

SOCIAL LEARNING WITH A HIDDEN ACTION

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ABSTRACT. We analyze a model of observational learning in which of the two actions taken by agents only one is observable, say, the positive action (investment, henceforth). The negative action (no investment) is hidden and can only be inferred by the lack of investments through time. We investigate the possible consequences of this lack of observability. We find our model has qualitatively the same asymptotic properties as the standard herding model in which both actions are observable (benchmark, henceforth). Namely, there is incomplete learning in the limit and possible settling on the wrong action if and only if private beliefs are bounded. However and perhaps surprisingly, there is always a higher chance of investment than in the benchmark whether the state is good or bad. More likely investment, although welfare improving in the good state, is “overinvestment” nonetheless as it always leads to lower ex ante welfare than in the benchmark.

Keywords: Informational herding

1. INTRODUCTION

Part of the social learning literature has focused on situations in which a countable number of individuals must take sequentially a binary decision under uncertainty. This framework was first introduced by Banerjee [4] and Bickchandani, Hirshleifer and Welch [5] (henceforth BHW) with a discrete signal space. Smith and Sorensen [7] extended and generalized the BHW model to a general signal space and in several other directions. Still, the above models, while differing mainly in the signal structure, assume that each decision maker observes all the sequence of actions taken by the previous decision makers. Indeed, the initial illustrative example, used by Banerjee [4] and still often used today to motivate a lot of the literature is the choice between two restaurants, in which prospective clients observe the sequence of choices of all previous clients. However, we may often want to think of a binary decision as an investment decision instead. In this case, for instance,

Date: First Version: December 1th 2007, This Version: 11th August 2008.

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we may want to relax some assumptions on what is observable about the predecessors' actions. Namely, if an agent has an idea about investing in a certain project, then he will certainly make inferences based on how many other agents chose to undertake a similar project in the past. However, not all the past investment decisions will typically be observable: the decisions to invest are observable, but the decisions not to are much harder to observe. In other words, there should be some observational bias towards one of the two decisions. It is certainly of much less use to know that ten agents invested in the past if we ignore how many agents did not invest having had the opportunity to do so. This lack of observability of 'negative' actions seems not uncommon in many situations, so a natural question arises: what happens in a herding model when agents can observe the positive past decisions, but can only conjecture about the negative ones? In particular, we may want to know whether good investment opportunities are never undertaken because all agents may think that, had it been a good opportunity, then somebody would have undertaken it already. For example, under what conditions can a good arbitrage opportunity in the market be left unexploited ("a dollar bill is left on the ground as everybody thinks that it would have been picked up already if it were not a fake bill") or a seemingly hard but actually manageable theoretical problem be left unsolved because nobody has attempted it?

We take a benchmark herding model with a general signal space à la Smith and Sorensen [7] (B model henceforth) and incorporate to that model additional uncertainty about the number of agents that made the binary decision in the past. At every instant, with some probability, no decision-making agent arrives, or one agent arrives and must take a decision immediately. This probability is constant and independent of the past, so the arrival process is memoryless or stationary. If an agent arrives at a given instant, his decision is observed only if it is, say, positive. So, if in a given interval no investment is observed, then it could either mean that nobody came along, or that some agent arrived and chose not to invest.

We cast this stationary arrival process in continuous time, so agents with the idea to invest on a certain project arrive according to a Poisson process. When making the decision an agent observes how much time has elapsed and the timing of past investments if there has been investments in the past. The mere passage of time is informative in this model because, as time passes without investment news, agents become more pessimistic. Our goal is to shed light on the informativeness of time. The main question we want to answer is whether the herding

results are exacerbated or mitigated when rational agents observe just one of the actions and the passage of time.

The main result is that herds on the wrong action can only occur if the private beliefs are bounded. So if beliefs are unbounded, eventually every agent will invest if the state is good and eventually agents will stop investing if the state is bad. A corollary to this result is that situations in which nobody ever invests in a good project may only arise if beliefs are bounded. Namely, if time passes without investment news, agents become progressively more pessimistic and the belief decreases to the lower bound but, if beliefs are unbounded, the rate of convergence of the belief to zero is low. This rate is low enough that with probability one agent with a sufficiently high signal will arrive in finite time and invest.

Another aspect of the model we focus on is the histories up to the first investment. In many applications this game does not continue past the first investment as all or most of the rewards are given just to the first investor. This is often the case in the case of innovation, certainly in the case when making an academic discovery and in the case of unexploited arbitrage opportunities that disappear when they are exploited the first time. Contrary to what we initially expected, we find that the lack of observability does not discourage investment, on the contrary, the chance of investment is higher than in the case where the no investment action is observable, moreover this happens regardless of the state of nature, i.e. whether investment is the good or the bad decision.

