

QUORUM AND TURNOUT IN REFERENDA

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ABSTRACT. We analyze the effect of turnout requirements in referenda in the context of a group turnout model. We show that a participation quorum requirement may reduce the turnout so severely that it generates a “quorum paradox”: in equilibrium, the expected turnout exceeds the participation quorum only if this requirement is not imposed. Moreover, a participation quorum does not necessarily imply a bias for the status quo. We also show that in order to induce a given expected turnout, the quorum should be set at a level that is lower than half the target, and the effect of a participation quorum on welfare is ambiguous. On the one hand, the quorum decreases voters’ welfare by misrepresenting the will of the majority. On the other hand, it might also reduce the total cost of voting. Finally, we show that an approval quorum is essentially equivalent to a participation quorum.

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“Last June, the church played a role in a referendum that sought to overturn parts of a restrictive law on in vitro fertilization [...]. To be valid, referendums in Italy need to attract the votes of more than half the electorate. Apparently fearing defeat, Cardinal Camillo Ruini called on Catholics to stay away so that the initiative would be thwarted with the help of the merely apathetic. His move was so blatantly tactical (and questionably democratic) that it prompted criticism from believers [...]. But it worked. Only 26% of the electorate turned out to vote, so the legislation remained in force.” *The Economist*, Dec. 10th 2005¹

1. INTRODUCTION

Direct democracy is firmly established in many democratic countries, and the use and scope of direct democracy institutions are increasing all around the world. In Europe, for instance, the average number of referenda held every year was 0.18 in the 80s, 0.39 in the 90s, and is around 0.27 in the current decade.²

In many countries and in some US states referenda have to meet certain turnout requirements in order to be valid. Typically, the status quo can be overturned only if a majority of voters is in favor of it, *and* the turnout reaches a certain level, that is a “participation quorum” is met. In some cases an “approval quorum” is required, i.e., the turnout of the majority that is against the status quo has to reach a certain level.³

The common rationale for a turnout requirement is that: “a low turnout in referendums is seen as a threat to their legitimacy” (Qvortrup (2002)). In other words, to change the status quo policy a large proportion of citizens should take part in this decision and a high turnout reflects the fact that enough citizens care about the issue at stake. However, the extent to which citizens “care” about an issue depends on the mobilization effort of political parties. In fact, political parties

¹The strategy of the status quo supporters was to encourage citizens to forget about the vote and the issue. As it turned out, the strategy was successful: the status quo got only 12% of the votes but the total turnout was only 26%. Of all the Italian referenda over the past 10 years, the only one that had a participation above 50% was the last 2006 referendum on the constitution, namely, the only referendum that had no participation quorum requirement. Note also that Italy is known to have one of the highest turnouts in national elections (typically above 80% of eligible voters) as compared to all other countries where voting is not mandatory.

²For evidence on the increasing use of referenda as tools for policy-making see e.g. Casella and Gelman (2005) and references therein, Qvortrup (2002), Matsusaka (2005a, 2005b).

³A list of countries that have participation and/or approval quorum requirements is provided in Table 2 in the appendix.

and allied interest groups spend a great deal of effort and huge amounts of campaign money in order to encourage citizens to vote for one of the alternatives.⁴

The existence of a turnout requirement introduces a crucial asymmetry in the campaign strategy of organized groups, by allowing those in favor of the status quo issue to use a “quorum-busting” strategy. Instead of devoting resources to increase the turnout of voters opposing the reform, the status quo party can exploit the group of apathetic citizens. In fact, if a significant fraction of voters abstains, the referendum will fail due to lack of quorum, and the status quo will remain in place.⁵

In this paper we analyze the effect of turnout requirements in referenda in the context of a simple game theoretic model of group turnout.

The main results of the paper are the following: First, we show that the introduction of a participation quorum requirement, which validates the referendum only if participation is high enough, may generate in equilibrium a “quorum paradox”, i.e., the equilibrium expected turnout may be smaller than the quorum itself. Interestingly enough, this could occur even if the expected turnout that would result in equilibrium in the absence of a participation quorum was greater than the required quorum. In other words, we show that there are levels of the quorum requirement such that in equilibrium the expected turnout exceeds the participation quorum only if the requirement is not imposed.

Second, a participation quorum requirement does not necessarily imply a bias for the status quo issue. In fact, the expected probability that the status quo is overturned may be higher in the presence of a participation quorum requirement than in its absence. Indeed, there are levels of the quorum requirement such that in equilibrium, either the equilibrium expected turnout is smaller than the quorum, or the

⁴The amount of resources political parties spend in order to mobilize voters has increased significantly in the last twenty years. For example, the National Election Studies [10] (Tables 6C.1a, 6C.1b and 6C.1c) provide evidence of a sharp increase in the percentage of respondents contacted by either party since 1990.

⁵Examples of referenda on salient policy issues that failed for lack of quorum abound also outside Italy. A recent controversial case was the 1998 referendum on abortion legalization in Portugal. The same referendum was repeated in February 2007. After the referendum failed pass due to lack of quorum for the second time, in July 2007 the Portuguese parliament approved a law to legalize abortion.

The historical evidence goes back to the Weimar republic. In 1926 and 1929 two referenda respectively on the confiscation of royal property and on the repudiation of the war guild obtained a yes vote of 93.3% and 94.5% respectively. Both referenda were declared void because the Weimar constitution required a majority not only of the votes but also of the eligible voters (see Qvortrup (2002)).

probability that the status quo is overturned is strictly higher than when the quorum requirement is absent.

