THE JOURNAL OF FINANCE • VOL. LV, NO. 2 • APRIL 2000

A Model of Returns and Trading in Futures Markets

HARRISON HONG*

ABSTRACT

This paper develops an equilibrium model of a competitive futures market in which investors trade to hedge positions and to speculate on their private information. Equilibrium return and trading patterns are examined. (1) In markets where the information asymmetry among investors is small, the return volatility of a futures contract decreases with time-to-maturity (i.e., the Samuelson effect holds). (2) However, in markets where the information asymmetry among investors is large, the Samuelson effect need not hold. (3) Additionally, the model generates rich time-to-maturity patterns in open interest and spot price volatility that are consistent with empirical findings.

An issue central to the analysis of futures markets is the relationship between speculation and futures price volatility. A long line of models (see, e.g., Grossman (1977), Bray (1981)) have tried to understand the effects of speculation on futures price volatility. This line of research is economically relevant as speculative trades appear to be an important determinant of volatility in futures markets. For instance, Roll (1984) finds that public information accounts for only a fraction of the movement in orange juice futures prices, which suggests that investors bring their own private information into the market through their trades. Unfortunately, existing models of speculation in futures are essentially static ones and cannot speak to a number of interesting aspects of returns and trading in futures markets.

An example is the relationship between the price volatility and the time-to-maturity of a futures contract. The analysis of this issue is an important one and has a long history. Assuming the existence of a representative investor and an exogenous spot price process, Samuelson (1965) shows that when there is a mean-reverting component in the spot price process and no

* Graduate School of Business, Stanford University. I am deeply indebted to Jiang Wang for his advice, encouragement, and insights. I am also very thankful to Jeremy Stein for his many comments. I also thank Hank Bessembinder, Darrell Duffie, Glenn Ellison, P. Pfleiderer, Ken Singleton, René Stulz (the editor), two anonymous referees, and A. Subrahmanyam for helpful comments. I have also received many helpful comments from seminar participants at Baruch College, Carnegie Mellon, CBOT Seminar Series, Columbia, Harvard Business School, London Business School, MIT, Northwestern, New York University, Queen's-Cornell Conference, the University of British Columbia, the University of California at Berkeley, the University of Chicago, the University of Michigan, Stanford, Wharton, and WFA 1997. I acknowledge financial support from the World Economy Laboratory at MIT.

arbitrage, the return volatility of a futures contract monotonically rises as the contract expires. This monotonic time-to-maturity pattern has come to be known as the "Samuelson effect" or the "Samuelson hypothesis." An alternative to the Samuelson hypothesis is the state variable hypothesis, which argues that heterogeneous information flows lead to violations of the Samuelson effect (see, e.g., Richard and Sundaresan (1981), Anderson and Danthine (1983)).[1] Despite the popularity of these two hypotheses (see textbooks on futures markets such as Kolb (1991)), little is known about the robustness of these models to speculation by investors since the models generally assume that investors are symmetrically informed.

In this paper, I develop a dynamic model to study the effects of speculation on various aspects of returns and trading in futures. In my model, investors trade in futures to hedge positions and to speculate on their private information. Futures are not redundant securities and cannot be priced by riskless arbitrage through trading in the risky and the risk-free asset (cash and carry).[2] Rather, spot and futures prices are simultaneously determined in equilibrium. Interestingly, the model is capable of producing a variety of return and trading patterns in futures that are consistent with existing empirical evidence. I will review this evidence in detail in Section I below.

My model features two classes of investors: the "informed" and the "uninformed." These investors trade competitively in a risky asset (the spot) and a futures contract written on the spot. The payoff to holding spot positions changes over time, depending on the "fundamental" of the risky asset. To fix ideas, think of the futures as a one-year silver futures contract and the fundamental as the expected growth rate of the supply of silver. The fundamental is stochastic and mean reverting. Only informed investors have private information on how the fundamental evolves. They trade in futures to speculate on their private information. Because they are risk averse, they also trade in futures to hedge their spot positions as well as nonmarketed risks, which have varying degrees of persistence.[3] The uninformed investors trade only for hedging reasons. They are willing to trade with informed investors since the informed may be trading for hedging reasons as well. Uninformed investors try to learn about the fundamental by rationally extracting information from prices. Their learning is incomplete, however, because of the noise in prices generated by hedging trades.

I begin by showing that when there are only informed investors in the economy, the Samuelson effect holds. The intuition is simple. Consider one-year silver futures. Suppose there is a mean-reverting, negative supply shock

[1] Anderson and Danthine (1983) offer grain trade as an example where more production uncertainty about harvests tends to be resolved during certain times of the year. Since these times of the year need not coincide with the expiration dates of futures contracts, they point out that this can lead to higher price volatility for a contract far from as opposed to near expiration.

[2] This differs from many of the existing dynamic models of futures prices, which tend to use the representative agent approach to price futures (see, e.g., Jagannathan (1985), Schwartz (1997)).

[3] These nonmarketed risks can arise, for example, from their positions in other markets.

today with a half-life of one month. Holding demand fixed, the spot price of silver will no doubt rise. However, the one-year futures price is largely unchanged because much of the shock dies away by the time the contract expires. As the futures contract approaches its expiration (from a one-year to a one-month contract), its price elasticity to such shocks increases and so its return volatility rises. So the Samuelson effect is really just a "price elasticity effect." This result merely extends Samuelson (1965) to a more general setting that allows for heterogeneity among investors in nonmarketed risks.

Things get more interesting when there are also uninformed investors in the economy (so informed investors trade to both hedge positions and speculate on their private information). In this instance, I show that the Samuelson effect need not hold. To see why, consider again one-year silver futures. Suppose that shocks to the fundamental are strongly persistent (close to a random walk), but shocks to the nonmarketed risks are highly mean reverting (a half-life of one month). The futures price when time-to-maturity is large is insensitive to nonmarketed risk shocks; uninformed investors can then infer that movements in the futures price are most likely due to changes in the fundamental and hence they learn the private information of the informed investors. So there is little information asymmetry when time-to-maturity is large.

As the futures contract rolls to its expiration date, however, its sensitivity to the nonmarketed risk shocks increases and uninformed investors can learn less about the fundamental—so information asymmetry rises. As a result, less private information is impounded into the futures price and so, all else equal, the futures price moves less as the contract expires. This effect, which I term the "speculative effect," can overwhelm the price elasticity effect identified by Samuelson (1965) and lead to rich, nonmonotonic time-to-maturity patterns in futures return volatility.[4]

Beyond return volatility, this model also generates a number of other auxiliary predictions. For instance, I show that open interest can take on rich time-to-maturity patterns. When information asymmetry among investors is important, uninformed investors face an adverse selection cost in trading with informed investors. The higher the information asymmetry, the higher is this cost and the lower the open interest. The variation in information asymmetry that affects the term structure of futures return volatility is also an important determinant of open interest. Additionally, the model can also speak to the effects of new futures on spot price volatility. In my model, new futures allow investors to better hedge spot price risk and hence increase the willingness of investors to take on larger spot positions. As such, new futures tend to result in lower spot price volatility.

In what follows, I develop a simple infinite horizon model that captures these ideas and explore their empirical content. I begin in Section I by discussing the empirical evidence that motivates my work. I present my model

[4] Unlike the state variables hypothesis, this occurs even assuming homogeneous information flow.

in Section 2 and the definition and solution of the equilibrium in Section 3. I then discuss the solution in Sections IV and V. I draw out the model's empirical implications in Section VI. I conclude in Section VII.

I. Empirical Findings

A. Futures Price Volatility

A long line of empirical work has tried to document the Samuelson effect and the state variable hypothesis. The Samuelson effect has been documented in a number of markets such as crude oil and many agriculturals like wheat.[5] For many of these markets, there is still a Samuelson effect after normalizing for nonstationarities in the information flow. Interestingly, there are exceptions. For example, Anderson (1985) finds little support for the Samuelson effect in corn even after controlling for seasonality. Khoury and Yougourou (1993) find the Samuelson effect in a number of commodities but not in canola. This evidence suggests that heterogeneous information flow may not be the only reason behind violations of the Samuelson effect. The model is capable of generating nonlinear time-to-maturity patterns in futures return volatility even in the presence of homogeneous information flow (see Section VI.A).

B. Open Interest

Although extensive empirical studies have been made of futures price volatility, relatively little empirical work has been done on open interest. Two exceptions are Bessembinder (1992) and Milonas (1986). Using data on hedging demand from many futures markets, Bessembinder finds that, conditional on the demand for hedging being net short (long), mean returns to trading in futures tend to be positive (negative). This evidence is consistent with the predictions of the model in Section VI.B. Milonas considers the time-to-maturity patterns in open interest for various markets. He finds that for the liquid contracts of intermediate maturities, there can be different time-to-maturity patterns, with more distant contracts having more or less open interest than those nearer to expiration.[6] He does not explain why such systematic patterns vary across different markets. The proposed model provides one explanation for such differences across markets (see Section VI.B).

[5] See, for example, Bessembinder et.al. (1996) for a current study and review of the literature. The Samuelson effect is weak in some precious metals and financials. This is not necessarily evidence consistent with my model since these markets are near cost-of-carry and my model is really about non-cost-of-carry markets.

[6] The very distant and the nearest contracts tend to have the least open interest because they are highly illiquid.

C. New Futures and Spot Price Volatility

Finally, the other aspect of futures markets that has received some attention is the effect of new futures on spot price volatility. A number of these studies are surveyed in Damodaran and Subrahmanyam (1992). Their study concludes, "In summary, there seems to be a consensus that listing futures on commodities has resulted in lower variance in commodity prices." Recently, Netz (1995) finds that this decrease in volatility in the corn market coincides with the increased sensitivity of spot holdings (inventory) to supply and demand shocks. This finding suggests that the channel in which new futures lead to lower spot price volatility is through the increased willingness of storage companies to absorb supply and demand shocks by taking on larger inventories as they can better hedge these risks with new futures. These findings are consistent with the predictions of the model documented in Section VI.B.

II. The Model

The economy is defined on a discrete time, infinite horizon $\mathcal{T} = \{0,1,2,\ldots,\infty\}$ with a single good that can be consumed or invested. There are two classes of investors denoted by $i = a, b$. Investors are identical within each class but different across classes in physical endowments and private information. Let the population weights of these two classes be ω and $1 - \omega$, respectively, where $\omega \in [0,1]$. Investors in class-i are also referred to as investor-i. The economy is further specified as follows.

A. Investment Opportunities

There are three publicly traded assets in the economy: a riskless asset, a risky asset ("spot"), and a futures contract written on the spot. The riskless asset is assumed to have an infinitely elastic supply at a positive constant rate of return r. Let $R = 1 + r$ be the constant gross rate of return per period on the riskless asset.

Investors can trade shares of the spot in a competitive spot market. The shares of the spot are perfectly divisible and traded at no cost. Each share of the spot generates a payoff of D_t at time t. D_t is governed by the process

$$D_t = Z_t + \epsilon_{D,t}, \tag{1}$$

where Z_t follows an AR(1) process

$$Z_t = a_Z Z_{t-1} + \epsilon_{Z,t}, \qquad 0 \leq a_Z \leq 1. \tag{2}$$

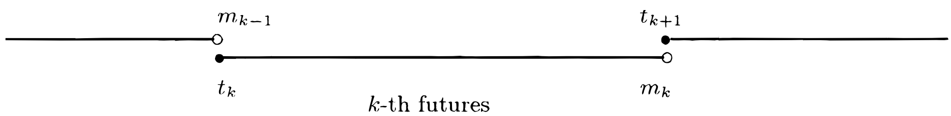


Figure 1. Periodic introduction of futures. The $(k - 1)$th contract expires at m_{k-1}, the kth contract is introduced at t_k and expires M periods later at m_k, and the $(k + 1)$th contract is introduced at t_{k+1}.

Here $\epsilon_{D,t}$ and $\epsilon_{Z,t}$ are independent and identically distributed (i.i.d.) shocks to D_t and Z_t respectively. Z_t is the persistent component in the spot payoff and $\epsilon_{D,t}$ is the idiosyncratic component. Z_t will be referred to as the "fundamental" of the spot as it fully determines the expectation of future spot payoffs.[7] I define S_t to be the spot price at time t.

