Stochastic Convenience Yield, Optimal Hedging and the Term Structure of Open Interest and Futures Prices

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Abstract

This paper develops a dynamic, equilibrium model of a futures market to study optimal hedging and the term structure of open interest and futures prices. Investors continuously face spot price risk over time and attempt to hedge this risk using futures. Convenience yield shocks generate basis risk to rolling over near-to-maturity futures. Hence, investors need to simultaneously trade far-from-maturity futures. The model predicts that in markets with substantial and mean reverting convenience yield shocks (e.g. energy futures), open interest is evenly distributed among contracts of different maturities. In markets where these shocks are persistent (e.g. metal futures), open interest is concentrated in near-to-maturity futures. The model generates additional implications regarding how the term structure of futures price volatility and the futures risk premium depend on the nature of convenience yield shocks.

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There is substantial variation across futures markets in the pattern of open interest with respect to contract maturity. In futures markets such as S&P and currencies, most of the open interest is concentrated in the near-to-maturity (or nearby) contracts, with the front two contracts typically accounting for about 80% of the total open interest in these markets. In markets such as metals and agriculturals, open interest tends to be less concentrated in nearby contracts. And in oil and other energy markets, open interest is evenly distributed between nearby and far-from-maturity (distant) futures.[1]

What accounts for this cross-sectional variation in the term structure of open interest? This question is an important one for at least a couple of reasons. First, these equilibrium open interest patterns reflect the variation of optimal hedging and speculation strategies across markets. So, an answer to this question will enhance our understanding of the determinants of optimal hedging and speculation in futures markets. Importantly, a number of papers have documented that the net holdings of hedgers, after controlling for systematic risk, significantly affect futures returns (see, e.g., Chang (1985), Carter, Rausser and Schmitz (1983), Bessembinder (1992), de Roon, Nijman and Veld (2000)). Hence an understanding of the determinants of the term structure of open interest will enhance our understanding of the term structure of futures prices.

The analysis of the determinants of optimal hedging, speculation and futures prices has a long and distinguished history. In the original normal backwardation theory of Keynes and Hicks (see, e.g., Keynes (1923) and Hicks (1939)), producers short futures to hedge their initially long positions in the underlying spot. Their supply of futures, or hedging pressure, tends to drive down the futures price relative to the expected value of the later spot price. Hence, risk averse speculators who long the futures to share the risk are provided with a positive expected return.

A substantial literature has since generalized the theory of Keynes-Hicks in a number of directions such as allowing for 1) producers to face quantity risks as well as price risks (see, e.g., Rolfo (1980), Newberry and Stigliz (1981)); 2) multiple consumption goods (see, e.g., Stiglitz (1983), Britto (1984)); and 3) a stock market or the tradeability of equity claims to producers' future revenues (see, e.g., Stoll (1979), Hirshleifer (1988, 1989)).

[1] These patterns are apparent from looking up futures prices and open interest in the Wall Street Journal on any given day. Futures textbooks also discuss these patterns.

Papers in the literature have also addressed how uncertainty resolution affects optimal dynamic hedging decisions (see, e.g., Anderson and Danthine (1983), Duffie and Jackson (1990), Hirshleifer (1991)) and how asymmetric information among market participants impacts equilibrium outcomes (see, e.g., Grossman (1977), Bray (1981), Hong (2000)).[2]

However, this literature is largely silent on why the term structure of open interest varies across futures markets. Most of the models in this literature allow hedgers to trade only one futures contract at any point in time. It is often assumed that market participants want to hedge against spot price risk at a future date and that a futures is available to hedge the risk on this particular date. So hedgers typically do not need to simultaneously trade futures of differing maturities. Within this canonical framework, differences in open interest patterns across futures markets need to be traced to differing hedging horizons among market participants. It is not clear, however, if and why hedgers in oil markets have a longer hedging horizon than hedgers in metals or currencies.

What does differ substantially across futures markets is the stochastic nature of convenience yield. In S&P and currency futures, there is a negligible convenience yield. In agricultural and metal markets, there is a more substantial convenience yield that fluctuates over time. And in energy futures, especially oil, convenience yield fluctuations are notoriously large. Not only does the variance of convenience yield shocks vary across markets, but so does the persistence of these shocks (see, e.g., Bessembinder e.t.al. (1995)). For instance, convenience yield shocks in energy markets tend to be very transitory, whereas such shocks in metal markets tend to be persistent.

The stochastic nature of convenience yield can have important effects on the term structure of open interest. The following examples bring out the intuition for why this is the case. In markets such as S&P futures, which are essentially cost-of-carry, even if one wanted to hedge over a 2 year horizon, one can do a pretty good job by rolling the 3-month contract. One does not really need a 2-year contract in the absence of any convenience yield shocks because there is negligible basis risk. In contrast, oil futures of differing maturities are not good substitutes because of the tremendous convenience yield

[2]Models in which futures prices depend only on the degree of covariation between futures prices and changes in economic state variables (systematic risk) without a role for hedging pressure include Black (1976), Breeden (1980), and Richard and Sundaresan (1981). Evidence for these models is mixed at best (see, e.g., Dusak (1973), Hodrick and Srivastava (1983), Jagannathan (1985)).

fluctuations. Trying to hedge over a long horizon by rolling the short-term contract can be quite costly because of basis risk, as the case of Metallgesellschaft (or MG Corp) makes clear. In a sense, these two examples make clear that to optimally hedge spot positions, it is more essential to trade a range of maturities in oil than in S&P.

In this paper, I develop a model of a futures market to explore these ideas. In my model, holding a spot position (as opposed to futures positions) provides the investor with an exogenously specified convenience yield, which is stochastic and mean reverting. Market participants are infinitely lived and are not just exposed to spot price risk at a particular future date. Instead, their spot positions are continually subject to price risk over time. Investors can hedge their spot positions with two futures contracts (a nearby and a distant). It is assumed that the available set of securities does not span fluctuations in the convenience yield. Hence futures are non-redundant securities and cannot be priced by arbitrage. Instead, prices are determined jointly by the hedging and speculative trades of investors.

The term structure of open interest along with the resulting futures prices depend on the properties of the convenience yield. Because of a fluctuating convenience yield, positions in the nearby are subject to basis risk. Hence, investors optimally hedge their positions in the nearby using the distant futures. This basis risk to trading in the nearby depends on the persistence of convenience yield shocks. I show that when these shocks are very persistent, there is less basis risk in rolling over the nearby. So rolling over the nearby is a good substitute for trading in the distant contract, and the distant futures atttracts less open interest. One testable implication then is that all else equal, markets with more mean reverting convenience yield shocks should have open interest more evenly distributed between nearby and distant contracts.

More importantly, since all prices in the economy are determined endogenously, the model can speak to a number of features of the term structure of prices. For instance, in the presence of mean-reverting convenience yield shocks, the price of a given futures is less sensitive to contemporaneous shocks than the spot price. These mean-reverting shocks are less reflected in the price of the futures than the spot because they will die away by the time the futures expires. By the same logic, the distant contract is less sensitive to underlying shocks than the nearby contract. So, the return of the nearby contract is more volatile than that of the distant contract. This term structure of volatility pattern is known as the "Samuelson effect" (see Samuelson (1965)). The Samuelson effect is more pronounced in markets with more mean reverting spot price shocks.

Combining the results for the term structure of open interest and price volatility, my model predicts that markets with mean reverting convenience yield shocks should have open interest evenly distributed among contract maturities and should also exhibit a pronounced Samuelson Effect. Furthermore, I also show that markets with more volatile convenience yield shocks are more likely to have a larger futures risk premium. And the futures risk premium increases with the persistence of convenience yield shocks.

My model is related to some previous work on the term structure of open interest and futures prices.[3] A number of other papers consider the behavior of optimal hedging over time (see, e.g., Anderson and Danthine (1983), Kamara (1993), Hong (2000)). These papers consider a futures contract expiring a number of periods in the future. They analyze the behavior of optimal hedging and futures prices as the contract expires. They do not generally allow trading in multiple futures contracts at the same time. In contrast, investors need to simultaneously trade multiple futures contracts at the same time in my model. This feature is crucial since the empirical findings regarding the term structure of open interest and futures prices really speak to the simultaneous trading of futures of different maturities at a point in time.

One paper that does allow for trading in multiple futures is Duffie and Jackson (1990), who develop a model of optimal hedging and equilibrium in a dynamic futures market. The intent of their paper is to solve for (in closed-form) optimal hedging formulas under a variety of settings. Their model is more general than mine in some ways such as allowing investors to trade more than two futures of different maturities at the same time. Unlike my model, the spot price in their model is exogenously specified and the non-marketed risk process that my investors face is different than theirs. As such, they do not consider the effects of convenience yield fluctuations on optimal hedging and equilibrium and my focus differs from theirs.

[3]My model is also related to a number of papers on optimal hedging of long-term commitments with short-term futures inspired in part by the MG Corp debacle (see, e.g., Mellon and Parsons (1995), Neuberger (1999)). These papers do not derive equilibrium open interest and futures prices.

In what follows, I develop a simple infinite horizon model that captures these ideas and to explore their empirical content. I present my model in Section I, the definition of equilibrium in Section II and the solution to the model in Section III. I then discuss the solution in Section IV. I conclude in Section V.

I The Model

The economy is defined on a continuous time, infinite horizon, $\mathcal{T} = [0, \infty)$, with a single good which is also used as the numeraire. The underlying uncertainty of the economy is characterized by a n-dimensional Weiner process w_t, $t \in \mathcal{T}$. There are two classes of investors denoted by $i = 1, 2$. Investors are identical within each class but different across classes in endowments. Let the population weights of these two classes be ω and $1 - \omega$, respectively, where $\omega \in [0, 1]$. Investors in class-i will also be referred to as investor-i. The economy is further specified as follows.

A Investment Opportunities

There are four publicly traded assets in the economy. Investors have access to a risk-free money-market account that pays a positive, constant rate of return $r > 0$. In addition, investors can trade shares of the spot in a competitive spot market, along with two futures written on the spot in a competitive futures market. The shares of the spot are perfectly divisible and all assets can be traded at no cost.