Third, we provide some normative results. We show that in order to induce in equilibrium a given expected turnout, the quorum should be set at a level that is lower than half the target. Regarding the welfare implications of introducing a participation quorum requirement, we show that it decreases voters' welfare by misrepresenting the will of the majority. However, since it may also reduce the total cost of voting, the overall effect on voters' welfare is ambiguous. Finally, we show under what conditions a participation quorum requirement is essentially equivalent to an approval quorum requirement in terms of parties' mobilization incentives.

Since our goal is to analyze how a participation requirement affects the distribution of voting outcomes in large elections, our approach is to consider a framework common in the literature on large elections, and extend it to the case where a turnout requirement is introduced. In particular, our model is based on the group-based model of turnout first developed by Snyder (1989) and Shachar & Nalebuff (1999).⁶ In these models two opposed parties spend effort to mobilize their supporters to the polls, while facing aggregate uncertainty on the voters' preferences.

To the best of our knowledge, the only paper that analyzes turnout requirements in referenda is Corte-Real and Pereira (2004). In this paper they study the effects of a participation quorum using a decision-theoretic axiomatic approach. Contrary to our paper, they do not explore how the incentives of parties and interest groups to mobilize voters depend on the turnout requirements.⁷

The remainder of the paper is organized as follows. In Section 2 we present the basic model and we introduce the main results through a simple example. Section 3 contains the full equilibrium characterization. In Section 4 we present the comparative statics of the model. Section 5 provides a normative analysis, and in Section 6 we discuss some generalizations and extensions of the basic model. Section 7 concludes. All proofs are in the appendix.

⁶See also Morton (1987, 1991) for other group-based models.

⁷For a review of the advantages and limits of decision-theoretic models of turnout as compared to mobilization models of turnout, see Feddersen (2004).

2. THE MODEL

Consider a simple model of direct democracy where individuals have to choose between two alternatives: r (“reform”) and s (“status quo”). The voting rule is simple majority and ties are broken randomly. Let $q \in [0, 1]$ denote a participation quorum requirement, i.e. the status quo can be replaced if and only if: i) at least a fraction q of the population shows up at the polls *and* ii) a majority of voters vote in favor of r .

There are two exogenously given parties supporting issues r and s , and a continuum of voters of measure 1, of which a proportion $\tilde{r} \in [0, 1]$ supports issue r , while the remaining support issue s . Slightly abusing notation, we will use the same symbol (e.g. s) to denote an issue and the party supporting that issue. We assume that, from the parties’ point of view, \tilde{r} is a random variable with uniform distribution.⁸ Each voter has a personal cost of voting $c \in [0, 1]$ that is also drawn from a uniform distribution.

Parties decide simultaneously the amount of campaign funds to spend (equivalently the amount of effort to exert) to mobilize voters in order to win the referendum. Parties’ objective functions are

$$\begin{aligned}\pi_r(S, R) &= BP - R \\ \pi_s(S, R) &= B(1 - P) - S,\end{aligned}$$

where P is the (endogenous) probability that alternative r is selected, R (S) is the spending of the party r (s) respectively, and $B > 0$ is the payoff to parties if their preferred alternative is chosen.⁹

Since our focus is on the strategic interactions between parties, we depart from the pivotal voter approach in modelling voters’ behavior.¹⁰ We assume that voters receive a benefit from voting their preferred issue that is strictly concave in parties’ mobilization efforts. In particular, if party r (s) spends x , the benefit of voters supporting issue r (s) is captured by the function $\rho(x) : \mathbb{R}_+ \rightarrow [0, 1]$, which is continuous for $x \geq 0$, twice differentiable for $x > 0$, strictly increasing and strictly concave, and satisfies the properties

$$\lim_{x \rightarrow 0} (x\rho'(x)) = 0, \quad \lim_{x \rightarrow 0} (x\rho''(x)) = 0, \quad \lim_{x \rightarrow \infty} \rho'(x) = 0.$$

⁸In Section 6 we consider different distributional assumptions.

⁹See Section 6 for the case in which parties have different payoffs.

¹⁰See, e.g., Börgers (2004) for a pivotal voter model of costly voting.

This specification is equivalent to having parties' expenditures affect individual cost of voting.¹¹ Finally, for the sake of simplicity, for most of the paper we will assume that $\rho(0) = 0$.¹²

For a given level of spending R , a voter that supports issue r and has a voting cost equal to c votes for alternative r if and only if $\rho(R) \geq c$, (likewise, $\rho(S) \geq c$ for a supporter of issue s). Hence,

$$\Pr(\rho(R) \geq c) = \rho(R),$$

and the vote shares for each alternative are,

$$v_R = \tilde{r}\rho(R), \quad v_S = (1 - \tilde{r})\rho(S).$$

Notice that P is the joint probability that the vote share in favor of alternative r is greater than the vote share of alternative s and that the total turnout exceeds the quorum q . Formally,

$$\begin{aligned} P &= \Pr((v_R \geq v_S), (v_R + v_S \geq q)) \\ &= \Pr\left(\tilde{r} \geq \frac{\rho(S)}{\rho(R) + \rho(S)}, (\rho(R) - \rho(S))\tilde{r} \geq q - \rho(S)\right). \end{aligned}$$

By defining

$$Q = \frac{q - \rho(S)}{\rho(R) - \rho(S)}, \quad K = \frac{\rho(S)}{\rho(R) + \rho(S)},$$

we can represent P as a function of $\rho(R)$ and $\rho(S)$ for any given q . In particular, P takes the values shown in Figure 1 (see the appendix for the construction of the Figure).