Investors can also trade in a competitive futures market. They can start trading in the first futures contract at time $t_1 = 0$. This contract expires M periods later at time $m_1 = t_1 + M$. N periods later, at time $t_2 = m_1 + N$, investors can trade in another futures contract of maturity M. This second contract expires at $m_2 = t_2 + M$. Additional futures are then periodically introduced into the economy in a similar manner. That is, the $(k - 1)$th contract expires at m_{k-1} and the kth contract starts trading at $t_k = m_{k-1} + N$ and expires at $m_k = t_k + M$ for $k = 1,2,3,\ldots$. When $N = 0$, we have $t_k = m_{k-1}$ and $m_k = t_{k+1}$. Figure 1 illustrates this special case.

Since there is at most one futures contract traded at any point in time, I define H_t to be the price of the contract at time t. It is then understood that the price of the kth contract is H_t for $t \in \mathcal{H}_k$, where $\mathcal{H}_k = \{t_k, \ldots, m_k\}$.[8]

B. Endowments

Without loss of generality, I assume that each investor is endowed with one share of the spot and that the futures contract is in zero net supply. In addition to these publicly traded securities, it is assumed that investors in

[7] D_t is often referred to as a "convenience yield" (see, e.g., Brennan (1991))—the value of any benefits that inventories provide, including the ability to smooth production or facilitate the scheduling of production and sales. See Schwartz (1997) for models of the convenience yield and evidence that the convenience yield is in fact mean reverting. Think of the risky asset as a claim to an apple tree that exogenously pays out dividends each period in the form of apples (the consumption good and numéraire). These dividends can be consumed, invested in the risk-free asset, or invested to purchase more shares of the stock. The risky asset can be interpreted as a stock (which is a claim on dividends) or a currency (which is a claim on the foreign interest rate) or a storable commodity like silver (which is a claim on convenience yields as defined above).

[8] Note that at $t = m_k$, the price of the kth contract has to equal the spot price, given no arbitrage. Thus, there is no ambiguity if the $(k + 1)$th contract is introduced at the same date on which the kth contract expires.

class-a receive a nonmarketed income with the following payoffs. The nonmarketed income q_t is given by

$$q_{t+1} = Y_t + Y_t \epsilon_{q,t+1}, \tag{3}$$

where Y_t determines the expected return and volatility of the nonmarketed risk. Here $\epsilon_{q,t+1}$ is the i.i.d. idiosyncratic shock to the return. Furthermore, I assume $Y_t = 1 + Y_{1,t} + Y_{2,t}$, where $Y_{i,t}$ $(i = 1,2)$ follows an AR(1) process

$$Y_{i,t} = a_{Y_i} Y_{i,t-1} + \epsilon_{Y_i,t}, \qquad 0 \leq a_{Y_i} < 1. \tag{4}$$

The innovation $\epsilon_{Y_i,t}$ is assumed to be i.i.d. over time. Clearly, $Y_{i,t}$ for $i = 1,2$ fully determines the evolution of investor-a's nonmarketed risk.

In addition to having different physical endowments, investors also have different informational endowments. It is assumed that all investors observe the history of payoffs to the spot and market prices. That is, at time t, let $\mathcal{N}_t = \{D_s, S_s, H_s : s \leq t\}$ be the set of publicly available information. In general, only class-a investors additionally observe the fundamental as well as the expected return on their nonmarketed incomes. In other words, class-a investors observe $\mathcal{I}_t = \{\mathcal{N}_s, Z_s, q_s, Y_{1,s}, Y_{2,s} : s \leq t\}$, the complete set of information in the economy. Thus, class-a investors in general have superior information to class-b investors about future payoffs to holding the spot.

C. Policies, Preferences, and Distributional Assumptions

Each investor chooses consumption and investment policies to maximize the expected utility over lifetime consumption. For investor-i $(i = a,b)$, let $c_{i,t}$ be his consumption at date t, $\theta^S_{i,t}$ the number of shares of the spot asset he holds, and $\theta^H_{i,t}$ for $t \in \mathcal{M}_k = \{t_k, \ldots, m_k - 1\}$ his position in the kth contract. Then investor-i's consumption policy and investment policy in the spot asset are given by $\{c_{i,t} : t \in \mathcal{T}\}$ and $\{\theta^S_{i,t} : t \in \mathcal{T}\}$, respectively; his investment policy in futures is given by $\{\theta^H_{i,t} : t \in \mathcal{M}\}$, where $\mathcal{M} = \cup_k \mathcal{M}_k$. Investors' policies are adapted to their information sets.

For tractability, I assume that all investors have constant absolute risk aversion (CARA):

$$i = a,b: \qquad \mathrm{E}_{i,t}\left[-\sum_{s=t}^{\infty} \rho^{(s-t)} e^{-\gamma c_{i,s}}\right], \tag{5}$$

where $\mathrm{E}_{i,t}$ is the expectation operator conditional on investor-i's information set, ρ is the time discount factor, and γ is the risk-aversion coefficient.

Finally, it is assumed that all ϵ-shocks are jointly normal and i.i.d. over time. All shocks in the economy are uncorrelated except for innovations to the spot payoffs and investor-a's nonmarketed risks, $\epsilon_{D,t}$ and $\epsilon_{q,t}$, respectively. These two shocks are assumed to be positively correlated, $\mathrm{Corr}(\epsilon_{D,t}, \epsilon_{q,t}) = \kappa_{Dq} > 0$. For future convenience, I denote the volatility of a normally distributed variable $\epsilon_{X,t}$ as σ_X. For example, the volatility of innovation $\epsilon_{D,t}$ is σ_D.

D. Comments on the Model

Many of the assumptions of the model, such as preferences, resemble Wang (1994) and are easy to justify given the goals of the paper.[9] The ones that deserve more comments are the following. First, the assumption that the spot pays an exogenous convenience yield is for convenience of modeling, bypassing the difficulty of dealing with the consumption and production of commodities. However, Routledge, Seppi, and Spatt (1998) point out that the convenience yield may arise endogenously from a nonnegativity constraint on inventory, in which case stockouts then play an important role in generating state dependent correlation between spot and futures prices.[10] Although my model does not account for nonnegativity constraints on inventory, the results are not likely to be colored since the results of the model are really time-to-maturity effects; stockouts are likely to be seasonal effects.[11]

The assumption of a periodic market structure for futures is merely for tractability. My paper is really about what happens when a typical contract expires. Because of the periodic structure, I can focus on the behavior of the kth contract (since each contract has qualitatively the same behavior). I allow N to be an arbitrary parameter and, by setting N large, the solution is similar to a terminal period model (as is typically considered in the literature) in which there is only one contract traded and no other opportunities after the contract expires.

Furthermore, the assumption that investors in class-a face a nonmarketed risk is a simple way to generate hedging trades in futures. The important component of this assumption is that $\kappa_{Dq} > 0$.[12] Investor-a uses the spot to hedge his nonmarketed risks. This therefore generates different demands for the spot asset. In equilibrium, the two classes of investors end up holding different shares of the spot asset, which leads them to have different demands for and hence to trade in the contract to hedge.[13]

[9] For instance, the constant interest rate is reasonable given Fama and French (1987) who show that a stochastic interest rate is an unimportant factor driving weekly to monthly movements in futures prices in most futures markets.

[10] A stockout occurs when the inventory of spot is close to running out.

[11] Since the existing empirical studies control for month and year effects, the empirical evidence suggests then that the violations of the Samuelson effect are not likely to be due to stockouts. Additionally, there is no obvious reason to believe that our other results on open interest and the effect of new futures on spot price volatility are affected.

[12] The case of $\kappa_{Dq} = 0$ reduces to a representative investor economy reminiscent of those in the existing literature. It is unimportant, however, whether this correlation is positive or negative.

[13] Think of investors in class-a as utility companies and investors in class-b as storage companies (or outsiders) who make the market for a discount in the price. These companies hold inventories of natural gas and hedge in natural gas futures. One can think of nonmarketed risks as the utilities' positions in coal. Fluctuations in the coal price then affect their spot and futures positions in natural gas. The other component of this assumption is that the nonmarketed risks are driven by the two state variables $Y_{i,t}$ $(i = 1,2)$. This is merely to keep the equilibrium prices from fully revealing all of investor-a's private information to investor-b.

III. Equilibrium

A. Definition of Equilibrium

In order to derive an equilibrium of the economy, I begin by stating each investor's optimization problem under given spot and futures prices. The investment policies of investors depend on the returns. I define Q_t^S to be the excess return on one share of the spot (the return minus the financing cost at the risk-free rate) and Q_t^H to be the return on the futures contract (the change in the futures price from time $t-1$ to t). That is,

$$Q_t^S \equiv S_t + D_t - RS_{t-1}, \qquad Q_t^H \equiv H_t - H_{t-1}. \tag{6}$$

Note that Q_t^S is the excess return on one share of the spot instead of the excess return on one dollar invested in the spot. The former is the excess share return, the latter is the excess rate of return.

Given the investors' preferences in equation (5), investors' endowments in equations (3) and (4), and the return processes defined in equation (6), the investors' optimization problems are given by, for $i = a, b$:

$$J_{i,t} \equiv \sup_{\{c_i, \theta_i^S, \theta_i^H\}} \mathrm{E}_{i,t}\left[-\sum_{s=t}^{\infty} \rho^{(s-t)} e^{-\gamma c_{i,s}} \right] \tag{7a}$$

subject to

$$W_{i,t+1} = (W_{i,t} - c_{i,t})R + \theta_{i,t}^S Q_{t+1}^S + \theta_{i,t}^H Q_{t+1}^H + \delta_i q_{t+1}, \tag{7b}$$

where $J_{i,t}$ is investor-i's value function at time t, $W_{i,t}$ is investor-i's wealth at time t, and δ_i is an index function where $\delta_i = 1$ if $i = a$ and $\delta_i = 0$ if $i = b$.

For there to be an equilibrium, the following two conditions must hold. First, the price of the kth contract must equal the spot price at its expiration date, m_k. That is, for $k = 1, 2, \ldots,$

$$H_{m_k} = S_{m_k}. \tag{8}$$

This is simply a no-arbitrage condition. Second, spot and futures prices are such that investors follow their optimal policies and markets clear:

$$\begin{aligned} t \in \mathcal{T}: &\quad \omega\theta_{a,t}^S + (1-\omega)\theta_{b,t}^S = 1, \\ t \in \mathcal{M}: &\quad \omega\theta_{a,t}^H + (1-\omega)\theta_{b,t}^H = 0. \end{aligned} \tag{9}$$

The resulting equilibrium prices and investors' optimal policies can in general be expressed as a function of the state of the economy and time. The state of the economy is determined by the investors' wealth and their expectations about current and future investment opportunities. But due to the assumptions of a constant risk-free rate and constant absolute risk aversion in preferences, investors' demand of risky investments will be independent of their wealth (see, e.g., Merton (1971)). Thus I seek an equilibrium in which the market prices are independent of investors' wealth. Let $\bullet$ denote the relevant state variables. Then one can write $S_t = S(\bullet;t)$, $H_t = H(\bullet;t)$, and $\{c_i(\bullet;t), \theta_i^S(\bullet;t), \theta_i^H(\bullet;t)\}$.

Due to the nature of the periodic introduction of futures, I consider periodic equilibria in which the equilibrium price processes and investors' optimal policies exhibit periodicity in time. Furthermore, I restrict myself to linear equilibria in which the price functions are linear in $\bullet$. M and N stay the same across time.

Definition 1: In the economy defined above, a linear, periodic equilibrium is defined by the price functions $\{S(\bullet;t), H(\bullet;t)\}$ and policy functions $\{c_i(\bullet;t), \theta_i^S(\bullet;t), \theta_i^H(\bullet;t)\}$, $i = a,b$, such that (a) the policies maximize investors' expected utility, (b) all markets clear, (c) the price functions are linear in the state variables $\bullet$ and periodic in time with periodicity $M + N$, and (d) investors' policy functions are also periodic in time.