1.1. Related Literature. This first models of sequential decisions and observational learning by Banerjee [4] and Bickchandani, Hirshleifer and Welch [5] and Smith and Sorensen [7] assumed that a wealth of actions were observed by the decision makers. Namely, a decision maker could observe the exact sequence of decisions made by all the predecessors. This has the advantage of making the model more tractable. After that other work by Celen and Kariv [2], Callander and Horner [1] Smith and Sorensen [6] assume that only a subsample of such actions is observable and indeed if that is the case the convergence properties of the model can radically change. In fact, Celen and Kariv [2] show that if agents can only observe the action of their immediate predecessor beliefs do not converge and so complete learning never arises as beliefs and actions continue to cycle. Callander and Horner [1] show that if agents can only observe the mass of agents that took each action and not the sequence of decisions then an agent may want to follow the action taken by the minority of predecessors. In sum, by assuming

imperfect observability of previous actions the properties of the model can radically change. Along similar lines a natural step is to study the properties of a model in which one of the two possible decisions is not observable and can only be inferred. This question is also intriguing since, given the unexpected results obtained in the related papers, it is hard to predict *ex ante* what the answer could be.

There is also a relation between our model and endogenous timing models such as Chamley and Gale [3]. In this model an inefficient lack of investment may arise because all agents may decide to wait for others to reveal some positive information by investing and as a result a no investment history may occur even if the investment is profitable. The link between their model and ours is that in both cases a lack of investment is necessarily partly interpreted as a consequence of agents' pessimistic beliefs, and this may not be the case. In fact, in Chamely and Gale [3] the lack of investment could also be due to strategic delay and in our model lack of investment could be due to absence of agents having the investment idea.

2. SETUP

Time is continuous and agents arrive according to a Poisson process with rate λ

$$t \sim \lambda e^{-\lambda t}$$

Each agent at the time of arrival must take one of two actions

$$j \in \{N, I\} := \{\text{Not Invest}, \text{Invest}\}$$

There are two states of the world

$$\theta \in \{\text{Bad}, \text{Good}\} := \{0, 1\}$$

There is an investment cost c , agents are risk-neutral and their utility u is

$$\begin{aligned} u(N) &= 0 \\ u(I) &= -c + \begin{cases} 0 & \text{if } \theta = 0 \\ 1 & \text{if } \theta = 1 \end{cases}, \quad c \in (0, 1) \end{aligned}$$

Hence an agent will take the investment action if their belief that the state is good is above c . Each agent has a private signal $x \in [0, 1]$ which, conditionally on the state θ , is extracted from the following distribution

$$\text{pdf: } f_\theta(x), \quad \text{cdf: } F_\theta(x)$$

The signal distributions satisfy the monotone likelihood ratio property (MLRP henceforth)

$$l(x) := \frac{f_1(x)}{f_0(x)} \quad \text{with } l'(x) > 0$$

Without loss of generality we assume

$$l(1/2) = 1$$

so the signal $x = 1/2$ is neutral news as Bayesian updating after such a signal leaves the prior unchanged.

We start from an initial belief at $t = 0$

$$p_0 := \Pr(\theta = 1 | t = 0)$$

2.1. Threshold Signal and Public Belief. An agent that arrives at time t will have a posterior belief based on two elements: his private signal x and the history publicly observed up to time t . The information gathered from the history can be summarized by the public belief p_t (which from now on we just call the belief). Define the likelihood ratio (LR henceforth) of public belief as

$$L_t := \frac{p_t}{1 - p_t} := \frac{\Pr(\theta = 1 | t)}{\Pr(\theta = 0 | t)}$$

We call L_t or p_t simply as the belief from now on.

For any given belief p_t there is a threshold signal x_t that makes an agent indifferent between investing or not

$$x_t := x \text{ solves } \mathbb{E}(\theta | x, t) = \Pr(\theta = 1 | x, t) = c$$

An agent with a signal above that threshold will invest and below that threshold will not invest. Bayesian updating implies that given a belief L_t at time t , then the threshold x solves

$$c = \Pr(\theta = 1 | x, t) = \frac{f_1(x) p_t}{f_1(x) p_t + f_0(x) (1 - p_t)}$$

or

$$(1) \quad x_t : l(x_t) L_t = \frac{c}{1 - c}$$

The belief evolves according to the history, which consists of intervals of time without news and instants of time when an investment news arrives.

2.2. Evolution without Investment. Here we study the evolution of the belief when no news arrives.

Let

$$G_{\theta,t} := \Pr \{I = 0 | \theta, t\}$$

be the probability of no investment ($I = 0$) conditional on state θ from the initial time zero to time t . Supposing the threshold is x_t , we have

$$G_{\theta,t+dt} = G_{\theta,t} (F_{\theta}(x_t) \cdot \lambda dt + 1 \cdot (1 - \lambda dt))$$

namely, the chance that nobody invested up to an instant after t is the chance that nobody invested up to t times the chance that either some agent arrived in that instant with a signal below the threshold or no agent arrived in that instant. Differentiating we have

$$\frac{G'_{\theta,t}}{G_{\theta,t}} = -\lambda (1 - F_{\theta}(x_t)) \implies G_{\theta,t} = e^{-\lambda \int_0^t (1 - F_{\theta}(x_s)) ds}$$

In a zero investment history the threshold x solves

$$c = \frac{f_1(x) G_{1,t} p_0}{f_1(x) G_{1,t} p_0 + f_0(x) G_{0,t} (1 - p_0)}$$

or calling

$$\gamma := \frac{1 - p_0}{p_0} \frac{c}{1 - c}$$

we have

$$(2) \quad x_t : l(x_t) = \gamma \frac{G_{0,t}}{G_{1,t}} = \gamma e^{\lambda \int_0^t (F_0(x_s) - F_1(x_s)) ds}$$

Differentiating with respect to t we have

$$\frac{l'(x) x'_t}{l(x)} = \lambda (F_0(x) - F_1(x))$$

and integrating we obtain the implicit equation for the threshold signal

$$t = g(x_t) := \frac{1}{\lambda} \int_{x_0}^{x_t} \frac{l'(x)}{l(x)} (F_0(x) - F_1(x))^{-1} dx$$

with initial condition: $x_0 = l^{-1}(\gamma)$.