Each share of the spot pays a cumulative "convenience yield" D_t.[4] D_t is governed by the process

$$dD_t = Z_t dt + b_{\mathsf{D}} dw_t, \tag{1}$$

where $Z_t = Z_{1,t} + Z_{2,t}$ and $Z_{j,t}$ $(j = 1, 2)$ follows

$$dZ_{j,t} = -a_{\mathsf{z}_j} Z_{j,t} dt + b_{\mathsf{z}_j} dw_t. \tag{2}$$

Here a_{z_j} $(j = 1, 2)$ are non-negative constants and b_{D} and b_{z_j} $(j = 1, 2)$ are matrices of proper order. We will also call Z_t the "fundamental" of the spot as it fully determines the

[4] I use the term "convenience yield" in the sense of Brennan (1991) and Pindyck (1993)—the value of any benefits that inventories provide, including the ability to smooth production or facilitate the scheduling of production and sales net storage and other costs. That is, D_t is a "net" convenience yield. The analogy here is that of dividends to holding equity.

expectation of future spot payoffs. While our model can be solved for arbitrary values of a_{z_j}, for simplicity, we set $a_{z_1} = 0$, so that $Z_{1,t}$ follows a random walk. Then $Z_{1,t}$ is the permanent component of the fundamental, while $Z_{2,t}$ is the mean-reverting or transitory component. So when we speak of convenience yield shocks being more transitory, we are referring to a_{z_2} increasing in magnitude. Let S_t denote the spot price.

Figure 1 illustrates the time intervals in which the two futures with identical maturity lengths of M can be traded. The top line is the maturity cycle for the first contract, while the bottom line represents the second contract. At $m_{1,k-1}$, the $(k-1)$-th replication of the the first contract has matured. Since the $(k-1)$-th replication of the second contract is initiated after the first, it does not mature until $m_{2,k-1}$. The k-th replication of the first contract starts trading at $t_{1,k} = m_{1,k-1}$ and expires at $m_{1,k}$ at which time the $(k+1)$-th contract starts trading at $t_{1,k+1} = m_{1,k}$. The k-th replication of the second contract starts trading at $t_{2,k}$, a period of length L after the first contract started trading, and matures at $m_{2,k}$. This maturity and replication process cycles periodically from the k-th replication to the $(k+1)$-th replication and so on.

Our discussion will primarily focus on the solution in the time interval between $t = t_{2,k}$ and $t = m_{1,k}$. In this interval, the first contract is nearest-to-maturity. So, for convenience, we will refer to the first contract as the "nearby" contract and the second contract as "distant" contract. Let $H_{1,t}$ be the price of the nearby and $H_{2,t}$ be the price of the distant.

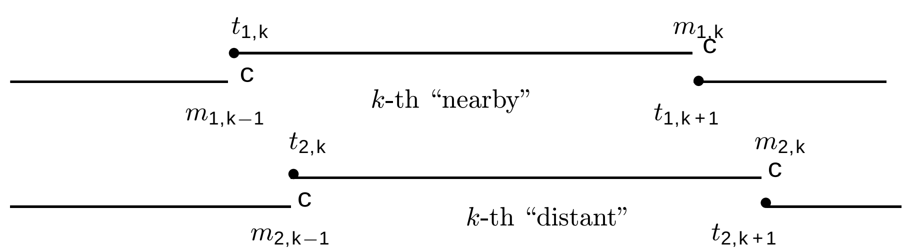


Figure 1: **Futures Maturity Cycle**

B Endowments

Without loss of generality, I assume that each investor is endowed with 1 share of the spot and the futures are in zero net supply. In addition to these publicly traded securities, it is assumed that investor-1 receives a non-marketed income with a cumulative excess rate of return q_t. q_t follows

$$dq_t = X_t{}' a_q X_t dt\, + e' X_t b_q dw_t, \tag{3}$$

where $X_t = [1,\, Y_t]'$ and Y_t follows

$$dY_t = -a_{\mathsf{Y}}\, Y_t dt + b_{\mathsf{Y}}\, dw_t. \tag{4}$$

Here a_q and a_{Y} are positive, b_q and b_{Y} are matrices of proper order and $e = [1,\, 1]'$. Y_t is a state variable which leads to time-variation in the expected return and volatility of the non-marketed income.

For $t \geq 0$, let $\mathcal{I}_t \equiv \{D_s,\, Z_{1,s},\, Z_{2,s},\, q_s,\, Y_s,\, S_s,\, H_{1,s},\, H_{2,s} \,:\, s \leq t\}$ denote the full information set about the economy. All investors are endowed with $\mathcal{I}_t$, that is, they all have complete and symmetric information sets.

C Preferences

All investors have constant absolute risk aversion (CARA). They maximize the expected utility of the following form:

$$i = 1, 2: \quad \mathrm{E}_{i,t}\left[-\int_t^{\infty} e^{-\rho(s-t)-\gamma c_{i,s}}\, ds \right], \tag{5}$$

where ρ and γ (both positive) are the time discount coefficient and the relative risk aversion coefficient, respectively, and $c_{i,s}$ is consumption at time s.

D Distributional Assumptions

I further assume that $w_t = [w_{\mathsf{D},t},\, w_{\mathsf{Z}_1,t},\, w_{\mathsf{Z}_2,t},\, w_{\mathsf{Y},t},\, w_{q,t}]$ and

$$b_{\mathsf{D}} = [\sigma_{\mathsf{D}},\, 0,\, 0,\, 0,\, 0],\; b_{\mathsf{Z}_1} = [0,\, \sigma_{\mathsf{Z}_1},\, 0,\, 0,\, 0],\; b_{\mathsf{Z}_2} = [0,\, 0,\, \sigma_{\mathsf{Z}_2},\, 0,\, 0],$$

$$b_{\mathsf{Y}} = [0,\, 0,\, 0,\, \sigma_{\mathsf{Y}},\, 0],\; b_q = [\sigma_q \kappa_{\mathsf{D}q},\, 0,\, 0,\, 0,\, \sigma_q \sqrt{1-\kappa_{\mathsf{D}q}^2}].$$

This specification of underlying shocks has a simple interpretation. For instance, instantaneous shocks to the convenience yield D_t are characterized by $w_{\mathsf{D},t}$ and σ_{D} gives its instantaneous volatility. All shocks in the economy are uncorrelated except for convenience yield and investor-1's non-marketed risks. These two shocks are assumed to be positive correlated, $\mathrm{Corr}(dw_{\mathsf{D},t}, dw_{q,t}) = \kappa_{\mathsf{D}q} > 0$.

E Discussion of Key Assumptions

My goal in this paper is to understand the effects of convenience yield fluctuations on optimal hedging and the term structure of open interest and futures prices. To do so, I make a number of simplifying assumptions for tractability. A number of these are fairly standard in the literature. For instance, like many other models of futures markets, we assume that the interest rate, r, is exogenously specified and assumed to be constant.

The assumption that the spot pays an exogenously specified convenience yield bypasses the difficulty of dealing with the consumption and production of commodities. This assumption is in other studies of futures markets such as Brennan (1991), Pindyck (1993), Gibson and Schwartz (1990) and Schwartz (1997). However, Williams and Wright (1991) and Routledge, Seppi and Spatt (1996) point out that the convenience yield may also arise endogenously from a non-negativity constraint on inventory, in which case stockouts then play an important role in generating state dependent correlation between spot and futures prices. Although my model does not account for such constraints, my results are not likely to be colored since the results are about the behavior of the term structure of open interest and futures prices on average (i.e. on a typical day) while stockouts are likely to be seasonal effects. Related, the empirical studies which document the stylized patterns that my model is trying to address are careful to control for seasonal effects.

Importantly, note that I need both σ_{z_1} and σ_{z_2} to be greater than zero. That is, I need the convenience yield fluctuations to be a two-factor model: $Z_{1,t} + Z_{2,t}$. Otherwise, trading in the existing securities can span convenience yield fluctuations and markets are effectively complete. In a sense, if there is not sufficiently rich convenience yield shocks, investors could do a pretty good job of hedging by trading just the nearby. This seems consistent with what we see in futures markets such as oil (with rich convenience yield shocks and lots of trading in a large menu of futures of different maturities) and S&P (with negligible convenience yield fluctuations and little trading other than in the nearest-to-maturity futures). If I added more futures of different maturities, I would need to add additional factors to the convenience yield—an additional factor for every new futures contract added.

Furthermore, I assume that investors in class-1 are endowed with non-marketed income q_t. Since investors are identical in every other way, this non-marketed risk is a simple way to generate heterogeneity among investors and trade in futures. The important component of this assumption is that $\kappa_{\mathsf{D}q} > 0$. (The sign of the correlation does not matter as long as it is not zero.) Because investor-1's non-marketed income is positively correlated with spot payoffs, investor-1 prefers to reduce his spot holdings so as to hedge his non-marketed risk. Investor-2 ends up taking on a larger spot position. As a result, investors then want to hedge their differing spot positions with futures.[5]

II Definition of Equilibrium

Given the investment opportunities specified in (1)-(2) and endowments in (3)-(4), investors choose consumption and investment policies to maximize their expected utility over life-time consumption. Let the consumption policy of investor-i be given by $\{c_{i,t} : t \in \mathcal{T}\}$. His investment policies in the spot, nearby and distant are given by $\{\theta^{\mathsf{S}}_{i,t} : t \in \mathcal{T}\}$, $\{\theta^{\mathsf{H}_1}_{i,t} : t \in \mathcal{T}\}$ and $\{\theta^{\mathsf{H}_2}_{i,t} : t \in \mathcal{T}\}$ respectively. Furthermore, the excess return on one share of the spot, which is the return minus the financing cost at the risk-free rate, follows

$$dQ_{0,t} = dS_t - rS_t dt + dD_t. \tag{6}$$

Note that $dQ_{0,t}$ is the excess return on one share of the spot instead of the excess return on one dollar invested in the spot. The former is the excess share return, while the latter is the excess rate of return. And the return on the first futures contract is

$$dQ_{1,t} = dH_{1,t}. \tag{7}$$

Similarly, the return on the second futures contract is

$$dQ_{2,t} = dH_{2,t}. \tag{8}$$

For future convenience, define $\theta_{i,t}$ to be the vector of investor-i's investment policies. That is, $\theta_{i,t} = [\theta^{\mathsf{S}}_{i,t}, \theta^{\mathsf{H}_1}_{i,t}, \theta^{\mathsf{H}_2}_{i,t}]'$. Further define Q_t to be the vector of returns. That is $Q_t = [Q_{0,t}, Q_{1,t}, Q_{2,t}]'$.

[5]One can think of the class-1 investors as utility companies that use (spot) natural gas in the production of electricity. These companies hold inventories of natural gas. One can think of the non-marketed risk as the positions that these utilities have in coal, another input in their production process. Fluctuations in coal price then affect their spot positions. The companies manage the price risk of these positions with futures. Think of the class-2 investors (who are not exposed to non-marketed risk) as natural counterparties who make the market for the trades of the class-1 investors.