Under Definition 1, the kth contract depends on the underlying uncertainty in the economy in the same way as the $(k+1)$th contract and so forth. That is, for $k = 1,2,\ldots,$

$$S(\bullet;t_k) = S(\bullet;t_{k+1}), \qquad H(\bullet;t_k) = H(\bullet;t_{k+1}). \tag{10}$$

Realized values of the spot and futures contracts can be different from period to period and contract to contract as the state variables change. Furthermore, periodicity in each investor's optimization problem and policy functions yield, for $i = a,b$,

$$J_i(\bullet;t_k) = J_i(\bullet;t_{k+1}). \tag{11}$$

Thus, a periodic equilibrium is given by periodic price functions that satisfy equation (8) such that investors optimally solve equations (7) and (11), and the markets clear—equation (9) holds. The periodicity conditions for the prices and value functions, equations (10) and (11), provide the necessary boundary conditions we need to solve for a periodic equilibrium.

B. Equilibrium Solution

In solving for an equilibrium, I proceed as follows: First, conjecture a particular equilibrium, then characterize the investors' optimal policies and the market clearing conditions under the conjectured equilibrium, and finally verify that the conjectured equilibrium in fact exists.

Let $\boldsymbol{G}_t = [Z_t, Y_{1,t}, Y_{2,t}]'$ denote the vector of these state variables. In general, the equilibrium also depends on the uninformed investors' expectations of these variables. Since $\boldsymbol{G}_t$ is not publicly observable, the uninformed investors rationally extract information about their values using public signals. Let $\hat{\boldsymbol{G}}_t = \mathrm{E}_{b,t}[\boldsymbol{G}_t]$ be the uninformed investors' conditional expectations of $\boldsymbol{G}_t$ based on their information set. The uninformed investors form their demands on the basis of their forecasts. Consequently, the equilibrium prices depend not just on $\boldsymbol{G}_t$ but also on $\hat{\boldsymbol{G}}_t$.

As such, I conjecture that the equilibrium asset prices have the following linear form:

CONJECTURE 1: *A linear periodic equilibrium is* $\{S_t, H_t\}$ *such that, for* $k = 1, 2, \ldots,$

$$\begin{aligned} t \in \mathcal{T}: \quad S_t &= \lambda_{S,Z}(t) Z_t - \boldsymbol{\lambda}_{S,X}(t) \boldsymbol{X}_t \\ t \in \mathcal{H}_k: \quad H_t &= \lambda_{H,Z}(t) Z_t - \boldsymbol{\lambda}_{H,X}(t) \boldsymbol{X}_t, \end{aligned} \tag{12}$$

where $\boldsymbol{X}_t = [1, Y_{1,t}, Y_{2,t}, (\boldsymbol{G}_t - \hat{\boldsymbol{G}}_t)']'$ *and equation (8) holds—that is,*

$$\lambda_{H,Z}(m_k) = \lambda_{S,Z}(m_k), \quad \boldsymbol{\lambda}_{H,X}(m_k) = \boldsymbol{\lambda}_{S,X}(m_k). \tag{13}$$

For the conjectured price function, I have imposed the no-arbitrage condition, equation (8), that the price of the kth contract equal the spot price at its expiration date. For future convenience, I define $\boldsymbol{\lambda}(t)$ to be the time-varying vector with the price elasticities (λs) as its elements.

To characterize the equilibrium, I take the price function in equation (12) as given and derive each investor's conditional expectations, policies, and the market-clearing conditions.

B.1. Conditional Expectations

I now calculate the evolution of the conditional expectations and conditional variances formed by investor-b: $\hat{\boldsymbol{G}}_t = \mathrm{E}_{b,t}[\boldsymbol{G}_t]$ and $\boldsymbol{o}(t) = \mathrm{E}_{b,t}[(\boldsymbol{G}_t - \hat{\boldsymbol{G}}_t)(\boldsymbol{G}_t - \hat{\boldsymbol{G}}_t)']$. Calculating conditional expectations and variances is just a Kalman filtering problem whose solution is given in Lemma 1 (see the Appendix).

LEMMA 1: *In a linear periodic equilibrium of the form of equation (12),* $\hat{\boldsymbol{G}}_t$ *is a linear, Gaussian Markov process under investor-b's information set. And* $\boldsymbol{o}(t)$ *evolves deterministically according to a system of difference equations:*

$$\boldsymbol{o}(t) = \boldsymbol{g}_o(\boldsymbol{o}(t-1); \boldsymbol{\lambda}(t)), \tag{14}$$

where $\boldsymbol{g}_o$ *is given in the Appendix.*

Given the price coefficients, $\boldsymbol{\lambda}(t)$, equation (14) guides the deterministic evolution of $\boldsymbol{o}(t)$ given an initial value. For instance, given an initial value $\boldsymbol{o}(t_k)$, I can integrate equation (14) to get $\boldsymbol{o}(t)$ for some $t > t_k$. This is the standard initial value problem. A periodic solution to equation (14), and hence to the filtering problem, further requires that for $k = 1, 2, \ldots$

$$\boldsymbol{o}(t_k) = \boldsymbol{o}(t_{k+1}). \tag{15}$$

B.2. Optimal Policies

Given the price functions and conditional expectations, I can solve for the optimal policies of the investors. Investor-i's control problem as defined in equation (7) can be solved explicitly. The following lemma summarizes the results.

LEMMA 2: *Let $\boldsymbol{X}_{i,t} = \mathrm{E}_{i,t}[\boldsymbol{X}_t]$, $i = a, b$. Given the price functions in system (12), investor-i's value function has the form:*

$$t \in \mathcal{T}: \quad J_{i,t} = -\rho^t \exp\left\{-\frac{r\gamma}{R} W_{i,t} - \frac{1}{2}\left(\boldsymbol{X}'_{i,t}\boldsymbol{v}_i(t)\boldsymbol{X}_{i,t}\right)\right\}, \qquad (i = a, b), \tag{16}$$

where $\boldsymbol{v}_i(t)$ are symmetric matrices that satisfy a system of difference equations given by

$$\boldsymbol{v}_i(t-1) = \boldsymbol{g}_{i,v}(\boldsymbol{v}_i(t); \boldsymbol{o}(t); \boldsymbol{\lambda}(t)), \tag{17}$$

where $\boldsymbol{g}_{i,v}$ is given in the Appendix. His optimal investment policies in the spot $\{\theta^S_{i,t}: t \in \mathcal{T}\}$ and futures contract $\{\theta^H_{i,t}: t \in \mathcal{M}\}$ are linear in the state variables:

$$\theta^S_{i,t} = \boldsymbol{h}^S_i \boldsymbol{X}_{i,t}, \qquad \theta^H_{i,t} = \boldsymbol{h}^H_i \boldsymbol{X}_{i,t}, \tag{18}$$

where $\boldsymbol{h}^S_i$ and $\boldsymbol{h}^H_i$ are functions of $\boldsymbol{o}$, $\boldsymbol{v}_i$, and $\boldsymbol{\lambda}$ and are given in the Appendix. Moreover, his optimal consumption policy is

$$t \in \mathcal{T}, \quad c_{i,t} = -\frac{1}{\gamma}\log\left[\frac{1}{\gamma}\frac{\partial J_{i,t}}{\partial W_i}\right].$$

Given $\boldsymbol{\lambda}(t)$ and $\boldsymbol{o}(t)$, Lemma 2 expresses investor-i's optimal policies as functions of the matrices $\boldsymbol{v}_i(t)$, to be solved from system (17). A periodic solution for investor-i's control problem further requires that, for $k = 1, 2, \ldots,$

$$\boldsymbol{v}_i(t_k) = \boldsymbol{v}_i(t_{k+1}). \tag{19}$$

B.3. Market Clearing

In equilibrium, the markets must clear. From equation (9) and Lemma 2, the market clearing conditions define a system of difference equations for $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}(t-1) = \boldsymbol{g}_{\lambda}(\boldsymbol{\lambda}(t); \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{o}), \tag{20}$$

where $\boldsymbol{g}_\lambda$ is given in the Appendix. The periodic condition for prices further requires that

$$\boldsymbol{\lambda}(t_k) = \boldsymbol{\lambda}(t_{k+1}). \tag{21}$$

B.4. Existence and Computation of Equilibrium

The previous discussion characterizes the investors' expectations and optimal policies in a linear, periodic equilibrium of equation (12). Solving for such an equilibrium now reduces to solving equations (14), (17), and (20), a system of first-order difference equations, for $\boldsymbol{o}$, $\boldsymbol{v}_a$, $\boldsymbol{v}_b$, and $\boldsymbol{\lambda}$ subject to boundary conditions (15), (19), and (21). The solution of a system of difference equations depends on the boundary condition. For the familiar initial-value problem, the boundary condition is simply the initial value of the system. It seeks a solution given its value at a fixed point in time. My problem, however, has a different boundary condition. I need to find particular initial values $\boldsymbol{o}(t_k)$, $\boldsymbol{v}_i(t_k)$ $(i = a, b)$, $\lambda_{S,Z}(t_k)$, $\boldsymbol{\lambda}_{S,X}(t_k)$, $\lambda_{H,Z}(t_k)$, and $\boldsymbol{\lambda}_{H,X}(t_k)$ such that the periodic conditions hold. This is known as a two-point boundary value problem, which seeks a solution of the system with values at two given points in time satisfying a particular condition.

Theorem 1 states the result on the existence of a solution to the given system, which gives a linear periodic equilibrium of the economy.

THEOREM 1: *For ω (the fraction of informed investors in the economy) close to one or σ_{Y_1} and σ_{Y_2} (the volatility of innovations to nonmarketed risks) close to zero, a linear periodic equilibrium of the form in equation (12) exists generically in which the uninformed investors' expectations are given by Lemma 1 and the optimal policies of both investors are given by Lemma 2.*

Here, the conditions on the parameters ω, σ_{Y_1}, and σ_{Y_2} arise from the particular approach I use in the proof as opposed to economic rationales (see the Appendix).[14] In general, the model needs to be solved numerically. In my numerical calculations, I make sure that the solution I obtain is robust to a variety of standard numerical checks. The numerical methods used to solve this system of difference equations are standard (see the Appendix). Once these coefficients are established, we can numerically calculate the unconditional return volatility of the spot and futures, $\text{Var}(\boldsymbol{Q}_t^S)$ and $\text{Var}(\boldsymbol{Q}_t^H)$, and the open interest in a contract—investor-a's average (over all shocks) position in the futures scaled by the population weight of class-a investors.

[14] My proof relies on a continuity argument. For instance, it is first shown that a solution to the given system exists for $\sigma_{Y_1} = \sigma_{Y_2} = 0$ (ω and the rest of the parameters of the system can take on arbitrary values). Since the system is smooth with respect to σ_{Y_1} and σ_{Y_2}, it is then shown that a solution also exists for σ_{Y_1} and σ_{Y_2} close to zero. I do not specify in the proof, however, how close it has to be. Analogously, I use the same proof technique to show that there exists a solution for ω close to one but the rest of the parameters can take on arbitrary values.

IV. A Special Case: Symmetric Information

To build intuition, I first consider a special case of the model in which investors differ only in their exposures to nonmarketed risks—that is, all investors are completely and symmetrically informed.

The solution to this special case follows from the general solution given in Section III.B by setting $\hat{\boldsymbol{G}}_t = \boldsymbol{G}_t$. First, consider the spot price. S_t is given by

$$t \in \mathcal{T}: \quad S_t = \lambda_{S,Z} Z_t - (\lambda_{S,0} + \lambda_{S,1} Y_{1,t} + \lambda_{S,2} Y_{2,t}), \tag{22}$$

where $\lambda_{S,Z} = a_Z/(R - a_Z)$ is the spot price elasticity to the fundamental Z_t and the remaining price coefficients, $\lambda_{S,i} > 0$ $(i = 0,1,2)$, are determined numerically (see Section III.B.4). The equilibrium futures price is given by

$$t \in \mathcal{H}_k: \quad H_t = \lambda_{H,Z} Z_t - (\lambda_{H,0} + \lambda_{H,1} Y_{1,t} + \lambda_{H,2} Y_{2,t}), \tag{23}$$

where $\lambda_{H,Z}(t) = \lambda_{S,Z} a_Z^{(m_k - t)}$ is the futures price elasticity to the fundamental Z_t and $\lambda_{H,i} > 0$ $(i = 0,1,2)$ are also determined numerically.