¹¹As Shachar and Nalebuff (1999) among others point out, party spending is effective in driving voters to the polls in several ways: campaign spending decreases the voters' cost of acquiring information, it decreases the direct cost of voting, it increases the cost of abstaining, and it signals the closeness and importance of the alternatives at stake.

¹²See Section 6 for the case in which $\rho(0) > 0$.

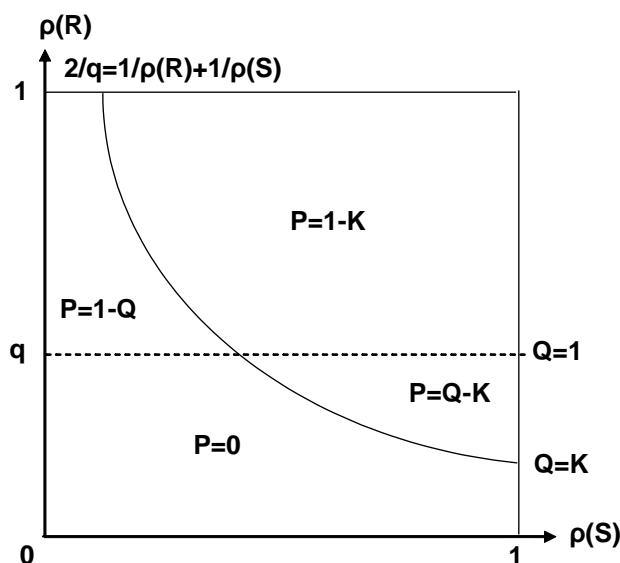


Figure 1

Note that P is continuous in its arguments on the whole space $(\rho(S), \rho(R)) \in [0, 1]^2$. If $q = 0$, i.e. there is no participation quorum requirement, the curved line collapses on the axes, and $P = 1 - K$ on the whole space. In this region the probability that the reform issue is selected is only a function of parties' mobilization efforts. However, as q increases, the curved line moves northeast continuously, and below the curved line the probability that the reform issue is selected also depends on the quorum requirement. Clearly, whenever $\rho(R) < q$, and $\rho(S)$ is sufficiently small, the reform issue cannot prevail in the referendum. When $q = 1$ the curved line collapses to the point $(1, 1)$ and P converges to zero.

Before characterizing the equilibria of this game, it might be useful to consider a simple numerical example that illustrates our results.

2.1. An Example. Consider the case in which $B = 4$, and $\rho(x) = 1 - e^{-x}$. As we will prove in the next section, depending on the level of q , this game has only three possible Nash equilibria in pure-strategies which are represented in Figure 2: two symmetric equilibria denoted by O and C , and an asymmetric equilibrium denoted by A .

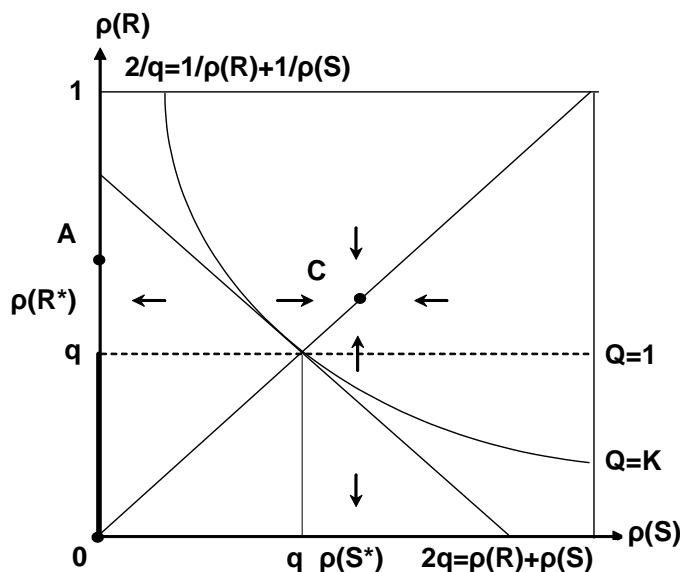


Figure 2

Table 1 below summarizes the equilibrium level of parties' spending, expected turnout $E(T)$, and expected probability $P(q)$ that the reform issue wins a majority of votes for different levels of participation quorum q .

Table 1

q	Equilibrium	S	R	$E(T)$	$P(q)$
< 0.1634	C	0.69	0.69	0.50	0.50
$= 0.25$	A	0.00	0.96	0.31	0.59
$= 0.40$	A	0.00	1.19	0.35	0.43
> 0.4910	O	0.00	0.00	0.00	0.00

If the quorum requirement is sufficiently low, the probability of winning will not depend on q . Given that there are no asymmetries, it is not surprising that the equilibrium will be symmetric and, in the unique positive spending Nash equilibrium, parties spend $R = S = \ln 2 \simeq 0.69$. Moreover, the expected turnout equals 0.5, the expected probability of issue r being selected equals 0.5, and the expected profit of each party equals $2 - \ln 2 \simeq 1.3$. This equilibrium exists if and only if $q < 0.1634$, and is represented by point C in Figure 2.

