where, $\hat{v}(c) = c + \frac{F(c)}{f(c)}$, (a) follows from facts that $\pi_{i,t}(\beta(\bar{c}), \bar{c}, \beta) = \mathbb{E}[A_i(1 - \Pi^{NFA})((\alpha - 1)(\bar{c} + X_t) + \alpha Z_{i,t})]$ and $r_{i,t}(\beta(c), p_i) = \mathbb{I}\{\beta(c_i) < \beta(c_j), \forall j \neq i, \beta(c_i) < p(c, X_t, Z_t)\}$, (b) follows by $\bar{q}_0(c, X_t) = \sum_{i=1}^{M} A_i(c_i + X_t)$ and $\bar{v}_{(1:N)}(c_{(1:N)}) = \hat{v}(c_{(1:N)}) + (\alpha - 1)A_{(1:N)} \frac{F(c_{(1:N)})}{f(c_{(1:N)})}$ by definition of
\( \bar{v}_i(c_i) \). Thus, substituting the expression of \( P_t^{FPA} \), one obtains

\[
P_t^{NFA} - P_t^{FPA} = \sum_{i=1}^{N} \mathbb{E}[A_i(1 - \mathbb{1}_{NFA})(\alpha - 1)(\bar{\epsilon} + X_t) + \alpha Z_{i,t}] - \sum_{i=1}^{N}(\alpha - 1)q_{i,M}(\bar{\epsilon} - \mu_c) \\
+ \mathbb{E} \left[ \sum_{i=1}^{M} A_i(c_i + X_t)(1 - \mathbb{1}_{NFA}) \right] + \mathbb{E} \left[ \mathbb{1}_{NFA} \left( \hat{\nu}(c_{(1:N)}) + (\alpha - 1)A_{(1:N)} \frac{F(c_{(1:N)})}{f(c_{(1:N)})} + X_t \right) \right] \\
- \mathbb{E} \left[ \hat{\nu}(c_{(1:N)}) + X_t \right] \\
\equiv (c) \left[ \left( \sum_{i=1}^{N} A_i Z_{i,t} + \sum_{i=1}^{M} A_i c_i - \hat{\nu}(c_{(1:N)}) \right) (1 - \mathbb{1}_{NFA}) \right] \\
+(\alpha - 1)\mathbb{E} \left[ \sum_{i=1}^{N} A_i(1 - \mathbb{1}_{NFA})(\bar{\epsilon} + Z_{i,t} + X_t) - \sum_{i=1}^{N} q_{i,M}(\bar{\epsilon} - \mu_c) + A_{(1:N)} \frac{F(c_{(1:N)})}{f(c_{(1:N)})} \right] \mathbb{1}_{NFA}
\]  

(A-1)

Here, (c) follows by the fact that \( \sum_{i=1}^{M} A_i = 1 \).

1. **Concentrated Markets.** First, we show the results for concentrated markets with \( \alpha = 1 \).

Without loss of generality, we assume \( M = N \). By the above argument, one has

\[
\lim_{N \to \infty} \left[ P_t^{NFA} - P_t^{FPA} \right] = \lim_{N \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{N} A_i Z_{i,t} + \sum_{i=1}^{N} A_i c_i - \hat{\nu}(c_{(1:N)}) \right) (1 - \mathbb{1}_{NFA}) \right] \\
\equiv (d) \lim_{N \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{N} A_i (Z_{i,t} + c_i) - \zeta \right) (1 - \mathbb{1}_{NFA}) \right] \\
\geq 0,
\]

where (d) follows from applying the Bounded convergence theorem to \( \lim_{N \to \infty} \mathbb{E} \left[ \left( \hat{\nu}(c_{(1:N)}) - \zeta \right) (1 - \mathbb{1}_{NFA}) \right] = 0 \), because \( c_{(1:N)} \), the lowest private cost among the \( N \) participants of the auction stage, converges to \( \zeta \) in probability when \( N \to \infty \). The inequality follows from the fact that \( \sum_{i=1}^{N} A_i (Z_{i,t} + c_i) - \zeta = \sum_{i=1}^{N} A_i (Z_{i,t} + c_i - \zeta) \geq 0 \) and \( 1 - \mathbb{1}_{NFA} \geq 0 \) for any sample path.

2. **Diffused Markets.** We note that \( c_{(1:N)} \), the lowest private cost among the \( N \) participants of the auction stage, converges to \( \underline{\zeta} \) in probability as \( N \to \infty \). Also, \( \lim_{M \to \infty} \sum_{i=1}^{N} q_{i,M} = 0 \) and \( \frac{F(c_{(1:N)})}{f(c_{(1:N)})} \to 0 \) in probability when \( N \to \infty \). By Bounded Convergence Theorem, one has
\[
\lim_{N \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \sum_{i=1}^{N} A_i (1 - \mathbb{I}^{\text{NFA}}) (\bar{\beta} + Z_{i,t} + X_t) - \sum_{i=1}^{N} q_{i,M}(\bar{\beta} - \mu_c) + A_{(1:N)} \frac{F(c_{(1:N)})}{f(c_{(1:N)})} \mathbb{I}^{\text{NFA}} \right] = 0,
\]

\[
\lim_{N \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \sum_{i=1}^{N} A_i Z_{i,t} (1 - \mathbb{I}^{\text{NFA}}) \right] = 0,
\]

(A-2)

where the order of the limits is consistent with the diffused market assumption.

Now, note that \( \hat{\beta}(c_{(1:N)}) - c_{(1:N)} = \frac{F(c_{(1:N)})}{f(c_{(1:N)})} \) converges in probability to 0 (recall that \( f(c) > 0 \)).

By bounded convergence theorem, we have:

\[
\lim_{N \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{M} A_i \bar{\beta}_i - \hat{\beta}(c_{(1:N)}) \right) (1 - \mathbb{I}^{\text{NFA}}) \right] = \lim_{N \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{M} A_i \bar{\beta}_i - c_{(1:N)} \right) (1 - \mathbb{I}^{\text{NFA}}) \right].
\]

Note that:

\[
\lim_{N \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{M} A_i \bar{\beta}_i - c_{(1:N)} \right) (1 - \mathbb{I}^{\text{NFA}}) \right] = \lim_{N \to \infty} \mathbb{E} \left[ (c - c_{(1:N)}) \mathbb{I} \{ \beta_{\infty}^{\text{NFA}}(c_{(1:N)}) \geq c + Z_t + X_t \} \right] = \mathbb{E} \left[ (c - c_{(1:N)}) \mathbb{I} \{ \beta_{\infty}^{\text{NFA}}(c) \geq c + Z_t + X_t \} \right] \quad (A-3)
\]

Here, \( \beta_{\infty}^{\text{NDA}}(\cdot) \) is the bidding strategy under naive FA with \( M \) potential suppliers. The first equitation follows because \( (c, Z_t) \) has the same distribution as \( (c_i, Z_{i,t}) \) (which for all \( i \) share the same marginal distribution), and it is independent of \( c_{(1:N)} \), because of the diffused market assumption.

The second equation follows by the bounded convergence theorem.

Now, extending Proposition 1 to multiple periods:

\[
\beta_{\infty}^{\text{NFA}}(c) = c + \sum_{t=1}^{T} \mathbb{E} \left[ X_t \mathbb{I} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \} \right] \frac{\mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \}}{\mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \}}
\]

\[
+ \sum_{t=1}^{T} \int_{c}^{c_{(1:N)}} \mathbb{E}^{\text{NFA}}(y) \cdot \mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(y) \leq \alpha(c_j + X_t + Z_{j,t}) \} dy \frac{\mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \}}{\mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \}}
\]

\[
\geq c + \min_{t} \frac{\mathbb{E} \left[ X_t \mathbb{I} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \} \right]}{\mathbb{P} \{ \beta_{\infty}^{\text{NFA}}(c) \leq \alpha(c_j + X_t + Z_{j,t}) \}} \geq c + \mathbb{E}[X_t], \quad (A-4)
\]

where \( \hat{t} \) achieves the minimum. Hence:
\[ \lim_{N \to \infty} \mathbb{E} \left[ P^{NDFA} - P^{FPA} \right] \geq \lim_{N \to \infty} \sum_{t=1}^{T} P^{NFA}_t - P^{FPA}_t \]
\[ = \sum_{t=1}^{T} \mathbb{E} \left[ (c - \xi) \mathbb{I} \{ \beta^{NDFA}_\infty(\xi) \geq c + Z_t + X_t \} \right] \]
\[ \geq \mathbb{E} \left[ (c - \xi) \mathbb{I} \{ \beta^{NDFA}_\infty(\xi) \geq c + Z_t + X_t \} \right] \]
\[ \geq \mathbb{E} \left[ (c - \xi) \mathbb{I} \{ \xi + \mathbb{E}[X_t] \geq c + Z_t + X_t \} \right] > 0, \quad (A-5) \]

where the second expression follows by (A-1), (A-2) and (A-3), the third because \( c - \xi \geq 0 \), the fourth by (A-4), and the last because \( z + x < \mathbb{E}[X_t] \). This concludes the proof. \( \square \)

**Proof of Proposition 8.** The proof follows ideas similar to the ones described in the proof of Proposition 6. Consider the expected total profits from period 1 to period \( T \) for bidder \( i \) when he bids \( b_i \), his cost is \( c_i \), and his competitors use equilibrium strategy \( \beta_{-i} \):

\[ \pi_i(b_i, c_i, \beta_{-i}) = \sum_{t=1}^{T} \pi_{i,t}(b_i, c_i, \beta_{-i}), \quad \text{where,} \]
\[ \pi_{i,t}(b_i, c_i, \beta_{-i}) = \mathbb{E}_{-i} \left[ (b_i - c_i - X_t) r_{i,t}(b_i, \beta_{-i}(c_{-i}), p_i(c, X_t, Z_t)) \right. \]
\[ + \left. [p_i(c, X_t, Z_{i,t}) - c_i - X_t] \mathbb{I} \{ b_i < \beta(c_j), \forall j \neq i \} \mathbb{I} \{ b_i \geq p_i(c, X_t, Z_{i,t}) \} \right] \quad (A-6) \]

where the first term is the profit in the event that bidder \( i \) is the FA winner and his bid is below his own spot market price, and the second term is the profit when he is the FA winner but loses to his own spot market price. Substituting \( p_i(c, X_t, Z_t) = \alpha (c_i + X_t + Z_{i,t}) \) into (A-6), one has:

\[ \pi_{i,t}(b_i, c_i, \beta_{-i}) = \mathbb{E}_{-i} \left[ (b_i - c_i - X_t) \mathbb{I} \{ b_i < \beta(c_j), \forall j \neq i \} \mathbb{I} \{ b_i < \alpha (c_i + X_t + Z_{i,t}) \} \right. \]
\[ + \left. ((\alpha - 1)(c_i + X_t) + \alpha Z_{i,t}) \cdot \mathbb{I} \{ b_i < \beta(c_j), \forall j \neq i \} \mathbb{I} \{ b_i \geq \alpha (c_i + X_t + Z_{i,t}) \} \right] \]
\[ = \mathbb{E}_{-i} \left[ \mathbb{I} \{ b_i < \beta(c_j), \forall j \neq i \} \cdot \int \left( \int_{b_i - c_i - x}^{b_i} f_{X_t}(x)dx \right) f_{Z_{i,t}}(z)dz \right] \]
\[ + \mathbb{E}_{-i} \left[ \mathbb{I} \{ b_i < \beta(c_j), \forall j \neq i \} \cdot \int \left( \int_{b_i - c_i - x}^{b_i} [(\alpha - 1)(c_i + x) + \alpha z] f_{X_t}(x)dx \right) f_{Z_{i,t}}(z)dz \right]. \]