Given the investors' preferences in (5), the return processes defined in (6)-(8), the investors' optimization problems are given by: for $i = 1, 2$

$$J_{i,t} \;\equiv\; \sup_{\{c_i,\theta_i\}} \; \mathrm{E}_{i,t}\left[-\int_t^\infty e^{-\rho(s-t)-\gamma c_{i,s}}\,ds\right] \tag{9a}$$

$$\text{subject to}\;\; dW_{i,t} = (rW_{i,t} - c_{i,t})\,dt + \theta_{i,t}{}'dQ_t + \delta_i dq_t, \tag{9b}$$

where $J_{i,t}$ is investor-i's value function at time t, $W_{i,t}$ is investor-i's wealth and δ_i is an index function, $\delta_i = 1$ if $i = 1$ and $\delta_i = 0$ if $i = 2$.

An equilibrium is given by spot and futures prices such that investors follow their optimal policies and markets clear:

$$t \in \mathcal{T}: \quad \omega\theta_{1,t} + (1-\omega)\theta_{2,t} = [1,\,0,\,0]'. \tag{10}$$

The resulting equilibrium prices and investors' optimal policies can in general be expressed as a function of the state of the economy and time. The state of the economy is determined by the investors' wealth and their information on current and future investment opportunities. And due to the assumptions of a constant risk-free rate and constant absolute risk aversion in preferences, investors' demand of risky investments will be independent of their wealth (see, e.g., Merton (1971)). Thus I seek an equilibrium in which the market prices are independent of investors' wealth. Let $\bullet$ denote the relevant state variables. Then one can write $S_t = S(\bullet; t)$, $H_{j,t} = H_j(\bullet; t)$, and $\{c_i(\bullet; t),\, \theta_i(\bullet; t)\}$.

Due to the nature of the periodic replication of futures after expiration, I will in this paper consider periodic equilibria in which the equilibrium price processes and investors' optimal policies exhibit periodicity in time. (See also Hong and Wang (2000) for an example of periodic equilibria.) Specifically, since M, the length of a futures, stays the same across time, one has

Definition 1 *In the economy defined above, a periodic equilibrium is defined by the price functions* $\{S(\bullet; t),\, H_j(\bullet; t)\,(j = 1, 2)\}$ *and policy functions* $\{c_i(\bullet; t),\, \theta_i(\bullet; t)\}$, $i = 1, 2$, *such that (a) the policies maximize investors' expected utility, (b) all markets clear, and (c) the price functions are periodic in time with periodicity M, and (d) investors' policy functions are also periodic in time.*

Under Definition 1, the k-th replication of a futures contract depends on the underlying uncertainty in the economy in the same way as the $(k+1)$-th replication and so forth:

for $k = 1, 2, \ldots$.

$$S(\bullet; t_{1,k}) = S(\bullet; t_{1,k+1}), \quad H_j(\bullet; t_{j,k}) = H_j(\bullet; t_{j,k+1}). \tag{11}$$

Realized values of the spot and futures can be different over time as the state variables change. Furthermore, periodicity in each investor's optimization problem and policy functions yield: for $i = 1, 2$

$$J_i(\bullet; t_{1,k}) = J_i(\bullet; t_{1,k+1}). \tag{12}$$

The periodicity conditions for the prices and value functions, (11) and (12), provide the necessary boundary conditions we need to solve for a periodic equilibrium. Thus, a periodic equilibrium is given by periodic price functions (11) such that investors optimally solve (9), (12), and the markets clear—(10) holds.

Furthermore, I restrict myself to the linear equilibria in which the price functions are linear in $\bullet$.

Definition 2 *A linear, periodic equilibrium is a periodic equilibrium in which: (a) $S_t = S(\bullet; t)$ and $H_{j,t} = H_j(\bullet; t)$ for $j = 1, 2$, (b) these price functions are linear in $\bullet$ and (c) time dependent with periodicity given in (11).*

Finally, I also impose the following no-arbitrage condition, which states that the futures price equals the spot price at the expiration date, $m_{j,k}$, for $j = 1, 2$ and $k = 1, 2, \ldots$

$$H_j(\bullet; m_{j,k}) = S(\bullet; m_{j,k}). \tag{13}$$

This condition provides another set of boundary conditions needed to solve for equilibrium prices.

III Solution of Equilibrium

In solving for an equilibrium, I proceed as follows: first, conjecture a particular equilibrium, then characterize the investors' optimal policies and the market clearing conditions under the conjectured equilibrium, and finally verify that the conjectured equilibrium in fact exists.

To begin with, I calculate the expected present discounted value of the convenience yield to holding the spot:

$$t \in \mathcal{T} : F_{0,t} = \mathrm{E}_t\left[\int_t^{\infty} e^{-r(s-t)} dD_s ds\right] = \lambda_{0,\mathsf{z}_1} Z_{1,t} + \lambda_{0,\mathsf{z}_2} Z_{2,t}, \tag{14}$$

where $\lambda_{0,\mathsf{z}_1} = 1/r$ and $\lambda_{0,\mathsf{z}_2} = 1/(r + a_{\mathsf{z}_2})$. $F_{0,t}$ is simply the spot price in a hypothetical economy with a risk neutral agent. In this risk neutral economy, I can apply the cost-of-carry formula to calculate the corresponding futures prices:

$$\begin{aligned} t \in [t_{j,k},\, m_{j,k}] : F_{j,t} &= e^{r(m_{\mathsf{j,k}}-t)}\left\{S_t - \mathrm{E}_t\left[\int_t^{m_{\mathsf{j,k}}} e^{-r(t-s)} dD_s ds\right]\right\} \\ &= \lambda_{j,\mathsf{z}_1}(t) Z_{1,t} + \lambda_{j,\mathsf{z}_2}(t) Z_{2,t}, \end{aligned} \tag{15}$$

where $\lambda_{j,\mathsf{z}_1}(t) = \lambda_{0,\mathsf{z}_1}$ and $\lambda_{j,\mathsf{z}_2} = \lambda_{0,\mathsf{z}_2} e^{-a_{\mathsf{z}_2}(m_{\mathsf{j,k}}-t)}$. Notice that $\lambda_{j,\mathsf{z}_1}(t)$ and $\lambda_{j,\mathsf{z}_2}(t)$ are periodic and converge (as required by no-arbitrage) to the spot price coefficients, λ_{0,z_1} and λ_{0,z_2} respectively, at $t = m_{j,k}$

With these calculations in mind, I conjecture that the equilibrium asset prices have the following linear form:

Conjecture 1 *A linear, periodic equilibrium is* $\{S_t,\, H_{1,t}\, H_{2,t}\}$ *such that:*

$$\begin{aligned} S_t &= F_{0,t} - \lambda_{0,\mathsf{x}} X_t, \\ H_{1,t} &= F_{1,t} - \lambda_{1,\mathsf{x}} X_t, \\ H_{2,t} &= F_{2,t} - \lambda_{2,\mathsf{x}} X_t, \end{aligned} \tag{16}$$

where $X_t = [1,\, Y_t]'$, $\lambda_{0,\mathsf{x}} = [\lambda_{0,0},\, \lambda_{0,\mathsf{Y}}]$ *and for* $j = 1, 2$, $\lambda_{j,\mathsf{x}} = [\lambda_{j,0},\, \lambda_{j,\mathsf{Y}}]$. $\lambda_{0,\mathsf{x}}$ *and* $\lambda_{j,\mathsf{x}}$ $(j = 1, 2)$ *are deterministic, time-dependent matrices with appropriate periodicities.*

For convenience, define $\lambda(t)$ to be the following time-varying column vector:

$$\lambda = \mathrm{stack}\{\lambda_{0,\mathsf{x}}{}',\, \lambda_{1,\mathsf{x}}{}',\, \lambda_{2,\mathsf{x}}{}'\}, \tag{17}$$

where the function stack makes a large column vector out of the elements within the braces.

To characterize the equilibrium, I take the price function in (16) as given and derive each investor's conditional expectations, policies and the market clearing conditions.

A Optimal Policies

Given the price functions, I can solve for the optimal policies of the investors. Investor-i's control problem as defined in (9) can be solved explicitly. The following lemma summarizes the results.

Lemma 1 *Given the price functions in (16), investor-i's value function has the form:*

$$t \in \mathcal{T}: \quad J_{i,t} = \exp\left\{-\rho t - r\gamma W_{i,t} - \frac{1}{2}\Big(X_t{}' v_i(t) X_t\Big)\right\}, \quad (i = 1, 2), \tag{18}$$

where $v_i(t)$ are symmetric matrices which satisfy a system of ordinary differential equations given by

$$\dot{v}_i = g_{i,v}(t; v_i(t); \lambda(t)), \tag{19}$$

where $g_{i,v}$ is given in the proof in the Appendix. Furthermore, her optimal consumption policy is

$$c_{i,t} = -\frac{1}{\gamma}\log(r) + rW_{i,t} + \frac{1}{2}X_t{}' v_i(t) X_t. \tag{20}$$

And her optimal investment policy is

$$\theta_{i,t} = \frac{1}{r\gamma} h_i(t) X_{i,t}, \tag{21}$$

where h_i is given in the proof in the Appendix.

Given $\lambda(t)$, the above lemmas expresses investor-i's optimal policies as functions of the matrices $v_i(t)$, to be solved from (19), which is a (vector) first-order ordinary differential equation. A periodic solution for investor-i's control problem further requires that for $k = 0, 1, 2, \ldots$:

$$v_i(t_{1,k}) = v_i(t_{1,k+1}) \tag{22}$$

B Market Clearing

In equilibrium, the markets must clear. From (10) and Lemma 1, the market clearing condition requires that

$$\omega \frac{h_1}{r\gamma} + (1-\omega)\frac{h_2}{r\gamma} = \text{stack}\{[1,\ 0],\ [0,\ 0],\ [0,\ 0]\}. \tag{23}$$

Given h_i $(i = 1, 2)$, (23) defines a (vector) first order differential equation for λ:

$$\dot{\lambda} = g_{\lambda}(t; \lambda(t); v_1(t), v_2(t)), \tag{24}$$

where g_{λ} is given in the Appendix. The periodicity condition for spot price requires that

$$\lambda_{0,\mathsf{x}}(t_{1,k}) = \lambda_{0,\mathsf{x}}(t_{1,k+1}). \tag{25}$$

And the periodicity condition for the futures prices implies that

$$\lambda_{j,\mathsf{x}}(t_{j,k}) = \lambda_{j,\mathsf{x}}(t_{j,k+1}). \tag{26}$$

Additionally, the convergence of futures to spot prices at maturity given in (13) implies that

$$\lambda_{0,\mathsf{x}}(m_{j,k}) = \lambda_{j,\mathsf{x}}(m_{j,k}). \tag{27}$$

(27) provides an additional set of boundary conditions that an equilibrium solution must satisfy.