Suppose now that $q = 0.25$. In this case the status quo party will exploit the asymmetry introduced by the participation quorum requirement. In fact, by choosing not to mobilize its supporters, i.e. by choosing $S = 0$, party s might be successful in “busting” the quorum at zero

cost. In this case, party r 's probability of winning is either $1 - F(Q)$ or 0 depending on $1 - e^{-R}$ being greater than or smaller than q (see Figure 1). In the unique pure-strategy Nash equilibrium outcome party r spends $R \simeq 0.96$, and party s spends $S = 0$. In this “quorum busting” equilibrium the expected turnout drops to 0.31, the expected probability of issue r being selected is strictly *bigger* than 0.5, and expected profits are such that $E(\pi_r) > 1.3 > E(\pi_s)$.

If the participation quorum requirement is higher, say $q = 0.4$, in equilibrium party r increases its spending to $R \simeq 1.19$, while party s -a fortiori- will spend $S = 0$. The resulting expected turnout is 0.35, the expected probability of issue r being selected is now strictly *smaller* than 0.5, and expected profits are such that $E(\pi_s) > 1.3 > E(\pi_r)$. The asymmetric pure-strategy Nash equilibrium exists if and only if $q \in (0.1655, 0.491)$, and it is represented by the point A in Figure 2.

Finally when q is high enough, since party r expected profits in the asymmetric equilibrium are clearly decreasing in q , and they become eventually negative, the unique equilibrium will be trivially $R = S = 0$. This equilibrium is represented by point O in Figure 2, and it is the unique equilibrium when $q > 0.491$.

Several interesting observations can be derived from this numerical example.

First, the introduction of a participation quorum requirement is usually motivated by the idea of validating the referendum results only if participation is high enough. However, a participation quorum may generate in equilibrium less participation if voters that turn out to the polls respond to parties' mobilization efforts. The drop in participation can be so large that the equilibrium expected turnout may be smaller than the quorum itself. Moreover, this “quorum paradox” is not a trivial consequence of the no spending equilibrium. In fact, it may also occur in the asymmetric equilibrium, i.e. when the quorum requirement is such that at least one party finds it profitable to devote resources in mobilizing voters. Interestingly enough, the quorum paradox could occur even for values of B and q such that the expected turnout that results in equilibrium holding B constant and removing the quorum requirement is greater than q . Stated differently, there are values of B such that the symmetric spending profile cannot be supported as an equilibrium even if the expected turnout that generates it is greater than q .

Second, a quorum requirement does not necessarily imply a bias for the status quo issue. In fact, the expected probability that the status quo is replaced may be higher in the presence of a participation quorum requirement as compared to the case where the quorum requirement

is absent. Moreover, the increase in the expected probability that the status quo is overturned may be associated with a smaller amount of total spending, (e.g., see the case of $q = 0.25$ in the numerical example).

Finally, for any level of the payoff B there always exists a range of q where there is no equilibrium in pure-strategy. The intuition is simple. There is a level of q such that party s is indifferent between playing $S = R$ when r is playing R (i.e., the symmetric equilibrium strategy profile), and trying to bust the quorum by playing $S = 0$. However, the symmetric spending strategy cannot be a best response to $S = 0$. In fact, when q is positive and $S = 0$, party r has a higher marginal return from spending. An increase in party r 's spending breaks party s 's indifference. Therefore, for a subset of (q, B) , the game parties are playing can be seen as a ‘‘matching pennies’’ game. We will show that there is at least one natural mixed strategy equilibrium in that region which smoothens the transition from the equilibrium in C to the equilibrium in A .

In the next section we characterize the Nash equilibria of this game for all values of the exogenous parameters (q, B) .

3. EQUILIBRIUM CHARACTERIZATION

We start by focusing on pure-strategy Nash equilibria. There are only three possible equilibria in pure-strategies: two symmetric equilibria, and an asymmetric equilibrium. These equilibria never overlap so we never have multiple equilibria in pure-strategies. However, as we will show, a pure-strategy equilibrium may not exist. Our first proposition characterizes the unique symmetric positive-spending Nash equilibrium of this game.

Proposition 1. *There exists a \underline{q} such that if $q \leq \underline{q}$ a unique positive-spending Nash equilibrium exists in which parties spend $S^* = R^* > 0$, where R^* solves*

$$\frac{\rho'(R^*)}{4\rho(R^*)} = \frac{1}{B},$$

and

$$(1) \quad \underline{q} = \left(\frac{1}{2} - \frac{R^*}{B} \right) \rho(R^*).$$

The proof is in the appendix.¹³ The spending profile (S^*, R^*) is an equilibrium as long as q is such that party s does not want to deviate to $S = 0$.

If instead $q > \underline{q}$, then the only candidate pure-strategy equilibrium has party s spending 0. Let $(0, \widehat{R})$ denote the quorum busting asymmetric spending profile: party s spends zero because its optimal strategy is trying to keep the total turnout below quorum, whereas party r spends a positive amount \widehat{R} in an effort to mobilize enough supporters to push the turnout above quorum with some probability. A quorum busting spending profile $(0, \widehat{R})$ is an equilibrium if and only if the following conditions hold:

$$\begin{aligned}\pi_r(0, \widehat{R}) &\geq \pi_r(0, 0) = 0 \\ \pi_s(0, \widehat{R}) &\geq \pi_s(\widehat{S}, \widehat{R})\end{aligned}$$

where \widehat{R} and \widehat{S} are functions of (q, B) implicitly defined by

$$(2) \quad \widehat{R} = \arg \max \left(B \left(1 - \frac{q}{\rho(\widehat{R})} \right) - \widehat{R} \right)$$

$$(3) \quad \widehat{S} = \arg \max \left(B \frac{\rho(\widehat{S})}{\rho(\widehat{R}) + \rho(\widehat{S})} - \widehat{S} \right),$$

and \widehat{S} is the best response of party s to party r spending \widehat{R} inside the $P = 1 - K$ region.¹⁴ Intuitively, a quorum busting equilibrium can exist if and only if q is not so high to make party r 's profits negative, and q is not too small to make party s worse off by spending zero than spending \widehat{S} . Note that \widehat{R} and \widehat{S} both depend on B and q . By defining