If we consider the evolution starting from any date τ and public belief p_{τ} , the law of motion of $x_{t+\tau}$ without investment remains the same with the initial γ replaced by: $\gamma' = \frac{1-p_{\tau}}{p_{\tau}} \frac{c}{1-c}$.

We also obtain the following expression for the chance of zero investment

$$G_{\theta,t} = e^{-\int_{x_0}^{x_t} \frac{l'(x)(1-F_{\theta}(x))}{l(x)(F_0(x)-F_1(x))} dx}$$

as the change of variables $t = g(x_t)$ gives

$$\int_0^t (1 - F_\theta(x_s)) ds = \int_{x_0}^{x_t} (1 - F_\theta(x)) g'(x) dx = \frac{1}{\lambda} \int_{x_0}^{x_t} \frac{l'(x)(1 - F_\theta(x))}{l(x)(F_0(x) - F_1(x))} dx$$

Finally, knowing the evolution of the threshold signal x_t from 1, we can determine the evolution of the belief.

$$L_t = \frac{c}{1 - cl(x_t)} \frac{1}{l(x_t)}$$

2.3. Evolution after Investment. If an investment occurs at time τ then the belief increases discontinuously to

$$L_{\tau+} = \frac{1 - F_1(x_\tau)}{1 - F_0(x_\tau)} L_\tau$$

and the threshold decreases discontinuously to the value $x_{\tau+}$ that solves

$$l(x_{\tau+}) = \frac{1 - F_0(x_\tau)}{1 - F_1(x_\tau)} l(x_\tau)$$

In sum, the general belief evolution is

$$\begin{aligned} L_{\tau+} &= \frac{1 - F_1(x_\tau)}{1 - F_0(x_\tau)} \frac{G_{1,\tau}}{G_{0,\tau}} L_0 \\ &= \frac{1 - F_1(x_\tau)}{1 - F_0(x_\tau)} \left(e^{-\lambda \int_0^\tau (F_0(x_s) - F_1(x_s)) ds} \right) L_0 \end{aligned}$$

3. EXAMPLE

A closed form example can be obtained with the following cdf

$$F_1 = \frac{e^{ax} - 1}{e^a - 1}, \quad F_0 = \frac{e^a - e^{a(1-x)}}{e^a - 1}$$

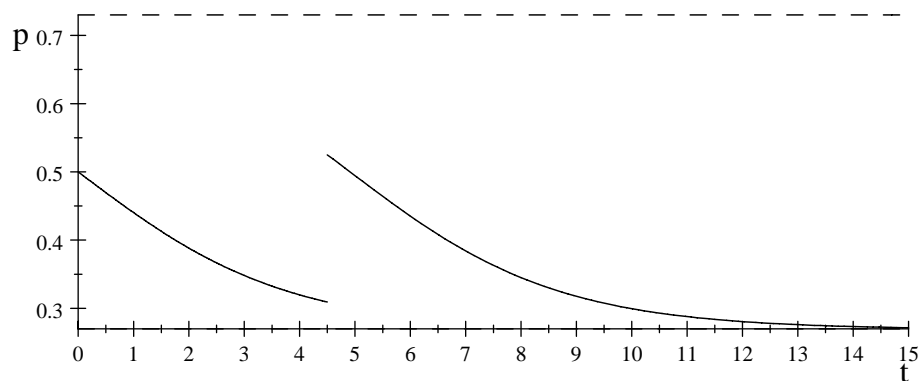
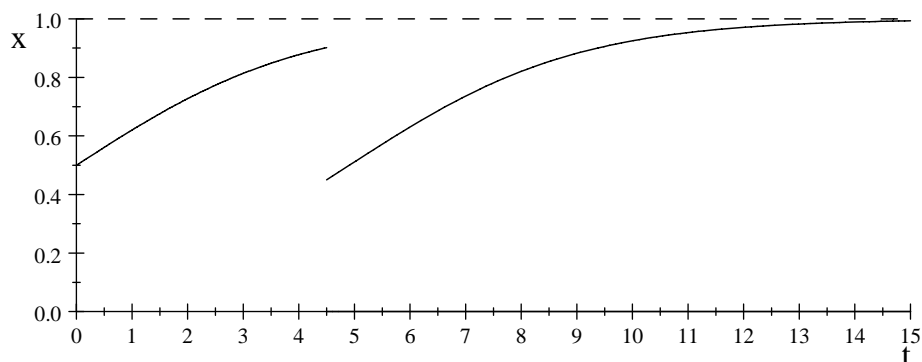
where $a > 0$ is a measure of how much information the private signals carry: the range of the likelihood ratio is $l(x) \in [e^{-a}, e^a]$. For initial condition $\gamma = 1$ (e.g. uniform prior $p_0 = 1/2$ and $c = 1/2$), the evolution of the belief and of the threshold up to the first investment is (the derivations for example are in the Appendix).