The spot price is simply the value of expected future convenience yields, $(\lambda_{S,Z} Z_t)$, minus a risk discount $(\lambda_{S,0} + \sum_i \lambda_{S,i} Y_{i,t})$. The risk discount naturally depends on uncertainty in the convenience yield since investors are risk averse. Additionally, it also depends on the covariance between convenience yields (D_{t+1}) and nonmarketed risks (q_{t+1}), given by $(1 + Y_{1,t} + Y_{2,t})\kappa_{Dq}\sigma_D\sigma_q$. Suppose $Y_{1,t} > 0$, then, all else equal, the covariance of D_{t+1} and q_{t+1} becomes more positive, which leads class-a investors to reduce their demands for the spot to hedge their nonmarketed risks. When class-a investors sell the spot to rebalance their portfolios, the spot price has to drop to attract class-b investors. The price drop rewards class-b investors for bearing additional risks by taking on more shares of the spot. Note that this price change occurs without any change in the spot's payoffs. Therefore, the equilibrium spot price depends not only on convenience yield movements but also on class-a investors' nonmarketed risks.

The futures price also consists of two parts. The first part, $\lambda_{H,Z} Z_t$, is simply the futures price in a risk-neutral world (i.e., obtained from the cost-of-carry formula). The second part, which depends on $Y_{i,t}$ $(i = 1,2)$, arises from the hedging demand of investors. Since class-a and b investors have different spot positions, they trade in the futures to hedge spot price risk due to fluctuations in Z_t. Thus, the investors' policies given in equation (18) depend on nonmarketed risks.

A. Nonmarketed Risks are i.i.d.

To see the behavior of return volatility and open interest, suppose that nonmarketed risks are i.i.d. Then we obtain closed-form solutions and establish the following proposition.

PROPOSITION 1: *When nonmarketed risks in the economy are i.i.d. (i.e., $\sigma_{Y_1} = \sigma_{Y_2} = 0$), the Samuelson effect holds and open interest monotonically declines as the futures expires.*

In this instance, $\mathrm{Var}(Q_t^H)$ only depends on innovations to Z_t. Since the futures price elasticity to Z_t, $\lambda_{H,Z}(t)$, declines with time-to-maturity when $a_Z < 1$, $\mathrm{Var}(Q_t^H)$ increases monotonically as the futures expires. So the Samuelson effect is just a "price elasticity effect."

Furthermore, investor-a's position in the futures given in equation (18) reduces to a simple mean variance hedge:

$$t \in \mathcal{M}_k: \quad \theta_{a,t}^H = \beta_{S,H}\Delta, \tag{24}$$

where $\beta_{S,H}$ is the regression coefficient of spot returns on futures returns and Δ is a constant spot position. The optimal hedge ratio $\beta_{S,H}$ naturally increases with the covariance of q_t and D_t, now given by $\kappa_{Dq}\sigma_D\sigma_q$. Moreover, it also increases with time-to-maturity when the fundamental Z_t is mean reverting. The intuition is simple. The farther the contract is from expiration, the less sensitive is its price to Z_t, and hence a larger position in the contract is required to hedge a given spot position. It follows that open interest increases with time-to-maturity.

B. Nonmarketed Risks Are Persistent

The behavior of return volatility and open interest established in Proposition 1 remains qualitatively the same in the general case of persistent nonmarketed risks. I make this point by numerically calculating the equilibrium for a set of parameters.[15]

Figure 2 (A) illustrates the time-to-maturity patterns of the futures price elasticities, λ_{H,Y_1} and λ_{H,Y_2}, for an arbitrary kth contract.[16] They increase monotonically as the contract expires. For this figure, I have set $a_{Y_1} = 0.97$ to be greater than $a_{Y_2} = 0.8$, so $Y_{1,t}$ is more persistent than $Y_{2,t}$. Not surprisingly, λ_{H,Y_2} is nearly zero far from expiration and rises dramatically as the futures nears expiration, whereas λ_{H,Y_1} exhibits a less dramatic decline with time-to-maturity. For brevity, I leave out the corresponding figures for the spot in this section (see Section VI.B for a discussion).

[15] I want each period of trading to correspond roughly to a week so, throughout, I set the constant risk-free rate per period, r, to be 0.05 percent. The maturity length for a given contract, M, will be set to thirty periods. This approximately corresponds to a typical six-month futures. I choose $\omega = 0.05$ so that the number of class-a traders is a relatively small fraction of the population. This parameter is chosen merely for ease of comparison with the asymmetric information case in Section V. For simplicity, I set N to zero in this section. Setting N small has the disadvantage of interacting the pure effect of a contract expiring with the "introduction effect" (see Section VI.B below). In later sections, I set $N = 30$ to separate these two effects. The remaining parameters are chosen to illustrate the main insights of the model.

[16] I omit the behavior of $\lambda_{H,0}(t)$. Its time-to-maturity pattern is similar to the closed form solution obtained for the i.i.d. case given in the Appendix. It falls as the contract rolls to expiration.

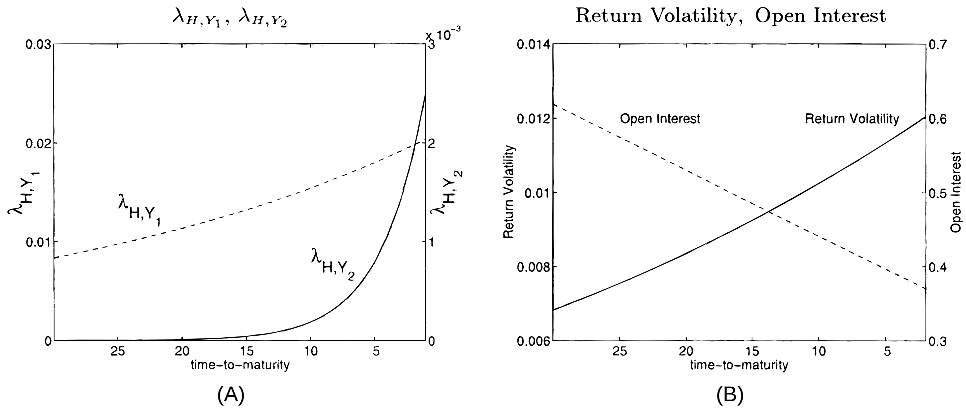


Figure 2. Time-to-maturity patterns of futures price elasticities, return volatility, and open interest under symmetric information. The figures plot the time-to-maturity patterns of (A) the futures price elasticities, λ_{H,Y_1} and λ_{H,Y_2}, and (B) the return volatility and open interest of the kth contract. The parameters are: $\kappa_{Dq} = 0.5$, $\sigma_D = 0.2$, $\sigma_q = 0.025$, $a_Z = 0.98$, $\sigma_Z = 0.025$, $a_{Y_1} = 0.97$, $\sigma_{Y_1} = 0.075$, $a_{Y_2} = 0.8$, $\sigma_{Y_2} = 0.05$.

Figure 2(B) illustrates the concomitant return volatility and open interest patterns for the kth contract. Since $\lambda_{H,Z}$ is also monotonically increasing as the contract expires, it follows from the discussion above on λ_{H,Y_1} and λ_{H,Y_2} that the return volatility of the contract increases as it expires. In other words, there is now an additional channel for the Samuelson effect—through the risk premium portion of prices. And the open interest of the contract continues to decline.

V. General Case of Asymmetric Information

I now assume that investors in class-b no longer have complete information about the state variables in the economy. Instead, they can only noisily estimate the true values of Z_t, $Y_{1,t}$, and $Y_{2,t}$ using public information, while class-a investors continue to observe these values. I refer to class-a investors as the "informed" and class-b investors as the "uninformed."

The futures price function, under asymmetric information, is now

$$t \in \mathcal{H}_k: \quad H_t = \lambda_{H,Z} Z_t - \lambda_{H,0} - \lambda_{H,1} Y_{1,t} - \lambda_{H,2} Y_{2,t} + \lambda_{H,\hat{Z}}(Z_t - \hat{Z}_t), \quad (25)$$

where $\lambda_{H,Z} = \lambda_{S,Z} a_Z^{(m_k - t)}$ and $\lambda_{H,i} > 0$ $(i = 0,1,2)$ and $\lambda_{H,\hat{Z}} > 0$ are determined numerically. The first four terms of the price function are similar to the case of symmetric information. The last term, reflecting differences in the informational endowments of the investors, arises from the speculative trades of the informed on their private information. For example, when the uninformed investors underestimate the fundamental, $Z_t - \hat{Z}_t > 0$, the informed investors purchase futures in expectation of future price increases, thereby driving up the futures price. This speculative motive for trade is also reflected in the holdings of the informed investors from equation (18) since they now depend on $(Z_t - \hat{Z}_t)$.[17]

A. Information Asymmetry, Return Volatility, and Open Interest

Both the return volatility and open interest of a futures contract depend crucially on the degree of information asymmetry in the economy. A natural measure of this information asymmetry among investors is

$$t \in \mathcal{T}: \quad \delta(t) = \sqrt{\mathrm{E}_{b,t}[(Z_t - \hat{Z}_t)^2]}, \tag{26}$$

the conditional standard deviation of the uninformed investor's estimation error of the fundamental. $\delta(t)$ is deterministic through time and is obtained easily from the solution of $o(t)$ from equation (14). All else equal, a larger δ implies that uninformed investors have less information about Z_t and hence that less of the informed investor's private information is impounded into prices, so the lower is the return volatility. Similarly, a larger δ implies that there is higher information asymmetry (and so higher adverse selection cost to trading for the uninformed), so the lower is the open interest.

B. Shocks to the Fundamental Are More Persistent than Nonmarketed Risks

In this section, I consider the case in which a_Z is larger than either a_{Y_1} or a_{Y_2}. So shocks to the fundamental are longer lived than those to nonmarketed risks (or the "noise" in prices). In Figure 3, I illustrate the time-to-maturity pattern for δ and the concomitant patterns for return volatility and open interest. I use the same parameters as described in Section IV.B. Note that a_Z is set to 0.98, so Z_t is highly persistent—close to a random walk. The nonmarketed risks are relatively more transitory, with a_{Y_1} and a_{Y_2} set to 0.97 and 0.8, respectively.

Consider Figure 3(A). On the x-axis is time-to-maturity and on the y-axis is δ, the measure of information asymmetry. Notice that information asymmetry is monotonically increasing as the futures contract expires. The reason is as follows. From the analysis of the Samuelson effect above, it follows that the the futures price is equally sensitive to Z_t across the life of the contract since Z_t is close to a random walk. However, the futures price elasticity to nonmarketed risk shocks falls with the time-to-maturity of the contract more quickly than the elasticity to the fundamental. Hence, with large time-to-maturity, the futures price is very informative about Z_t since it is

[17] An uninformed investor cannot perfectly identify the informed investors' trading motives. His market making trade is based only on his expectation about the informed investors' hedging needs, as seen from the dependence on $\hat{Y}_{1,t}$ and $\hat{Y}_{2,t}$ and not on the actual values, $Y_{1,t}$ and $Y_{2,t}$. Part of the market making trade actually corresponds to the informed investors' speculative trade.