¹³Note that, since \underline{q} is always smaller than $1/2$, Proposition (1) implies that if the voting quorum is set at $q = 1/2$, the symmetric spending profile cannot be an equilibrium. This is due to the simplifying assumption that there are no ‘‘strong partisan’’ voters, i.e., $\rho(0) = 0$. If instead $\rho(0) > 0$, and some voters vote even if parties are not mobilizing, it is straightforward to show that

$$\underline{q} = \left(\frac{1}{2} - \frac{R^*}{B} \right) \rho(R^*) + \rho(0) \left(\frac{1}{2} + \frac{R^*}{B} \right),$$

and the symmetric equilibrium can survive even if $q > \frac{1}{2}$.

¹⁴Note that the assumptions on $\rho(\cdot)$ guarantee that \widehat{R} and \widehat{S} are well-defined.

the two thresholds on q as

$$(4) \quad \bar{q}(B) : \pi_r(0, \hat{R}) = \pi_r(0, 0)$$

$$(5) \quad \hat{q}(B) : \pi_s(0, \hat{R}) = \pi_s(\hat{S}, \hat{R}),$$

we have the following proposition,

Proposition 2. *The quorum busting profile $A \equiv (0, \hat{R})$ is an equilibrium if and only if $q \in [\hat{q}, \bar{q}]$.*

Proving this proposition amounts to showing that for any B the thresholds \bar{q} , and \hat{q} are uniquely defined. We postpone the argument to the proof of Proposition (4). As it will be clear later, the quorum busting equilibrium may not always exist, since for low values of B we may have that $\hat{q} > \bar{q}$.

Finally, the zero spending profile $O \equiv (0, 0)$ is an equilibrium if and only if it is optimal for r to spend zero when s spends zero, that is $0 = \pi_r(0, 0) \geq \pi_r(0, \hat{R})$. For this to be true q has to be high enough. In fact, it is immediate to see that for $q = 0$ the zero spending profile cannot be an equilibrium since, for any B , party r can spend an arbitrarily small amount and increase its probability of winning discretely from one half to one. Moreover, for all $\rho(R) < q$, $S = 0$ is a dominant strategy for s (the boldfaced line in Figure 2) as s can guarantee itself that $P = 0$. In other words, s can win the referendum with probability one at no cost attaining the maximum possible payoff $\pi_s = B$. Hence, no strictly positive equilibrium spending profile can be in the interior of the $\rho(R) < q$ region. If \bar{q} is uniquely defined, the next proposition follows immediately.

Proposition 3. *The zero spending profile $O \equiv (0, 0)$ is an equilibrium if and only if $q \in [\bar{q}, 1]$.*

In order to completely characterize the pure-strategy equilibria of this game we need to show that the thresholds on q implicitly defined by (1), (4), and (5) are unique. This is what we show in the next proposition that also shows that pure-strategy equilibria never coexist, and that they may fail to exist.

Proposition 4. *For all B the thresholds $\underline{q}(B)$, $\hat{q}(B)$, and $\bar{q}(B)$ are uniquely defined. Moreover, $\underline{q} < \hat{q}$, and $\underline{q} < \bar{q}$.*

The asymmetric equilibrium may not exist for low B , in fact it can be that $\hat{q} > \bar{q}$ and the interval $[\hat{q}, \bar{q}]$ disappears. However, this never occurs if B is large enough.

Since $\underline{q} < \hat{q}$, there is always a region of non-existence in pure-strategies. However, there is a natural mixed strategy equilibrium in that region which smoothens the transition from the pure-strategy symmetric equilibrium to the non-zero pure-strategy asymmetric equilibrium. Let (\tilde{S}, \tilde{R}) be defined as

$$\tilde{S} = \arg \max \left(B \frac{\rho(S)}{\rho(S) + \rho(\tilde{R})} - S \right)$$

$$\pi_s(0, \tilde{R}) = \pi_s(\tilde{S}, \tilde{R}).$$

In words, \tilde{S} is the best response to \tilde{R} in the $P = 1 - K$ region, and \tilde{R} is the level of spending by party r such that party s is indifferent between spending \tilde{S} and 0.¹⁵ The next proposition characterizes the mixed strategy equilibrium.

Proposition 5. *If $q \in (\underline{q}, \hat{q})$, there exists a mixed strategy equilibrium where r plays the pure-strategy \tilde{R} while s plays the mixed strategy*

$$S = \begin{cases} 0 & \text{with prob. } \alpha \\ \tilde{S} & \text{with prob. } 1 - \alpha, \end{cases}$$

where

$$\alpha = \frac{\frac{1}{B} - \frac{\rho'(\tilde{R})\rho(\tilde{S})}{(\rho(\tilde{R}) + \rho(\tilde{S}))^2}}{\frac{q\rho'(\tilde{R})}{(\rho(\tilde{R}))^2} - \frac{\rho'(\tilde{R})\rho(\tilde{S})}{(\rho(\tilde{R}) + \rho(\tilde{S}))^2}}.$$

Moreover, $\alpha(\underline{q}) = 0$, $\alpha(\hat{q}) = 1$.