Therefore, taking derivative with respect to \( c_i \), one has:
\[\frac{\partial \pi_{i,t}(b_i, c_i, \beta_{-i})}{\partial c_i}\bigg|_{b_i = \beta(c_i)} = (a) \mathbb{E}_{-i}\left[\mathbb{I}\{c_i < c_j, \forall j \neq i\} \cdot \left(-FX_t\left(\frac{b_i}{\alpha} - c_i - Z_{i,t}\right) + \left(\frac{(\alpha - 1)b_i}{\alpha} + Z_{i,t}\right) f_X(t) \left(\frac{b_i}{\alpha} - c_i - Z_{i,t}\right)\right) + \mathbb{I}\{c_i < c_j, \forall j \neq i\} \cdot (\alpha - 1)FX_t\left(\frac{b_i}{\alpha} - c_i - Z_{i,t}\right) - \left(\frac{(\alpha - 1)b_i}{\alpha} + Z_{i,t}\right) f_X(t) \left(\frac{b_i}{\alpha} - c_i - Z_{i,t}\right)\right]\]

\[= (b) \alpha \mathbb{E}_{-i}\left[-\mathbb{I}\{c_i < c_j, \forall j \neq i\} \cdot \left(FX_t\left(\frac{b_i}{\alpha} - c_i - Z_{i,t}\right) - (\alpha - 1)/\alpha\right)\right]\]

\[= (c) \alpha \mathbb{E}_{-i}\left[-r_i,t(\beta(c), p_i(c, X_t, Z_t)) + \mathbb{I}\{c_i < c_j, \forall j \neq i\} (\alpha - 1)/\alpha\right],\]

where: (a) follows from applying Leibniz rule; (b) follows by re-arranging and simplifying terms; and (c) follows from the definition of the equilibrium allocation in a monitored FA, in which the FA winner is the lowest cost supplier given that equilibrium bids are strictly increasing. Let \(p_{i,t} = p_i(c, X_t, Z_t)\), applying the envelop theorem yields:

\[\pi_i(\beta, \bar{c}, \beta_{-i}) - \pi_i(\beta(c_i), c_i, \beta_{-i}) = \int_{c_i}^\bar{c} \frac{\partial \pi_i(b_i, s, \beta)}{\partial s} |_{b_i = \beta(s)} ds = \sum_{t=1}^T \int_{c_i}^\bar{c} \mathbb{E}_{-i} \left[ -\alpha r_{i,t}(\beta(s), \beta_{-i}(c_{-i}), p_{i,t}) + \mathbb{I}\{s < c_j, \forall j \neq i\} (\alpha - 1)\right] ds,\]

where we omit the arguments of \(p(\cdot)\) to simplify notation. Note that

\[\pi_i(\beta(c_i), c_i, \beta_{-i}) = \sum_{t=1}^T \mathbb{E}_{-i} \left[ N_{i,t}(\beta(c), p_{i,t}) - (c_i + X_t)r_{i,t}(\beta(c), p_{i,t}) + \mathbb{I}\{p_{i,t} - c_i - X_t\} \mathbb{I}\{c_i < c_j, \forall j \neq i\} \mathbb{I}\{\beta(c_i) \geq p_{i,t}\} \right].\]

Combining the above two equations, and using the fact that \(\pi_{i,t}(\beta, \bar{c}, \beta_{-i}) = 0\), one has

\[\sum_{t=1}^T \mathbb{E}_{-i} \left[ N_{i,t}(\beta(c), p_{i,t}) \right] = \sum_{t=1}^T \left\{ \mathbb{E}_{-i} \left[ (c_i + X_t)r_{i,t}(\beta(c), p_{i,t}) - \mathbb{I}\{p_{i,t} - c_i - X_t\} \mathbb{I}\{c_i < c_j, \forall j \neq i\} \mathbb{I}\{\beta(c_i) \geq p_{i,t}\} \right] \right.\]

\[\left. + \int_{c_i}^\bar{c} \mathbb{E}_{-i} \left[ -\alpha r_{i,t}(\beta(s), \beta_{-i}(c_{-i}), p_{i,t}) + \mathbb{I}\{s < c_j, \forall j \neq i\} (\alpha - 1)\right] ds \right\}.\]

If the auctioneer does not buy from one of the FA bidders in the FA, she buys from the spot market. Therefore, the expected total buying price for the monitored FA from period 1 to period \(T\) is:

\[\mathbb{E}[P_{MFA}] = \mathbb{E} \left[ \sum_{t=1}^T \sum_{i=1}^N m_{i,t}(\beta(c), p_{i,t}) + \sum_{t=1}^T \sum_{i=1}^N \mathbb{I}\{c_i < c_j, \forall j \neq i\} \mathbb{I}\{\beta(c_i) \geq p_{i,t}\} \right] = \sum_{t=1}^T P_{t,MFA}^M,\]

where
Since \( \sum_{t=1}^{N} \{ \mathbb{E} \left[ (c_i + X_t) r_{i,t}(\beta(e), p_{i,t}) - [p_{i,t} - c_i - X_t] \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} \right] + \int_{c_i}^{E} \mathbb{E} \left[ \alpha r_{i,t}(\beta(s), \beta_{-i}(e_{-i}), p_{i,t}) - \mathbb{I} \{ s < c_j, \forall j \neq i \} (\alpha - 1) \right] ds + \mathbb{E} \left[ \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} p_{i,t} \right] \right) \)

\[ P_t^{MFA} = \sum_{i=1}^{N} \left\{ \mathbb{E} \left[ (c_i + X_t) r_{i,t}(\beta(e), p_{i,t}) - [p_{i,t} - c_i - X_t] \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} \right] + \int_{c_i}^{E} \mathbb{E} \left[ \alpha r_{i,t}(\beta(s), \beta_{-i}(e_{-i}), p_{i,t}) - \mathbb{I} \{ s < c_j, \forall j \neq i \} (\alpha - 1) \right] ds + \mathbb{E} \left[ \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} p_{i,t} \right] \right\} \]

\[ = \mathbb{E} \left[ \sum_{i=1}^{N} r_{i,t}(\beta(e), p_{i,t}) (c_i + X_t) \right] + \mathbb{E} \left[ \sum_{i=1}^{N} (c_i + X_t) \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} \right] + \mathbb{E} \left[ \sum_{i=1}^{N} \int_{c_i}^{E} \alpha r_{i,t}(\beta(s), \beta_{-i}(e_{-i}), p_{i,t}) ds \right] - (\alpha - 1) \sum_{i=1}^{N} \mathbb{E}[c_{(1:N),-i} - c_i]^+] \]

\[ \equiv (a) \mathbb{E} \left[ \sum_{i=1}^{N} (c_i + X_t) \mathbb{I} \{ c_i < c_j, \forall j \neq i \} \left( 1 - \sum_{j=1}^{N} r_{j,t}(\beta(e), p_{i,t}) \right) \right] + \mathbb{E} \left[ \sum_{i=1}^{N} r_{i,t}(\beta(e), p_{i,t}) \left( c_i + \frac{F(c_i)}{f(c_i)} + X_t \right) \right] - (\alpha - 1) \sum_{i=1}^{N} \mathbb{E}[c_{(1:N),-i} - c_i]^+] , \]

where the last equation established by changing the order of integration and since

\[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(e), p_{i,t})) \mathbb{I} \{ c_i < c_j, \forall j \neq i \} = \mathbb{I} \{ \beta(c_i) \geq p_{i,t} \} \mathbb{I} \{ c_i < c_j, \forall j \neq i \} . \]

Since \( \sum_{i=1}^{N} (c_i + X_t) \mathbb{I} \{ c_i < c_j, \forall j \neq i \} = c_{(1:N)} + X_t = q_0(e, X_t) \), one has,

\[ P_t^{MFA} = - (\alpha - 1) \sum_{i=1}^{N} \mathbb{E}[c_{(1:N),-i} - c_i]^+] + \mathbb{E} \left[ q_0(e, X_t) + \sum_{i=1}^{N} r_{i,t}(\beta(e), p_{i,t}) (v(c_i) + X_t - q_0(e, X_t)) \right] . \]

This concludes the proof.
Proof of Theorem 1. By Propositions 6 and 8 (also refer to proof of Proposition 7), we have

\[ E[P_{NFA}^t] - E[P_{MFA}^t] \geq \sum_{t=1}^{T} (P_{NFA}^t - P_{MFA}^t), \]

where:

\[ P_{NFA}^t - P_{MFA}^t \]

\[ \overset{(a)}{=} \sum_{i=1}^{N} E[A_i(1 - 1_{NFA})(\alpha - 1)(\bar{e} + X_t + \alpha Z_{i,t})] - \sum_{i=1}^{N} (\alpha - 1)q_{i,M}(\bar{e} - E[c_i]) \]

\[ + E \left[ \sum_{i=1}^{M} A_i(c_i + X_t)(1 - 1_{NFA}) \right] + E \left[ E_{NFA} \left( v(c_{(1:N)}) - (\alpha - 1)(1 - A_{(1:N)}) \frac{F(c_{(1:N)})}{f(c_{(1:N)})} + X_t \right) \right] \]

\[ + (\alpha - 1) \sum_{i=1}^{N} E[c_{(1:N),-i} - c_i] + E[c_{(1:N)} + X_t] - E \left[ E_{MFA} \left( v(c_{(1:N)}) - c_{(1:N)} \right) \right] \]

\[ = E \left[ \sum_{i=1}^{N} A_iZ_{i,t} + \sum_{i=1}^{M} A_i(c_i - c_{(1:N)}) \right] (1 - 1_{NFA}) + E \left[ (v(c_{(1:N)}) - c_{(1:N)})(1_{NFA} - 1_{MFA}) \right] \]

\[ + (\alpha - 1) E \left[ \sum_{i=1}^{N} A_i(1 - 1_{NFA})(\bar{e} + X_t + Z_{i,t}) - \sum_{i=1}^{N} q_{i,M}(\bar{e} - E[c_i]) \right] \]

\[ + (\alpha - 1) E \left[ \sum_{i=1}^{N} [c_{(1:N),-i} - c_i] + (1 - A_{i,N}) \frac{F(c_{(1:N)})}{f(c_{(1:N)})} 1_{NFA} \right], \]

where \( 1_{NFA} = I\{\beta_{NFA}(c_{(1:N)}) < p(c, X_t, Z_t)\}, 1_{MFA} = I\{\beta_{MFA}(c_{(1:N)}) < \alpha(c_{(1:N)} + X_t + Z_{(1:N),t})\}. \)

(a) follows by \( \bar{e}_i(c) = c + (1 + (\alpha - 1)A_i)\frac{F(c)}{f(c)} = c + \alpha\frac{F(c)}{f(c)} - (\alpha - 1)(1 - A_i)\frac{F(c)}{f(c)} = v(c) - (\alpha - 1)(1 - A_i)\frac{F(c)}{f(c)}. \)

1. Concentrated Markets. First, we show the results for concentrated markets with \( \alpha = 1. \)

Without loss of generality, we assume \( M = N. \) One has

\[ P_{NFA}^t - P_{MFA}^t = E \left[ \left( \sum_{i=1}^{N} A_iZ_{i,t} + \sum_{i=1}^{M} A_i(c_i - c_{(1:N)}) \right) (1 - 1_{NFA}) \right] + E \left[ (v(c_{(1:N)}) - c_{(1:N)})(1_{NFA} - 1_{MFA}) \right]. \]

We note that \( c_{(1:N)}, \) the lowest private cost among the \( N \) participants of the auction stage, converges to \( c \) in probability as \( N \to \infty. \) In addition, \( v(c_{(1:N)}) - c_{(1:N)} = \frac{F(c_{(1:N)})}{f(c_{(1:N)})} \) converges in probability to 0 (recall that \( f(c) > 0 \)). Since \( |1_{NFA} - 1_{MFA}| \leq 2, \) by bounded convergence theorem, we have:

\[ E \left[ (v(c_{(1:N)}) - c_{(1:N)})(1_{NFA} - 1_{MFA}) \right] \to 0 \text{ as } N \to \infty. \]

(A-7)

Using a similar argument to Proposition 7, one then shows that \( \lim_{N \to \infty} [P_{NFA}^t - P_{MFA}^t] \geq 0. \)

2. Diffused Markets. Using the diffused market assumption in which \( \sum_{i=1}^{N} q_{i,M} \to 0 \) as \( M \) goes infinity, one can show that:

\[ \lim_{N \to \infty} \lim_{M \to \infty} E \left[ \sum_{i=1}^{N} A_i(1 - 1_{NFA})(\bar{e} + X_t + Z_{i,t}) - \sum_{i=1}^{N} q_{i,M}(\bar{e} - c_i) \right] = 0, \]
where the order of the limits is consistent with the diffused market assumption. The previous
equation, together with (A-7), that \( \frac{F(c_{i,1:N})}{f(c_{i,1:N})} \) converges in probability to 0, and that \([c_{i,1:N} - c_i] + 0 \), implies that:

\[
\lim_{N \to \infty} \lim_{M \to \infty} P_{t}^{MFA} - P_{t}^{NFA} \leq \lim_{N \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \left( \sum_{i=1}^{M} A_i c_i - c_{1:N} \right) \right] (1 - \mathbb{I}_{NFA})
\]

The rest of the proof follows like in Proposition 7.