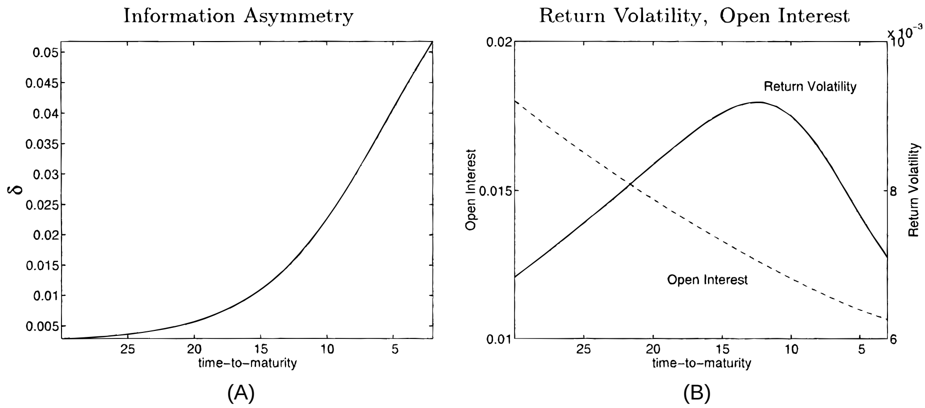
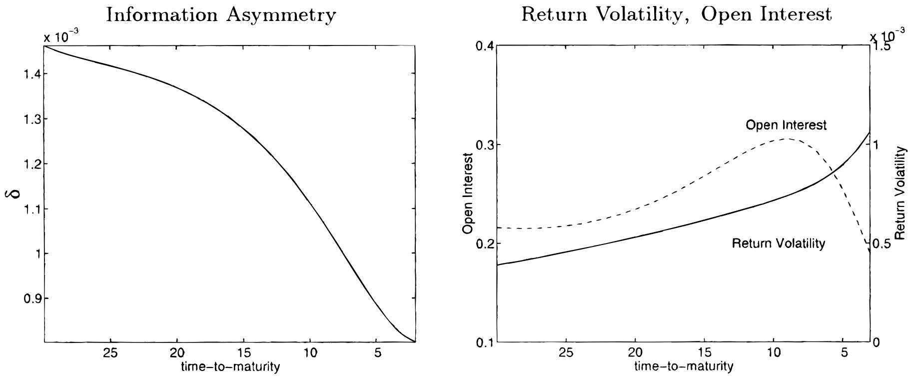


Figure 3. Time-to-maturity patterns of information asymmetry, futures return volatility and open interest. The figures plot the time-to-maturity patterns of (A) information asymmetry, δ, and (B) the return volatility and open interest of the k-th contract. The parameters are set at the following values: $\kappa_{Dq} = 0.5$, $\sigma_D = 0.2$, $\sigma_q = 0.025$, $a_Z = 0.98$, $\sigma_Z = 0.025$, $a_{Y_1} = 0.97$, $\sigma_{Y_1} = 0.075$, $a_{Y_2} = 0.8$, $\sigma_{Y_2} = 0.05$.

relatively inelastic to nonmarketed risk shocks (the noise in the futures price signal). As the contract rolls to its expiration, its sensitivity to these noise shocks increases and information asymmetry rises. I term this variation in information asymmetry with the time-to-maturity of the futures contract "the speculative effect" of futures expiration.

Next consider Figure 3(B). Notice that the Samuelson effect need no longer hold. The reason is due to the interaction of the price elasticity effect pointed out by Samuelson (1965) and the speculative effect. With lots of time to expiration, there is little information asymmetry between the two classes. Hence, the return volatility is driven mainly by the price elasticity effect—the Samuelson effect holds with lots of time to expiration. But as the contract expires, information asymmetry, in this case, rises substantially. All else equal, this implies that the return volatility of the contract tends to fall. In this instance, the speculative effect outweighs the Samuelson effect and the return volatility of the futures falls near expiration. In general, a number of other time-to-maturity patterns can occur. For one, the two effects may just offset each other, leaving only a relatively flat time-to-maturity pattern. Regardless, the point is that the speculative effect works against the Samuelson effect in this instance.

Now observe that open interest is monotonically declining as the contract expires. From the discussion in Section IV, this is to be expected. Under asymmetric information, open interest also depends on information asymmetry, or the degree of adverse selection. The higher the adverse selection, the lower the open interest. In this instance, since information asymmetry (adverse selection) is lowest with lots of time to maturity and increases as the futures expires, open interest also falls as the futures expires because of an increase in adverse selection.



(A) (B)

Figure 4. Time-to-maturity patterns of information asymmetry, futures return volatility, and open interest. The figures plot the time-to-maturity patterns of (A) information asymmetry, δ, and (B) the return volatility and open interest of the kth contract. The parameters are set at the following values: $\kappa_{Dq} = 0.5$, $\sigma_D = 0.2$, $\sigma_q = 0.025$, $a_Z = 0.95$, $\sigma_Z = 0.025$, $a_{Y_1} = 0.98$, $\sigma_{Y_1} = 0.075$, $a_{Y_2} = 0.95$, $\sigma_{Y_2} = 0.05$.

C. Shocks to the Fundamental Are Less Persistent than Nonmarketed Risks

Here I consider the case in which a_Z is smaller than either a_{Y_1} or a_{Y_2}, so shocks to the fundamental are shorter lived than those to nonmarketed risks. In Figure 4, I illustrate the time-to-maturity pattern for δ and the concomitant patterns for return volatility and open interest. I use the same parameters as in Figure 3, except that a_Z is set to 0.95 and a_{Y_1} and a_{Y_2} are set to 0.98 and 0.97, respectively. So Z_t is relatively more transitory, and $Y_{1,t}$ and $Y_{2,t}$ are more persistent.

Consider Figure 4(A). Since fundamental shocks now are less persistent than nonmarketed risk shocks, information asymmetry now monotonically decreases as the contract expires.[18] Next consider Figure 4(B). Notice that the return volatility of the futures contract is now increasing everywhere as the contract expires. This is to be expected since the speculative effect in this instance reinforces the Samuelson effect. As the contract expires, uninformed investors are able to better track Z_t and hence revise their expectations more frequently. So return volatility increases because of both the price-elasticity and the speculative effect.

Now notice that the open interest in the contract takes on an inverted U-shaped pattern, rising initially and then decreasing as the contract expires. The reason is simple. The price elasticity effect implies that open interest increases with time-to-maturity. However, since information asymmetry rises with time-to-maturity, it follows that the adverse selection cost to trad-

[18] The time variation in information asymmetry is now the opposite of that considered in Section V.B.

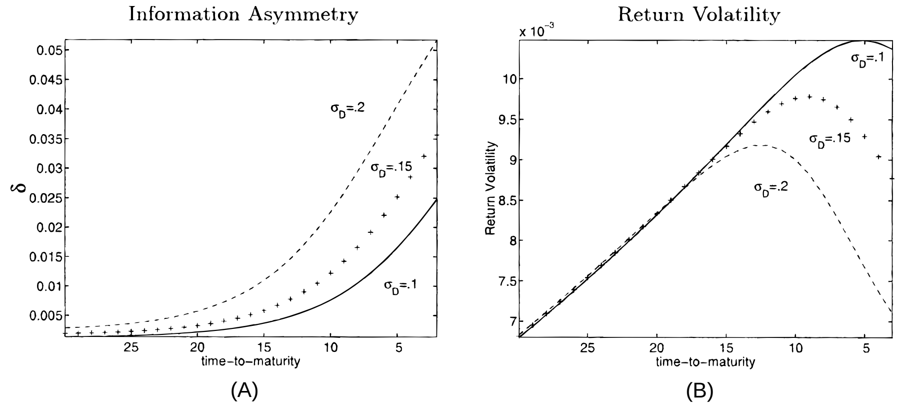


Figure 5. Time-to-maturity patterns of information asymmetry and futures return volatility for various values of σ_D—the standard deviation of convenience yield innovations. The figures plot the time-to-maturity patterns of (A) information asymmetry, δ, and (B) the return volatility of the kth contract. The parameters are set at the following values: $\kappa_{Dq} = 0.5$, $\sigma_q = 0.025$, $a_Z = 0.98$, $\sigma_Z = 0.025$, $a_{Y_1} = 0.97$, $\sigma_{Y_1} = 0.075$, $a_{Y_2} = 0.8$, $\sigma_{Y_2} = 0.05$.

ing is higher the farther the contract is from expiration. All else equal, this tends to lead to lower open interest with time-to-maturity. The interaction of these two effects produces the nonlinear pattern shown.

VI. Empirical Implications

In this section, I turn to some comparative static exercises, in order to draw out the testable implications of the model.

A. Futures Return Volatility

From the analysis above, one out-of-sample prediction of the model is that the Samuelson effect is more likely to be violated in high information asymmetry markets with fundamental shocks that are relatively more persistent than noise shocks.

First, I highlight how the level of information asymmetry is crucial to producing nonmonotonic patterns by varying σ_D (0.1, 0.15, and 0.2), the volatility of payoffs to holding the spot in Figure 5. The parameters are the same as in Figure 3. Figure 5(A) illustrates the time-to-maturity pattern of information asymmetry for various levels of σ_D. Notice that as σ_D rises, the level of information asymmetry (across the entire life of the futures) increases since payoffs to the spot provide noisier signals about Z_t. Also, information asymmetry increases as a contract expires and does so at a faster rate when σ_D is large.

Figure 5(B) shows the corresponding futures return volatility pattern. The level of futures return volatility (across the entire life of the futures) falls with σ_D for the following reason. Since a larger δ implies that uninformed

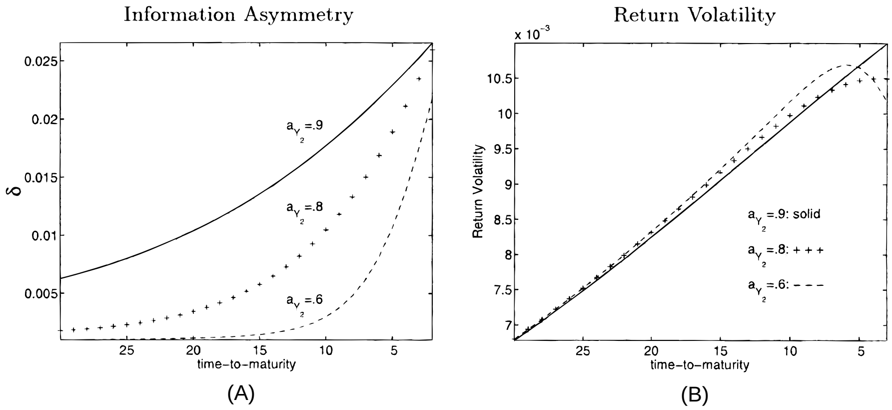


Figure 6. Time-to-maturity patterns of information asymmetry and futures return volatility for various values of a_{Y_2}—persistence of nonmarketed risk shocks. The figures plot the time-to-maturity patterns of (A) information asymmetry, δ, and (B) the return volatility of the kth contract. The parameters are set at the following values: $\kappa_{Dq} = 0.5$, $\sigma_D = 0.2$, $\sigma_q = 0.025$, $a_Z = 0.98$, $\sigma_Z = 0.025$, $a_{Y_1} = 0.97$, $\sigma_{Y_1} = 0.075$, $\sigma_{Y_2} = 0.05$.

investors are less certain about future spot payoffs, futures prices move less. When σ_D is small, information asymmetry is less important and the Samuelson effect dominates throughout most of the life of the contract. When σ_D is large, information asymmetry increases dramatically near expiration and counteracts the Samuelson effect, leading to futures return volatility that actually falls even when the contract is reasonably far from expiration.

Next, I demonstrate the importance of the persistence of shocks to investor-a's hedging trades, a_{Y_1} and a_{Y_2}. Without loss of generality, I assume that $Y_{1,t}$ is more persistent than $Y_{2,t}$, $a_{Y_1} > a_{Y_2}$. Holding a_{Y_1} fixed, decreasing a_{Y_2} implies that the noise in the price signals becomes less serially correlated. Figure 6(A) illustrates the time-to-maturity pattern of δ for various values of a_{Y_2} (0.9, 0.8, and 0.6). All other parameters are similar to Figure 3. Notice that for smaller values of a_{Y_2}, the level of information asymmetry over the entire life of the contract falls. The reason is that the uninformed investors learn more from a sequence of prices at lower levels of a_{Y_2}.[19] What is most prominent about the figure is that for smaller values of a_{Y_2}, information asymmetry increases at a much faster rate as the contract expires.

Figure 6(B) shows the corresponding time-to-maturity patterns for futures return volatility. For larger values of a_{Y_2}, the Samuelson effect remains. However, we obtain an inverted U-shaped time-to-maturity pattern for smaller values of a_{Y_2}. This is because information asymmetry needs to increase sufficiently rapidly as the contract expires to counteract the Samuelson

[19] For instance, in the extreme case when a_{Y_2} is close to zero, neighboring prices provide close to independent signals (about Z_t), which tend to reveal more of the informed investors' private information about Z_t.

effect. In this sense, we are more likely to see an inverted pattern in futures return volatility in markets whose fundamental shocks are relatively more persistent compared to those for nonmarketed risk.

B. Open Interest and Spot Return Volatility

My model also provides a number of auxiliary implications for the open interest and spot return volatility. For brevity, I leave out the figures illustrating the comparative statics.

First, the current level of open interest helps predict the expected return to trading futures. This follows from equations (12) and (18) since the risk premium of the futures and the holdings of investors depend on nonmarketed risks $Y_{i,t}$. This aspect of the model is consistent with the evidence cited in Section I.B.