There might be other mixed strategy equilibria. However, an appealing feature of the equilibrium described in proposition (5) is that as q increases from \underline{q} to \hat{q} we move gradually and continuously from the pure-strategy symmetric equilibrium (S^*, R^*) to the pure-strategy asymmetric equilibrium $(0, \hat{R})$. We summarize the set of pure-strategy Nash equilibria and the mixed strategy equilibrium described above as a function of q and B in Figure 3 below, (see the appendix for the construction of the Figure).

¹⁵Showing that \tilde{R} and \tilde{S} are indeed well-defined is part of the proof of Proposition 5.

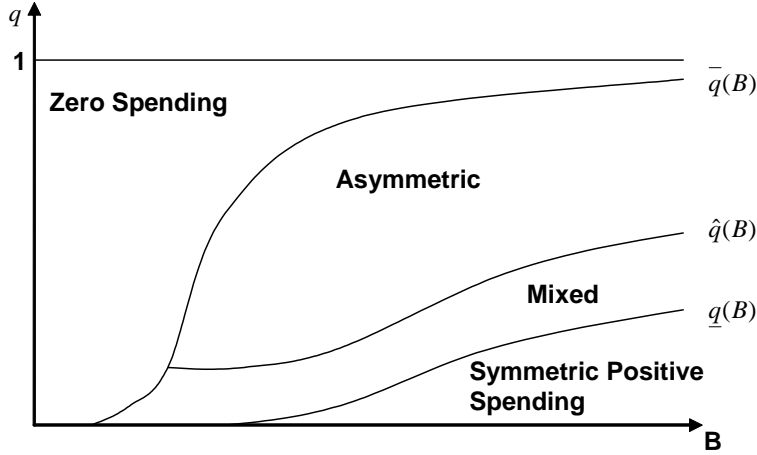


Figure 3

In the next section we study how the expected turnout, the expected probability of winning, and the expected profits change in the different equilibria as a function of q and B .

4. EXPECTED TURNOUT AND PROBABILITY OF REFERENDUM APPROVAL

In this section we show that the conclusions drawn from the example in Section 2 are general. The introduction of a quorum requirement motivated by the idea of validating the referendum results only if participation is high enough, may generate in equilibrium less participation. Moreover, a quorum requirement does not necessarily imply a bias for the status quo issue. Indeed, when $q \in (\hat{q}, \bar{q})$, either the equilibrium expected turnout is smaller than the quorum, or the equilibrium probability that the reform issue is adopted is strictly bigger than the case where the quorum requirement is absent.

We start analyzing how the expected turnout $E(T)$ varies depending on which region of the parameter space we are in.¹⁶

In the positive spending symmetric equilibrium region, expected turnout is constant in q , increasing in B , and always above q . Namely, when $q \in (0, \underline{q})$, we have that $E(T) = \rho(R^*) > q$. Clearly, the symmetric spending profile cannot be supported in equilibrium if the expected turnout that generates it is not high enough to meet the quorum, i.e.,

¹⁶In this section we will assume that B is such that $\hat{q} < \bar{q}$. This is always true when B is large enough as it is shown in the appendix (Construction of Figure 3).

when $q > \rho(R^*)$. However, if the quorum requirement is in the interval $q \in (\underline{q}, \rho(R^*))$, the symmetric spending profile cannot be supported in equilibrium even if the expected turnout that generates it is greater than q . This is precisely what we call the “quorum paradox”: in equilibrium the expected turnout exceeds the participation quorum only if the requirement is not imposed.

If $q \in (\underline{q}, \hat{q})$, and parties are playing the mixed strategy equilibrium described in Proposition (5) above, we have that the expected turnout is equal to

$$E(T) = \frac{\rho(\tilde{R})}{2} + (1 - \alpha(q)) \frac{\rho(\tilde{S})}{2},$$

and satisfies the properties that are summarized in the next claim.

Claim 1 *If $q \in (\underline{q}, \hat{q})$ and parties are playing the mixed strategy equilibrium of Proposition (5), then $E(T) > q$, and*

$$\begin{aligned} \lim_{q \rightarrow \underline{q}} E(T) &= \rho(R^*) > \frac{\rho(\hat{R}(\hat{q}))}{2} = \lim_{q \rightarrow \hat{q}} E(T) \\ \lim_{q \rightarrow \underline{q}^+} \frac{\partial E(T)}{\partial q} &< 0. \end{aligned}$$

The proof is in the appendix. Claim (1) shows that the expected turnout in the mixed equilibrium is smaller than the expected turnout in the symmetric positive spending equilibrium and it is decreasing in q , for some $q \in (\underline{q}, \hat{q})$.¹⁷

If $q \in (\hat{q}, \bar{q})$, i.e., in the region where parties are playing the asymmetric equilibrium, we have that $E(T) = \rho(\hat{R})/2$, the expected turnout is increasing in q and B , and satisfies the properties that are summarized in the next claim.

Claim 2 *If $q \in (\hat{q}, \bar{q})$, then $E(T)|_{q=\hat{q}} > \hat{q}$. in addition, there exist \bar{B} and q' such that for $B > \bar{B}$, $E(T) > q$ if and only if $q < q'$.*

In other words, when the benefit is high enough, there always exists an interval where q belongs to, such that the equilibrium expected turnout is strictly smaller than the quorum itself.

