Online Appendix for Quiet Bubbles

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1. Proof of Proposition 1

As shown in the text, mispricing can be written as:

$$\text{mispricing} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\pi^{-1}[\pi(y)-\frac{2Q}{\gamma}]} \left(\pi(y) - \pi(x) - \frac{2Q}{\gamma} \right) \phi(x)dx \right) \phi(y)dy + \int_{-\infty}^{\infty} \left(\pi(y) - \pi(y-b)\right) \phi(y)dy$$

Note that $x < \pi^{-1}\left[\pi(y) - \frac{2Q}{\gamma}\right] \Rightarrow x < y$. Moreover, $\frac{\partial(\pi(y)-\pi(x))}{\partial D} = \Phi(D-G-b-x) - \Phi(D-G-b-y)$. Thus, for all $x < \pi^{-1}\left[\pi(y) - \frac{2Q}{\gamma}\right]$, $\frac{\partial(\pi(y)-\pi(x))}{\partial D} > 0$. Similarly, as $b > 0$, $\frac{\partial(\pi(y)-\pi(y-b))}{\partial D} = \Phi(D-G-y) - \Phi(D-G-b-y) > 0$. Thus, the derivative of mispricing w.r.t. D is strictly positive:

$$\frac{\partial(\text{mispricing})}{\partial D} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\pi^{-1}[\pi(y)-\frac{2Q}{\gamma}]} \underbrace{\frac{\partial(\pi(y)-\pi(x))}{\partial D}}_{>0} \phi(x)dx \right) \phi(y)dy + \int_{-\infty}^{\infty} \underbrace{\frac{\partial(\pi(y)-\pi(y-b))}{\partial D}}_{>0} \phi(y)dy \tag{1}$$

Thus, as D increases, both the resale option and the mispricing due to aggregate optimism increases, so that overall mispricing increases.

We now turn to expected turnover. To save on notations, call $\bar{x}(y)$ the unique real number such that: $\pi(\bar{x}(y)) = \pi(y) + \frac{2Q}{\gamma}$. Similarly, call $\underline{x}(y)$ the unique real number such that: $\pi(\underline{x}(y)) = \pi(y) - \frac{2Q}{\gamma}$. Obviously,

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$\underline{x}(y) < y < \bar{x}(y)$. Expected turnover is:

$$\mathbb{T} = \int_{-\infty}^{\infty} \left(\underbrace{\int_{-\infty}^{\underline{x}(y)} Q\phi(x)dx}_{\text{A short-sales constrained}} + \underbrace{\int_{\underline{x}(y)}^{\bar{x}(y)} \gamma \frac{|\pi(y) - \pi(x)|}{2} \phi(x)dx}_{\text{no short-sales constraint}} + \underbrace{\int_{\bar{x}(y)}^{\infty} Q\phi(x)dx}_{\text{B short-sales constrained}} \right) \phi(y)dy$$

We can take the derivative of the previous expression w.r.t. D. Note that the derivative of the bounds in the various integrals cancel out, so that:

$$\frac{\partial \mathbb{T}}{\partial D} = \int_{-\infty}^{\infty} \left(\int_{\underline{x}(y)}^{\bar{x}(y)} \frac{\gamma}{2} \frac{\partial |\pi(y) - \pi(x)|}{\partial D} \phi(x)dx \right) \phi(y)dy \tag{2}$$

If $y > x$, $\frac{\partial|\pi(y)-\pi(x)|}{\partial D} = |\Phi(D - G - b - x) - \Phi(D - G - b - y)| > 0$. Thus turnover is strictly increasing with D.