**Proof of Theorem 2.** From Proposition 8 and Proposition 10, one has \( \mathbb{E}[P_{t}^{FLE} - P_{t}^{MFA}] = \sum_{t=1}^{T} (P_{t}^{FLE} - P_{t}^{MFA}) \), where

\[
P_{t}^{FLE} - P_{t}^{MFA} = \mathbb{E}[Z_t] + \mathbb{E} \left[ \mathbb{I}_{t}^{FLE}(v(c_{1:N}) - Z_t) \right] - \mathbb{E} \left[ c_{1:N} \right] - \mathbb{E} \left[ \mathbb{I}_{t}^{MFA}(v(c_{1:N}) - c_{1:N}) \right]
\]

\[
= \mathbb{E} \left[ (1 - \mathbb{I}_{t}^{FLE})(Z_t - c_{1:N}) \right] + \mathbb{E} \left[ \mathbb{I}_{t}^{FLE} - \mathbb{I}_{t}^{MFA} \right] (v(c_{1:N}) - c_{1:N})
\]

Where \( \mathbb{I}_{t}^{FLE} = \mathbb{I} \{ c_{1:N} \leq Z_t \} + \mathbb{I} \{ \beta^{FLE}(c_{1:N}) \leq Z_t + X_t, c_{1:N} > Z_t \} \) and

\( \mathbb{I}_{t}^{MFA} = \mathbb{I} \{ \beta^{MFA}(c_{1:N}) \leq c_{1:N} + X_t + Z_{(1:N),t} \} \). Since \( c_{1:N} \) converges to \( c \) in probability as \( N \to \infty \) and \( c \leq \bar{z} \) (recall \( Z_t = \sum_{j=1}^{\infty} A_j (c_j + Z_{j,t}) \)), \( 1 - \mathbb{I}_{t}^{MFA} \) converges in probability to zero, thus, by bounded convergence theorem, \( \mathbb{E} \left[ (1 - \mathbb{I}_{t}^{FLE})(Z_t - c_{1:N}) \right] \) converges to zero. In addition, \( v(c_{1:N}) - c_{1:N} = \frac{F(c_{1:N})}{f(c_{1:N})} \) converges in probability to 0 (recall that \( f(c) > 0 \)), and \( \| \mathbb{I}_{t}^{FLE} - \mathbb{I}_{t}^{MFA} \| \leq 2 \), by bounded convergence theorem, \( \mathbb{E} \left[ (\mathbb{I}_{t}^{FLE} - \mathbb{I}_{t}^{MFA}) (v(c_{1:N}) - c_{1:N}) \right] \) converges to zero.

Next, we show that \( \mathbb{E}[P_{t}^{MFA} - P_{t}^{FPA}] = \sum_{t=1}^{T} (P_{t}^{MFA} - P_{t}^{FPA}) \to 0 \) as \( N \to \infty \). From the proof of Proposition 9, one has

\[
P_{t}^{MFA} - P_{t}^{FPA} = \mathbb{E} \left[ (\mathbb{I}_{t}^{MFA} - 1)F(c_{1:N})/f(c_{1:N}) \right]
\]

Again, as \( N \to \infty \), \( P_{t}^{MFA} - P_{t}^{FPA} \to 0 \) by the bounded convergence theorem. Finally, we show that \( \mathbb{E}[P_{t}^{FLE} - P_{t}^{OPT}] \to 0 \) as \( N \to \infty \). According to the Appendix E, one has

\[
\mathbb{E}[P_{t}^{OPT}] = \sum_{t=1}^{T} P_{t}^{OPT}, \quad \text{where,} \quad P_{t}^{OPT} = \mathbb{E} \left[ Z_t + X_t + \min \{0, v(c_{1:N}) - Z_t \} \right]
\]

Together with Proposition 10, one has

\[
P_{t}^{FLE} - P_{t}^{OPT} = \mathbb{E} \left[ \mathbb{I}_{t}^{FLE}(v(c_{1:N}) - Z_t) \right] - \mathbb{E} \left[ \min \{0, v(c_{1:N}) \} - Z_t \right]
\]

\[
= \mathbb{E} \left[ (\mathbb{I}_{t}^{FLE} - \mathbb{I}_{t}^{OPT})(v(c_{1:N}) - Z_t) \right]
\]

where \( \mathbb{I}_{t}^{OPT} = \mathbb{I} \{ v(c_{1:N}) \leq Z_t \} \). As \( N \to \infty \), since \( v(c_{1:N}) \) convergence to \( c \) in probability, and \( c \leq \bar{z} \), \( \mathbb{I}_{t}^{OPT} \to 1 \) and \( \mathbb{I}_{t}^{FLE} \to 1 \) in probability. Thus, \( P_{t}^{FLE} - P_{t}^{OPT} \to 0 \), by the bounded convergence theorem. This completes the proof. \( \Box \)
B Additional Proofs

Proof of Proposition 2. Taking derivatives of bidder $i$’s profit with respect to $c_i$, and using

$$\frac{\partial \mathbb{E} \left[ \min \{b_i, \alpha(X + c_i + Z_i)\} - c_i - X \right]}{\partial c_i} = \mathbb{E} \left[ -\mathbb{I} \{b_i \leq \alpha(X + c_i + Z_i)\} + (\alpha - 1)\mathbb{I} \{b_i > \alpha(X + c_i + Z_i)\} \right]$$

one obtains the partial derivative specified in the proposition. Applying the envelope theorem, one has that:

$$\pi_i (\beta(c_i), c_i, \beta) = \pi_i (\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i (\beta(c), c, \beta)}{\partial c} dc.$$

The bidder’s equilibrium profit can be simplified as:

$$\pi_i (\beta(c_i), c_i, \beta) = \mathbb{E} \left[ \mathbb{I} \{c_i \leq c_{(1:N),-i}\} \left( \sum_{j=1}^{M} q_{j,M} \mathbb{I} \{\beta(c_i) \leq \alpha(X + c_j + Z_j)\} \right) \cdot (\beta(c_i) - c_i - X) 
+ q_{i,M} \mathbb{I} \{c_i \leq c_{(1:N),-i}, \beta(c_i) > \alpha(X + c_i + Z_i)\} (\alpha(X + c_i + Z_i) - c_i - X) 
+ q_{i,M} \mathbb{I} \{c_i > c_{(1:N),-i}, \beta(c_{(1:N),-i}) > \alpha(X + c_i + Z_i)\} (\alpha(X + c_i + Z_i) - c_i - X) \right]$$

Also, the profit associated with the highest cost $\bar{c}$ is strictly positive because of the potential value available for suppliers at the spot market:

$$\pi_i (\beta(\bar{c}), \bar{c}, \beta) = q_{i,M} \mathbb{E} \left[ (\alpha(\bar{c} + X + Z_i) - \bar{c} - X) \mathbb{I} \{\beta(c_{(1:N),-i}) > \alpha(X + \bar{c} + Z_i)\} \right].$$

All together, one obtains the integral equations in the proposition. This concludes the proof.

Proof of Proposition 3. Under monitored FA, bidder’s profit can be written as
\[ \pi_i(b_i, c_i, \beta) = \mathbb{E} \left[ \mathbb{I}\{b_i \leq \beta(c_{(1:N)}, -i)\}\mathbb{I}\{b_i \leq \alpha(c_i + X + Z_i)\}(b_i - c_i - X) \right] \\
+ \mathbb{E} \left[ \mathbb{I}\{b_i \leq \beta(c_{(1:N)}, -i)\}\mathbb{I}\{b_i > \alpha(c_i + X + Z_i)\}(\alpha(c_i + X + Z_i) - c_i - X) \right] \\
= \bar{F}^{N-1}(\beta^{-1}(b_i)) \int_{\alpha - c_i - Z}^{b_i} (b_i - c_i - x) f_X(x)f_Z(z)dx dz \\
+ \bar{F}^{N-1}(\beta^{-1}(b_i)) \int_{\alpha - c_i - Z}^{b_i} (\alpha(c_i + x + z) - c_i - x) f_X(x)f_Z(z)dx dz. \]

Taking derivative with respect to \( c_i \), one has

\[ \frac{\partial \pi_i(b_i, c_i, \beta)}{\partial c_i} \bigg|_{b_i=\beta(c_i)} = \bar{F}^{N-1}(c_i) \mathbb{E} \left[ -\mathbb{I}\{c_i \leq \alpha(c_i + X + Z_i)\} + (\alpha - 1)\mathbb{I}\{c_i > \alpha(c_i + X + Z_i)\} \right] \]

Applying the envelop theorem, one has:

\[ \pi_i(\beta(c_i), c_i, \beta) = \pi_i(\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(\beta(c), c, \beta)}{\partial c} dc \\
= \pi_i(\beta(\bar{c}), \bar{c}, \beta) \\
+ \int_{c_i}^{\bar{c}} \bar{F}^{N-1}(c) \cdot \mathbb{E} \left[ \mathbb{I}\{c \leq \alpha(c + X + Z_i)\} - (\alpha - 1)\mathbb{I}\{c > \alpha(c + X + Z_i)\} \right] dc. \]

Bidder’s equilibrium profit is given by:

\[ \pi_i(\beta(c_i), c_i, \beta) = \bar{F}^{N-1}(c_i) \mathbb{E} \left[ \mathbb{I}\{c_i \leq \alpha(c_i + X + Z_i)\}(\beta(c_i) - c_i - X) \right] \\
+ \bar{F}^{N-1}(c_i) \mathbb{E} \left[ \mathbb{I}\{c_i > \alpha(c_i + X + Z_i)\}(\alpha(c_i + X + Z_i) - c_i - X) \right]. \]

Taking \( \pi_i(\beta(\bar{c}), \bar{c}, \beta) = 0 \), one obtains:

\[ \beta^{MFA}(c_i) = c_i + \frac{\mathbb{E} \left[ X \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]}{\mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]} \\
+ \frac{\int_{c_i}^{\bar{c}} \bar{F}^{N-1}(c) \cdot \mathbb{E} \left[ \mathbb{I}\{c \leq \alpha(c + X + Z_i)\} \right] dc}{\bar{F}^{N-1}(c_i) \mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]} \left( \frac{\bar{F}^{N-1}(c_i) \mathbb{E} \left[ (\alpha - 1)\mathbb{I}\{\beta(c) \geq \alpha(c + X + Z_i)\} \right]}{\bar{F}^{N-1}(c_i) \mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]} \right) \\
- \frac{\mathbb{E} \left[ \mathbb{I}\{\beta(c_i) > \alpha(c_i + X + Z_i)\}(\alpha(c_i + X + Z_i) - c_i - X) \right] \mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]}{\mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]} \mathbb{E} \left[ \mathbb{I}\{\beta(c_i) \leq \alpha(c_i + X + Z_i)\} \right]. \]

This concludes the proof. \( \square \)

**Proof of Proposition 4.** First, we introduce the following definition. A twice-differentiable function \( h : \mathbb{R}^2 \rightarrow \mathbb{R} \) is called supermodular or log-supermodular, respectively, if for all \( x \) and \( \theta \):

\[ \frac{\partial^2}{\partial x \partial \theta} h(x, \theta) \geq 0, \text{ or if } h > 0 \quad \frac{\partial^2}{\partial x \partial \theta} \ln(h(x, \theta)) \geq 0. \]

Note that these are sufficient conditions for supermodularity. There are weaker related conditions that do not require differentiability and use function differences for the case of discrete actions.
Definition 1. (Athey 2001) The Single Crossing Condition (SCC) for games of incomplete information is satisfied if for each \( i = 1, 2, \ldots, N \), whenever every opponent \( j \neq i \) uses a strategy \( \beta_j \) that is increasing, player \( i \)'s profit function, \( \pi_i(b, c_i, \beta_{-i}) \) is supermodular or log-supermodular in \((b_i, c_i)\).

Now, we are ready to prove the proposition.

(i) Establishing SCC for monitored FA. By Proposition 3, one has

\[
\pi_i(b, c, \beta) = \mathbb{P}[i \text{ wins with } b] \cdot \mathbb{E}[(b - c - X)I\{b \leq \alpha(c + X + Z_i)\} + (\alpha - 1)I\{b > \alpha(c + X + Z_i)\}].
\]

Taking derivatives with respect to \( c \), one obtains:

\[
\frac{\partial \pi_i(b, c, \beta)}{\partial c} = \mathbb{P}[i \text{ wins with } b] \cdot \mathbb{E}[\left(-I\{b \leq \alpha(c + X + Z_i)\} + (\alpha - 1)I\{b > \alpha(c + X + Z_i)\}\right)]
\]

where \( \mathbb{P}[i \text{ wins with } b] \) is the probability bidder \( i \) defeats its competitors’ with a bid \( b \):

\[
\mathbb{P}[i \text{ wins with } b] = \mathbb{P}(\beta_j(c_j) > b, \text{ for all } j \neq i) + \sum_{k=1}^{N-1} \mathbb{P}[(\text{exactly } k \text{ bidders other than } i \text{ bid } b \text{ and the rest higher than } b)] \cdot \frac{k + 1}{k + 1}.
\]

It can be easily shown that the winning probability \( \mathbb{P}[i \text{ wins with } b] \) is decreasing in \( b \), i.e., a higher bid induces a lower winning probability. Obviously, \( \mathbb{P}[b \leq \alpha(c + X + Z_i)] \) is decreasing in \( b \) and \( \alpha \mathbb{P}[b \leq \alpha(c + X + Z_i)] - (\alpha - 1) \geq 0 \) since \( \alpha \leq 1 \). Thus, the partial derivative \( \frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} \) is increasing in \( b \). Therefore, by Definition 1, SCC is satisfied.