C Existence and Computation of Equilibrium

The previous discussion characterizes the investors' optimal policies in a linear, periodic equilibrium of (16). Solving for such an equilibrium now reduces to solving (19) and (24), a system of first-order differential equations, for v_1, v_2 and λ subject to boundary conditions (22), (25), (26) and (27). The solution of a system of differential equations depends on the boundary conditions. In the case of the familiar initial-value problem, the boundary condition is simply the initial value of the system. It seeks a solution given its value at a fixed point in time. My problem, however, has a different boundary condition. I need to find particular initial values $v_i(t_{1,k})$ $(i = 1, 2)$ and $\lambda(t_{1,k})$ such that the periodicity conditions hold. This is known as a two-point boundary value problem, which seeks a solution of the system with its values at two given points in time satisfying a particular condition.

Theorem 1 states the result on the existence of a solution to the given system, which gives a linear, periodic equilibrium of the economy. Here, the condition that ω be close to one arises from the particular approach I use in the proof as opposed to economic rationales (see Appendix). My proof relies on a continuity argument. It is first shown that at $\omega = 1$, a solution to the given system exists. Since the system is smooth with respect to ω, it is then shown that a solution also exists for ω close to one. I do not specify in the proof, however, how close it has to be.

Theorem 1 *For ω close to one, a linear periodic equilibrium of the form in (16) exists generically in which the optimal policies of both investors are given by Lemma 1.*

In general, the model needs to be solved numerically. The numerical method used to solve this system of nonlinear first order differential equations is standard and is discussed in Kubicek (1983).

IV The Term Structure of Open Interest and Futures Prices

In this section, I develop the implications of the model for the term structure of open interest and futures prices. I begin in Section IV.A below by setting the volatility of the persistent component of non-marketed income shock Y_t equal to zero, i.e. $\sigma_{\mathsf{Y}} = 0$. So the non-marketed income process is simply $dq_t = \sigma_q dw_{q,t}$, which is *i.i.d.* over time. This setting is interesting for a couple of reasons. First, I can obtain closed-form solutions for prices and holdings. These closed form solutions will help us greatly in interpreting our numerical solution for $\sigma_{\mathsf{Y}} > 0$. It turns out that a number of our key results derived for $\sigma_{\mathsf{Y}} = 0$ are qualitatively similar for non-zero values of σ_{Y}. When σ_{Y} is very large, we obtain a few new predictions. We present the numerical solution for the case $\sigma_{\mathsf{Y}} > 0$ in Section IV.B below. In the discussions of these two cases, we relate our results to the empirically documented variation in the term structure of open interest and futures prices across futures markets.

A I.I.D. Non-marketed Income Shocks: $\sigma_{\mathsf{Y}} = 0$

I begin with the the following proposition regarding equilibrium prices.

Proposition 1 *When non-marketed risk shocks in the economy are i.i.d, the spot price S_t is given by*

$$S_t = F_{0,t} - \bar{\lambda}_{0,0} \tag{28}$$

where $\bar{\lambda}_{0,0}$ *is given by*

$$\bar{\lambda}_{0,0} = \gamma(\lambda_{0,\mathsf{Z}_1}^2 \sigma_{\mathsf{Z}_1}^2 + \lambda_{0,\mathsf{Z}_2}^2 \sigma_{\mathsf{Z}_2}^2 + \sigma_{\mathsf{D}}^2) + \gamma\omega\sigma_{\mathsf{D}q}. \tag{29}$$

The futures prices $H_{j,t}\,(j = 1, 2)$ *are given by, for* $k = 1, 2, 3, ...,$

$$t \in [t_{j,k},\, m_{j,k}]: \quad H_{j,t} = F_{j,t} - \bar{\lambda}_{j,0}(t), \tag{30}$$

where $\bar{\lambda}_{j,0}(t)$ *is given by*

$$\bar{\lambda}_{j,0}(t) = \bar{\lambda}_{0,0} + \gamma\Big((m_{j,k} - t)\lambda_{0,\mathsf{Z}_1}^2 \sigma_{\mathsf{Z}_1}^2 + \frac{1 - e^{-a_{\mathsf{Z}_2}(m_{\mathsf{j},\mathsf{k}} - t)}}{a_{\mathsf{Z}_2}} \lambda_{0,\mathsf{Z}_2}^2 \sigma_{\mathsf{Z}_2}^2\Big). \tag{31}$$

Notice that the solution for S_t trivially satisfies the definition of a linear periodic equilibrium since $\bar{\lambda}_{0,0}$ is constant through time. Moreover, the solution for $H_{j,t}$ also satisfies the periodicity condition. As required for no-arbitrage, the futures price converges to the spot price at expiration: $F_{j,t} = F_{0,t}$ at expiration and $\bar{\lambda}_{j,0}(t) = \bar{\lambda}_{0,0}$ at $t = m_{j,k}$.

There is a simple economic interpretation for the spot and futures prices. Recall from the discussion in Section III of equation (14) that the first piece of the spot price, $F_{0,t}$, is simply the expected present discounted value of the payoffs to holding the spot. The second piece, $\bar{\lambda}_{0,0}$, is the price discount, which naturally increases with investor risk aversion γ and the variance of convenience yield shocks (σ_{D}, σ_{Z_1} and σ_{Z_2}).

Moreover, the price discount also increases with $\sigma_{\mathsf{D}q}$, the covariance of the spot payoff with investor-1's non-marketed income. Because of the positive correlation between investor-1's non-marketed income shocks and convenience yield shocks, investor-1 wants to reduce his initial holdings of the spot for diversification reasons. Given a fixed supply of the spot and given that the demand of investor-2 has not changed, for markets to clear, the spot price has to drop (or the expected return to holding the spot increase) to induce investor-2 to hold more shares of the spot in equilibrium.

We next turn to the futures prices $H_{j,t}\,(j = 1, 2)$. From the discussion in Section III of equation (15), the first piece, $F_{j,t}$, is simply the futures price in a risk neutral economy. The second piece, $\bar{\lambda}_{j,0}(t)$, is the price discount for being long the futures. It increases with investors' risk aversion and volatility of convenience yield shocks. In other words, there is a positive futures risk premium to being long futures, just as in other models with normal backwardation along the lines of Keynes and Hicks.

We next consider the holdings of the investors at these prices. Without loss of generality, we will focus our analysis on the holdings of investor-1. The following proposition establishes the equilibrium holding patterns.

Proposition 2 *Given the equilibrium prices specified in Proposition 1, we can establish the following results regarding the equilibrium holdings. The spot position of investor-*1 *is given by*

$$\theta^{\mathsf{S}}_{1,t} = 1 - (1 - \omega)\frac{\sigma_q \kappa_{\mathsf{D}q}}{\sigma_{\mathsf{D}}}. \tag{32}$$

*And investor-*1 *takes a long position in the nearby futures* $\theta^{\mathsf{H}_1}_{1,t} > 0$ *and simultaneously a short position in the distant futures* $\theta^{\mathsf{H}_2}_{1,t} < 0$.

Both investors-1 and 2 are endowed with one share of the spot. Because investor-1's non-marketed income is positively correlated with payoffs to the spot, investor-1 wants to reduce his initial spot holdings (away from one) to hedge his non-marketed income. This reduction is proportional to the regression coefficient of the instantaneous return to the non-marketed income dq_t on the convenience yield: $(\sigma_q \kappa_{\mathsf{D}q})/\sigma_{\mathsf{D}}$. The larger is this coefficient, the more aggressively investor-1 hedges his non-marketed income risk using his spot holdings. In equilibrium, investor-1 holds less than one share and investor-2 ends up holding more than one share, by an amount equal to

$$(1 - \omega)\frac{\sigma_q \kappa_{\mathsf{D}q}}{\sigma_{\mathsf{D}}}.$$

Note that if $\omega = 1$ (all investors are from class-1), then no hedging is possible.

As a result, investors-1 and 2 will have different demands for the futures. In particular, investor-1's spot position is subject to spot price risk in the form of convenience yield fluctuations driven by $Z_{1,t}$ and $Z_{2,t}$. As a result, investor-1 hedges these risks with futures contracts. The natural thing for investor-1 to do is to hedge his (short) spot position with a long position in the nearby futures. However, when the convenience yield growth rate follows a two factor model, positions in the nearby is subject to a basis risk as well. So, the distant contract provides a vehicle to hedge this basis risk. Since she goes long in the nearby contract, the natural way to hedge this long position in the nearby is with a short position in the distant contract.

Having established the equilibrium prices and holdings in Propositions 1 and 2, we now derive a number of results regarding the term structure of open interest. We calculate the open interest in each futures by taking the absolute value of investor-1's position and weighing it by the proportion of class-1 investors in the economy, ω. The following proposition summarizes the main results on open interest.

Proposition 3 *When non-marketed income shocks are i.i.d., the open interest of the nearby futures is greater than that of the distant futures. And the ratio of the open interest in the distant to the nearby futures is increasing in the mean reversion of convenience yield shocks.*

These results reflect the fact that when shocks to $Z_{2,t}$ are more persistent (or $a_{z_2} \to 0$), spot and futures prices are more correlated and hence the nearby contract is more effective in hedging both convenience yield shocks, $Z_{1,t}$ and $Z_{2,t}$. As a_{z_2} increases, the nearby contract is a less effective hedge and there is more need to use the distant contract to hedge the basis risk.

Because prices are all determined endogenously, our model also have a number of implications for the term structure of futures prices.

Proposition 4 *When non-marketed income shocks are i.i.d., the return volatility of the nearby futures is greater than that of the distant futures, i.e. the Samuelon effect holds. And the ratio of the return volatility of the nearby to the distant futures is increasing in the mean reversion of convenience yield shocks.*

When the non-marketed risks are *i.i.d.*, the only uncertainty affecting the spot and futures prices are $Z_{1,t}$ and $Z_{2,t}$. When the convenience yield is mean reverting, $a_{z_2} > 0$, the price elasticity to $Z_{2,t}$ of the nearby futures is greater than that of the distant futures: $\lambda_{1,z_2} > \lambda_{2,z_2}$. The intuition is simple. Since shocks to $Z_{2,t}$ are transitory, contracts far from maturity will be less sensitive to these shocks as they will die out by the time the contract expires. Hence, the return volatility of the distant futures is less than that of the nearby, or the Samuelson Effect holds (see, e.g., Samuelson (1965)). An immediate implication of this analysis is that the more transitory are convenience yield shocks, the greater the ratio of the return volatility of the nearby to the distant futures.