2. Proof of Proposition 2

We first look at mispricing. Note that $\bar{P}_0$ is independent of b. Thus:

$$\frac{\partial \text{mispricing}}{\partial b} = \frac{\partial P_0}{\partial b} = \int_{-\infty}^{\infty} \left(\Phi\left(\underline{x}(y)\right) \Phi(D - G - b - y) + \int_{\underline{x}(y)}^{\infty} \Phi(D - G - b - x)\phi(x)dx \right) \phi(y)dy > 0$$

We now turn to the derivative of turnover w.r.t. b:

$$\frac{\partial \mathbb{T}}{\partial b} = \int_{-\infty}^{\infty} \left(\int_{\underline{x}(y)}^{\bar{x}(y)} \frac{\gamma}{2} \frac{\partial |\pi(y) - \pi(x)|}{\partial b} \phi(x)dx \right) \phi(y)dy$$

If $y > x$, then we have: $\frac{\partial|\pi(y)-\pi(x)|}{\partial b} = \Phi(D - G - b - y) - \Phi(D - G - b - x) < 0$. Thus: $\frac{\partial \mathbb{T}}{\partial b} < 0$

3. Proof of Proposition 3

Consider first mispricing. The formula for the derivative of mispricing w.r.t. D (equation 1) holds irrespective of the nature of the claim, i.e. whether $\pi = \pi^E$ or $\pi = \pi^D$. We simply remark that π^E, as π^D, is increasing with x (the investor's belief) and $\frac{\partial \pi^E(y) - \pi^E(x)}{\partial D} = \Phi(D - G - b - y) - \Phi(D - G - b - x) = -\frac{\partial \pi^D(y) - \pi^D(x)}{\partial D}$. And similarly: $\frac{\partial \pi^E(y) - \pi^E(y-b)}{\partial D} = \Phi(D - G - b - y) - \Phi(D - G - y) = -\frac{\partial \pi^D(y) - \pi^D(y-b)}{\partial D}$. Thus:

$$\frac{\partial(\text{mispricing on } \pi^E)}{\partial D} = -\frac{\partial(\text{mispricing on } \pi^D)}{\partial D} < 0$$

Thus mispricing of the equity tranche decreases with D. The difference between the mispricing on the equity claim and the mispricing on the debt claim decreases with D as well. When D goes to infinity, there is no mispricing on the equity claim (which is worth 0) so that the difference between the mispricing on the equity claim and the mispricing on the debt claim is strictly negative. Similarly, when D goes to $-\infty$, there is no mispricing on the debt claim (which is worth 0) so that the difference between the mispricing and the equity claim on the mispricing on the debt claim is strictly positive. Thus, there exists a unique $\bar{D}^1 \in \mathbb{R}$ such that for $D \geq \bar{D}^1$, there is a larger mispricing on the equity claim than on the debt claim.

Consider now turnover. The formula for the derivative of turnover w.r.t. D (equation 2) holds irrespective of the nature of the claim, i.e. whether $\pi = \pi^E$ or $\pi = \pi^D$. We simply remark that π^E, as π^D, is increasing with x (the investor's belief) and $\frac{\partial \pi^E(y) - \pi^E(x)}{\partial D} = \Phi(D - G - b - y) - \Phi(D - G - b - x) = -\frac{\partial \pi^D(y) - \pi^D(x)}{\partial D}$. Thus:

$$\frac{\partial \mathbb{T}(\pi^E)}{\partial D} = -\frac{\partial \mathbb{T}(\pi^D)}{\partial D} < 0$$

Thus, the turnover of the equity tranche decreases with D. The difference between the turnover on the equity claim and the turnover on the debt claim decreases with D as well. When D goes to infinity, there is no turnover on the equity claim (which is worth 0) so that the difference between the turnover on the equity claim and the turnover on the debt claim is strictly negative. Similarly, when D goes to $-\infty$, there is no turnover on the debt claim (which is worth 0) so that the difference between the turnover on the equity claim and the turnover on the debt claim is strictly positive. Thus, there exists a unique $\bar{D}^2 \in \mathbb{R}$ such that for $D \geq \bar{D}^2$, there is a larger mispricing on the equity claim than on the debt claim. The proof for the existence of $\bar{G}$ and $\bar{b}$ follows exactly the same logic.