(ii) Establishing SCC for naive FA in diffused market. Recall that the private costs \( \{c_j\} \) are i.i.d. and independent with the common cost \( X \). For any increasing profile \( \beta_{-i} \), we have

\[
\pi_i(b, c, \beta_{-i}) = \mathbb{P}[i \text{ wins with } b] \cdot \mathbb{E}[(b - c - X)I\{b \leq \alpha(X + c_j + Z_j)\}],
\]

where \( \mathbb{P}[i \text{ wins with } b] \) is same as the one given above in (i). Thus, taking the partial derivatives
with respect to \( c \), we have
\[
\frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} = -\mathbb{P}[i \text{ wins with } b] \cdot \mathbb{P}(b \leq \alpha(X + c_j + Z_j)).
\]
Since both \( \mathbb{P}(b \leq \alpha(X + c_j + Z_j)) \) and \( \mathbb{P}[i \text{ wins with } b] \) are decreasing in \( b \), the partial derivative \( \frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} \) is increasing in \( b \). Therefore, by Definition 1, SCC is satisfied. The existence of a symmetric BNE in increasing strategies follows by Theorem 1 in Athey (2001).

Proof of Proposition 5.

(i) Strictly monotonicity and continuity for monitored FA.

- **Part 1. Equilibrium is strictly increasing.** By Section 3.2, one has
  \[
  \pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins with } \beta(c)] \cdot \mathbb{E} \left[ \min\{\beta(c), \alpha(c + X + Z)\} - c - X \right].
  \]

  Now, we show that the equilibrium is strictly increasing by contradiction. Assume that there is an interval with positive length \([\hat{c}_1, \hat{c}_2]\), with \( \hat{c}_2 \leq \hat{c} \), such that \( \beta(c) = \hat{b} \) for all \( c \in [\hat{c}_1, \hat{c}_2] \). We consider two cases. First, suppose that \( \pi_i(\hat{b}, \hat{c}_2, \beta) > 0 \). In this case, \( \mathbb{E} \left[ \min\{\hat{b}, \alpha(\hat{c}_2 + X + Z)\} - \hat{c}_2 - X \right] > 0 \). It is simple to observe that the bidder with private cost \( \hat{c}_2 \) is strictly better off by unilaterally deviating from \( \hat{b} \) to \( \hat{b} - \delta \) for small enough \( \delta > 0 \).

  To see this, note that with this deviation, \( \mathbb{P}[i \text{ wins with } \beta(c)] \) increases by a strictly positive discrete amount and the second term \( \mathbb{E} \left[ \min\{\hat{b} - \delta, \alpha(\hat{c}_2 + X + Z)\} - \hat{c}_2 - X \right] \) remains essentially unchanged for small enough \( \delta > 0 \) by continuity.

  The second case we consider is \( \pi_i(\hat{b}, \hat{c}_2, \beta) = 0 \). In this case, it must be that \( \pi_i(\beta(c), c, \beta) > 0 \), for \( c \in [\hat{c}_1, \hat{c}_2] \), because the previous function is strictly decreasing in \( c \) for \( c \in [\hat{c}_1, \hat{c}_2] \) (for \( \alpha = 1 \)). Then, we can repeat the previous argument.

- **Part 2. Equilibrium is continuous.** We show it by contradiction using the first part of the proof. Assume there is a symmetric and strictly increasing equilibrium \( \beta(\cdot) \) and \( \hat{c}_1 \in [\hat{c}, \hat{c}] \) such that \( \beta(\cdot) \) has a jump at \( \hat{c}_1 \). Let the left-limit and right-limit of \( \beta \) at \( \hat{c}_1 \) be \( b_- = \lim_{c \searrow \hat{c}_1} \beta(c) \) and \( b_+ = \lim_{c \nearrow \hat{c}_1} \beta(c) \), respectively. Then, \( b_- < b_+ \). By (B-8) and the fact that the ties happens with probability zero, one has
  \[
  \pi_i(b, \hat{c}_1, \beta) = F^{N-1}(\hat{c}_1) \cdot \mathbb{E} \left[ \min\{b, \alpha(c + X + Z)\} - c - X \right], \quad \text{for any } b \in [b_-, b_+].
  \]

  This is impossible because \( \mathbb{E} \left[ \min\{b, \alpha(c + X + Z)\} - c - X \right] \) is strictly increasing in \( b \). Thus,
it must be continuous.

(ii) Strictly monotonicity and continuity for naive FA.

• **Part 1. Equilibrium is strictly increasing.** By Section 3.1.1, one has

\[
\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins with } \beta(c)] \cdot \mathbb{E}[(b - c - X) \cdot \mathbb{I}\{\beta(c) \leq c_j + Z_j + X\}].
\]

Where \((c_j, Z_j)\) is independent with \(c, X\). In the following, we let \(Z = c_j + Z_j\). We need the following Lemma for the proof.

**Lemma 1.** Let \(\beta(\cdot)\) be a symmetric and increasing BNE strategy of the naive FA model. Then, \(\beta(c) < \bar{z} + \bar{x}\), for all \(c < \bar{z}\).

**Proof.** We argue by contradiction. Suppose that \(\beta(c) = \bar{z} + \bar{x}\), for some \(c < \bar{z}\). Note that in this case, \(\pi_i(\beta(c), c, \beta) = 0\), because bidder \(i\) with private cost \(c\) has no chance of defeating the spot market. We show that \(\beta'(c) = \bar{z} + \bar{x} - \epsilon\), for small enough \(\epsilon > 0\) is a profitable unilateral deviation, so the initially proposed strategy cannot be a BNE. Let \(\{i \text{ wins}\}\) be the event in which \(\beta_i(c_i) \leq \beta_j(c_j)\), \(\forall j\) and bidder \(i\) is selected in case of a tie. Then,

\[
\pi_i(\beta'(c), c, \beta) = \mathbb{P}[i \text{ wins}] \cdot \mathbb{E}[(\beta'(c) - c - X) \cdot \mathbb{I}\{\beta'(c) \leq Z + X\}]
\]

\[
= \mathbb{P}[i \text{ wins}] \cdot \mathbb{E}[(\bar{z} - c - \epsilon + (\bar{x} - X)) \cdot \mathbb{I}\{\beta'(c) \leq Z + X\}]
\]

\[
= \mathbb{P}[i \text{ wins}] \cdot [(\bar{z} - c - \epsilon) \cdot \mathbb{P}[\beta'(c) \leq Z + X] + \mathbb{E}[(\bar{x} - X) \cdot \mathbb{I}\{\beta'(c) \leq Z + X\}]].
\]

Clearly, \(\mathbb{P}[i \text{ wins}] > 0\), \(\mathbb{P}[\beta'(c) \leq Z + X] > 0\), and \((\bar{x} - x) \geq 0\), for all realizations \(x\). Moreover, for small enough \(\epsilon\), \(\bar{z} - c - \epsilon > 0\). The result follows.

Now, we show that the equilibrium is strictly increasing by contradiction. Let us write:

\[
\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins}] \cdot (\beta(c) - c - \mathbb{E}[X | \beta(c) \leq Z + X]) \cdot \mathbb{P}[\beta(c) \leq Z + X].
\]

Assume that there is an interval with positive length \([\hat{c}_1, \hat{c}_2]\), with \(\hat{c}_2 < \bar{z}\), such that \(\beta(c) = \hat{b}\) for all \(c \in [\hat{c}_1, \hat{c}_2]\). We consider two cases. First, suppose that \(\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) > 0\). In this case, \(\beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E}_{Z,X}[X | \beta(\hat{c}_2) \leq Z + X]) \cdot \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0\). It is simple to observe that the bidder with private cost \(\hat{c}_2\) is strictly better off by unilaterally deviating from \(\hat{b}\) to \(\hat{b} - \delta\) for small enough \(\delta > 0\). To see this, note that with this deviation, \(\mathbb{P}[i \text{ wins}]\) increases by a strictly positive discrete amount and the other terms in \(\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta)\) remain essentially unchanged for small enough \(\delta > 0\) by continuity.
The second case we consider is \( \pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) = 0 \). Because \( \hat{c}_2 < \bar{z} \), \( \beta(\cdot) \) is increasing, and Lemma 1, it must be that \( \mathbb{P}[i \text{ wins}] \cdot \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0 \). Hence, it must be that 
\[
\beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E}[X|\beta(\hat{c}_2) \leq Z + X] = 0.
\]
Take small enough \( \epsilon \), for which \( \beta(\hat{c}_2 - \epsilon) = \beta(\hat{c}_2) \).

We have that \( \pi_i(\beta(\hat{c}_2 - \epsilon), \hat{c}_2 - \epsilon, \beta) > 0 \), and we can use the same argument regarding a unilateral deviation like in the first case. The result follows.

**Part 2. Equilibrium is continuous.** We show it by contradiction using the first part of the proof. Assume there is a symmetric and strictly increasing equilibrium \( \beta(\cdot) \) and \( \hat{c}_1 \in [c, \bar{z}] \) such that \( \beta(\cdot) \) has a jump at \( \hat{c}_1 \). Let the left-limit and right-limit of \( \beta \) at \( \hat{c}_1 \) be 
\[
b_\leftarrow = \lim_{c \to \hat{c}_1^-} \beta(c)\text{ and } b_\rightarrow = \lim_{c \to \hat{c}_1^+} \beta(c),
\]
respectively. Then, \( b_\leftarrow < b_\rightarrow \). By (B-8) and the fact that the ties happens with probability zero, one has 
\[
\pi_i(b, \hat{c}_1, \beta) = \mathbb{P}(b < \beta(c_j), j \neq i) \cdot \mathbb{E}[(b - \hat{c}_1 - X)\mathbb{I}\{b \leq X + Z\}]
\]
\[
= \hat{F}^{N-1}(\hat{c}_1) \cdot \mathbb{E}[(b - \hat{c}_1 - X)\mathbb{I}\{b \leq X + Z\}], \text{ for any } b \in [b_\leftarrow, b_\rightarrow]. \tag{B-10}
\]

There are two cases to consider. Suppose \( \beta(\hat{c}_1) = b_\leftarrow \). Then, \( b_\leftarrow \) must be the maximum of 
\[
\mathbb{E}[(b - \hat{c}_1 - X)\mathbb{I}\{b \leq X + Z\}] \text{ by the previous equation. Moreover, by continuity and taking the limit } \lim_{c \to \hat{c}_1^-}, \text{ } b_\rightarrow \text{ must also be the maximum of } \mathbb{E}[(b - \hat{c}_1 - X)\mathbb{I}\{b \leq X + Z\}]. \text{ This contradicts our assumption of the unique maximum of } \mathbb{E}[(b - c - X)\mathbb{I}\{b \leq X + Z\}] \text{ for any } c. \text{ The second case is analogous, proving the result.}
\]

\[\Box\]