$$x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right), \quad p_t = \frac{e^a}{e^a + \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right)^2}$$

If an investment occurs at time τ , then the belief and the threshold jump. For instance with $\lambda = a = 1$, if one investment occurs at $\tau = 4.5$, we have the following discontinuity

$$\tau = 4.5 \implies \begin{cases} x_\tau \simeq 0.9, & x_{\tau+} \simeq 0.45 \\ p_\tau \simeq 31\%, & p_{\tau+} \simeq 52.5\% \end{cases}$$

For that history, the total evolution of the threshold and of the belief can be seen below 1



In general, the ex-ante (i.e. assessed at $t = 0$) chance of zero investment in the good state is

$$G_{1,t} = \left(\frac{e^{-\frac{a}{2}} + e^{-\frac{\lambda t}{2}}}{e^{-\frac{a}{2}} + 1} \right)^2 \rightarrow \left(\frac{e^{-\frac{a}{2}}}{e^{-\frac{a}{2}} + 1} \right)^2 > 0$$

so, even if the project is profitable, there is a positive chance that everybody decides not to invest in it. This chance decreases as the informativeness of the signals a increases, but for all $a > 0$ it remains bounded away from zero.

If we take another example with the following cdf

$$F_1 = x^2, \quad F_0 = 1 - (1 - x)^2$$

The the chance nobody invests in the good state is (see Appendix)

$$G_{1,t} = \frac{1 - x_t}{x_t} e^{\frac{1}{2x_t} - 1} \rightarrow 0$$

So, in a zero investment history we would have as $t \rightarrow \infty$

$$x_t \rightarrow 1 \quad \implies \quad G_{1,t} \rightarrow 0$$

Namely, a zero investment history in the good state is a zero probability event, so eventually a profitable project is always undertaken.

4. ASYMPTOTIC BEHAVIOR

From equation (1) we see that boundaries of the LR-belief are L^- and L^+ are defined by the threshold being respectively one or zero, namely.

$$L^- := \frac{c}{1 - cl(1)} \frac{1}{1} < \frac{c}{1 - c} < \frac{c}{1 - cl(0)} \frac{1}{0} := L^+$$

or $p \in [p^-, p^+]$ such that

$$\begin{aligned} p^- &= \frac{L^-}{1 + L^-}, & p^+ &= \frac{L^+}{1 + L^+} \\ 0 &\leq p^- < c < p^+ \leq 1 \end{aligned}$$

If signals are unbounded the weak inequalities become strict, whereas with bounded signals: the weak inequalities become equalities. Signals are unbounded if

$$f_0(1) = f_1(0) = 0 \quad \iff \quad l(0) = 0, \quad l(1) = \infty$$

so agents receiving the most pessimistic or the most optimistic signal are certain about what the state is.

Proposition 1. *Beliefs converge to the boundaries of the belief interval*

$$p_t \rightarrow p_\infty \in \{p^-, p^+\}$$

(i.e. to $p_\infty \in \{0, 1\}$ complete learning with unbounded signals. The threshold signal x_t converges to

$$x_t \rightarrow x_\infty \in \{0, 1\}$$

Proof. See Appendix. □

Proposition 2. *For any initial interior prior $p_0 \in (0, 1)$ and interior cost $c \in (0, 1)$, if the signals are unbounded the chance there is no investment in the good state is zero, and conversely.*

Proof. See Appendix. \square

Lemma 3. *Investment is finite (infinite) if and only if the beliefs converge to the lower bound p^- (upper bound p^+).*

Proof. If investments are finite, then after the last investment beliefs must decrease monotonically to the lower bound p^- .

If the number of investments were infinite, beliefs could not converge to the lower bound p^- , as after an investment the public belief must jump to some value above the cost: $p_t \geq c$, hence beliefs must converge to the upper bound p^+ . \square

Proposition 4. *The belief converges to the correct value almost surely if and only if private signals/beliefs are unbounded.*

Proof. See Appendix. \square

5. ONE INVESTMENT GAME

We now analyze a special case of this problem. We focus on situations in which the game ends after the first investment, so that only the first one to invest may obtain a payoff. In this problem the exogenous timing assumption is not restrictive; it is not necessary to assume that agents are impatient and have a once and for all chance to invest when they have the idea to do so.

Proposition 5. *In the endogenous timing game, i.e. if agents have the option to wait to invest, the optimal strategy in any equilibrium is the same as in the exogenous timing game.*

Proof. See Appendix. \square

5.1. Comparison with BHW Model Benchmark. Now we know that in the good state a first investment will eventually happen with probability one in our hidden action framework. Since this is also the case in the B framework, i.e., with observable actions, then we would like to know if in one of the two models the investment is more likely.