4. Proof of Proposition 5

We consider the case where group A has prior $G + b + \sigma$ and group B has prior $G + b - \sigma$. Thus, at date 1, beliefs are given by $(G + b + \sigma + \eta^A)$ for group A, with $\eta^A \sim \Phi()$ and $(F - \sigma + \eta^B)$ for group B, with $\eta^B \sim \Phi()$. Agents also receive at date 1 an interim payoff proportional to $\pi()$ from holding the asset at date-0. We first start by solving the date-1 equilibrium. At date 1, three cases arise:

1. Both groups are long. Thus demands are:

$$\begin{cases} n_1^A = n_0^A + \gamma \left(\pi(\sigma + \eta^A) - P_1 \right) \\ n_1^B = n_0^B + \gamma \left(\pi(-\sigma + \eta^B) - P_1 \right) \end{cases}$$

The date-1 price in this case is: $P_1 = \frac{1}{2}\left(\pi(\sigma+\epsilon^A)+\pi(-\sigma+\epsilon^B)\right)$. This is an equilibrium if and only if: $\frac{2n_0^A}{\gamma} > \pi(-\sigma+\epsilon^B)-\pi(\sigma+\epsilon^A)$ and $\frac{2n_0^B}{\gamma} > \pi(\sigma+\epsilon^A)-\pi(-\sigma+\epsilon^B)$

2. Only A group is long. The date-1 equilibrium price is then simply: $P_1 = \pi(\sigma+\epsilon^A)-\frac{n_0^B}{\gamma}$

   This is an equilibrium if and only if $\pi(\sigma+\epsilon^A)-\pi(-\sigma+\epsilon^B) > \frac{2n_0^B}{\gamma}$.

3. Only B group is long. The date-1 equilibrium price is then simply: $P_1 = \pi(-\sigma+\epsilon^B)-\frac{n_0^A}{\gamma}$. This is an equilibrium if and only if $\pi(-\sigma+\epsilon^B)-\pi(\sigma+\epsilon^A) > \frac{2n_0^A}{\gamma}$

At date 0, group A program can be written as [1] :

$$\max_{n_0}\left\{n_0\pi(\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\pi^{-1}[\pi(y-\sigma)-\frac{2n_0^A}{\gamma}]-\sigma}\left(n_0\left(\pi(y-\sigma)-\frac{n_0^A}{\gamma}\right)-\frac{n_0^2}{2\gamma}\right)\phi(x)dx+\int_{\pi^{-1}[\pi(y-\sigma)-\frac{2n_0^A}{\gamma}]-\sigma}^{\infty} n_0\pi(\sigma+x)\phi(x)dx\right]\phi(y)dy - \left(n_0P_0+\frac{n_0^2}{2\gamma}\right)\right\}$$

The F.O.C. of group A's agents program is given by (substituting n_0^A for n_0 in the F.O.C.):

$$0=\pi(\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\pi^{-1}[\pi(y-\sigma)-\frac{2n_0^A}{\gamma}]-\sigma}\left(\pi(y-\sigma)-\frac{2n_0^A}{\gamma}\right)\phi(x)dx+\int_{\pi^{-1}[\pi(y-\sigma)-\frac{2n_0^A}{\gamma}]-\sigma}^{\infty}\pi(\sigma+x)\phi(x)dx\right]\phi(y)dy-\left(P_0+\frac{n_0^A}{\gamma}\right)$$

Similarly, at date 0, group B agents' program can be written as:

$$\max_{n_0}\left\{\pi(-\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\pi^{-1}[\pi(y+\sigma)-\frac{2n_0^B}{\gamma}]+\sigma}\left(n_0\left(\pi(y+\sigma)-\frac{n_0^B}{\gamma}\right)-\frac{n_0^2}{2\gamma}\right)\phi(x)dx+\int_{\pi^{-1}[\pi(y+\sigma)-\frac{2n_0^B}{\gamma}]+\sigma}^{\infty} n_0\pi(-\sigma+x)\phi(x)dx\right]\phi(y)dy - \left(n_0P_0+\frac{n_0^2}{2\gamma}\right)\right\}$$

Group B agents' F.O.C.:

$$0=\pi(-\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\pi^{-1}[\pi(y+\sigma)-\frac{2n_0^B}{\gamma}]+\sigma}\left(\pi(y+\sigma)-\frac{2n_0^B}{\gamma}\right)\phi(x)dx+\int_{\pi^{-1}[\pi(y+\sigma)-\frac{2n_0^B}{\gamma}]+\sigma}^{\infty}\pi(-\sigma+x)\phi(x)dx\right]\phi(y)dy-\left(P_0+\frac{n_0^B}{\gamma}\right)$$