**Proof of Proposition 6.** Note that the total profit for bidder \( i \) from periods 1 to period \( T \) equals to the sum of the profits across all periods, and same is true for buyer’s expected buying prices, namely, 
\[
\pi_i = \sum_{t=1}^{T} \pi_{i,t} \text{ and } \mathbb{E}[P_{NFA}] = \sum_{t=1}^{T} P_{t,NFA}^{NFA}.
\]
Consider the expected profits for bidder \( i \) at period \( t \) when he bids \( b_i \), his cost is \( c_i \), and his competitors use equilibrium strategy \( \beta_{-i} \):
\[
\pi_{i,t}(b_i, c_i, \beta_{-i}) = \mathbb{E}_{-i} \left[ (b_i - c_i - X_i) r_{i,t}(b_i, \beta_{-i}(c_{-i}), p(c, X_t, Z_t)) \right. \\
\left. + [p_i(c, X_t, Z_{i,t}) - c_i - X_i] A_i \left( 1 - \sum_{j=1}^{N} r_{j,t}(b_j, \beta_{-i}, p(c, X_t, Z_t)) \right) \right],
\]

\[
= \sum_{j=1}^{M} q_{j,M} \mathbb{E}_{-i} \left[ (b_i - c_i - X_i) \mathbb{1}(b_i < \beta_t(c_t), \forall t \neq i, b_i < \alpha(c_j + X_t + Z_{j,t})) \right. \\
\left. + q_{i,M} \mathbb{E}_{-i} \left[ (\alpha(c_i + X_t + Z_{i,t}) - c_i - X_t) \mathbb{1}\{\min\{b_i, \beta_t(c_t)\}_{t \neq i} \geq \alpha(c_i + X_t + Z_{i,t})\} \right] \right],
\]

where the second equation follows from \( p(c, X_t, Z_t) = \sum_{j=1}^{M} A_j \alpha(c_j + X_t + Z_{j,t}) \), and \((1-\sum_{j=1}^{N} r_{j,t}(b, p_t)) = \mathbb{I}\{\min\{b_j\}_{j=1}^{N} \leq p_t\}\). Taking derivative with respect to \( c_i \), one has:

\[
\frac{\partial \pi_{i,t}(b_i, c_i, \beta)}{\partial c_i} \bigg|_{b_i = \beta_t(c_t)} = \sum_{j \neq i} q_{j,M} \mathbb{E}_{-i} \left[ -\mathbb{1}\{c_i < c_t, \forall t \neq i, \beta_t(c_t) < \alpha(c_j + X_t + Z_{j,t})\} \right. \\
\left. + q_{i,M} \mathbb{E}_{-i} \left[ (\alpha(c_i + X_t + Z_{i,t}) - c_i - X_t) \mathbb{1}\{\min\{b_i, \beta_t(c_t)\}_{t \neq i} \geq \alpha(c_i + X_t + Z_{i,t})\} \right] \right] \\
\leq \mathbb{E}_{-i} \left[ -r_{i,t}(\beta(c), p(c, X_t, Z_t)) + A_i (\alpha - 1) \mathbb{1}\{\beta(c_{11}) \geq \alpha(c_i + X_t + Z_{i,t})\} \right] \\
\leq \mathbb{E}_{-i} \left[ -r_{i,t}(\beta(c), p(c, X_t, Z_t)) + A_i (\alpha - 1) \mathbb{1}\{\beta(c_{11}) \geq \alpha(c_i + X_t + Z_{i,t})\} \right] \cdot
\]

where (a) follows from the definition of the allocation in the monitored FA \((r_{i,t}(b, p) = \mathbb{I}\{b_i < b_j, \forall j \neq i, b_i < p\})\), which in equilibrium implies the FA winner is the lowest cost suppliers given that equilibrium strategies are strictly increasing, and the factor that probability of \( A_i \) equals to \( q_{i,M} \), and (b) follows from \((\frac{\alpha-1}{\alpha}\beta(c_{11}) + Z_{i,t}) f_{X_t} \left( \frac{\beta(c_{11})}{\alpha} - c_i - Z_{i,t} \right) \mathbb{1}\{c_i > c_{11}\} \geq 0 \) for any sample path. Because \((1-\sum_{j=1}^{N} r_{j,t}(\beta(c), p)) = \mathbb{I}\{\beta(c_{11}) \geq \alpha\}\), one has:

\[
\frac{\partial \pi_{i,t}(b_i, c_i, \beta)}{\partial c_i} \bigg|_{b_i = \beta_t(c_t)} \leq \mathbb{E}_{-i} \left[ -r_{i,t}(\beta(c), p(c, X_t, Z_t)) + A_i (\alpha - 1) \left( 1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t) \right) \right].
\]

Thus, applying the envelop theorem to \( \pi_i \) yields:

\[
\pi_i(\beta(c), c, \beta) - \pi_i(\beta(c_i), c_i, \beta) = \int_{c_i}^{c} \frac{\partial \pi_i(b_i, s, \beta)}{\partial s} \bigg|_{b_i = \beta(s)} ds = \int_{c_i}^{c} \sum_{t=1}^{T} \frac{\partial \pi_{i,t}(b_i, s, \beta)}{\partial s} \bigg|_{b_i = \beta(s)} ds.
\]

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Note that:

\[ \pi_i(\beta(c_i), c_i, \beta) = \sum_{t=1}^{T} \mathbb{E}_{-i} \left[ m_{i,t}(\beta(c), p_t) - (c_i + X_t) r_{i,t}(\beta(c), p_t) + [p_{i,t} - c_i - X_t] A_i(1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \right]. \]

Combining the above two equations, one has:

\[
\mathbb{E}_{-i} \left[ \sum_{t=1}^{T} m_{i,t}(\beta(c), p_t) \right] = \sum_{t=1}^{T} \left[ \pi_{i,t}(\beta(c), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_{i,t}(\beta(s), s, \beta)}{\partial s} ds \right] \\
+ \sum_{t=1}^{T} \mathbb{E}_{-i} \left[ (c_i + X_t) r_{i,t}(\beta(c), p_t) - [p_{i,t} - c_i - X_t] A_i(1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \right].
\]

If the auctioneer does not buy from one of the FA bidders in the FA, he buys from the spot market. Therefore, the expected buying price for Naive FA is:
\[
\begin{align*}
\mathbb{E}[P^{NFA}] & = \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{t=1}^{T} m_{i,t}(\beta(c), p_t) + \sum_{t=1}^{T} (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) p(c, X_t, Z_t) \right] \\
& = \sum_{t=1}^{T} \left\{ \mathbb{E} \left[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) p(c, X_t, Z_t) \right] + \sum_{i=1}^{N} \pi_{i,t}(\beta(\bar{c}), \bar{c}, \beta) - \mathbb{E} \left[ \sum_{i=1}^{N} \int_{c_i}^{\bar{c}} \frac{\partial \pi_{i,t}(\beta(s), s, \beta)}{\partial s} ds \right] \right\} \\
& + \mathbb{E} \left[ \sum_{i=1}^{N} \left( (c_i + X_t) r_{i,t}(\beta(c), p_t) - [p_{i,t} - c_i - X_t] A_i(1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \right) \right] \\
\end{align*}
\]
changing the order of integration, and the definition of $\tilde{v}(\cdot)$, and the fact that

$$
\begin{align*}
\mathbb{E} \left[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \cdot \left( \sum_{j=N+1}^{M} A_{j,p_j}(c, X_t, Z_t) \right) + \sum_{i=1}^{N} A_i(c_i + X_t) (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \right] \\
= \mathbb{E} \left[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \cdot \left( \sum_{j=N+1}^{M} A_{j}(c_j + Z_{j,t} + X_t) + \sum_{i=1}^{N} A_i(c_i + X_t) \right) \right] \\
\geq \mathbb{E} \left[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \cdot \sum_{i=1}^{N} A_i(c_i + X_t) \right] = \mathbb{E} \left[ (1 - \sum_{j=1}^{N} r_{j,t}(\beta(c), p_t)) \cdot \tilde{q}_0(c, X_t) \right],
\end{align*}
$$

by the definition of $\tilde{q}_0(c, X_t) = \sum_{i=1}^{M} A_i(c_i + X_t)$. This concludes the proof. □

Proof of Proposition 9. Using the envelope theorem we obtain the expression for $E[P_{FPA}] = \sum_{t=1}^{T} P_t^{FPA}$, where $P_t^{FPA} = \mathbb{E} \left[ c_{(1,N)} + X_t + F(c_{(1,N)})/f(c_{(1,N)}) \right]$. Furthermore, by Proposition 8, $E[P_{MFA}] = \sum_{t=1}^{T} P_t^{MFA}$, thus, $E[P_{MFA} - P_{FPA}] = \sum_{t=1}^{T} (P_t^{MFA} - P_t^{FPA})$. In the following, we show that there exists $\bar{\alpha}_t \geq 1$ such $E[P_{MFA} - P_t^{FPA}] \leq 0$ for any $\alpha \in [1, \bar{\alpha}_t]$. One has

$$
P_t^{MFA} - P_t^{FPA} = - (\alpha - 1) \sum_{i=1}^{N} \mathbb{E}[c_{(1,N),i} - c_i]^+ + \mathbb{E} \left[ q_0(c, X_t) \right]$$

$$
= - (\alpha - 1) \sum_{i=1}^{N} \mathbb{E}[c_{(1,N),i} - c_i]^+ + \mathbb{E} \left[ (\alpha \mathbb{I}_t^{MFA} - 1)F(c_{(1,N)})/f(c_{(1,N)}) \right],
$$

where $\mathbb{I}_t^{MFA} = \mathbb{I}\{\beta^{MFA}(c_{(1,N)}) < \alpha(c_{(1,N)} + X_t + Z_{(1,N),t})\}$ is the event that FA winner wins over spot market. The last equations follows from the fact that $q_0(c, x) = c_{(1,N)} + x$, $v(c_i) = c_i + \alpha F(c_i)/f(c_i)$, and $r_{i,t}(\beta(c), p(c, X_t, Z_t)) = \mathbb{I}\{c_i < c_j, \forall j \neq i, \beta(c_i) < \alpha(c_i + X_t + Z_{i,t})\}$. Note that, when $\alpha = 1$, $P_t^{MFA} - P_t^{FPA} = \mathbb{E} \left[ (\mathbb{I}_t^{MFA} - 1)F(c_{(1,N)})/f(c_{(1,N)}) \right]$, it suffices to show that this expression is negative by the continuity of function $P_t^{MFA} - P_t^{FPA}$ in $\alpha$. It is sufficient to show that $\mathbb{P}(\mathbb{I}_t^{MFA} < 1) > 0$ or $\mathbb{P}(\beta^{MFA}(c_{(1,N)}) < c_{(1,N)} + X_t + Z_{(1,N),t}) < 1$, which holds if $\beta(c) > c$ for all $c$. Next, we show $\beta(c) > c$ for all $c$ by contradiction. Assume there exists $c_0$ such that $\beta(c_0) \leq c_0$. The profit of bidder $i$ with a cost equals to $c_0$ is given by $\pi_i(\beta(c_0), c_0, \beta) = \mathbb{E} \left[ \mathbb{I}\{\beta(c_0) \leq \beta(c_{(1,N),i})\} \cdot (\beta(c_0) - c_0 - X) \right]$, because, given $\beta(c_0) \leq c_0$, the bid is always smaller than the spot market price. Obviously, $\beta(c_0)$ could not be equilibrium, because $\beta(c_0) - c_0 - X < 0$ for any $X > 0$ and $\pi_i(\beta(c_0), c_0, \beta) < 0$, therefore, the bidder could be better off to deviate unilaterally by bidding high enough so that he will never win.

Now, let $\bar{\alpha}_t > 1$ be the smallest value such that $P_t^{MFA} - P_t^{FPA}$, the result follows with $\bar{\alpha}_t =
\[ \min \{ \bar{x}_t \}. \]

**Proof of Proposition 10.** Bidder \( i \)'s total profit in a flexible FA is given by:

\[
\pi_i(\beta(c_i), c_i, \beta_{-i}) = \sum_{t=1}^T \pi_{i,t}(b_i, c_i, \beta_{-i}). \quad \text{where,} \\
\pi_{i,t}(b_i, c_i, \beta_{-i}) = \mathbb{E} \left[ (b_i - c_i - X_t) \cdot \mathbb{I} \{ b_i < \beta_j(c_j), \ j \neq i, b_i < Z_t + X_t \} \right] \\
+ \mathbb{E} \left[ (Z_t - c_i) \cdot \mathbb{I} \{ b_i < \beta_j(c_j), \ j \neq i, b_i \geq Z_t + X_t \geq c_i + X_t \} \right].
\]

Taking derivative w.r.t \( c_i \), one has

\[
\frac{\partial \pi_{i,t}(b_i, c_i, \beta)}{\partial c_i} \bigg|_{b_i=\beta(c_i)} = -\mathbb{E} \left\{ \mathbb{I} \{ c_i < c_j, \ j \neq i, \beta(c_i) < Z_t + X_t \} + \mathbb{I} \{ c_i < c_j, \ j \neq i, \beta(c_i) \geq Z_t + X_t \geq c_i + X_t \} \right\},
\]

Define:

\[
r_i(b, X_t, Z_t) = \mathbb{I} \{ b_i < b_j, \ j \neq i, b_i < Z_t + X_t \}; \quad n_i(c_i, b, X_t, Z_t) = \mathbb{I} \{ b_i < b_j, \ j \neq i, b_i \geq Z_t + X_t \geq c_i + X_t \}.
\]

Applying the Envelop Theorem to \( \pi_i \) yields

\[
\pi_i(\beta(\bar{c}), \bar{c}, \beta_{-i}) - \pi_i(\beta(c_i), c_i, \beta_{-i}) = \int_{c_i}^{\bar{c}} \sum_{t=1}^T \mathbb{E}_{-i} \left[ -r_i(\beta(s), \beta_{-i}(c_{-i}), X_t, Z_t) - n_i(s, \beta(s), \beta_{-i}(c_{-i}), X_t, Z_t) \right] ds,
\]

Note that:

\[
\pi_{i,t}(\beta(c_i), c_i, \beta_{-i}) = \mathbb{E}_{-i} \left[ m_i(\beta(c), X_t, Z_t) - (c_i + X_t)r_i(\beta(c), X_t, Z_t) + [Z_t - c_i] n_i(c_i, \beta(c), X_t, Z_t) \right].
\]