Let $B_{\theta,n}$ be the B model analogous of $G_{\theta,t}$ in the hidden action model, i.e., the chance that, in state θ , no investment has occurred after n agents had the idea, namely the chance that the first n agents did not invest.

$$B_{\theta,n} := \prod_{k=0}^{n-1} F_{\theta}(x_k)$$

where, since

$$x_{n+1} : \gamma = l(x_{n+1}) \frac{\prod_{k=0}^n F_1(x_k)}{\prod_{k=0}^n F_0(x_k)}$$

the thresholds x_n are given by the recursion

$$x_0 : l(x_0) = \gamma, \quad x_{n+1} : l(x_{n+1}) = \frac{F_0(x_n)}{F_1(x_n)} l(x_n)$$

Take a no investment history in the hidden action model and let t_n be the time in which the threshold reaches x_n defined by the above recursion. We have

$$G_{\theta, \infty} = e^{-\int_{x_0}^1 \frac{1-F_{\theta}(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx}, \quad B_{\theta, \infty} = \prod_{k=0}^{\infty} F_{\theta}(x_k)$$

Note the same constant of proportionality between the chance of no investment conditional on the different states in the two models. Namely,

Lemma 6.

$$G_{1, \infty} = \frac{\gamma}{l(1)} G_{0, \infty}, \quad B_{1, \infty} = \frac{\gamma}{l(1)} B_{0, \infty}$$

Proof.

$$\begin{aligned} G_{1, \infty} &= e^{-\int_{x_0}^1 \left(1 + \frac{1-F_0(x)}{F_0(x)-F_1(x)}\right) \frac{l'(x)}{l(x)} dx} = \frac{\gamma}{l(1)} G_{0, \infty} \\ B_{1, \infty} &= \left(\frac{\prod_{k=0}^{\infty} F_1(x_k)}{\prod_{k=0}^{\infty} F_0(x_k)} \right) B_{0, \infty} = \frac{\gamma}{l(1)} B_{0, \infty} \end{aligned}$$

□

Define the hazard function $h_{\theta}(x)$ and the hazard ratio $m^{-1}(x)$ as

$$h_{\theta}(x) := \frac{f_{\theta}(x)}{1 - F_{\theta}(x)}, \quad m^{-1}(x) := \frac{h_1(x)}{h_0(x)}$$

If $m^{-1}(x)$ is increasing we say the signal distribution satisfies the increasing hazard ratio property (IHRP henceforth). IHRP is not implied by MLRP (see Appendix).

Proposition 7. *Assume IHRP. Whichever the state is, the chance of investment is always larger in the hidden action model.*

$$G_{\theta,\infty} < B_{\theta,\infty}$$

Proof. See Appendix. □

The proof hinges on the inequality

$$F_{\theta}(x_k) > e^{-\int_{x_k}^{x_{k+1}} \frac{1-F_{\theta}(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx}$$

Intuitively, if we start from a same (initial public belief and hence) initial threshold x_k in both models, define x_{k+1} as the new threshold after negative news in the BHW model and define Δt_k as the time it takes the threshold to go from x_k to x_{k+1} in the hidden action model. Whichever the state is, the chance there is one investment in the hidden action model in the time interval Δt_k is always larger than the chance that the next agent invests in the BHW model. So the comparison is between what the next agents does in the BHW model and what none, one, or many agents do if they arrive in the time interval Δt_k in the hidden action model.

5.1.1. Welfare Comparison.

Proposition 8. *The welfare in the hidden action model is lower than in the benchmark model.*

Proof. The welfare in the hidden action model is

$$\begin{aligned} W(G) &: = \mathbf{E}_{\theta} [(1 - G_{\theta}) u(I) + (G_{\theta}) u(N)] \\ &= p_0 (1 - G_1) (1 - c) + (1 - p_0) (1 - G_0) (-c) \\ &= p_0 (1 - c) \left((1 - G_1) - \gamma \left(1 - \frac{l(1)}{\gamma} G_1 \right) \right) \\ &= p_0 (1 - c) (1 - \gamma + (l(1) - 1) G_1) \end{aligned}$$

and likewise for $W(B)$. Since $l(1) > 1$, as 1 is the highest signal and must be larger the neutral news signal we have

$$W(G) < W(B)$$

if the IHRP is satisfied. □

So even though the hidden action model performs better conditional on the good state as it always has higher chance of investment, unconditionally it has lower welfare than the benchmark model.

6. CONCLUSIONS

At least as far as convergence is concerned, we have shown that relaxing the assumptions that all types of decisions are observable does not change significantly the asymptotic learning properties of the model. Beliefs always converge and will converge to the true value for sure if and only if private beliefs are unbounded. So, even if one action is hidden and can only be inferred, the market aggregates the information correctly anyway when beliefs are unbounded.

It would be interesting to see if these properties hold in situations in which even less information is available to the decision makers. A natural extension is to assume that of the investment actions only the ones that are successful investments are observed and the investments that resulted into failures can only be inferred. So failures of investments, no investments and lack of arrivals would be observationally equivalent.

7. APPENDIX

7.1. Proofs.

Proof of Proposition 1. Since the public belief p_t is a Martingale bounded on $[0, 1]$, then the Martingale convergence theorem implies almost sure convergence to a value $p_\infty \in [p^-, p^+]$.

Beliefs cannot converge to any interior value:

$$p_\infty \in (p^-, p^+)$$

namely, beliefs cannot settle within any $\varepsilon > 0$ of any interior value $p_\infty \in (p^-, p^+)$, as beliefs would jump discretely after any investment, or would decrease to p^- otherwise.

The threshold signal x_t must converge because it is a continuous function of the belief.