Finally for $q \in (\bar{q}, 1)$, we have that $E(T) = 0$. Figure 4 below summarizes the results.

¹⁷In the case of $\rho(x) = 1 - e^{-x}$ the expected turnout is decreasing in q for any $q \in (\underline{q}, \hat{q})$.

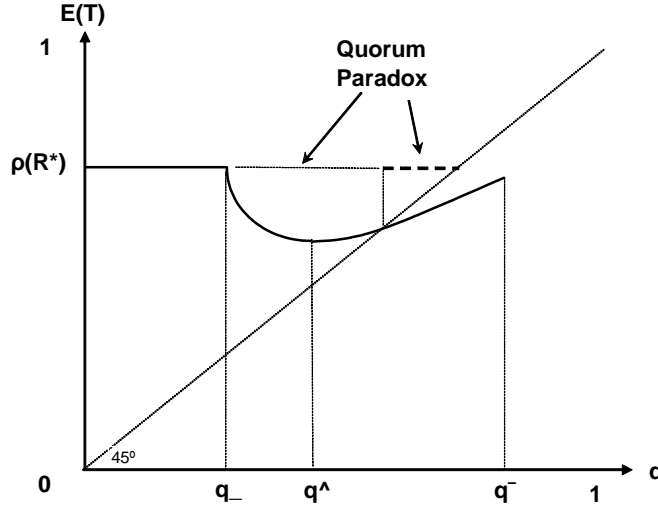


Figure 4

As Figure 4 shows, when $q > \underline{q}$, the introduction of a quorum requirement decreases the expected turnout in equilibrium. More importantly, in the region represented by the dashed boldfaced segment, the equilibrium expected turnout is smaller than the quorum itself (even if, the expected turnout that results in equilibrium holding B constant and removing the quorum requirement is greater than q). Claim (2) guarantees that for B high enough such a region always exists.

Similarly to the expected turnout, the expected probability $P(q)$ that the reform issue wins a majority of votes varies depending on which region of the parameter space we are in. In particular, $P(q)$ is continuous for $q \neq \bar{q}$, and it is equal to:

$$P(q) = \begin{cases} \frac{1}{2} & q \in (0, \underline{q}) \\ \alpha \left(1 - \frac{q}{\rho(\hat{R})}\right) + (1 - \alpha) \left(\frac{\rho(\hat{R})}{\rho(\hat{R}) + \rho(\hat{S})}\right) & q \in (\underline{q}, \hat{q}) \\ 1 - \frac{q}{\rho(\hat{R})} & q \in (\hat{q}, \bar{q}) \\ 0 & q \in (\bar{q}, 1). \end{cases}$$

For $q \in (\hat{q}, \bar{q})$, the expected probability $P(q)$ is decreasing in q , as we proved in Lemma 1 in the appendix. Also, it must be that $P(\hat{q}) > 1/2$. In fact, by the definition of \hat{q} , the status quo party is indifferent between playing $S = 0$ and \hat{S} at $q = \hat{q}$. Hence, since its profits are equal, the chance of winning must be higher in the case s is spending a positive amount \hat{S} . Namely,

$$1 - P(\hat{q}) < 1 - P(\hat{S}, \hat{R}) < \frac{1}{2},$$

where the last inequality comes from the fact that, when $q \geq \hat{q}$, $\hat{S} < R^* < \hat{R}$.

Finally, if $B > \bar{B}$, since $\rho(\hat{R}(\bar{q}))/2 < \bar{q}$, it follows that

$$P(\bar{q}) = 1 - \frac{\bar{q}}{\rho(\hat{R}(\bar{q}))} < \frac{1}{2} = P(q'),$$

where \bar{B} and q' are defined in Claim (2).

We conclude this section by analyzing parties' expected profit $E(\pi)$ as a function of q . If $q \leq \underline{q}$, parties' expected profits are equal and do not depend on q . Namely, $E(\pi)|_{q \leq \underline{q}} = B/2 - R^*$. If instead $q \in (\underline{q}, \hat{q})$, it is immediate to show that $E(\pi_s) < E(\pi)|_{q \leq \underline{q}}$. Moreover, if $\tilde{R} < 2R^*$ then $E(\pi)|_{q \leq \underline{q}} < E(\pi_r)$.¹⁸ For $q \in (\hat{q}, \bar{q})$, when parties are playing the asymmetric pure-strategies equilibrium, we have that

$$E(\pi_s) = B \frac{q}{\rho(\hat{R})}, \quad E(\pi_r) = B \left(1 - \frac{q}{\rho(\hat{R})} \right) - \hat{R}.$$

Not surprisingly, the expected profits of the status quo party are strictly smaller than those of the reform party when $q = \hat{q}$, and they are increasing in the quorum requirement. The expected profits of the reform party are instead decreasing in q . Finally, for $q \in (\bar{q}, 1)$, the reform issue cannot win, expected profits equal actual profits, and $\pi_s = B > 0 = \pi_r$.

5. NORMATIVE ANALYSIS

A common rationale for the use of a participation quorum requirement is to make sure that, for a referendum to be valid, there is enough popular "interest" in the issue at stake. Since this interest is typically associated with the voter turnout, the quorum requirement should take into account that, if voters responds to parties' mobilization efforts, turnout is endogenous. In this section we address three issues. First, we show that in order to induce an expected equilibrium turnout of q , the participation quorum requirement should be set at a level that is *less than half* of q . Second, we try to assess the welfare gains/losses of introducing a participation quorum requirement relative to the case in which the quorum is absent. Third, we argue that a super majority

¹⁸This follows from

$$E(\pi_r) + E(\pi_s) = B - \tilde{R} > 2E(\pi)|_{q \leq \underline{q}}.$$

requirement to overturn the status quo is never equivalent to a participation quorum (in the sense of yielding the same Nash equilibrium outcomes).