Combining the above two equations, and using the fact that \( \pi_i(\beta(\bar{c}), \bar{c}, \beta_{-i}) = 0 \), one has

\[
\sum_{t=1}^T \mathbb{E}_{-i} [m_i(\beta(c), X_t, Z_t)] = \sum_{t=1}^T \left\{ \mathbb{E}_{-i} \left[ (c_i + X_t)r_i(\beta(c), X_t, Z_t) - [Z_t - c_i] n_i(c_i, \beta(c), X_t, Z_t) \right] \right. \\
+ \int_{c_i}^{\bar{c}} \mathbb{E}_{-i} \left[ r_i(\beta(s), \beta_{-i}(c_{-i}), X_t, Z_t) + n_i(s, \beta(s), \beta_{-i}(c_{-i}), X_t, Z_t) \right] ds \right\}.
\]

If the auctioneer does not buy from the FA winner, she buys from the spot market. Therefore, the expected buying price in the flexible FA is:

\[
\mathbb{E}[P^{FLE}] = \mathbb{E} \left[ \sum_{i=1}^N \sum_{t=1}^T m_i(\beta(c), X_t, Z_t) + \sum_{t=1}^T \left( 1 - \sum_{i=1}^N r_i(\beta(c), X_t, Z_t) \right) (Z_t + X_t) \right] = \sum_{t=1}^T P_t^{FLE}.
\]

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Where

\[ p_t^{FLE} = \sum_{i=1}^{N} \mathbb{E} \left[ (c_i + X_t) r_i(\beta(c), X_t, Z_t) - [Z_t - c_i] n_i(c_i, \beta(c), X_t, Z_t) \right] \]

\[ + \sum_{i=1}^{N} \mathbb{E}_{c_i} \left[ \int_{c_i}^{\zeta} \mathbb{E}_{c_{i-1}} \left[ r_i(\beta(s), \beta_{i-1}(c_{i-1}), X_t, Z_t) + n_i(s, \beta(s), \beta_{i-1}(c_{i-1}), X_t, Z_t) \right] ds \right] \]

\[ + \mathbb{E} \left[ \left( 1 - \sum_{i=1}^{N} r_i(\beta(c), X_t, Z_t) \right) (Z_t + X_t) \right] \]

\[ = \sum_{i=1}^{N} \mathbb{E} \left[ (c_i + X_t) r_i(\beta(c), X_t, Z_t) - [Z_t - c_i] n_i(c_i, \beta(c), X_t, Z_t) \right] \]

\[ + \sum_{i=1}^{N} \mathbb{E} \left[ \frac{F(c_i)}{f(c_i)} r_i(\beta(c), X_t, Z_t) + \frac{F(c_i)}{f(c_i)} n_i(c_i, \beta(c), X_t, Z_t) \right] + \mathbb{E} \left[ \left( 1 - \sum_{i=1}^{N} r_i(\beta(c), X_t, Z_t) \right) (Z_t + X_t) \right] \]

\[ \overset{(a)}{=} \mathbb{E}[X_t + Z_t] + \mathbb{E} \left[ \sum_{i=1}^{N} (v(c_i) - Z_t) \cdot (r_i(\beta(c), X_t, Z_t) + n_i(c_i, \beta(c), X_t, Z_t)) \right] \]

\[ \overset{(b)}{=} \mathbb{E}[X_t + Z_t] + \mathbb{E} \left[ \sum_{i=1}^{N} (v(c_i) - Z_t) \cdot \mathbb{I} \{ c_i < c_j, \ j \neq i \} \cdot (\mathbb{I} \{ \beta(c_i) < Z_t + X_t \} + \mathbb{I} \{ \beta(c_i) \geq Z_t + X_t \geq c_i + X_t \}) \right] \]

Where (a) follows by the definition \( v(c) = c + \frac{F(c)}{f(c)} \), and (b) follows from definition of \( r_i, n_i \) in (B-12). This concludes the proof.

**Proof of Proposition 11.** We show the ODE and boundary conditions for naive FA, monitored FA, and flexible FA separately in the follows.

(i) **ODE for Naive FA in diffused market.** First, we show that the bidding strategy satisfies the ODE equation. Taking derivative of profit function w.r.t \( b \) and using the fact that \( \frac{\partial \beta^{-1}(b)}{\partial b} \big|_{b=\beta(c)} = \frac{1}{\beta'(c)} \), one has

\[ \frac{\partial \pi^{\text{NDFA}}(b, c, \beta)}{\partial b} \bigg|_{b=\beta(c)} = -\frac{(N - 1)f(c)F^{N-2}(c)}{\beta'(c)} \cdot \int_{\beta(c) - z_0}^{\bar{c}} (\beta(c) - x) f_X(x) dx \]

\[ + F^{N-1}(c) \cdot \left[ F_X(\beta(c) - z_0) - (z_0 - c) f_X(\beta(c) - z_0) \right] . \]

Thus, by first-order-condition \( \frac{\partial \pi^{\text{NDFA}}(b, c, \beta)}{\partial b} = 0 \), we have shown that any symmetric equilibrium satisfies the ODE (7).

Next, we show the boundary condition. In the proof, we will omit the superscript “NDFA”.


Taking limit \( c \not\to z_0 \) in the integral equation in Proposition 1, one obtains

\[
\beta(z_0) = z_0 + \mathbb{E}\left[X \mid \beta(z_0) \leq z_0 + X\right].
\]

It is easy to verify that the only solution to the above equation is \( \beta(z_0) = z_0 + \bar{x} \) by applying L'Hôpital's rule. In what follows, we show \( \beta'(z_0) = 0 \). In (7), by taking limit \( c \to z_0 \), one obtains:

\[
\beta'(z_0) = \lim_{c \to z_0} \left[ \frac{(N - 1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \{\beta(c) - c - X\} \cdot \mathbb{I}\{\beta(c) \leq z_0 + X\} \right] \right]
\]

\[
= \lim_{c \to z_0} \left[ \frac{(N - 1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \frac{\int_{\beta(c) - z_0}^{\beta(c) - c - x} f_X(x)dx}{\beta(c) - z_0 - (z_0 - c)f_X(\beta(c) - z_0)} \right] \right]
\]

\[
= \lim_{c \to z_0} \left[ \frac{(N - 1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \frac{\int_{\beta(c) - z_0}^{\beta(c) - c - x} f_X(x)dx}{\beta(c) - z_0 - (z_0 - c)f_X(\beta(c) - z_0)} \right] \right]
\]

\[
\stackrel{(a)}{=} \lim_{K \to \beta} \left[ \frac{(N - 1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \frac{\int_{\beta(c) - z_0}^{\bar{K} - x} f_X(x)dx}{\bar{K} - \beta(c) - (\beta(c) - \bar{K})f_X(\beta(c) - z_0)} \right] \right]
\]

\[
\stackrel{(b)}{=} \lim_{K \to \beta} \left[ \frac{(N - 1)f(c)}{F(c)} \cdot \mathbb{E}_X \left[ \frac{\int_{\beta(c) - z_0}^{\bar{K} - x} f_X(x)dx}{\bar{K} - \beta(c) - (\beta(c) - \bar{K})f_X(\beta(c) - z_0)} \right] \right]
\]

where (a) holds since \( \beta(z_0) = z_0 + \bar{x} \), and (b) is obtained by L'Hôpital's rule.

(ii) ODE for Monitored FA. Recall that, bidder’s profit is

\[
\pi_i(b, c, \beta) = \mathbb{E}[\{b \leq \beta(c_{(1:N)}, -i)\}] \cdot \min\{b, c + X + \Delta\} - c - X
\]

\[
= \bar{F}^{N-1}(b) \cdot \left[ \Delta \mathbb{P}(b > c + X + \Delta) + \int_{b-c-\Delta}^{\min\{b, c + X + \Delta\}} (b - c - x)f_X(x)dx \right]
\]

Taking derivatives w.r.t \( b \), one has

\[
\frac{\partial \pi_i(b, c, \beta)}{\partial b} \bigg|_{b=\beta(c)} = -(N - 1)\bar{F}^{N-2}(c)f(c) \cdot \left( \frac{\partial \beta^{-1}(b)}{\partial b} \bigg|_{b=\beta(c)} \right) \cdot \mathbb{E}_X \left[ \min\{\beta(c) - c - X, \Delta\} \right]
\]

\[
+ \bar{F}^{N-1}(c) \cdot \mathbb{P}(X \geq \beta(c) - c - \Delta).
\]

Since \( \frac{\partial \beta^{-1}(b)}{\partial b} \bigg|_{b=\beta(c)} = 1/\beta'(c) \), \( \frac{\partial \pi_i(b, c, \beta)}{\partial b} \bigg|_{b=\beta(c)} = 0 \) yields the ODE.

Next, we derive the boundary condition. From Proposition 3, the bidding strategy also satisfies integral equation, i.e., taking limit at \( \bar{c} \), one has

\[
\beta(\bar{c}) = \bar{c} + \mathbb{E}\left[X \mid \beta(\bar{c}) \leq \bar{c} + X + \Delta\right] - \Delta \frac{1 - \mathbb{P}(\beta(\bar{c}) \leq \bar{c} + X + \Delta)}{\mathbb{P}(\beta(\bar{c}) \leq \bar{c} + X + \Delta)}.
\]

Let \( \beta(\bar{c}) = \bar{c} + \Delta + K \), obviously, \( K \leq \bar{x} \) since the above equation is not well-defined otherwise,
then \( K \) satisfies
\[
K + \Delta = \mathbb{E}\left[ X \middle| K \leq X \right] - \Delta \frac{1 - \mathbb{P}(X \geq K)}{\mathbb{P}(X \geq K)}.
\]

Let
\[
H(k) = \begin{cases} 
  k + \Delta - \mathbb{E}[X], & k \leq 0 \\
  (k + \Delta)\bar{F}_X(k) - \int_{k}^{x} x f_X(x) dx + \Delta \cdot F_X(k), & k \in [0, \bar{x}]
\end{cases}
\]
then \( K \in [0, \bar{x}] \) is solution to \( H(K) = 0 \). Note that, \( H'(k) = \bar{F}_X(k) - (k + \Delta)f_X(k) + kf_X(k) + \Delta f_X(k) = \bar{F}_X(k) > 0 \), thus, \( H(k) \) is increasing when \( k \leq \bar{x} \) and there is a unique solution to \( H(K) = 0 \). This completes the proof.

(iii) ODE for flexible FA. Similar to the naive FA in diffused market, one could show that the bidding strategy under flexible FA satisfies (7) and it satisfies the boundary conditions, we omit the details.

C Revenue Equivalence Between FPA without Reserve Price and FPA with Reserve Price

In this section we show that FPA without reserve prices is asymptotically equivalent to FPA with a random reserve price that equals to the spot market price determined as in the diffused naive FA.

Under the FPA with random reserve price (denoted as “RFPA”), the allocation function is
\[
r_{i,t}(b_i, b_{-i}, p) = \mathbb{I}\{b_i \leq b_j, \forall j \neq i\} \mathbb{I}\{b_i \leq p(c, X_{t}, Z_{t})\},
\]
where \( p(c, X_{t}, Z_{t}) = c + Z_{t} + X_{t} \) is spot market price under the diffused naive FA, and \( (c, Z_{t}) \) is independent to \( \{(c_i, Z_{i,t}), i = 1, 2, \ldots, N\} \), because of the diffused market assumption. However, note that \( (c, Z_{t}) \) has the same marginal distribution to \( (c_i, Z_{i,t}) \). We have the following result.

Proposition 12. (Revenue Equivalence) The expected buying price (measured at \( t = 0 \)) for running the FPA without reserve price at every time period is given by:
\[
\mathbb{E}[P^{FPA}] = \sum_{t=1}^{T} \mathbb{E}\left[ c_{(1:N)} + X_{t} + F(c_{(1:N)})/f(c_{(1:N)}) \right]
\]
The expected buying price (measured at \( t = 0 \)) for running the RFPA is given by:
\[
\mathbb{E}[P^{RFPA}] = \sum_{i=1}^{T} \mathbb{E}\left[ \min\left\{ c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) + X_{t}, c + Z_{t} + X_{t} \right\} \right].
\]
In addition, we have

$$\lim_{N \to \infty} E[P^{FPA} - P^{RFPA}] = 0.$$ 

**Proof.** By standard arguments based on the envelope theorem, one can easily get the expressions for $E[P^{FPA}]$ and for $E[P^{RFPA}]$ (Milgrom 2004). Next, we show $\lim_{N \to \infty} E[P^{FPA} - P^{RFPA}] = 0$. Substituting the equations for $E[P^{FPA}]$ and $E[P^{RFPA}]$, one gets

$$E[P^{FPA} - P^{RFPA}] = \sum_{t=1}^{T} E \left[ c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) - c - Z_t \right]^{+}.$$ 

Since $v(c_{(1:N)}) = c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)})$ converges to $c$ in probability as $N \to \infty$, one has

$$\lim_{N \to \infty} E \left[ c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) - c - Z_t \right]^{+} = 0,$$

by the bounded convergence theorem. Thus, $\lim_{N \to \infty} E[P^{FPA} - P^{RFPA}] = 0$. \hfill \square

### D Existence of Equilibrium for Compact Space

In this appendix, we extend the existence of the equilibrium under the naive FA with diffused market from a finite set (in Proposition 4) to a compact space. To that end, we need to verify that our basic FA game satisfies some technical conditions required for a limiting argument used by Athey (2001) to pass from games with finite action spaces to games with continuous action space. To simplify, we abuse notation and denote the sum of the random variables $c + Z$ as just $Z$, and we assume $\alpha = 1$.