The threshold must converge to its boundaries

$$x_t \rightarrow x_\infty \in \{0, 1\}.$$

as the limit threshold cannot be any interior value $x_\infty \in (0, 1)$ because, if it so were, at some point an investment will occur with probability one making the belief and the threshold jump. \square

Proof of Proposition 2. In a no investment history the belief would converge to zero and the threshold to one. We now show that with unbounded beliefs this is a zero chance history, and conversely:

$$l(1) = +\infty \iff \lim_t G_{1,t} = 0$$

Observe that the probability of no investment ever under state 1 converges, as it is an increasing and bounded function. Since

$$F_0(x_t) - F_1(x_t) \leq 1 - F_1(x_t)$$

it follows that from (2) that

$$G_{1,t} = e^{-\int_0^t \lambda(1-F_1(x_s))ds} \leq \frac{c}{1-c} \frac{1-p_0}{p_0} \frac{1}{l(x_t)},$$

so

$$\lim_t G_{1,t} = 0 \quad \text{if } l(1) = +\infty.$$

Conversely, if beliefs are bounded we have that $l(1) < \infty$. Then

$$h(x) := \frac{l(x)}{l'(x)} \lambda (F_0(x) - F_1(x))$$

converges to 0, and, so for all x sufficiently close to 1,

$$x'_t = h(x) > -h'(1)(1-x_t) - M(1-x_t)^2,$$

with $h'(1) < 0$ as $h(x) > 0$, for some $M > 0$. This implies that, for all t sufficiently large,

$$1 - x_t \leq \frac{-h'(1)}{M + C_1 e^{-h'(1)t}},$$

for some constant C_1 . K . Since

$$1 - F_1(x_s) \leq f_1(1)(1-x_s) + C_2(1-x_s)^2,$$

and

$$\int^t \frac{-h'(1) ds}{M + C_1 e^{-h'(1)s}} = \ln \left(C_1 + M e^{-h'(1)t} \right) / M < \ln(C_1 + M) / M,$$

it follows that $G_{1,t}$ is bounded below, so that

$$l(1) < +\infty \Rightarrow \lim_t G_{1,t} > 0.$$

$$p_\infty(\theta = 1) = 1 \quad \Rightarrow \quad p_\infty(\theta = 0) = 0$$

□

Proof of Proposition 3. Assume signals are unbounded.

If the state is good, then the public belief converges to one. Suppose not, then beliefs must converge to zero and investment would stop at some point. But, by the Lemma 1, finite investment is incompatible with the state being 1.

If the state is bad, then the public belief converges to zero. By the martingale property if the belief converges to 1 in the good state it must converge to zero in the bad state, namely

$$\begin{aligned} p_t &= E_t(p_\infty) = (p_\infty(\theta = 1))p_t + (p_\infty(\theta = 0))(1 - p_t) \\ p_\infty(\theta = 1) &= 1 \implies p_\infty(\theta = 0) = 0 \end{aligned}$$

Assume signals are bounded.

In the good state beliefs can converge to the wrong value p^- because there is a positive chance of zero investment history.

In the bad state, beliefs converge to p^+ with positive probability. Indeed, if they did not, then it would be the case that $p^+ = 1$ (by definition of p^+), and this is impossible with bounded beliefs. \square

Proof of Proposition 5. Let $\tau(x, t)$ denote the stopping time of a player arriving at instant t with signal x , and let $F(s; t, x)$, $s \geq t$, denote the corresponding c.d.f. Fix an equilibrium and suppose for the sake of contradiction that $0 < F(s; t, x) < 1$ for some x, t and $s > t$, for some finite s .¹ Let q_τ denote the private belief of this agent at time $\tau \geq t$, given his signal x and the equilibrium strategies (conditional on the event E_τ that no one invested up to τ). Observe that q_τ is non-increasing, and constant over some interval of time $[t', t'']$ if and only if $F(t''; s, x) = F(t'; s, x)$ for all $s \leq t'$ and $x \in [0, 1]$. Assume that it is strictly profitable to invest at time t with signal x . Then, because the payoff from investing is strictly increasing in q_t , $F(s; t, x) < 1$ is only possible if $q_s = q_t$, and player i assigns probability one to no one investing before (or at the same) time than he does. In particular, any other player arriving in the interval of time (t, s) must invest with probability zero in that interval of time. Consider the event that some player arrives in this interval of time with a signal $x \geq x_t$, i.e. a player whose payoff from investing immediately is strictly positive. (x_t here depends obviously upon the equilibrium strategies.) This event has strictly positive probability, and thus, given the equilibrium strategies, there exists a player arriving at some time $t' \in (t, s)$ whose probability of investing first (after s) is strictly less than 1. This player would profitably gain from investing immediately at time t' . Assume now that it is strictly unprofitable to invest at time t with signal x . Plainly it remains unprofitable to invest at any later time, and so it cannot be that $0 < F(s; t, x)$ for some finite s . Therefore, a player is

¹This formulation seems to assume that the strategies are symmetric, i.e. only depend on arrival time and signal, not on the specific player; the logic applies equally to the case in which strategies are not symmetric, by simply adding the appropriate subscript i to refer to a specific player i .

unwilling to delay unless $x = x_t$, i.e. he is indifferent between investing immediately or never. This event has zero probability, and so does not affect the analysis, in particular the determination of x_t . \square

Proof of Proposition 6. Monotone hazard ratio means that

$$m(x) := \frac{f_0(x)/(1 - F_0(x))}{f_1(x)/(1 - F_1(x))}$$

is decreasing, so that, for $m = m(x_k)$,

$$\frac{1 - F_1(x)}{1 - F_0(x)} \leq ml(x),$$

for all $x \in [x_k, x_{k+1}]$, with equality for $x = x_k$. That is, we have

$$\frac{F_0(x) - F_1(x)}{1 - F_0(x)} \leq ml(x) - 1,$$

which implies that the right-hand side is positive, as the left-hand side is.