Second, the time-to-maturity pattern of open interest depends on information asymmetry in the economy. When investors only differ in physical endowments, open interest tends to monotonically decline as the contract expires. When private information is important and information asymmetry increases with time-to-maturity, the analysis shows that open interest can take on nonmonotonic patterns. It follows that this nonmonotonic pattern is more likely in markets where private information is more important and fundamental shocks are shorter lived relative to noise shocks. This may explain the documented variation in time-to-maturity patterns in open interest across futures markets (see Section I.B).

Up to this point, I have focused mainly on the implications of the model for the return and trading patterns in futures. Because spot and futures prices are determined endogenously, this model can also be used to study the effects of new futures on the underlying spot return process. In this model, the introduction of new futures tends to lower spot return volatility. The reason is as follows. Recall that the spot price in equation (25) is sensitive to $Y_{i,t}$ ($i = 1,2$). The effect of introducing new futures is to decrease the sensitivity of the spot price to $Y_{i,t}$ ($i = 1,2$)—the reason being that the futures allows investors to better hedge their nonmarketed risks, so investors are willing to absorb more of the nonmarketed risk shocks in their spot holdings. That is, the hs in equation (18) increase. This finding is consistent with the empirical studies on the effect of new futures on spot price volatility (see Section I.C).

VII. Conclusions

In this paper, I build a parsimonious model of returns and trading in futures markets. In the model, investors differ in both physical and informational endowments. I show that although the Samuelson effect survives heterogeneity in physical endowments, it need not hold when investors also differ in informational endowments. I relate this violation of the Samuelson effect to attributes of markets such as the importance of information asymmetry and persistence of underlying shocks. I also develop auxiliary predictions for (1) the time-to-maturity patterns of open interest (hence providing a link between prices and quantities in futures markets) and (2) the effect of new futures on spot price volatility.

Appendix

I define the following notation for future convenience. For a set of elements $e_1, e_2, \ldots, e_m$ (of proper order), let diag$\{e_1, e_2, \ldots, e_m\}$ denote a diagonal matrix, stack$\{e_1, e_2, \ldots, e_m\}$ the column matrix, and $[e_1, e_2, \ldots, e_m]$ the row matrix, from these elements. Let $\boldsymbol{i}_{i,j}^{(l,k)}$ be a $l \times k$ matrix with its (i,j)th element equal to one and all of its other elements equal to zero. Let $\mathbf{1}^{(m)}$ be the identity matrix of rank m and $\mathbf{0}^{(m,n)}$ be the zero matrix of dimension $m \times n$. For a matrix $\boldsymbol{m}$, $\boldsymbol{m}_{(i,j)}$ denotes its (i,j)th element, $\boldsymbol{m}_{(i,\cdot)}$ its ith row, and $\boldsymbol{m}_{(p:q,\cdot)}$ the matrix comprised of the pth to qth rows. Given two vectors of random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ of appropriate order, $\boldsymbol{\sigma}_{X,Y}$ denotes their covariance matrix. Let $\boldsymbol{\epsilon}_t = [\epsilon_{D,t}, \epsilon_{q,t}, \epsilon_{Z,t}, \epsilon_{Y_1,t}, \epsilon_{Y_2,t}]'$. Recall that all shocks are jointly normally distributed and i.i.d. over time. Define $\boldsymbol{b}_D = \sigma_D \boldsymbol{i}_{1,1}^{(1,5)}$, $\boldsymbol{b}_q = \sigma_q \boldsymbol{i}_{1,2}^{(1,5)}$, $\boldsymbol{b}_Z = \sigma_Z \boldsymbol{i}_{1,3}^{(1,5)}$, $\boldsymbol{b}_{Y_1} = \sigma_{Y_1} \boldsymbol{i}_{1,4}^{(1,5)}$, and $\boldsymbol{b}_{Y_2} = \sigma_{Y_2} \boldsymbol{i}_{1,5}^{(1,5)}$. I prove results for the case of $N = 0$. The case of $N > 0$ follows easily.

Proof of Lemma 1: The system of interest is the vector of state variables $\boldsymbol{G}_t = [Z_t, Y_{1,t}, Y_{2,t}]'$, which follows the process

$$\boldsymbol{G}_t = \boldsymbol{a}_G \boldsymbol{G}_{t-1} + \boldsymbol{b}_G \boldsymbol{\epsilon}_t, \tag{A1}$$

where $\boldsymbol{a}_G = \text{diag}\{a_Z, a_{Y_1}, a_{Y_2}\}$ and $\boldsymbol{b}_G = \text{stack}\{\boldsymbol{b}_Z, \boldsymbol{b}_{Y_1}, \boldsymbol{b}_{Y_2}\}$. Given the conjectured price function in equation (12), the effective spot price signal received by investor-b is $\tilde{S}_t = \boldsymbol{\lambda}_{S,G} \boldsymbol{G}_t$ where $\boldsymbol{\lambda}_{S,G} = [\lambda_{S,\hat{Z}}, -\lambda_{S,Y_1}, -\lambda_{S,Y_2}]$. The effective futures price signal is $\tilde{H}_t = \boldsymbol{\lambda}_{H,G} \boldsymbol{G}_t$ where $\boldsymbol{\lambda}_{H,G} = [\lambda_{H,\hat{Z}}, -\lambda_{H,Y_1}, -\lambda_{H,Y_2}]$. Let $\boldsymbol{N}_t = [\tilde{S}_t, \tilde{H}_t, D_t]$ be the signal vector received by investor-b, which follows the process

$$\boldsymbol{N}_t = \boldsymbol{a}_N \boldsymbol{G}_t + \boldsymbol{b}_N \boldsymbol{\epsilon}_t, \tag{A2}$$

where $\boldsymbol{a}_N = \text{stack}\{\boldsymbol{\lambda}_{S,G}, \boldsymbol{\lambda}_{H,G}, \boldsymbol{i}_{1,1}^{(1,3)}\}$ and $\boldsymbol{b}_N = \text{stack}\{\mathbf{0}^{(1,5)}, \mathbf{0}^{(1,5)}, \boldsymbol{i}_{1,1}^{(1,5)}\}$.

Calculating $\hat{\boldsymbol{G}}_t$ and $\boldsymbol{o}_t$ is a Kalman filtering problem. From Theorem 7.2 in Jazwinski (1970),

$$\hat{\boldsymbol{G}}_t = \boldsymbol{a}_G \hat{\boldsymbol{G}}_{t-1} + \boldsymbol{k}(t-1)(\boldsymbol{N}_t - \mathrm{E}_{b,t}[\boldsymbol{N}_t]), \tag{A3}$$

where

$$\boldsymbol{k}(t-1) = (\boldsymbol{a}_G \boldsymbol{o}(t-1)\boldsymbol{a}_G' + \boldsymbol{\sigma}_{GG})\boldsymbol{a}_N'[\boldsymbol{a}_N(\boldsymbol{a}_G \boldsymbol{o}(t-1)\boldsymbol{a}_G' + \boldsymbol{\sigma}_{GG})\boldsymbol{a}_N' + \boldsymbol{\sigma}_{NN}]^{-1}.$$

Equation (14), which guides the evolution of $\boldsymbol{o}(t)$, is just

$$\boldsymbol{g}_o(t,\boldsymbol{o}(t-1)) = (\mathbf{1}^{(3)} - \boldsymbol{k}(t-1)\boldsymbol{a}_N)(\boldsymbol{a}_G\boldsymbol{o}(t-1)\boldsymbol{a}_G' + \boldsymbol{\sigma}_{GG}). \tag{A4}$$

Furthermore, $\hat{\boldsymbol{G}}_t$ is a Gaussian Markov process under investor-b's information set. Q.E.D.

Proof of Lemma 2: Given the evolution of the uninformed investors' conditional expectations, it is easy to derive the dynamics of $\boldsymbol{X}_{i,t}$. Let $\boldsymbol{\epsilon}_{a,t} = \boldsymbol{\epsilon}_t$ and

$$\boldsymbol{\epsilon}_{b,t} = [(\boldsymbol{G}_{t-1} - \hat{\boldsymbol{G}}_{t-1})', \boldsymbol{\epsilon}_t']'. \tag{A5}$$

It follows from equations (A2) and (A3) that

$$\hat{\boldsymbol{G}}_t = \boldsymbol{a}_G\hat{\boldsymbol{G}}_{t-1} + \boldsymbol{b}_{\hat{G}}(t)\boldsymbol{\epsilon}_{b,t}, \tag{A6}$$

where $\boldsymbol{b}_{\hat{G}}(t) = [\{\boldsymbol{k}(\mathrm{t})\boldsymbol{a}_{\mathrm{N}}(\mathrm{t})\boldsymbol{a}_{\mathrm{G}}\},\{\boldsymbol{k}(\mathrm{t})(\boldsymbol{a}_{\mathrm{N}}(\mathrm{t})\boldsymbol{b}_{\mathrm{G}} + \boldsymbol{b}_{\mathrm{N}}(\mathrm{t}))\}]$. Let $\boldsymbol{\Delta}_t \equiv \boldsymbol{G}_t - \hat{\boldsymbol{G}}_t$. Equations (A1) and (A6) together imply that

$$\boldsymbol{\Delta}_t = \boldsymbol{a}_\Delta(t)\boldsymbol{\Delta}_{t-1} + \boldsymbol{b}_\Delta(t)\boldsymbol{\epsilon}_t, \tag{A7}$$

where $\boldsymbol{a}_\Delta(t) = (\mathbf{1}^{(3)} - \boldsymbol{k}(t)\boldsymbol{a}_N(t))\boldsymbol{a}_S$ and $\boldsymbol{b}_\Delta(t) = \boldsymbol{b}_G - \boldsymbol{k}(t)(\boldsymbol{a}_N(t)\boldsymbol{b}_S + \boldsymbol{b}_N(t))$. Hence, $\boldsymbol{X}_{i,t}$ follows

$$\boldsymbol{X}_{i,t} = \boldsymbol{a}_{i,X}\boldsymbol{X}_{i,t-1} + \boldsymbol{b}_{i,X}\boldsymbol{\epsilon}_{i,t}, \tag{A8}$$

where

$$\boldsymbol{a}_{a,X}(t) = \mathrm{diag}\{1, a_{Y_1}, a_{Y_2}, \boldsymbol{a}_\Delta(t)\}, \qquad \boldsymbol{b}_{a,X}(t) = \mathrm{stack}\left\{\mathbf{0}^{(1,5)}, \boldsymbol{b}_{Y_1}, \boldsymbol{b}_{Y_2}, \boldsymbol{b}_\Delta(t)\right\},$$

$$\boldsymbol{a}_{b,X}(t) = \mathrm{diag}\{1, a_Y\}, \qquad \boldsymbol{b}_{b,X}(t) = \mathrm{stack}\left\{\mathbf{0}^{(1,8)}, [\boldsymbol{b}_{\hat{G}}(t)]_{(2:3,\cdot)}\right\}.$$

Observe that $\boldsymbol{X}_{i,t}$ follows a Gaussian Markov process under investor-i's information set.

I next derive the return process, Q_t^S and Q_t^H. Let $\boldsymbol{\eta} = \mathrm{stack}\left\{\mathbf{0}^{(1,3)}, \boldsymbol{i}_{1,2}^{(1,3)}, \boldsymbol{i}_{1,3}^{(1,3)}, \mathbf{1}^{(3)}\right\}$. Define the following time-varying matrices:

$$\boldsymbol{a}_{a,Q}(t) = \mathrm{stack}\left\{R\boldsymbol{\lambda}_{S,X}(t-1) - \boldsymbol{\lambda}_{S,X}(t)\boldsymbol{a}_{a,X}(t), \boldsymbol{\lambda}_{H,X}(t-1) - \boldsymbol{\lambda}_{H,X}(t)\boldsymbol{a}_{a,X}(t)\right\},$$

$$\boldsymbol{b}_{a,Q}(t) = \mathrm{stack}\left\{\boldsymbol{b}_D + \boldsymbol{b}_Z + \lambda_{S,Z}\boldsymbol{b}_Z - \boldsymbol{\lambda}_{S,X}(t)\boldsymbol{b}_{a,X}, \lambda_{H,Z}(t)\boldsymbol{b}_Z - \boldsymbol{\lambda}_{H,X}(t)\boldsymbol{b}_{a,X}\right\},$$

$$\boldsymbol{a}_{b,Q} = \boldsymbol{a}_{a,Q}\mathrm{stack}\left\{\mathbf{1}^{(3)}, \mathbf{0}^{(3,3)}\right\}, \qquad \boldsymbol{b}_{b,Q} = [\boldsymbol{a}_{a,Q}\boldsymbol{\eta}, \boldsymbol{b}_{a,Q}].$$

I then define the return process (comprising of the spot and futures) for each investor as $\boldsymbol{Q}_{i,t}$, which follows

$$\boldsymbol{Q}_{i,t} = \boldsymbol{a}_{i,Q}\boldsymbol{X}_{i,t-1} + \boldsymbol{b}_{i,Q}\boldsymbol{\epsilon}_{i,t}. \tag{A9}$$

Let $\boldsymbol{e} = [1,1,1,0,0,0]$, then q_{t+1} defined in equation (3) can be rewritten as

$$q_{t+1} = \boldsymbol{e}\boldsymbol{X}_{a,t} + \boldsymbol{e}'\boldsymbol{X}_{a,t}\boldsymbol{b}_q\boldsymbol{\epsilon}_{a,t+1}. \tag{A10}$$

Given these processes, I now derive the system of nonlinear difference equations that govern $\boldsymbol{v}_i(t)$. Suppose that at t, the value function is given by $J_i(W_{i,t};\boldsymbol{X}_{i,t};t)$. Let $\boldsymbol{\theta}_{i,t} = \text{stack}\{\theta^S_{i,t},\theta^H_{i,t}\}$.