Before presenting the existence proof, we present and prove the following lemma that we use below; we also referred to this result in the main body of the paper.

**Lemma D1.** For any random variable $Y$ with pdf $h(\cdot)$, cdf $H(\cdot)$ and support $[y, \bar{y}]$ (possibly $y = -\infty$ and/or $\bar{y} = \infty$), let

$$B(a,b) = E \left[ Y \middle| a < Y < b \right] = \frac{\int_{a}^{b} t dH(t)}{H(b) - H(a)}, \quad y \leq a < b \leq \bar{y}.$$ 

Then, $B(a,b)$ is increasing in $a$ and $b$.

**Proof.** For any $y \leq a < b \leq \bar{y}$, we have
Next, we show that an increasing symmetric pure strategy BNE exists for the naive FA with a diffused market, by applying Theorem 6 in Athey (2001). For self-completeness, we briefly summarize notation and assumptions made in Theorem 6 of Athey (2001). After introducing the theorem, we then show that all conditions are satisfied for the naive FA model.

Part 1. Restatement of results in Athey (2001). Consider a game of incomplete information between $I$ players, $i = 1, \ldots, I$, where each player first observes his own type $t_i \in T_i = [l_i, \bar{l}_i]$ and then takes an action $a_i$ from a compact set $A_i \in \mathbb{R}$. Let $A = A_1 \times \cdots \times A_I$, $T = T_1 \times \cdots \times T_I$, $a_i = \min A_i$, and $\bar{a}_i = \max A_i$. The joint density over player types is $f(\cdot)$, with the conditional density of $t_{-i}$ given $t_i$ denoted $f(t_{-i}|t_i)$. Player $i$’s payoff function is $u_i : A \times T \to \mathbb{R}$. Given any set of strategies for the opponents, $\alpha_j : T_j \to A_j, j \neq i$, player $i$’s objective function is defined as follows (using the notation $(a_i, \alpha_{-i}(t_{-i})) = (\cdots, a_{i-1}(t_{i-1}), a_i, a_{i+1}(t_{i+1}), \cdots)$):

$$U_i(a_i, t_i, \alpha_{-i}(t_{-i})) = \int_{t_{-i}} u_i ((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}|t_i) dt_{-i}.$$ 

Assumption D1. The types have joint density with respect to Lebesgue measure, $f(\cdot)$, which is bounded and atomless. Further, $\int_{t_{-i} \in S} u_i ((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}|t_i) dt_{-i}$ exists and is finite for all convex $S$ and all increasing functions $\alpha_j : T_j \to A_j, j \neq i$.

For games with finite action spaces, say $A_i = \{A_0, A_1, \ldots, A_M\}$, she shows that the Kakutani’s fixed point theorem is applicable when SCC is satisfied. Thus, a pure strategy BNE exists for games with finite action spaces. For games with compact action spaces, she assumes that player $i$’s payoff, given a realization of types and actions, has the following form

$$u_i(a, t) = \varphi_i(a) \cdot \bar{v}_i(a_i, t) + (1 - \varphi_i(a)) \cdot \underline{v}_i(a_i, t) = \underline{v}_i(a_i, t) + \varphi_i(a) \cdot \Delta v_i(a_i, t),$$  

(D-1)

where $\Delta v_i(a_i, t) = \bar{v}_i(a_i, t) - \underline{v}_i(a_i, t)$. Intuitively, the winners receive payoffs $\bar{v}_i(a_i, t)$ with probability $\varphi_i(a)$, while losers receive payoffs $\underline{v}_i(a_i, t)$ with probability $1 - \varphi_i(a)$. In most auction models, participation is voluntary: there is some outside option such as not placing a bid that provides a fixed certain utility to the agent, typically normalized to zero. We refer to this action as $Q$. We introduce the following assumption.
Assumption D2. There exists $\lambda > 0$ such that, for all $i = 1, \ldots, I$, all $a_i \in [a_i, \bar{a}_i]$ and all $t \in T$: (i) the types have support $T_1 \times \cdots \times T_I$; (ii) $\bar{v}_i(a_i, t)$ and $\underline{v}_i(a_i, t)$ are bounded and continuous in $(a_i, t)$; (iii) $\bar{v}_i(Q, t) = 0$, $\underline{v}_i(Q, t) = 0$, $\underline{v}_i(a_i, t) \leq 0$, and $\Delta v_i(\bar{a}_i, \bar{t}) < 0$; (iv) $\Delta v_i(a_i, t)$ is strictly increasing in $(-a_i, t_i)$; (v) for all $\varepsilon > 0$, $\Delta v_i(a_i, t_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, t_{-i}, t_i) \geq \lambda \varepsilon$.

Let $W_i(a_i, \alpha_{-i})$ denote the event that the realization of $t_{-i}$ and the outcome of the tie-breaking mechanism are such that player $i$ wins with $a_i$, when opponents use strategies $\alpha_{-i}$ with the realization of $t_{-i}$. Thus,

$$\mathbb{P}(W_i(a_i, \alpha_{-i})|t_i) = \int \phi_i((a_i, \alpha_{-i}(t_{-i})) \cdot f(t_{-i}|t_i) dt_{-i}. \quad (D-2)$$

Assumption D3. For all $i = 1, \ldots, I$, all $a_i, a_i' \in [a_i, \bar{a}_i]$, and whenever every opponent $j \neq i$ uses a strategy $\alpha_j$ that is increasing, $E_{t_{-i}} \left[ \Delta v_i(a_i, t)|t_i, W_i(a_i', \alpha_{-i}) \right]$ is strictly increasing in $t_i$ and increasing in $a_i'$.

Theorem D1. (Athey 2001) For all $i$, let $A_i = Q \cup [a_i, \bar{a}_i]$. Suppose Assumptions D1, D2, and D3 hold, and that the game satisfies the SCC. Then, there exists a pure strategy BNE in increasing strategies.

It is simple to to use the previous result to establish the existence of a symmetric BNE for symmetric games with incomplete information, which is our case of interest.

Part 2. Verifications of the Assumptions D1–D3. Now, we are ready to show the existence of a BNE in the naive FA model by verifying the conditions in Assumptions D1-D3. This together with the verification of SCC guarantee the existence of increasing BNE. Our proof is presented for the general case of random $Z$. For this we need one additional assumption:

Assumption D4. Assume that the random variables $X$ and $Z$ satisfy: $E[X|X+Z > b]$ is increasing in $b$, for all $b \geq 0$.

The above assumption is used to guarantee Assumption D3. It can be shown that the condition in the assumption is satisfied in the following cases: 1) if $Z$ and $X$ are independent and identically distributed; and 2) if $Z$ and $X$ are both uniformly distributed (with potentially different supports). Since the bid has to be at least the private cost, thus the lowest possible rational bid is $c_i$. For technical reasons, however, we define $b = c_i - \Delta$, $\Delta > 0$. Also, the bid will not be higher than $\bar{z} + \bar{x}$, namely, $\bar{b} = \bar{z} + \bar{x}$, which is the highest possible price in the spot market. To handle the two random components in the spot market price, we decompose the spot market into two “virtual” bidders: one has private cost $c_0^1 = x$ and the other has private cost $c_0^2 = z$, and they bid their true
cost, i.e., \( b_0^1 = x \) and \( b_0^2 = z \). The FA winner competes with the “aggregate” price of these two virtual bidders, with bid \( b_0 = b_0^1 + b_0^2 \). To be consistent with notations in Athey (2001), we make the following transformation of the private cost and the bids: for all bidders \( i = 1, 2, \ldots, N \) with private cost \( c_i \) and bid \( b_i \), let

\[
\begin{align*}
a_i &= \bar{c} + \bar{x} - b_i, \quad a_j = \bar{c} + \bar{x} - \bar{b} = \bar{c} - z_0, \quad \bar{a}_i = \bar{c} + \bar{x} - \bar{b} = \bar{c} + \bar{x} - \bar{c} + \Delta, \\
t_i &= \bar{c} + \bar{x} - c_i, \quad \bar{t}_i = \bar{c} + \bar{x} - \bar{c}, \\
a_0^1 &= t_0^1 = \bar{x} - x, \quad a_0^1 = t_0^1 = 0, \quad \bar{a}_0^1 = \bar{t}_0^1 = \bar{x} - x, \\
a_0^2 &= t_0^2 = \bar{c} - z, \quad a_0^2 = t_0^2 = \bar{c} - \bar{z}, \quad \bar{a}_0^2 = \bar{t}_0^2 = \bar{c} - \bar{z}.
\end{align*}
\] (D-3)

For any given \( \mathbf{a} = (a_0^1, a_0^2, a_1, \ldots, a_N), \mathbf{t} = (t_0^1, t_0^2, t_1, \ldots, t_N) \), corresponding to (D-1), our naive FA model can be specified as follows: For any \( i = 1, 2, \ldots, N \),

\[
\begin{align*}
\varphi_i(\mathbf{a}) &= \begin{cases} 
1, & \text{if } b_i < b_j, \forall j \neq i \\
0, & \text{otherwise}
\end{cases} = \mathbb{I} \{b_i < b_j, \forall j \neq i\} = \mathbb{I} \{a_i > a_j, \forall j \neq i\}, \\
v_i(\mathbf{a}, \mathbf{t}) &= 0, \quad \bar{v}_i(a_i, \mathbf{t}) = b_i - c_i - x = t_i - a_i - x = t_i - a_i - \bar{x} + t_0^1.
\end{align*}
\] (D-4) (D-5)

For simplification, we ignore ties in the winning probability in (D-4); a similar analysis applies if we consider them.

**Proposition D1.** Assume that Assumption D4 holds. Then, Assumptions D1-D3 hold for the naive FAs.

**Proof.** We will check all conditions in Assumptions D1-D3 hold for naive FAs.

- **Assumption D1:** Assumption D1 is trivially true since \( u_i(\mathbf{a}, \mathbf{t}) \) is bounded for any \( \mathbf{a}, \mathbf{t} \) by (D-1)-(D-5) and the fact that \( a_i, t_i \) is bounded for any \( i \).

- **Assumption D2:**
  - (i) and (ii) are trivial by (D-5) and the fact that \( a_i, t_i \) is bounded for any \( i \).
  - (iii). Let \( Q = \bar{c} - \bar{z} \), i.e., the bid equals to the highest possible spot market price \( \bar{z} + \bar{x} \).
    
    Thus, by bidding \( Q \), the bidder will never win against spot market and \( u_i(\mathbf{a}, \mathbf{t}|_{a_i=Q}) = 0 \),
    thus, \( a_i \geq Q \). \( \Delta v_i(\mathbf{a}, \mathbf{t}) = \bar{v}_i(a_i, \mathbf{t}) = \bar{t}_i - \bar{a}_i + \bar{t}_0^1 - \bar{x} = -\bar{x} - \Delta < 0 \) by (D-3).
  - (iv). By (D-5), \( \Delta v_i(a_i, \mathbf{t}) = \bar{v}_i(a_i, \mathbf{t}) = t_i - a_i + t_0^1 - \bar{x} \). Obviously, \( \Delta v_i(a_i, \mathbf{t}) \) is strictly increasing in \((-a_i, t_i)\).
\(- (v)\). For all \(\varepsilon > 0\), by (D-5), \(\Delta v_i(a_i, t_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, t_{-i}, t_i) = \varepsilon\). Thus, (v) in Assumption D2 is true for any \(\lambda \in (0, 1)\).