The proportionality result

$$G_{1,\infty} = \frac{\gamma}{l(1)} G_{0,\infty}, \quad B_{1,\infty} = \frac{\gamma}{l(1)} B_{0,\infty}$$

means we can just show $B_{\theta,\infty} > G_{\theta,\infty}$ for $\theta = 0$. We have

$$\prod_{k=0}^{\infty} F_0(x_k) > e^{-\int_{x_0}^1 \frac{1-F_0(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx} = \prod_{k=0}^{\infty} e^{-\int_{x_k}^{x_{k+1}} \frac{1-F_0(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx}$$

so it suffices to show that for all k that

$$\ln F_0(x_k) + \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1 - F_0(x)}{F_0(x) - F_1(x)} dx \geq 0$$

Therefore,

$$\begin{aligned} \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1 - F_0(x)}{F_0(x) - F_1(x)} dx &\geq \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)(ml(x) - 1)} dx \\ &= \ln \frac{ml(x_{k+1}) - 1}{l(x_{k+1})} - \ln \frac{ml(x_k) - 1}{l(x_k)} = \ln \frac{ml(x_k) - \frac{F_1(x_k)}{F_0(x_k)}}{ml(x_k) - 1}, \end{aligned}$$

using that $l(x_{k+1}) = \frac{F_0(x_k)}{F_1(x_k)} l(x_k)$. Therefore, the inequality

$$\ln F_0(x_k) + \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1 - F_0(x)}{F_0(x) - F_1(x)} dx \geq 0$$

would be implied by

$$\ln F_0(x_k) + \ln \frac{ml(x_k) - \frac{F_1(x_k)}{F_0(x_k)}}{ml(x_k) - 1} \geq 0,$$

Rearranging, this is equivalent to

$$\frac{1 - F_1(x_k)}{1 - F_0(x_k)} \geq ml(x_k),$$

but this is the case, since in fact both sides are equal, by definition of m . \square

7.2. Exponential Example. Take the following cdf

$$F_1 = \frac{e^{ax} - 1}{e^a - 1}, \quad F_0 = \frac{e^a - e^{a(1-x)}}{e^a - 1}$$

The threshold can be expressed explicitly as

$$\begin{aligned} \lambda t &= \int_{x_0}^x 2a \int \frac{e^a - 1}{1 + e^a - e^{as} - e^{a(1-s)}} ds = \left[2 \ln \left(\frac{e^{as} - 1}{e^a - e^{as}} \right) \right]_{x_0}^x \\ \lambda t &= 2 \ln \left(\frac{e^{ax} - 1}{e^a - e^{ax}} \right) - A_0 \implies x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + A_0}{2} + a} + 1}{e^{\frac{\lambda t - A_0}{2} + 1}} \right) \end{aligned}$$

With uniform prior we have

$$\left(x_0 = \frac{1}{2} \implies A_0 = -a \right) \iff x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + a}{2} + 1}}{e^{\frac{\lambda t - a}{2} + 1}} \right)$$

The likelihood ratio and the public belief are

$$\begin{aligned} l(x_t) &= e^{a(2x_t - 1)} = e^{-a} \left(\frac{e^{\frac{\lambda t + a}{2} + 1}}{e^{\frac{\lambda t - a}{2} + 1}} \right)^2 \rightarrow e^a \\ L_t &= e^a \left(\frac{e^{\frac{\lambda t - a}{2} + 1}}{e^{\frac{\lambda t + a}{2} + 1}} \right)^2 \rightarrow e^{-a} \implies p_t = \frac{L_t}{1 + L_t} \rightarrow \frac{1}{1 + e^a} \end{aligned}$$

Since

$$e^{ax_t} = \frac{e^{\frac{\lambda t + a}{2} + 1}}{e^{\frac{\lambda t - a}{2} + 1}}, \quad \int_0^t \frac{1}{e^{\frac{\lambda t - a}{2} + 1}} dt = t - \frac{2}{\lambda} \ln \left(\frac{e^{\frac{\lambda t - a}{2} + 1}}{e^{-\frac{a}{2} + 1}} \right)$$

then the chance of zero investment in the good state is

$$\begin{aligned} G_{1,t} &= e^{-\int_0^t \lambda(1-F_1)dt} = e^{-\lambda \int_0^t \left(1 - \frac{e^{ax}-1}{e^a-1}\right)dt} = e^{-\lambda \int_0^t \frac{1}{e^{\frac{\lambda t-a}{2}}+1} dt} \\ &= e^{-\lambda t} \left(\frac{e^{\frac{\lambda t-a}{2}} + 1}{e^{-\frac{a}{2}} + 1} \right)^2 = \left(\frac{e^{-\frac{a}{2}} + e^{-\frac{\lambda t}{2}}}{e^{-\frac{a}{2}} + 1} \right)^2 \rightarrow \left(\frac{e^{-\frac{a}{2}}}{e^{-\frac{a}{2}} + 1} \right)^2 \end{aligned}$$