Using the above notation, both the informed and the uninformed investors' optimization problems can be expressed in the form of the Bellman equation:

$$0 = \sup_{\{c_i,\theta_{i,t}\}}\{-\rho^t e^{-\gamma c_{i,t}} + \mathrm{E}_{i,t}[J(W_{i,t+1};\boldsymbol{X}_{i,t+1};t+1)] - J(W_{i,t};\boldsymbol{X}_{i,t};t)\} \tag{A11a}$$

subject to

$$W_{i,t+1} = (W_{i,t} - c_{i,t})R + \boldsymbol{\theta}'_{i,t}\boldsymbol{Q}_{i,t+1} + \delta_i q_t. \tag{A11b}$$

Consider the following trial solution for the value function:

$$J_i(W_{i,t};\boldsymbol{X}_{i,t};t) = -\rho^t\exp\{-\alpha W_{i,t} - \tfrac{1}{2}(\boldsymbol{X}'_{i,t}\boldsymbol{v}_i(t)\boldsymbol{X}_{i,t})\}, \tag{A12}$$

where α is a constant and $\boldsymbol{v}_i(t)$ is a symmetric matrix to be determined. I next define a number of time-varying matrices. I suppress the time index for simplicity. Let $\boldsymbol{v}_{i,aa} = \boldsymbol{a}'_{i,X}\boldsymbol{v}_i\boldsymbol{a}_{i,X}$, $\boldsymbol{v}_{i,ab} = \boldsymbol{a}'_{i,X}\boldsymbol{v}_i\boldsymbol{b}_{i,X}$, and $\boldsymbol{v}_{i,bb} = \boldsymbol{b}'_{i,X}\boldsymbol{v}_i\boldsymbol{b}_{i,X}$. Then define $\boldsymbol{\Omega}_i = [\boldsymbol{\sigma}^{-1}_{i,\epsilon\epsilon} + \boldsymbol{v}_{i,bb}]^{-1}$, where $\boldsymbol{\sigma}_{i,\epsilon\epsilon}$ is the covariance matrix of $\boldsymbol{\epsilon}_{i,t}$. Next let $\boldsymbol{\Sigma}_i = (\boldsymbol{b}_{i,Q}\boldsymbol{\Omega}_i\boldsymbol{b}'_{i,Q})^{-1}$. Also, let $d_i = |\boldsymbol{\Omega}_i^{-1}\boldsymbol{\sigma}_{i,\epsilon\epsilon}|^{-1/2}$. It follows from normality of $\boldsymbol{\epsilon}_{i,t+1}$ that

$$\begin{aligned}E_{i,t}[J_{i,t+1}] = -d\rho^{t+1}\exp\big\{&-\alpha R(W_{i,t} - c_{i,t}) - \alpha\boldsymbol{X}'_{i,t}\boldsymbol{g}'_i\boldsymbol{\theta}_{i,t} + \tfrac{1}{2}\alpha^2\boldsymbol{\theta}'_{i,t}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\theta}_{i,t}\\ &- \tfrac{1}{2}\boldsymbol{X}'_{i,t}[\boldsymbol{v}_{i,aa} - (\boldsymbol{v}_{i,ab} + \alpha\delta_i\boldsymbol{\eta}_q)\boldsymbol{\Omega}_i(\boldsymbol{v}_{i,ab} + \alpha\delta_i\boldsymbol{\eta}_q)'\\ &+ \delta_i\boldsymbol{a}_q]\boldsymbol{X}_{i,t}\big\},\end{aligned} \tag{A13}$$

where $\boldsymbol{g}_i = [\boldsymbol{a}_{i,Q} - \boldsymbol{b}_{i,Q}\boldsymbol{\Omega}_i(\boldsymbol{v}_{i,ab} + \alpha\delta_i\boldsymbol{\eta}_q)']$, $\boldsymbol{\eta}_q = \boldsymbol{e}'\boldsymbol{b}_q$, $\boldsymbol{a}_q = \text{stack}\{\boldsymbol{i}^{(1,6)}_{1,1}, \boldsymbol{i}^{(1,6)}_{1,1}, \boldsymbol{i}^{(1,6)}_{1,1}\}$, and δ_i is an index function that equals one if $i = a$ and zero if $i = b$. The first-order conditions for the optimal investment-consumption policies are

$$\boldsymbol{\theta}_{i,t} = \boldsymbol{h}_i\boldsymbol{X}_{i,t}, \qquad c_{i,t} = \bar{c}_i + \frac{\alpha R}{\gamma + \alpha R}W_{i,t} + \frac{1}{2(\gamma + \alpha R)}\boldsymbol{X}'_{i,t}m_i\boldsymbol{X}_{i,t}, \tag{A14}$$

where

$$\boldsymbol{h}_i = \frac{1}{\alpha}\boldsymbol{\Sigma}_i\boldsymbol{g}_i, \qquad \bar{c}_i = \frac{1}{\gamma + \alpha R}\log\left(\frac{\gamma}{\alpha\rho R d_i}\right)$$

and

$$\boldsymbol{m}_i = \boldsymbol{v}_{i,aa} - (\boldsymbol{v}_{i,ab} + \alpha\delta_i\boldsymbol{\eta}_q)\boldsymbol{\Omega}_i(\boldsymbol{v}_{i,ab} + \alpha\delta_i\boldsymbol{\eta}_q)' + \delta_i\boldsymbol{a}_q + \boldsymbol{g}_i'\boldsymbol{\Sigma}_i\boldsymbol{g}_i.$$

Substituting equation (A14) back into the Bellman equation, we obtain $\alpha = r\gamma/R$

$$\bar{c}_i = -\frac{1}{\gamma R}\log(r\rho d_i), \qquad \bar{\boldsymbol{v}}_i = \left[\gamma\bar{c}_i + \log\left(\frac{r}{R}\right)\right]\boldsymbol{i}_{1,1}^{(n_i,n_i)},$$
$$\frac{1}{R}\boldsymbol{m}_i(t) - \boldsymbol{v}_i(t-1) + \bar{\boldsymbol{v}}_i = 0, \tag{A15}$$

where $n_1 = 6$ and $n_2 = 3$. This gives the recursive relationship for $\boldsymbol{v}_i(t-1)$ given $\boldsymbol{v}_i(t)$. So we can define $\boldsymbol{g}_{i,v}$ in equation (17) as

$$\boldsymbol{g}_{i,v} = \frac{1}{R}\boldsymbol{m}_i(t) + \bar{\boldsymbol{v}}_i. \tag{A16}$$

$\boldsymbol{v}_i$ in Lemma 2 satisfies the following nonlinear two-point boundary value problem if for $k = 0,1,2,\ldots$ equation (19) holds. Q.E.D.

Proof of Theorem 1: First, I prove the result for ω close to one. The line of argument is as follows. I show that there exists a linear periodic equilibrium at $\omega = 1$. At $\omega = 1$, the price functions are determined purely by informed investors. I show that there exists a periodic solution to the uninformed investors' learning and control problem.[20] Having done so, I show that a solution exists generically for ω close to one.

The case $\omega = 1$ collapses to a representative investor economy. The existence of a linear periodic equilibrium of the form in equation (10) is straightforward from Wang (1994). Next, I show there exists a covariance matrix, $\boldsymbol{o}(t_k)$, such that $\boldsymbol{o}(t)$ follows dynamics specified in equation (A4) and $\boldsymbol{o}(t_{k+1}) = \boldsymbol{o}(t_k)$. Since $\boldsymbol{a}_G$ is diagonal, $|\boldsymbol{a}_G| \le 1$, $\boldsymbol{b}_G\boldsymbol{b}_G'$ is nonsingular, and the price coefficients of the signal vector are uniformly bounded in time, by Lemma 5.1 of Anderson and Moore (1981), we conclude that $|\boldsymbol{o}(t_{k+1})| \le \beta_0|\boldsymbol{o}(t_k)| + \beta_1$, where $0 < \beta_0 < 1$ and $\beta_1 > 0$ (β_0 and β_1 are functions only of the under-

[20] Given the price functions, uninformed investors can nonetheless learn from the prices and solve their control problem even if they have no impact on equilibrium prices.

lying parameters of the filtering problem and are found in Anderson and Moore. Given this bound, Brouwer's fixed-point theorem (see, e.g., Cronin (1994)) implies that there exists a symmetric, positive semidefinite periodic solution.

I next prove that there exists a periodic solution to the value function of the uniformed investors. I prove the existence of the terminal value problem for equation (A6) for $i = b$ by bounding (lower and upper) its solution by the solutions to two particular matrix Riccati difference equations. First the lower bound. Define the following auxiliary system, which is derived from equation (A16) for $i = b$:

$$\boldsymbol{v}_i^*(t-1) = \frac{1}{R}\left\{\boldsymbol{v}_{i,aa}^*(t) - \boldsymbol{v}_{i,ab}^*(t)\boldsymbol{\Omega}_i\boldsymbol{v}_{i,ab}^*(t)'\right\} + \bar{\boldsymbol{v}}_i^*, \tag{A17}$$

where the notation is the same as in equation (A16) for $i = b$ except that we have replaced the variable $\boldsymbol{v}_i$ by $\boldsymbol{v}_i^*$. This auxiliary system is a Riccati system (see, e.g., Caines and Mayne (1970)). Suppose $\boldsymbol{v}_{i,T} \geq \boldsymbol{v}_{i,T}^*$, then it follows that $\boldsymbol{v}_i(t) \geq \boldsymbol{v}_i^*(t)$, where $\boldsymbol{v}_i^*(t)$ satifies equation (A17). This follows from fact that $\boldsymbol{g}_i'\boldsymbol{\Sigma}_i\boldsymbol{g}_i \geq 0$. Next, an upper bound. Note that $\boldsymbol{\Omega}_{i,t} - (\boldsymbol{b}_{i,Q}\boldsymbol{\Omega}_{i,t})'\boldsymbol{\Sigma}_{i,t}(\boldsymbol{b}_{i,Q}\boldsymbol{\Omega}_{i,t}) > 0$, so that equation (A16) for $i = b$ is negative in the quadratic terms involving $\boldsymbol{v}_b(t)$. So, this keeps the solution bounded. It follows from Theorem 4.1 of Caines and Mayne that the solution is bounded by the solution to a matrix linear equation whose linear term is $\boldsymbol{a}_{i,X}'\boldsymbol{e}_t\boldsymbol{a}_{i,X}$ and the constant term is independent of $\boldsymbol{v}_{i,t}$ $\forall t$. Hence, since we assume that $|\boldsymbol{a}_{i,X}| < 1$, it follows that we can find β_0 and β_1 such that the result holds. Given this bound, Brouwer's fixed-point theorem shows that there exists a periodic solution.