In addition to these implications regarding the term structure of futures price volatility, our model also has a number of implications regarding the futures risk premium.

Proposition 5 *When non-marketed income shocks are i.i.d., the futures risk premium is increasing in the volatility and persistence of convenience yield shocks.*

It is easy to see this result from the closed-form solution for $\bar{\lambda}_{j,0}$ given in Proposition 1. Since $\bar{\lambda}_{j,0} > 0$, it follows that long positions earn a positive expected (excess) return (i.e, a futures risk premium). The greater is $\bar{\lambda}_{j,0}$, the greater the futures risk premium. It is easy to see that $\bar{\lambda}_{j,0}$ increases with σ_{z_j} $(j = 1, 2)$, the volatility of the convenience yield shocks. It is not hard to show that $\bar{\lambda}_{j,0}$ also increases with a_{z_2}.

In passing, it is worth noting that a number of the predictions of Propositions 2-5 fit nicely with existing evidence on hedging strategies, open interest and futures prices. To begin with, the hedging strategy described in Proposition 2 of being on opposite sides of nearby and distant futures (long nearby and short distant or vice versa) is known as a spread trade (see, e.g., Brown and Errera (1987) for a description of this strategy). Our model suggests that this hedging strategy is most likely to be used in futures markets with a substantial convenience yield and hence basis risk to rolling nearby contracts. Indeed, consistent with the model, such a strategy is most often used in energy futures, markets notorious for convenience yield fluctuations and basis risk associated with rolling over nearby contracts.

And consistent with the predictions of Proposition 3, oil markets, which have very mean reverting convenience yield shocks, have open interest evenly distributed among nearby and distant. In contrast, metals and agriculturals, which tend to have more persistent shocks, have more of their open interest concentrated in the nearby contracts.

It is important to note that the implications of Propositions 2-3 are not entirely consistent with empirical findings. In many markets, when the nearby contract is within a month or so of expiration, traders roll out of their positions in the nearby into the next closest contract. Many argue that this effect is likely due to liquidity considerations. Indeed, contracts within the last month of expiration tend to be very illiquid. My model does not capture this effect. That is, open interest in the nearby exceeds the distant futures until expiration at $t = m_{j,k}$, at which point the distant futures becomes the nearby and a new contract is added. Hence, one should really think of the predictions of Propositions 2-3 as pertaining to the relationship between contracts of different maturities at a single point in time that is far from the expiration of the nearby.

Consistent with some of the predictions of Proposition 4, there is a more pronounced Samuelson effect in futures markets with more mean reverting convenience yield shocks. For instance, the Samuelson effect is more pronounced in energy futures, less so in agriculturals, even less so in metals and almost non-existent in financial futures (see, e.g., Bessembinder, et.al. (1996)). There is less anecdotal evidence in support of Proposition 5. Its implications, however, are testable.

B Persistent Non-Marketed Income Shocks: $\sigma_{\mathsf{Y}} > 0$

We now discuss the solution when $\sigma_{\mathsf{Y}} > 0$. The equilibrium spot price is

$$S_t = F_{0,t} - \lambda_{0,0} - \lambda_{0,\mathsf{Y}_1} Y_t, \tag{33}$$

where $\lambda_{0,0} > 0$ and $\lambda_{0,\mathsf{Y}_1} > 0$ are periodic in time. The price discount now consists of two parts: (a) an unconditional part, $\lambda_{0,0}$, and (b) a conditional part that depends on the state variable driving non-marketed risk. The conditional covariance between convenience yield shocks and non-marketed income shocks becomes $(1 + Y_t)\sigma_{\mathsf{D}q}$. Y_t is a state variable which drives the amount of selling or buying that the investor-1 does since Y_t drives the sign and the magnitude of the correlation between investor-1's non-marketed income and spot payoffs. Again, the price of the spot has to drop to attract class-2 investors to take off-setting positions. Note that this price changes occurs without any change in the spot's payoffs. Therefore, the equilibrium spot price depends not only on the convenience yield but also on class-1 investors' non-marketed risk Y_t.

The equilibrium futures prices are now given by (for $j = 1, 2$):

$$H_{j,t} = F_{j,t} - \lambda_{j,0} - \lambda_{j,\mathsf{Y}_1} Y_t, \tag{34}$$

where $\lambda_{j,0} > 0$ and $\lambda_{j,\mathsf{Y}_1} > 0$ are periodic in time. The dependence of the futures prices on Y_t arises from the fact that investors trade in the futures to help allocate the risks associated their different spot positions.

Unfortunately, we are no longer able to obtain closed-form solutions. In general, the price coefficients need to be solved numerically. To discuss some of the properties of the model, I set the parameters of the model as follows for the numerical exercise. The parameter of absolute risk aversion, γ, is set to be 100. I choose $\omega = 0.5$ so that the number of class-1 traders is one-half of the population. The maturity length for a given contract, M, is set to be 0.6. The second contract will be staggered by a length of $L = 0.1$. I will choose the other parameters such that $M = 0.6$ corresponds roughly to a contract with seven months to expiration. I set the constant risk-free rate, r, to be .05. I set the mean reversion coefficient for the convenience yield, a_{Z_2}, to be 1 and the mean reversion coefficient for the non-marketed risk shocks, a_{Y}, to be 3. We set $\sigma_{\mathsf{Y}} = \sigma_{\mathsf{Z}_1} = \sigma_{\mathsf{Z}_2} = 0.05$, $\sigma_{\mathsf{D}} = \sigma_q = 0.25$ and $\kappa_{\mathsf{D}q} = 0.5$.

Figure 2 illustrates the price coefficients, solved for numerically at these parameters. Figure 2(a) shows the solutions for $\lambda_{0,0}$, $\lambda_{1,0}$ and $\lambda_{2,0}$. The solution for $\lambda_{0,0}$ is given by the solid line. The solutions for $\lambda_{1,0}$ (the nearby) and $\lambda_{2,0}$ (the distant) are given by dashed and '+' respectively. One the x-axis is the interval $[0, .5]$. Moving from $t = 0$ to $t = .5$ is the same as moving from $t = t_{2,k}$ to $t = m_{1,k}$ (see Figure 1). Notice that at the expiration of the nearby, $\lambda_{1,0}$ equals $\lambda_{0,0}$. Moreover, $\lambda_{2,0} > \lambda_{1,0} > \lambda_{0,0}$ as in the case of *i.i.d.* non-marketed income shocks.

Figure 2(b) shows the solutions for $\lambda_{0,\mathsf{Y}}$, $\lambda_{1,\mathsf{Y}}$ and $\lambda_{2,\mathsf{Y}}$, which are represented again by the solid, dashed and '+' lines, respectively. At the expiration of the nearby, $\lambda_{1,\mathsf{Y}} = \lambda_{0,\mathsf{Y}}$. Before the expiration date, the distant futures is less sensitive to Y_t shocks than the nearby futures: $\lambda_{2,\mathsf{Y}} < \lambda_{1,\mathsf{Y}} < \lambda_{0,\mathsf{Y}}$. One reason for this is the same as why the distant futures is less sensitive to mean reverting convenience yield shocks. The distant futures is less sensitive to mean reverting non-marketed income shocks than the nearby because these shocks are more likely to die away by the time the distant futures expires.

Next, I focus on the investors' optimal policies. First, the investors' optimal investment policies in the spot are given by

$$\theta^{\mathsf{S}}_{i,t} = h^{\mathsf{S}}_{i,0} + h^{\mathsf{S}}_{i,\mathsf{Y}} Y_t. \tag{35}$$

The first component $h^{\mathsf{S}}_{i,0}$ gives his unconditional spot position. The second component, $h^{\mathsf{S}}_{i,1} Y_t$, arises from hedging (market-making) trades associated with investor-1's non-marketed risk. The investors' optimal policies in nearby and distant futures are given

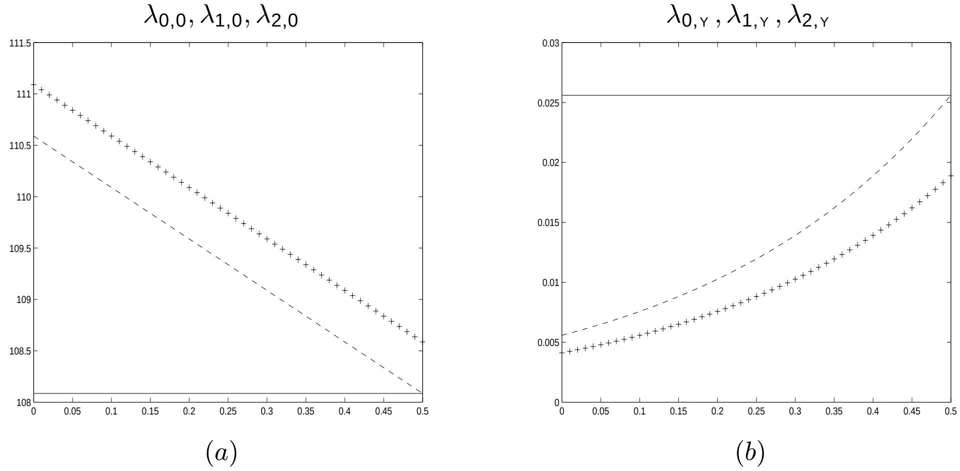


Figure 2: **Price Coefficients.** In figure (a) the spot and futures price risk discounts are shown: $\lambda_{0,0}$ is the solid line, $\lambda_{1,0}$ is the dashed line, $\lambda_{2,0}$ is the '+' line. In figure (b) the spot and futures price elasticities to Y_t are shown: $\lambda_{0,Y}$ is the solid line, $\lambda_{1,Y}$ is the dashed line, $\lambda_{2,Y}$ is the + line. On the x-axis is the time interval $[0, .5]$ which corresponds to $[t_{2,k}, m_{2,k}]$ (see Figure 1). The remaining parameters are set at the following values: $r = 0.05$, $a_{Z_2} = 1$, $a_Y = 3$, $\sigma_{Z_1} = 0.05$, $\sigma_{Z_2} = 0.05$, $\sigma_Y = 0.05$, $\sigma_D = 0.25$, $\sigma_q = 0.25$, $\kappa_{Dq} = 0.5$, $M = 0.6$, $L = 0.1$, $\omega = 0.5$, $\gamma = 100$.