Given the symmetry

$$F_0(x; a) = F_1(x; -a)$$

the chance of zero investment in the bad state is

$$G_{0,t} = \left(\frac{e^{\frac{a}{2}} + e^{-\frac{\lambda t}{2}}}{e^{\frac{a}{2}} + 1} \right)^2 \rightarrow \left(\frac{e^{\frac{a}{2}}}{e^{\frac{a}{2}} + 1} \right)^2$$

After a trade the threshold $x_{\tau+}$ decreases discontinuously (e.g. take $a = 1$)

$$\begin{aligned} l(x_{\tau+}) &= \frac{1 - F_0(x_\tau)}{1 - F_1(x_\tau)} l(x_\tau) = (e^{-x_\tau}) l(x_\tau) \\ e^{2x_{\tau+}-1} &= e^{x_\tau-1} \implies x_{\tau+} = \frac{x_\tau}{2} \end{aligned}$$

and the belief increases discontinuously

$$p_{\tau+} = \frac{(e^{x_\tau}) p_\tau}{(1 - p_\tau) + (e^{x_\tau}) p_\tau} > p_\tau$$

7.3. Algebraic Example. Take the following cdf

$$F_1 = x^2, \quad F_0 = 1 - (1 - x)^2$$

In this example beliefs are unbounded, namely: $l(x) \in [0, \infty]$. For $\lambda = 1$, $c = 1/2$ ($\implies x_0 = 1/2$) the threshold is determined implicitly by

$$t = \int_{0.5}^{x_t} \left(\frac{1}{2x^2(1-x)^2} \right) dx = \frac{1}{2} \left(\frac{1}{1-x_t} - \frac{1}{x_t} \right) + \ln \left(\frac{x_t}{1-x_t} \right)$$

This change of variables

$$t = \frac{1}{2} \left(\frac{1}{1-x} - \frac{1}{x} \right) + \ln \left(\frac{x}{1-x} \right) := g(x)$$

gives

$$\int_0^t (1 - F_1(x_s)) ds = \int_{0.5}^{x_t} (1 - F_1(x)) g'(x) dx = \ln \frac{x_t}{1-x_t} - \frac{1}{2x_t} + 1$$

so the chance that nobody invests in the good state is

$$G_{1,t} = e^{-\lambda \int_0^t (1-F_1(x_s)) ds} = \frac{1-x_t}{x_t} e^{\frac{1}{2x_t}-1} \rightarrow 0$$

and the chance nobody invests in the bad state is

$$G_{0,t} = e^{\frac{1}{2x_t}-1} \rightarrow e^{-\frac{1}{2}} \simeq 61\%$$

8. HAZARD FUNCTION AND HAZARD RATIO

For θ belonging to any ordered state space, define the hazard function h_θ , the likelihood ratio $l_{\theta,\theta'}$ and hazard ratio $m_{\theta,\theta'}^{-1}$ as

$$\begin{aligned} h_\theta & : = \frac{f_\theta(x)}{1-F_\theta(x)}, \\ \theta > \theta' & \quad l_{\theta,\theta'} := \frac{f_\theta(x)}{f_{\theta'}(x)}, \quad m_{\theta,\theta'}^{-1} := \frac{h_\theta(x)}{h_{\theta'}(x)} \end{aligned}$$

Note that the properties of increasing likelihood ratio property (ILRP), the decreasing hazard function property (DHFP) and the increasing hazard ratio property (IHRP) can be stated as

$$\begin{aligned} \text{ILRP} & \iff \frac{\partial^2}{\partial\theta\partial x} \ln(f_\theta(x)) > 0 \\ \text{DHFP} & \iff \frac{\partial^2}{\partial\theta\partial x} \ln(1-F_\theta(x)) > 0 \\ \text{IHRP} & \iff \frac{\partial^2}{\partial\theta\partial x} \ln\left(\frac{f_\theta(x)}{1-F_\theta(x)}\right) > 0 \end{aligned}$$

Lemma 9.

$$\begin{aligned} \text{IHRP} + \text{DHFP} & \implies \text{ILRP} \\ \text{ILRP} & \implies \text{DHFP} \end{aligned}$$

Proof. The first implication is trivial as IHRP is by definition the first inequality and DHFP by definition the second inequality:

$$\frac{\partial^2}{\partial\theta\partial x} \ln(f_\theta(x)) > \frac{\partial^2}{\partial\theta\partial x} \ln(1-F_\theta(x)) > 0$$

For the second implication we need to show that

$$\theta > \theta' \quad \frac{f_\theta(x)}{\int_x^1 f_\theta(z) dz} < \frac{f_{\theta'}(x)}{\int_x^1 f_{\theta'}(z) dz}$$

which implies

$$\int_x^1 (f_{\theta'}(x) f_{\theta}(z) - f_{\theta}(x) f_{\theta'}(z)) dz > 0$$

$$\int_x^1 f_{\theta'}(x) f_{\theta'}(z) (l_{\theta, \theta'}(z) - l_{\theta, \theta'}(x)) dz > 0$$

where the latter step is implied by ILRP. \square

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