Consider now an equilibrium where only group A is long, i.e. $n_0^A = 2Q$ and n_0^B=0. In this case, the date-0 price is given by:

$$P_0=\pi(\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\pi^{-1}[\pi(y-\sigma)-\frac{4Q}{\gamma}]-\sigma}\left(\pi(y-\sigma)-\frac{4Q}{\gamma}\right)\phi(x)dx+\int_{\pi^{-1}[\pi(y-\sigma)-\frac{4Q}{\gamma}]-\sigma}^{\infty}\pi(\sigma+x)\phi(x)dx\right]\phi(y)dy-\frac{2Q}{\gamma}$$

This is an equilibrium if and only if:

$$P_0 > \pi(-\sigma)+\int_{-\infty}^{\infty}\left[\int_{-\infty}^{y+2\sigma}\pi(y+\sigma)\phi(x)dx+\int_{y+2\sigma}^{\infty}\pi(-\sigma+x)\phi(x)dx\right]\phi(y)dy$$

[1] In group A agents' program, we note n_0^A group A agents aggregate holding – each agent in group A takes n_0^A as given.

We now show that P_0 is increasing with σ (noting $K = D - G - b$):

$$\begin{aligned}
\frac{\partial P_0}{\partial \sigma} &= \pi'(\sigma) + \int_{-\infty}^{\infty} \left[\int_{\pi^{-1}[\pi(y-\sigma)-\frac{4Q}{\gamma}]-\sigma}^{\infty} \Phi(K-\sigma-x)\phi(x)dx - \int_{-\infty}^{\pi^{-1}[\pi(y-\sigma)-\frac{4Q}{\gamma}]-\sigma} \Phi(K-y+\sigma)\phi(x)dx \right] \phi(y)dy \\
&= \pi'(\sigma) + \int_{-\infty}^{\infty} \phi\left(\pi^{-1}[\pi(y+\sigma)+\frac{4Q}{\gamma}]+\sigma\right)\Phi(K-\sigma-y)\phi(y)dy - \int_{-\infty}^{\infty} \phi\left(\pi^{-1}[\pi(y-\sigma)-\frac{4Q}{\gamma}]-\sigma\right)\Phi(K-y+\sigma)\phi(y)dy \\
&\geq \pi'(\sigma) + \int_{-\infty}^{\infty} \phi(y+2\sigma)\Phi(K-\sigma-y)\phi(y)dy - \int_{-\infty}^{\infty} \phi(y-2\sigma)\Phi(K-y+\sigma)\phi(y)dy \\
&\geq \pi'(\sigma) + \int_{-\infty}^{\infty} \phi(y+2\sigma)\Phi(K-\sigma-y)\phi(y)dy - \int_{-\infty}^{\infty} \phi(y)\Phi(K-y-\sigma)\phi(y+2\sigma)dy \\
&\geq \pi'(\sigma) + \int_{-\infty}^{\infty} \Phi(K-\sigma-y)\left[\phi(y+2\sigma)\phi(y) - \phi(y)\phi(y+2\sigma)\right]dy
\end{aligned}$$

Call $\psi(\sigma) = \phi(y+2\sigma)\phi(y) - \phi(y)\phi(y+2\sigma)$. $\psi'(\sigma) = 2\phi(y+2\sigma)\left(\phi(y) + (y+2\sigma)\phi(y)\right)$ Thus, ψ is increasing if and only if: $2\sigma > -y - \frac{\phi(y)}{\phi(y)}$. Now consider the function $\kappa : y \in \mathbb{R} \rightarrow y\phi(y) + \phi(y)$. $\kappa'(y) = \phi(y) > 0$. Thus, κ is increasing strictly with y. But $\lim_{y \to -\infty} \kappa'(y) = 0$. Thus: $\forall\ y,\ \kappa(y) > 0$. Thus, for all $\sigma > 0$, $-y - \frac{\phi(y)}{\phi(y)} = -\frac{\kappa(y)}{\phi(y)} < 0 < 2\sigma$ so that ψ is strictly increasing with σ, for all $\sigma > 0$ and $y \in \mathbb{R}$. Now $\psi(0) = 0$. Thus, $\psi(y) > 0$ for all $\sigma > 0$. As a consequence:

$$\frac{\partial P_0}{\partial \sigma} \geq \pi'(\sigma) + \int_{-\infty}^{\infty} \Phi(K-\sigma-y)\psi(y)dy > 0$$

Thus, when the equilibrium features only group A long at date 0, the price is strictly increasing with dispersion σ.

We now simply show that in this equilibrium, turnover is strictly increasing. Turnover is $2Q$ when group B only is long at date 1 (i.e. $\pi(-\sigma+\eta^B) > \pi(\sigma+\eta^A) + \frac{4Q}{\gamma}$), it is given by: $\gamma\frac{\pi(y-\sigma)-\pi(x+\sigma)}{2}$ when group B and group A are long at date 1 ((i.e. $\pi(-\sigma+\eta^B) < \pi(\sigma+\eta^A) + \frac{4Q}{\gamma}$ and $\pi(-\sigma+\eta^B) > \pi(\sigma+\eta^A)$ and it is 0 if only group A is long at date 1 (i.e. $\pi(-\sigma+\eta^B) < \pi(\sigma+\eta^A)$). Thus, conditioning over η^B, expected turnover can be written as:

$$\mathbb{T} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\pi^{-1}(\pi(-\sigma+y)-\frac{4Q}{\gamma})-\sigma} 2Q\phi(x)dx + \int_{\pi^{-1}(\pi(-\sigma+y)-\frac{4Q}{\gamma})-\sigma}^{y-2\sigma} \gamma\frac{\pi(y-\sigma)-\pi(x+\sigma)}{2}\phi(x)dx \right] \phi(y)dy$$

Again, the derivative of the bounds in the integrals cancel out and the derivative is simply:

$$\frac{\partial \mathbb{T}}{\partial \sigma} = \int_{-\infty}^{\infty} \int_{\pi^{-1}(\pi(-\sigma+y)-\frac{4Q}{\gamma})-\sigma}^{y-2\sigma} -\gamma\frac{\Phi(K-y+\sigma)+\Phi(K-x-\sigma)}{2}\phi(x)dx\phi(y)dy < 0$$

Now consider the equation defining the equilibrium where only group A is long at date 0. This condition is:

$$\delta(\sigma) = P_0(\sigma) - \left(\pi(-\sigma) + \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y+2\sigma} \pi(y+\sigma)\phi(x)dx + \int_{y+2\sigma}^{\infty} \pi(-\sigma+x)\phi(x)dx \right] \phi(y)dy \right) > 0$$

Notice that the derivative of the second term in the parenthesis can be written as:

$$\begin{aligned} & \int_{-\infty}^{\infty}\left[\int_{-\infty}^{y+2\sigma}\pi'(y+\sigma)\phi(x)dx - \int_{y+2\sigma}^{\infty}\pi'(-\sigma+x)\phi(x)dx\right]\phi(y)dy \\ = & \int_{-\infty}^{\infty}\phi(y+2\sigma)\pi'(y+\sigma)\phi(y)dy - \int_{-\infty}^{\infty}\phi(y-2\sigma)\pi'(y-\sigma)\phi(y)dy \\ = & \int_{-\infty}^{\infty}\Phi(K-y-\sigma)\left(\phi(y+2\sigma)\phi(y)-\phi(y)\phi(y+2\sigma)\right)dy \end{aligned}$$

We thus have:

$$\frac{\partial\delta}{\partial\sigma} \geq (\pi'(\sigma)+\pi'(-\sigma)) > 0$$

Thus, there is $\bar{\sigma} > 0$ such that for $\sigma \geq \bar{\sigma}$, the equilibrium has only group A long at date 0, the price increases with dispersion and turnover decreases with dispersion. QED.