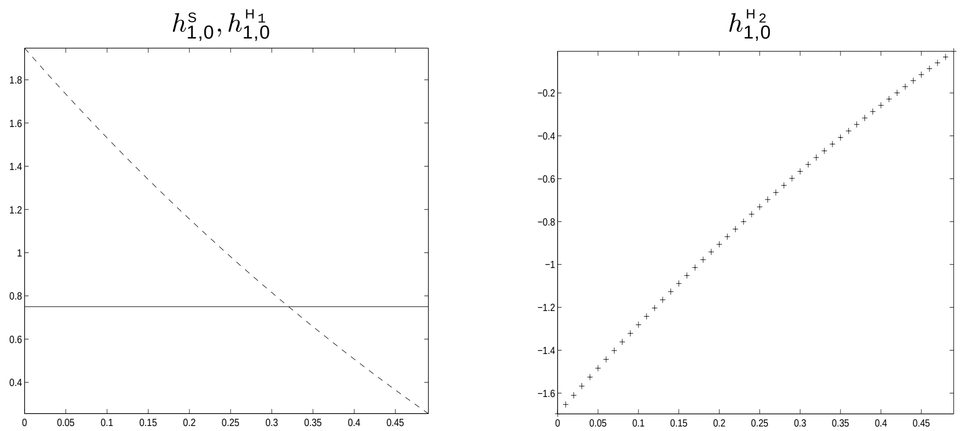
by

$$\theta_{i,t}^{H_1} = h_{i,0}^{H_1} + h_{i,Y}^{H_1} Y_t,$$
$$\theta_{i,t}^{H_2} = h_{i,0}^{H_2} + h_{i,Y}^{H_2} Y_t. \tag{36}$$

Since Y_t is on average zero, I will be particularly interested in the unconditional portion of the investors' holdings $h_{i,0}^S$, $h_{i,0}^{H_1}$ and $h_{i,0}^{H_2}$ to contrast with those derived for the case of $\sigma_Y = 0$.

Figure 3 shows these unconditional positions in spot, the nearby and the distant futures respectively. Figures 3(a) shows the solutions for $h_{1,0}^S$ and $h_{1,0}^{H_1}$. Figure 3(b) shows the solution for $h_{1,0}^{H_2}$. Notice that $h_{1,0}^S$ is less than 1, $h_{1,0}^{H_1} > 0$ and $h_{1,0}^{H_2} < 0$. Qualitatively, these patterns are quite similar to those described in Proposition 2. Investor-1 holds less than one share of the spot, longs the nearby futures and shorts the distant futures. Moreover, $|h_{1,1}^{H_1}| > |h_{1,2}^{H_2}|$ as in Proposition 3. So, when σ_Y is not very large, the qualitative patterns in Proposition 3 regarding open interest continue to hold.

When σ_Y is very large, this need no longer be the case. Investor-1's hedging problem becomes considerably more complex. In this instance, investor-1 may actually find it optimal to take a long position in the distant futures to hedge his spot position as opposed



(a) (b)

Figure 3: **Spot and Futures Holdings.** In figure (a) the spot and nearby futures holdings are shown: $h^{S}_{1,0}$ is the solid line, $h^{H_1}_{1,0}$ is the dashed line. In figure (b) the distant futures holding is shown: $h^{H_2}_{1,0}$ is the '+' line. On the x-axis is the time interval $[0, .5)$ which corresponds to $[t_{2,k}, m_{2,k})$, the interval in which contract 1 is the nearby and contract 2 the distant (see Figure 1). The remaining parameters are the same as in Figure 2.

to taking a long position in the nearby. We consider the solution for $\sigma_Y = 0.8$ in Figure 4. All the other parameters remain the same. Figure 4(a) shows the solutions for $h^{S}_{1,0}$ and $h^{H_1}_{1,0}$ and Figure 4(b) shows the solution for $h^{H_2}_{1,0}$. Notice that $h^{S}_{1,0}$ is still less than 1, but $h^{H_1}_{1,0} < 0$ and $h^{H_2}_{1,0} > 0$. The other interesting to note is that the absolute magnitude of the average position in the distant futures is now greater than in the nearby. As such, it turns out that the distant futures can actually attract more open interest than the nearby. Hence, in futures markets with substantial non-marketed risks (σ_Y large), open interest tends to be evenly distributed between nearby and distant futures. Indeed, the open interest in distant futures can even exceed that in the nearby.

Finally, when non-marketed income shocks are persistent, futures price volatility depends not only on the convenience yield shocks $Z_{i,t}$ $(i = 1, 2)$ but also on Y_t. The Samuelson Effect still characterizes the term structure of futures price volatility. And the more mean reverting the convenience yield and non-marketed income shocks, the more prominent the Samuelson effect. The futures risk premium now also depends the σ_Y in addition to the volatility and persistence of convenience yield shocks. Not surprisingly, numerical comparative statics indicate that the results of Proposition 5 continue to hold when $\sigma_Y > 0$.

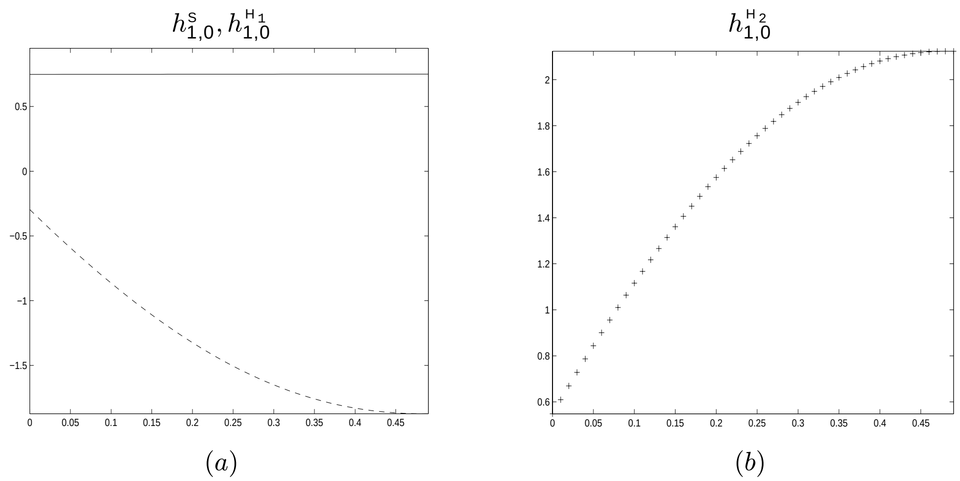


Figure 4: **Spot and Futures Holdings for σ_Y Large.** In figure (a) the spot and nearby futures holdings are shown: $h^{S}_{1,0}$ is the solid line, $h^{H_1}_{1,0}$ is the dashed line. In figure (b) the distant futures holding is shown: $h^{H_2}_{1,0}$ is the '+' line. On the X-axis is the time interval $[0, .5)$ which corresponds to $[t_{2,k}, m_{2,k})$, the interval in which contract 1 is the nearby and contract 2 the distant. The remaining parameters are the same as in Figure 2 except that $\sigma_Y = 0.8$.

V Conclusion

This paper develops a model of a futures market to study the effects of convenience yield shocks on optimal hedging and the resulting term structure of open interest and futures prices. Because convenience yield fluctuations generate basis risk to trading futures, investors, continuously facing spot price risk over time, have to simultaneously trade futures of different maturities to optimally hedge their spot positions. The model generates a number of predictions that are consistent with empirical findings on how the term structure of open interest and futures prices vary across futures markets.

An important avenue to consider in future research is to what extent liquidity effects play a role in shaping the term structure of open interest and futures prices. While our model can qualitatively deliver on different open interest and price patterns, one may need to account for the multiplier effects associated with liquidity to match these patterns quantitatively. For instance, if there is asymmetric information or some fixed cost of participation in particular futures, then there might be bunching of open interest in a particular contract (see, e.g., Admati and Pfleiderer (1988), Pagano (1989)). I leave this for future research.

Appendix

We begin by introducing some additional notation. Given a matrix m, let $\mathrm{tr}(m)$ be its trace, $[m]$ the column matrix consisting of its independent elements and $\|m\| = \max |m_{i,j}|$ its norm. When m is positive semi-definite (positive definite), we state $m \geq 0$ (> 0). Also, let $i_{ij}^{(m,n)}$ be an index matrix of order $m \times n$ with its (ij)-th element being 1 and all the other elements being zero. And let $i^{(n)}$ be an identity matrix of rank n. $\Theta = \{r > 0, \rho > 0, \gamma > 0, a_{\mathsf{Z}_1} \geq 0, a_{\mathsf{Z}_2} \geq 0, a_{\mathsf{Y}} \geq 0, \sigma_{\mathsf{D}} \geq 0, \sigma_{\mathsf{Z}_1} \geq 0, \sigma_{\mathsf{Z}_2} \geq 0, \sigma_{\mathsf{Y}} \geq 0, \sigma_q \geq 0, 0 < \kappa_{\mathsf{D}q} < 1, M > 0, L > 0\}$ denotes the set of parameter values and $\theta \in \Theta$ denotes an element.

A Mathematical Preliminaries

In deriving several results in the paper, we often encounter the two-point boundary-value problem for a (vector) first-order ODE. Here, we give a formal and relatively general definition of the two-point boundary-value problem and state some known results concerning its solution.

Definition A.1 *Let $f : \Re_+ \otimes \Re^n \otimes \Re^m \otimes \Re \rightarrow \Re^n$, and $g : \Re^n \otimes \Re^m \otimes \Re \rightarrow \Re^n$. A two-point boundary-value problem is defined as*

$$\begin{cases} \dot{z} = f(t, z; \theta, \omega) \quad \forall\ t \in [0,\ T] \\ 0 = g\left[z(0), z(T); \theta, \omega\right], \end{cases} \tag{A.1}$$

where $T > 0$, $\theta \in \Theta$ and $\omega \in [0,\ 1]$.