- Assumption D3:

\[
\mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a'_i, \alpha_{-i}) \right] = \frac{\mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \cdot \mathbb{I} \{ W_i(a'_i, \alpha_{-i}) \} \mid t_i \right]}{\mathbb{P} \left( W_i(a'_i, \alpha_{-i}) \mid t_i \right)}
\]

\[
= \mathbb{E}_{t_{-i}} \left[ (t_i - a_i + t_0^1 - x) \cdot \mathbb{I} \{ W_i(a'_i, \alpha_{-i}) \} \mid t_i \right] \frac{\mathbb{P} \left( W_i(a'_i, \alpha_{-i}) \mid t_i \right)}{\mathbb{P} \left( W_i(a'_i, \alpha_{-i}) \mid t_i \right)}
\]

\[
(t_i - a_i - x) + \frac{\mathbb{E}_{t_{-i}} \left[ t_0^1 \cdot \mathbb{I} \{ W_i(a'_i, \alpha_{-i}) \} \mid t_i \right]}{\mathbb{P} \left( W_i(a'_i, \alpha_{-i}) \mid t_i \right)}
\]

where \((a2)\) holds because \(t_{-i}\) and \(t_i\) are independent. By (D-2) and (D-4), we have

\[
\frac{\mathbb{E}_{t_{-i}} \left[ t_0^1 \cdot \mathbb{I} \{ W_i(a'_i, \alpha_{-i}) \} \mid t_i \right]}{\mathbb{P} \left( W_i(a'_i, \alpha_{-i}) \mid t_i \right)}
\]

\[
= \frac{\int_{t_{-i}} t_0^1 \cdot \mathbb{I} \{ a'_i > \alpha_j(t_j), \forall j \neq i \} f(t_{-i}) dt_{-i}}{\int_{t_{-i}} \mathbb{I} \{ a'_i > \alpha_j(t_j), \forall j \neq i \} f(t_{-i}) dt_{-i}}
\]

\[
= \frac{\mathbb{E}_{t_{-i}} \left[ t_0^1 \mathbb{I} \{ a'_i > a_0^1 + a_0^2 \} \right]}{\mathbb{E}_{t_{-i}} \left[ \mathbb{I} \{ a'_i > a_0^1 + a_0^2 \} \right]}
\]

\[
= \mathbb{E} \left[ t_0^1 \mid a'_i > t_0^1 + t_0^2 \right]
\]

where \((a3)\) follows from the fact that \(t_{-i}\) are independent and the last equality from the facts that \(a_0^1 = t_0^1\) and \(a_0^2 = t_0^2\). Thus,

\[
\mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a'_i, \alpha_{-i}) \right] = t_i - a_i - x + \mathbb{E} \left[ t_0^1 \mid a'_i > t_0^1 + t_0^2 \right].
\]

Obviously, \(\mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a'_i, \alpha_{-i}) \right]\) is strictly increasing in \(t_i\). Next, we show that
is increasing in \( a_i' \). By tower property of conditional expectation and independence of \( t_1^0 \) and \( t_2^0 \), one has

\[
\mathbb{E}_{t_1^0,t_2^0} \left[ t_i^0 \bigg| a_i' > t_1^0 + t_2^0 \right] = \mathbb{E} \left[ \bar{x} - X \bigg| a_i' > \bar{x} - X + \bar{c} - Z \right] = \bar{x} - \mathbb{E} \left[ X \bigg| X + Z > \bar{x} + \bar{c} - a_i' \right].
\]

The first equation follows by the definition \( t_1^0 = \bar{x} - X \) and \( t_2^0 = \bar{c} - Z \). Thus, by Assumption D4, one has \( \mathbb{E}_{t_1^0,t_2^0} \left[ t_i^0 \bigg| a_i' > t_1^0 + t_2^0 \right] \) is increasing in \( a_i' \). Thus, Assumption D3 is true, namely,

\[
\mathbb{E}_{t-i} \left[ \Delta v_i(a_i, t) \bigg| t_i, W_i(a_i', \alpha_{-i}) \right] \text{ is strictly increasing in } t_i \text{ and increasing in } a_i'.
\]

Therefore, we have shown conditions in Assumptions D1-D3 hold for the naive FAs.

Recall that Athey’s method is applicable to establishing the existence of a symmetric BNE when the game is symmetric. The above analysis establishes the conditions required to use Theorem D1 except SCC. This together with the SCC property established in Proposition 4 imply that an increasing symmetric BNE for naive FAs exists.

\[ \square \]

E The Optimal Mechanism

We consider the class of mechanisms defined in §5. The following analysis is similar to Milgrom (2004). The following proposition characterizes the expected buying price in the BNE of a given FA.

**Proposition E1.** Let \( \beta(\cdot) \) be a BNE strategy profile induced by a mechanism \( w = (r, m) \), such that equilibrium expected profits satisfy \( \pi_i(\beta_i(c), \bar{c}, \beta_{-i}) = 0 \), for all \( i \). Then, the expected total buying price for the auctioneer is given by:

\[
\mathbb{E}[P] = \sum_{t=1}^{T} \mathbb{E}[Z_t + X_t] + \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t)(v(c_i) - Z_t) \right], \tag{E-1}
\]

where the “virtual cost” function is \( v(c) = c + F(c)/f(c) \).

**Proof.** Recall that if the auctioneer does not buy from one of the FA bidders, she buys from the spot market. Therefore, the total expected payments for an FA given its equilibrium strategy can be expressed as:

\[
\mathbb{E}[P] = \mathbb{E} \left\{ \sum_{t=1}^{T} \sum_{i=1}^{N} m_{i,t}(\beta(c), X_t, Z_t) + \sum_{t=1}^{T} \left( 1 - \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t) \right) (Z_t + X_t) \right\}, \tag{E-2}
\]

where the expectation is taken with respect to the random variables \( X = (X_1, \ldots, X_T) \) and \( Z = (Z_1, \ldots, Z_T) \), and the random vector \( c \). In addition, throughout this proof we use the notation \( \mathbb{E}_{-i,t} \).
to denote expectation with respect to \( X_t, Z_t \), and the random vector \( c_{-i} \). Consider the equilibrium payoff for bidder \( i \):

\[
\pi_i(\beta_i(c_i), c_i, \beta_{-i}) = \sum_{t=1}^{T} E_{-i,t} \left[ m_{i,t}(\beta_i(c_i), \beta_{-i}(c_{-i}), X_t, Z_t) - (c_i + X_t)r_{i,t}(\beta_i(c_i), \beta_{-i}(c_{-i}), X_t, Z_t) \right].
\]  

(E-3)

Using the envelope theorem, and using the fact that \( \pi_i(\beta(c), \bar{c}, \beta_{-i}) = 0 \), we obtain:

\[
\pi_i(\beta_i(c_i), c_i, \beta_{-i}) = \sum_{t=1}^{T} \int_{c_i}^{\bar{c}} E_{-i,t} \left[ r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X_t, Z_t) \right] dy.
\]  

(E-4)

Equating (E-3) and (E-4) we obtain:

\[
\sum_{t=1}^{T} E_{-i} \left[ m_{i,t}(\beta_i(c_i), \beta_{-i}(c_{-i}), X_t, Z_t) \right] = \sum_{t=1}^{T} E_{-i,t} \left[ (c_i + X_t)r_{i,t}(\beta_i(c_i), \beta_{-i}(c_{-i}), X_t, Z_t) \right]
\]

\[+ \sum_{t=1}^{T} \int_{c_i}^{\bar{c}} E_{-i,t} \left[ r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X_t, Z_t) \right] dy.
\]

Replacing in equation (E-2), using the fact that the private costs \( c_i \) are independent across firms, we get:

\[
E[P] = \sum_{t=1}^{T} \left[ \sum_{i=1}^{N} \left( c_i + X_t \right) r_{i,t}(\beta(c), X_t, Z_t) + \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X_t, Z_t) dy \right]
\]

\[+ \sum_{t=1}^{T} \left[ \left( 1 - \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t) \right) (Z_t + X_t) \right]
\]

\[= \sum_{t=1}^{T} \left[ \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t)(c_i - Z_t) + \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X_t, Z_t) dy \right]
\]

\[+ \sum_{t=1}^{T} E[Z_t + X_t]
\]  

(E-5)

Next, note that

\[
E \left[ \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) dy \right] = E_{-i} \left[ \int_{c_i}^{\bar{c}} \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) dy f(c_i) dc_i \right]
\]

\[= E_{-i} \left[ \int_{c_i}^{\bar{c}} \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) f(c_i) dc_i dy \right]
\]

\[= E_{-i} \left[ \int_{c_i}^{\bar{c}} \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \beta_{-i}(c_{-i}), X, Z)(F(y)/f(y)) f(y) dy \right]
\]

\[= E \left[ r_{i,t}(\beta(c), X, Z)F(c_i)/f(c_i) \right],
\]  

(E-6)
where the first equation follows by the independence of the private costs and the second by changing
the order of integration. Replacing \((E-6)\) in \((E-5)\), we obtain:

\[
E[P] = \sum_{t=1}^{T} E[Z_t + X_t] + E \left[ \sum_{t=1}^{T} \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t)(c_i + \frac{F(c_i)}{f(c_i)} - Z_t) \right],
\]

proving the result.

We next consider the following structural assumption on the virtual cost.

**Assumption E1.** The virtual cost function \(v(c) = c + F(c)/f(c)\) is strictly increasing in \(c\), for all \(c \in [c, \bar{c}]\).

In the following result we provide a characterization of mechanisms that minimize the expected
buying price of the auctioneer.

**Proposition E2.** Suppose Assumption E1 holds. An augmented mechanism \((w, \beta)\) minimizes the
expected buying price for the auctioneer among all feasible augmented mechanisms if it satisfies
\(\pi_i(\beta_i(\bar{c}), \bar{c}, \beta_{-i}) = 0\), for all \(i\), and its allocation rule in period \(t = 1, 2, \ldots, T\) under the BNE
strategy profile \(\beta\) satisfies the following: (1) if \(v(c(1)) \leq z_t\), then buy from the lowest cost FA
supplier; and (2) if \(v(c(1)) > z_t\), then buy from the spot market. Moreover, there exists at least one
such augmented mechanism that achieves the optimum.

**Proof.** Following the same argument as Proposition E1, the expected buying price for an augmented
feasible mechanism \((w, \beta)\) satisfies:

\[
E[P] = \sum_{t=1}^{T} E[Z_t + X_t] + E \left[ \sum_{t=1}^{T} \sum_{i=1}^{N} r_{i,t}(\beta(c), X_t, Z_t)(v(c_i) - Z_t) \right] + \sum_{i=1}^{N} \pi_i(\beta_i(\bar{c}), \bar{c}, \beta_{-i}) \geq \sum_{t=1}^{T} E[Z_t + X_t] + E \left[ \sum_{t=1}^{T} \min_{i=1, \ldots, N} (v(c_i) - Z_t) \right],
\]

where the inequality follows because for a feasible mechanism \(\pi_i(\beta_i(c_i), c_i, \beta_{-i}) \geq 0\), for all \(i\) and \(c_i\),
and because \(\sum_{i=1}^{N} r_{i,t}(b, x, z) \leq 1\) and \(r_{i,t}(b, x, z) \geq 0\), for all \(b, x, z\). The right hand side of \((E-7)\)
provides a lower bound on the expected buying price for any feasible mechanism, therefore, a feasible
mechanism that achieves it must be optimal. Hence, using the fact that \(v(\cdot)\) is strictly increasing,
a feasible augmented mechanism with an allocation rule in equilibrium like the one proposed in the
statement of the proposition and that satisfies \(\pi_i(\beta_i(\bar{c}), \bar{c}, \beta_{-i}) = 0\), for all \(i\), must be optimal.

To prove the second part of the proposition we construct a mechanism that achieves the optimum.
Consider a “modified” second-price auction in which every period bidders submit bids \(b_i\) and the
spot market "submits" a bid equal to \( b_0^t = v^{-1}(z_t) \) after observing the realization of \( Z_t = z_t \). The lowest bid among \( b_0^t, b_1, \ldots, b_N \) wins and sells the object. If one of the bidders 1, \ldots, N wins, after observing the realization of \( X_t \), the auctioneer pays him \( b_{(2)} + x_t \), where \( b_{(2)} \) is the second lower order statistics among \( b_0^t, b_1, \ldots, b_N \). Loosing bidders do not receive payments. Therefore, the actual payoff for a winning bidder \( i \) is given by \( b_{(2)} + x_t - (c_i + x_t) = b_{(2)} - c_i \), which for every period is the same as the payoff in a standard second price auction. Hence, truthful bidding is a dominant strategy, so that bidder \( i \) submitting a bid \( b_i = c_i \) is a BNE. Clearly, a bidder with cost \( \bar{c} \) has no chance of winning and \( \pi_i(\beta_i, \bar{c}, \beta_{-i}) = 0 \). Moreover, the winning bidder is determined by the minimum between \( c_{(1)} \) and \( v^{-1}(z_t) \). Because \( v(\cdot) \) is strictly increasing, it follows that the allocation rule satisfies: (1) if \( v(c_{(1)}) \leq z_t \), then buy from the lowest cost FA supplier; and (2) if \( v(c_{(1)}) > z_t \), then buy from the spot market. These facts prove the result. \( \square \)