I next show that there exists a solution for ω close to one. The market-clearing conditions, equation (9), which determine the price coefficients $\boldsymbol{\lambda}(t)$, define the following relationship between $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\lambda}(t+1)$: $0 = \mathbf{F}(\boldsymbol{\lambda}(t), \boldsymbol{\lambda}(t+1); \vartheta; \omega)$, where $\vartheta \in \Theta$ with $\Theta = \{r > 0,\ \gamma > 0,\ 1 \geq a_Z \geq 0,\ 1 > a_{Y_i} \geq 0\ (i = 1,2),\ \sigma_D \geq 0,\ \sigma_q \geq 0,\ \sigma_Z \geq 0,\ \sigma_{Y_i} \geq 0\ (i = 1,2), \kappa_{Dq} > 0\}$. By the implicit function theorem (see, e.g., Protter and Morrey (1991)), $\mathbf{F}$ defines an implicit function: $\boldsymbol{\lambda}(t) = \boldsymbol{g}_\lambda(\boldsymbol{\lambda}(t+1); \vartheta, \omega)$ if $\nabla_{\boldsymbol{\lambda}}\mathbf{F}$ is nonsingular. Let $\vartheta_0 = [0.001, 100, 0.99, 0.95, 0.9, 0.05, 0.025, 0.05, 0.025, 0.5, 0.5]$. For $\vartheta = \vartheta_0$ and $\omega_0 = 1$, $\det(\nabla_{\boldsymbol{\lambda}}\mathbf{F}) \neq 0$. So, it follows from Lemma 4 of Huang and Wang (1997) that $\boldsymbol{\lambda}(t) = \boldsymbol{g}_\lambda(\boldsymbol{\lambda}(t+1); \vartheta, \omega)$ given in equation (20) exists generically. Let $\boldsymbol{u} = \text{stack}\{\boldsymbol{\lambda}', [\boldsymbol{o}], [\boldsymbol{v}_a], [\boldsymbol{v}_b]\}$. Then $\boldsymbol{u}(t-1) = \mathbf{f}(\boldsymbol{u}(t))$, where $\mathbf{f} = \text{stack}\{\boldsymbol{g}_\lambda, \boldsymbol{g}_o, \boldsymbol{g}_{a,v}, \boldsymbol{g}_{b,v}\}$. Let $\mathbf{b}(\boldsymbol{u}(t_k), \boldsymbol{u}(t_{k+1})) = \mathbf{0}$ denote the boundary conditions for the system. Starting at t_{k+1}, given a terminal value, we can integrate backward to get $\boldsymbol{u}(t_k)$. Let $\omega_0 = 1$. Since I have existence of $\boldsymbol{u}(t; \vartheta, \omega_0)$, it remains to verify that $\boldsymbol{u}(t; \vartheta, \omega_0)$ is an isolated solution (see Agarwal (1992) for definition). From Agarwal, this is equivalent to showing that $\boldsymbol{m}(\vartheta, \omega_0) = \nabla_{\boldsymbol{u}_{t_k}}\mathbf{b}(\boldsymbol{u}(t_k)) + \nabla_{\boldsymbol{u}_{t_{k+1}}}\mathbf{b}(\boldsymbol{u}(t_{k+1}))(\nabla_u\mathbf{f})^M$ is nonsingular. Clearly, $\boldsymbol{m}(\vartheta, \omega_0)$ is analytic. It is easy to show that $\det(\boldsymbol{m}(\vartheta_0, \omega_0)) \neq 0$. So $\boldsymbol{m}(\vartheta, \omega_0)$ is generically nondegenerate. The first part of Theorem 1 follows.

Next, I show that the theorem also holds for arbitrary ω but σ_{Y_1} and σ_{Y_2} close to zero. The line of proof is similar to the case of ω close to one. First, observe that when $\sigma_{Y_1} = \sigma_{Y_2} = 0$, we have a unique solution to the equilibrium from Proposition 1 below. Observe that the solution for $\boldsymbol{o}$ is merely the zero matrix as the equilibrium is fully revealing. It follows then from the arguments above that since there is a unique solution for the case of $\sigma_{Y_1} = \sigma_{Y_2} = 0$, it follows that there generically exists a solution for σ_{Y_1} and σ_{Y_2} close to zero though we do not specify how close they have to be. Q.E.D.

Proof of Proposition 1: Let $\sigma_S^2 = \sigma_D^2 + (1 + \lambda_{S,Z}^2)\sigma_Z^2$, $\sigma_{S,H} = (1 + \lambda_{S,Z})\lambda_{S,Z}\sigma_Z^2$, $\sigma_H^2 = \lambda_{H,Z}^2\sigma_Z^2$, $\boldsymbol{a}_Q = \text{stack}\{R\lambda_{S,0}(t) - \lambda_{S,0}(t-1), \lambda_{H,0}(t) - \lambda_{H,0}(t-1)\}$, and $\boldsymbol{\sigma}_{Q,Q} = \text{stack}\{[\sigma_S^2, \sigma_{S,H}], [\sigma_{S,H}, \sigma_H^2]\}$. Let $\alpha = r\gamma/R$, then the market-clearing condition implies

$$t \in \mathcal{M}_k: \quad \omega \frac{1}{\alpha} \boldsymbol{\sigma}_{Q,Q}^{-1}(\boldsymbol{a}_Q - \alpha\boldsymbol{\sigma}_{Dq}[1,0]') + (1-\omega)\frac{1}{\alpha}\boldsymbol{\sigma}_{Q,Q}^{-1}\boldsymbol{a}_Q = [1,0]'.$$

It follows that, in equilibrium, $\boldsymbol{a}_Q = \alpha\boldsymbol{\sigma}_{Q,Q}([1,0]' + \omega\boldsymbol{\sigma}_{Dq}[1,0]')$. The spot price is given by

$$t \in \mathcal{T}: \quad S_t = \lambda_{S,Z}Z_t - \bar{\lambda}_{S,0}, \tag{A18}$$

where $\lambda_{S,Z} = a_Z/(R - a_Z)$ and $\bar{\lambda}_{S,0} = (\gamma/R)[\sigma_D^2 + (1 + \lambda_Z^2)\sigma_Z^2 + \omega\sigma_D\sigma_q\kappa_{Dq}]$. The futures price is

$$t \in \mathcal{H}_k: \quad H_t = \lambda_{H,Z}(t)Z_t - \bar{\lambda}_{H,0}(t), \tag{A19}$$

where $\lambda_{H,Z}(t) = \lambda_{S,Z}a_Z^{(m_k - t)}$ and

$$\bar{\lambda}_{H,0}(t) = \bar{\lambda}_{S,0} + \frac{\gamma}{R}\frac{1 - a_Z^{(m_k - t)}}{1 - a_Z}(1 + \lambda_Z)\lambda_Z\sigma_Z^2.$$

Since $\lambda_{H,Z}(t)$ decreases with t, the Samuelson effect follows. Substituting a_Q into the investor-a's futures position and differentiating with respect to t gives the desired result for open interest. Q.E.D.

Numerical Procedure

I use the Newton–Kantorovich method to solve this problem numerically (see, e.g., Agarwal (1992)). This recursive method linearizes the system and the boundary conditions around a conjectured solution to the nonlinear problem at a discrete number of points in the interval $[t_k, t_{k+1}]$. Since the system is linearized, it is easy to calculate an updated solution that satisfies the linearized system and boundary conditions from the conjectured solution. The updated solution is then used as the conjectured solution to start the next recursion. It can be shown that the limit of this recursion converges to the solution of the nonlinear problem given that the initial conjectured solution is not too far away from the true solution.

This method requires a sufficiently accurate initial guess of the true solution. I obtain such a guess by starting the recursion at $\omega = 1$ for any given set of parameters since a solution exists at $\omega = 1$. In order to calculate a solution at $\omega_0 < 1$, I begin by using the solution at $\omega = 1$ as the initial guess to find a solution for an ω close to 1 and I repeat the same procedure to move toward ω_0. Since I have no knowledge about the uniqueness of the solution, the above procedure also guarantees that I stay on the same branch of solutions.[21] Additionally, I also check all solutions by recalculating them using the solution at $\sigma_{Y_1} = \sigma_{Y_2} = 0$ (but arbitrary ω) as the initial guess since we have the explicit solution for this case (see Proposition 1).

REFERENCES

Agarwal, Ravi P., 1992, *Difference Equations and Inequalities: Theory, Methods, and Applications* (M. Dekker, New York).

Anderson, Brian, and John B. Moore, 1981, Detectability and stabilizability of time-varying discrete-time linear systems, *SIAM Journal of Control and Optimization* 19, 20–32.

Anderson, Ronald, 1985, Some determinants of the volatility of futures prices, *Journal of Futures Markets* 5, 331–348.

Anderson, Ronald, and Jean-Pierre Danthine, 1983, The time pattern of hedging and the volatility of futures prices, *Review of Economic Studies* 50, 249–266.

Bessembinder, Hendrik, 1992, Systematic risk, hedging pressure, and risk premiums in futures markets, *Review of Financial Studies* 5, 637–667.

Bessembinder, Hendrik, Jay Coughenour, Paul Seguin, and Margaret Smoller, 1996, Is there a term structure of futures volatilities? Reevaluating the Samuelson hypothesis, *Journal of Derivatives* 4, 45–58.

Bray, Margaret, 1981, Futures trading, rational expectations, and the efficient markets hypothesis, *Econometrica*, 575–596.

Brennan, Michael J., 1991, The price of convenience and the valuation of commodity contingent claims; in D. Lund and B. Oksendal, eds.: *Stochastic Models and Option Values* (North Holland, Amsterdam).

Caines, Peter E., and David Q. Mayne, 1970, On the discrete time matrix Riccati equation of optimal control, *International Journal of Control* 12, 785–794.

Cronin, Jane, 1994, *Differential Equations: Introduction and Qualitative Theory* (M. Dekker, New York).

Damodaran, Aswath, and Marti G. Subrahmanyam, 1992, The effects of derivative securities on the markets for the underlying assets in the United States: A survey, *Financial Markets, Institutions and Instruments* 2, 1–21.

Fama, Eugene F., and Kenneth R. French, 1987, Commodity futures prices: Some evidence on forecast power, premiums, and the theory of storage, *Journal of Business* 60, 55–73.

Grossman, Sanford J., 1977, The existence of futures markets, noisy rational expectations and informational externalities, *Review of Economic Studies* 44, 431–449.

Huang, Jennifer, and Jiang Wang, 1997, Market structure, security prices and informational efficiency, *Macroeconomic Dynamics* 1, 431–449.

[21] In particular, the solution gives the expected results when I take the limit $\omega \to 1$. We have also checked the solution by taking other limits in the parameter space such as $\sigma_Z \to 0$ and obtained the expected results.

Jagannathan, Ravi, 1985, An investigation of commodity futures prices using the consumption-based intertemporal capital asset pricing model, *Journal of Finance* 40, 175–191.

Jazwinski, Andrew H., 1970, *Stochastic Processes and Filtering Theory* (Academic Press, New York).

Khoury, Nabil, and Pierre Yourougou, 1993, Determinants of agricultural futures price volatilities: Evidence from Winnipeg commodity exchange, *Journal of Futures Markets* 13, 345–356.

Kolb, Robert, 1991, *Understanding Futures Markets* (Kolb Publishing Company, Miami).

Merton, Robert, 1971, Optimal consumption and portfolio rules in a continuous time model, *Journal of Economic Theory* 3, 373–413.

Milonas, Nikolaos T., 1986, Liquidity and price variability in futures markets, *Financial Review* 21, 211–237.

Netz, Jane, 1995, The effect of futures markets and corners on storage and spot price variability, *American Journal of Agricultural Economics* 77, 182–193.

Protter, Murray H., and Charles B. Morrey, 1991, *A First Course in Real Analysis* (Springer-Verlag, New York).

Richard, Scott F., and M. Sundaresan, 1981, A continuous time equilibrium model of forward prices and futures prices in a multigood economy, *Journal of Financial Economics* 9, 347–371.

Roll, Richard, 1984, Orange juice and weather, *American Economic Review* 74, 861–880.

Routledge, Bryan R., Duane J. Seppi, and Chester S. Spatt, 1998, Equilibrium forward curves for commodities, Working paper, Carnegie-Mellon University.

Samuelson, Paul A., 1965, Proof that properly anticipated prices fluctuate randomly, *Industrial Management Review* 6, 41–50.

Schwartz, Eduardo S., 1997, The stochastic behavior of commodity prices: Implications for valuation and hedging, *Journal of Finance* 52, 923–974.

Wang, Jiang, 1994, A model of competitive stock trading volume, *Journal of Political Economy* 102, 127–168.