We also define the terminal value problem:

$$\begin{cases} \dot{z} = f(t, z; \theta, \omega) \quad \forall\ t \in [0,\ T] \\ z(T) = z_T. \end{cases} \tag{A.2}$$

Under appropriate smoothness conditions on $f(t, z; \theta, \omega)$, (A.2) has a unique solution $z = z(t; \theta, \omega; z_T)$, which is differentiable in z_T (see Keller (1992), Theorem 1.1.1). Solving the two-point boundary-value (A.1) is to seek a value for z_T that solves

$$0 = g\left[z(0; \theta, \omega; z_T), z_T; \theta, \omega\right] \equiv g \circ z(z_T; \theta, \omega). \tag{A.3}$$

The existence of a root to (A.3) relies on the properties of $g \circ z(z_T; \theta, \omega)$. Furthermore, let $(g \circ z + 1)(z_T; \theta, \omega) \equiv g \circ z(z_T; \theta, \omega) + z_T$.

Lemma A.1 *If* $(g \circ z + 1)(\cdot; \theta, \omega) : \Re^n \to \Re^n$ *is continuous and there exists a nonempty, closed, bounded, and convex subset of* $\Re^n$*,* L*, such that* $(g \circ z+1)(\cdot; \theta, \omega)$ *maps* L *into itself, then (A.3) has a root and the two-point boundary value problem (A.1) has a solution.*

Proof. Existence of a root to (A.3) follows from Brouwer's Fixed Point Theorem (see, e.g., Cronin (1994, p.352)).

The condition on $(g \circ z + 1)$ required by Lemma A.1 is not always easy to verify, in which case the existence of a solution to (A.1) is not readily confirmed. However, if a solution exists for ω_0, the existence of a solution for ω close to ω_0 is easy to establish.

Definition A.2 $z = z(t; \theta; \omega_0)$ *is an isolated solution of system (A.1) if the linearized system*

$$\begin{cases} \dot{y} = \nabla_z f(t, z; \theta, \omega_0) \quad \forall\, t \in [0, T] \\ 0 = \nabla_z g(z_0; \theta, \omega_0)\, y(0) + \nabla_z g(z_T; \theta, \omega_0)\, y(T) \end{cases} \tag{A.4}$$

has $y = 0$ *as the only solution, where* ∇ *denotes the partial derivative operator.*

Lemma A.2 *Suppose that (a) (A.1) has an isolated solution* $z = z(t; \theta, \omega_0)$ *for* $\omega = \omega_0$*, and (b)* $f(t, \cdot; \theta, \omega)$ *and* $g(\cdot; \theta, \omega)$ *are continuously differentiable in the neighborhood of* $(t, z(t; \theta, \omega_0), \omega_0)$*. Then, (A.1) has a solution for* ω *close to* ω_0*.*

Proof. See Keller (1992), p.199.

For future use, we also state two auxiliary lemmas, which are needed in proving Lemma 1 and Theorem 1.

Definition A.3 *Let* $a_0 \geq 0$ *and* $a_2 > 0$ *be constant, symmetric matrices. Also let* u *be the variable of interest, which is a symmetric matrix. Finally, let* $a_3(t, u)$ *be a positive, linear operator mapping symmetric matrices into themselves. A matrix Riccati differential equation is defined as*

$$\dot{u} + a_0 + (a_1' u + u a_1) - u a_2 u + a_3(t, u) = 0 \quad (a.e.) \quad \forall\, t \in [0,\, T] \tag{A.5}$$

where $\dot{u} = \frac{du}{dt}$ *and* $u(T) = u_T$*.*

Lemma A.3 *For any given terminal value $u_T \geq 0$, the matrix Ricatti equation (A.5) has a unique, symmetric, positive, semi-definite solution. Let m be an arbitrary (bounded measurable) matrix defined on $[0, T]$ and $k(t)$ be the solution of the following linear equation:*

$$\dot{k} + a_0 + (a_1 - m)'k + k(a_1 - m) + m'a_2 m + a_3(t, k) = 0 \quad \forall\, t \in [0, T] \tag{A.6}$$

where $k(T) = u_T$. If $u(t)$ is the solution of (A.5), then $u(t) \leq k(t) \quad \forall\, t \in [0, T]$.

Proof. See Wonham (1968).

B Solution of Investors' Control Problem

Let $a_X = \text{diag}\{0, a_Y\}$ and $b_X = \text{stack}\{0, b_Y\}$. Then

$$dX_t = -a_X X_t dt + b_X dw_t. \tag{A.7}$$

Given the price function in (16), the excess share return of the spot is

$$dQ_{0,t} = a_{0,Q} X_t dt + b_{0,Q} dw_t, \tag{A.8}$$

where $a_{0,Q} = \lambda_{0,X}\left(ri^{(2)} - a_X\right) - \dot{\lambda}_{0,X}$ and $b_{0,Q} = b_D + \lambda_{0,Z_1} b_{Z_1} + \lambda_{0,Z_2} b_{Z_2} - \lambda_{0,X} b_X$.

The return on futures is

$$dQ_{j,t} = a_{j,Q} X_t dt + b_{j,Q} dw_t \tag{A.9}$$

where $a_{j,Q} = -\lambda_{j,X} a_X - \dot{\lambda}_{j,X}$ and $b_{j,Q} = \lambda_{j,Z_1} b_{Z_1} + \lambda_{j,Z_2} b_{Z_2} - \lambda_{j,X} b_X$.

Then we re-write $Q_t = [Q_{0,t}, Q_{1,t}, Q_{2,t}]$ as

$$dQ_t = a_Q X_t dt + b_Q dw_t \tag{A.10}$$

where $a_Q = \text{stack}\{a_{0,Q}, a_{1,Q}, a_{2,Q}\}$ and $b_Q = \text{stack}\{b_{0,Q}, b_{1,Q}, b_{2,Q}\}$.

Investor-i's control problem can be solved explicitly. We start by conjecturing that his value function has the following form:

$$J_{i,t} = -\exp\left\{-\rho t - r\gamma W_{i,t} - \frac{1}{2} X_{i,t}' v_i X_t\right\} \tag{A.11}$$

where v_i are symmetric matrices. We show that the conjectured value function gives the solution to investor-i's control problem by verifying that it satisfies the Bellman equation:

$$
\begin{aligned}
t \in \mathcal{T}: \quad 0 &= \sup_{c_i, \theta_i} \left\{ -e^{-\rho t - \gamma c_i} + \mathrm{E}_{i,t}\left[dJ_{i,t} \right] / dt \right\} \\
&\text{s.t. } dW_{i,t} = (rW_{i,t} - c_{i,t})dt + \theta_{i,t}' dQ_t + \delta_i dq_{i,t}
\end{aligned} \tag{A.12}
$$

and the required boundary conditions.

Substituting the conjectured form of the value function into the Bellman's equation and applying Ito's lemma, we obtain the following expression for the optimal policies. The consumption policy is given by

$$
c_{i,t} = -\frac{1}{\gamma}\log(r) + rW_{i,t} + \frac{1}{2\gamma} X_t' v_i(t) X_t. \tag{A.13}
$$

And the investment policies are given by

$$
\theta_{i,t} = \frac{1}{r\gamma} h_i(t) X_t \tag{A.14}
$$

where

$$
h_i(t) = (b_Q b_Q')^{-1} \left[a_Q - b_Q (r\gamma e b_q \delta_i + v_i b_X)' \right].
$$

Substituting the optimal policies into the Bellman's equation, we obtain equality with the conjectured form gotten when v_i is defined:

$$
\begin{aligned}
0 = \dot{v}_i &- (r\gamma e b_q \delta_i + v_i b_X)(r\gamma e b_q \delta_i + v_i b_X)' + h_i' b_Q b_Q' h_i \\
&- r v_i - (a_X v_i + v_i a_X') + 2 a_q \delta_i + [\bar{v} + \mathrm{tr}(b_X b_X' v_i)] i_{11}^{(2,2)};
\end{aligned} \tag{A.15}
$$

where $\bar{v} = 2(\rho + r\log(r) - r)$. $g_{i,v}$ of equation (19) is given by (A.15). The periodic boundary condition is of course given by (22).

We now prove the existence of a periodic solution v_i for system (19) with boundary conditions (22) assuming that λ is periodic.

Lemma B.1 *There exist α_i and β_i with $\alpha_i < 1$ such that given the terminal value $v_{i,t_{1,k+1}}$, which is symmetric and semi-positive definite, (A.15) has a solution $v_i(t; v_{i,t_{1,k+1}})$ which is also symmetric, positive semi-definite and $\|v_i(t; v_{i,t_{1,k+1}})\| \leq \alpha_i \|v_{i,t_{1,k+1}}\| + \beta_i$. Moreover, the system (19) with boundary conditions (22) has a solution.*

Proof. For simplicity, we will assume that $a_q > (r\gamma)^2 e b_q {b_q}' e'$. Then using the notation of Lemma A.3, let $u = v_i$,

$$a_0 = (a_Q - r\gamma b_Q {b_q}' e'\delta_i)'(b_Q {b_Q}')^{-1}(a_Q - r\gamma b_Q {b_q}' e'\delta_i) + 2a_q\delta_i - (r\gamma)^2 e b_q {b_q}' e'\delta_i,$$

$$a_1 = -\frac{1}{2} r i^{(2)} - a_X - (a_Q - r\gamma b_Q {b_q}' e'\delta_i)'(b_Q {b_Q}')^{-1} b_Q b_X - r\gamma e b_q {b_X}'\delta_i,$$

$$a_2 = b_X {b_X}' - (b_Q {b_X}')(b_Q {b_Q}')^{-1}(b_Q {b_X}'),$$

and $a_3(t,u) = 2a_q + [\bar{v} + \mathrm{tr}(b_X {b_X}' u)]\, i_{11}^{(2,2)}$. It is easy to verify that a_0 and a_2 and positive definite. Assume $\bar{v} \geq 0$ (this lemma can easily be shown to hold for $\bar{v} < 0$). $a_3(t,x)$ is a linear positive operator since trace is a linear operator. Hence, (A.15) is a matrix Ricatti differential equation.

By Lemma A.3, given $v_i(t_{1,k+1}) = v_{i,t_{1,k+1}} > 0$, $v_i(t; v_{i,t_{1,k+1}})$ exists and is symmetric, positive semi-definite. Let $m = -\Big((a_Q - r\gamma b_Q {b_q}' e'\delta_i)'(b_Q {b_Q}')^{-1} b_Q b_X - r\gamma e b_q {b_X}'\delta_i\Big)$, then by Lemma A.3, $v_i(t; v_{i,t_{1,k+1}}) \leq k(t; v_{i,t_{1,k+1}})$, where k is the solution to a linear system given in Lemma A.3 with $a_1 - m = -\frac{1}{2} r i^{(2)} - a_X < 0$.

Letting $T = t_{1,k+1} - t_{1,k}$, by standard linear differential equation theory, it follows then that $\|k(t; v_{i,t_{1,k+1}})\| \leq \alpha_i \|v_{i,t_{1,k+1}}\| + \beta_i$, where

$$\alpha_i = \exp\{-(r+2a_Y)T\}; \quad \zeta_i = \exp\{[-\frac{1}{2} r i^{(2)} - a_X](T-s)\};$$

$$\beta_i = \left\| \int_0^T \zeta_i [m(T-s)' a_2^{-1} m(T-s) + a_3(T-s) + a_0(T-s)]\zeta_i ds \right\|.$$

Given the terminal boundary condition $v_{i,t_{1,k+1}}$, $v_i(t, v_{i,t_{1,k+1}})$ defines a mapping of $A_i : L_i \to L_i$ where L_i is a non-empty, closed, bounded convex subset of a finite dimensional normed vector space. It is easy to show that $||A_i(v_{i,t_{1,k+1}})|| \leq \beta_i/(1-\alpha_i)$. By Lemma A.1, it follows that system (19) with boundary condition (22) has a solution.

A solution to system (19) with boundary condition (22) gives a solution to the Bellman equation under the desired boundary condition. This completes our proof of Lemma 1.

C Proof of Theorem 1

I prove Theorem 1 as follows. First, I show that there exists a linear periodic equilibrium at $\omega = 1$. At $\omega = 1$, the value function of investor-1 is given by the following algebraic equation:

$$
\begin{aligned}
0 = & -(r\gamma e b_q + v_1 b_{\mathsf{X}})(r\gamma e b_q + v_1 b_{\mathsf{X}})' + \text{stack}\{[b_{0,\mathsf{Q}} b_{0,\mathsf{Q}}{}',\ 0],\ [0,\ 0]\} \\
& - r v_1 - (a_{\mathsf{X}} v_1 + v_1 a_{\mathsf{X}}{}') + 2a_q + [\bar{v} + \text{tr}(b_{\mathsf{X}} b_{\mathsf{X}}{}' v_1)] i_{11}^{(2,2)};
\end{aligned}
\tag{A.16}
$$

Given $\lambda_{0,\mathsf{X}}$, the equation for v_1 has only two roots, one positive and one negative. The positive root corresponds to the optimal solution of the investors' control problem since it gives a higher expected utility. λ can then be solved from

$$
\frac{h_1}{r\gamma} = \text{stack}\{[1,\ 0],\ [0,\ 0],\ [0,\ 0]\}.
$$

Given the solution for λ, then v_2 can be obtained as the solution to (A.15) and (22), which is guaranteed by Lemma 1. Hence there is a solution for v_1, v_2 and λ at $\omega = 1$.

For $i = 1, 2$, let $z_i = [v_i]$ for $t \in \mathcal{T}$. Let $z = \text{stack}\{z_1,\ z_2,\ \lambda\}$. Using the notation in Definition A.1, (19) and (24) define f. (22) and (25)-(27) define g. Then z is the solution to a two-point boundary value problem as defined in Definition A.1. Let $\omega_0 = 1$. Existence of $z(t;\theta,\omega_0)$ was just proved above. It remains to verify that $z(t;\theta,\omega_0)$ is an isolated solution. This is equivalent to showing that $m(\theta,\omega_0) = \nabla_z g(z_0) + \nabla_z g(z_T) \exp\{\int_{t_{1,\mathsf{k}}}^{t_{1,\mathsf{k}+1}} \nabla_z f\}$ is nonsingular [see Keller (1992), p.191]. It is easy to show that $\det(m(\theta_0,\omega_0)) \neq 0$ for various values of θ_0. By Lemma A.2, Theorem 1 holds.

D Proof of Propositions 1-5

Proof of Proposition 1

Take the solutions for S_t, $H_{1,t}$ and $H_{2,t}$ given in Proposition 1. One then calculates $a_{0,\mathsf{Q}}$ and $b_{0,\mathsf{Q}}$ specified in (A.8) and $a_{j,\mathsf{Q}}$ and $b_{j,\mathsf{Q}}$ specified in (A.9). With these calculations, calculate a_{Q} and b_{Q} specified in (A.10). With a_{Q} and b_{Q}, we then calculate $\theta_{i,t}$ given in (A.14). Since $\sigma_{\mathsf{Y}} = 0$, it follows that

$$
\theta_{i,t} = \frac{1}{r\gamma}(b_{\mathsf{Q}} b_{\mathsf{Q}}{}')^{-1}[a_{\mathsf{Q}} - b_{\mathsf{Q}}(r\gamma e b_q \delta_i)']. \tag{A.17}
$$

Then verify that

$$
\omega\theta_{1,t} + (1-\omega)\theta_{2,t} = \text{stack}\{[1,0],[0,0],[0,0]\}.
$$

It follows that the solutions for S_t, $H_{1,t}$ and $H_{2,t}$ given in Proposition 1 are indeed equilibrium prices.

Proof of Proposition 2

In proving Proposition 1, we had to calculate $\theta_{i,t}$ given in (A.17). Lets explicitly write down this calculation for investor-1 to prove Proposition 2. For simplicity, lets define $\hat{T} = m_{2,k+1} - t$ and $T^* = m_{1,k+1} - t$. Note that $\hat{T}$, the time-to-maturity of the distant contract is greater than T^*, the time-to-maturity of the nearby contract: $\hat{T} > T^*$.

The closed-form solution for $\theta_{1,t} = \text{stack}\{\theta^{\mathsf{S}}_{1,t},\, \theta^{\mathsf{H}_1}_{1,t},\, \theta^{\mathsf{H}_2}_{1,t}\}$ is given by the following. The spot holding is given by

$$\theta^{\mathsf{S}}_{1,t} = 1 - (1-\omega)\frac{\sigma_q \kappa_{Dq}}{\sigma_D}. \tag{A.18}$$

And the holdings in the nearby ($\theta^{\mathsf{H}_1}_{1,t}$) and the distant ($\theta^{\mathsf{H}_2}_{1,t}$) are given by

$$\begin{aligned} \theta^{\mathsf{H}_1}_{1,t} &= \frac{\sigma_q \kappa_{Dq}}{\sigma_D} \frac{1 - e^{-a_{\mathsf{Z}_2}\hat{T}}}{e^{-a_{\mathsf{Z}_2}T^*} - e^{-a_{\mathsf{Z}_2}\hat{T}}} \\ \theta^{\mathsf{H}_2}_{1,t} &= -\frac{\sigma_q \kappa_{Dq}}{\sigma_D} \frac{1 - e^{-a_{\mathsf{Z}_2}T^*}}{e^{-a_{\mathsf{Z}_2}T^*} - e^{-a_{\mathsf{Z}_2}\hat{T}}} \end{aligned} \tag{A.19}$$

From these closed-form expressions, it is easy to verify that $\theta^{\mathsf{H}_1}_{1,t} > 0$ and $\theta^{\mathsf{H}_2}_{1,t} < 0$.

Proof of Proposition 3

Using the expressions for $\theta^{\mathsf{H}_1}_{1,t}$ and $\theta^{\mathsf{H}_2}_{1,t}$ given in (A.19), the open interest in each futures is merely the absolute value of these expressions weighted by ω. Since $\theta^{\mathsf{H}_1}_{1,t} > 0$ and $\theta^{\mathsf{H}_2}_{1,t} < 0$, the ratio of the open interest in the distant over the nearby is given by:

$$\frac{-\theta^{\mathsf{H}_2}_{1,t}}{\theta^{\mathsf{H}_1}_{1,t}} = \frac{1 - e^{-a_{\mathsf{Z}_2}T^*}}{1 - e^{-a_{\mathsf{Z}_2}\hat{T}}} < 1. \tag{A.20}$$

Hence the open interest in the distant is less than in the nearby.

Now, take the derivative of this ratio of distant to nearby with respect to a_{Z_2}, which gives us

$$\frac{e^{a_{\mathsf{Z}_2}(-T^* + \hat{T})}(\hat{T}(1 - e^{a_{\mathsf{Z}_2}T^*}) + T^*(-1 + e^{a_{\mathsf{Z}_2}\hat{T}}))}{(-1 + e^{a_{\mathsf{Z}_2}\hat{T}})^2} \tag{A.21}$$

It is not hard to verify that the expression

$$\hat{T}(1 - e^{a_{\mathsf{Z}_2}T^*}) + T^*(-1 + e^{a_{\mathsf{Z}_2}\hat{T}}) \tag{A.22}$$

is positive. To see this, at $a_{\mathsf{Z}_2} = 0$, the expression is zero. Moreover, the derivative of this expression with respect to a_{Z_2} is given by

$$-e^{a_{\mathsf{Z}_2}T^*}T^*\hat{T} + e^{a_{\mathsf{Z}_2}\hat{T}}T^*\hat{T} > 0. \tag{A.23}$$

Hence, we conclude that as a_{z_2} increases, so does the ratio of open interest in the distant to the nearby.

Proof of Proposition 4

The ratio of the instantaneous return volatility of the nearby to distant is

$$\frac{b_{1,Q}b_{1,Q}'}{b_{2,Q}b_{2,Q}'} = \frac{\lambda_{0,z_1}^2\sigma_{z_1}^2 + \lambda_{1,z_2}^2\sigma_{z_2}^2}{\lambda_{0,z_1}^2\sigma_{z_1}^2 + \lambda_{2,z_2}^2\sigma_{z_2}^2} \tag{A.24}$$

The price elasticity of the futures with respect to $Z_{2,t}$ is $\lambda_{j,z_2} = e^{-a_{z_2}(m_{j,k}-t)}\lambda_{0,z_2}$ for $j = 1, 2$. Since $m_{1,k} - t < m_{2,k} - t$, it follows that $\lambda_{1,z_2} > \lambda_{2,z_2}$ and so the price volatility of the nearby is greater than that of the distant. Taking the derivative of the ratio with respect to a_{z_2} and with some tedious algebra, one can show that this derivative is positive, so the ratio of return volatility in the nearby to distant increases with a_{z_2}.

Proof of Proposition 5

From Proposition 1, take the derivative of $\bar{\lambda}_{j,0}(t)$ with respect to $\sigma_{z_j}^2$ $(j = 0, 1)$ and it is clear that the derivative is positive. So, $\bar{\lambda}_{j,0}$ increases with $\sigma_{z_j}^2$. Take the derivative again, but this time, with respect to a_{z_2}. It is easy to verify that the smaller the a_{z_2}, the higher is $\bar{\lambda}_{j,0}$.

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