Suppose that q is the expected equilibrium turnout that we want to induce in a given referendum. Ideally, a spending profile that is an equilibrium without the quorum requirement and yields an expected turnout above q should remain an equilibrium when the quorum requirement is imposed. This occurs if and only if the zero spending strategy is not a profitable deviation for the party supporting the status quo. In other words, to avoid the quorum paradox we have described in the previous section, the quorum busting strategy (which is always available to party s), should be used only when the interest in the issue at stake is low enough that the expected turnout without a quorum requirement is below q .

Recall from the previous section that in the symmetric positive spending equilibrium, the level of the exogenous benefit B determines the symmetric equilibrium spending $R^*(B)$ and the expected turnout $E(T)$. For any q , there exists a threshold value B_q below which, in the positive spending equilibrium, the expected turnout is below q . Namely, if $B < B_q$ then $E(T) < q$ in the positive spending equilibrium. This threshold is implicitly defined by

$$\rho(R^*(B_q)) = q.$$

Ideally, only when $B < B_q$ the status quo party should play $S = 0$. Since for given q the zero spending strategy is the best response of the status quo party for values of B such that

$$\left(\frac{1}{2} - \frac{R^*}{B}\right) \rho(R^*) \leq q,$$

we can map any participation quorum q into what we call an *effective* participation quorum q^e , where

$$q^e = \left(\frac{1}{2} - \frac{R^*(B_q)}{B_q}\right) q.$$

Therefore, in order to induce an expected equilibrium turnout of q , the participation quorum requirement should be set at q^e instead. This policy achieves two goals. First, the status quo party plays $S = 0$ whenever $B < B_q$ (which would imply $E(T) < q$ in the positive spending equilibrium). Second, the positive spending equilibrium survives if $B > B_q$ (which implies $E(T) > q$). The effective quorum target q^e corrects for the endogeneity of parties' mobilization efforts and is *less than half* of the original participation quorum q .

For example, in the case of $\rho(R) = 1 - e^{-R}$ it is easy to obtain that

$$q^e = \frac{2q + (1 - q) \ln(1 - q)}{4} < \frac{q}{2}.$$

In the case of $q = 0.4$, Figure 5 below shows how an effective quorum of $q^e(0.4) = 0.12$ can avoid the quorum paradox by inducing an expected turnout smaller than $q = 0.4$ only when expected turnout would have been below quorum anyway.

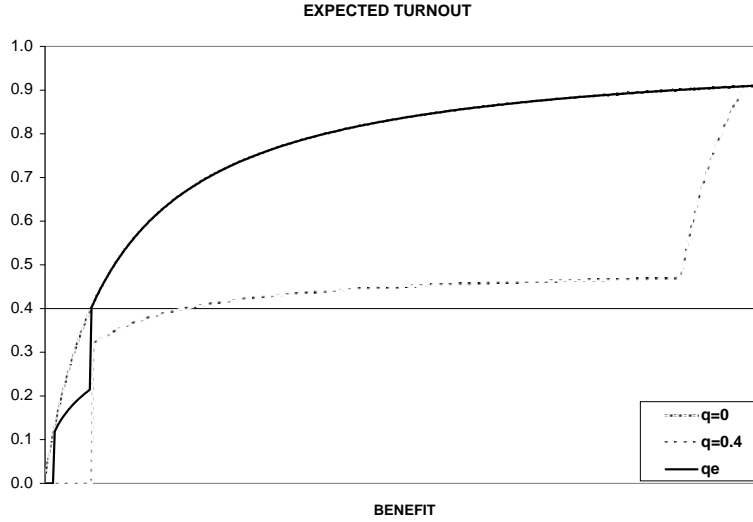


Figure 5

We now move to the welfare analysis. Since this is a model where voters are mobilized by the effort of political parties, welfare implications might be different depending on whether we focus on parties' expected profits or on the welfare of the voters. We have already analyzed in the previous section how parties' profits change with q . Here we focus on voters' welfare and analyze first the revenue side and then the cost side.

Suppose that every voter supporting the winning issue in the referendum obtains a payoff of B , and normalize to 0 the payoff of the voters supporting the losing issue. Define $r_q \in (0, 1)$ as the threshold such that if a proportion of voters $r > r_q$ prefers the reform issue than this issue is selected, and note that this threshold depends on the equilibrium played. For any realized voters' preference split \tilde{r} , let the ex-post revenue be

$$w_q(\tilde{r}) = B\tilde{r}I(\tilde{r} > r_q) + B(1 - \tilde{r})I(\tilde{r} < r_q),$$

where I is the indicator function.

When $q = 0$, in the unique positive spending symmetric equilibrium, the issue supported by the majority of the voters always wins, i.e. $r_0 = 1/2$. However, when $q > 0$, generically we have that $r_q \neq 1/2$, i.e. the issue that prevails in the referendum may not be the one supported by the majority of the citizens. To see this, note that in the asymmetric equilibrium, for uniformly distributed \tilde{r} , we have that $r_q = 1/2$ if and only if $E(T) = q$. In other words, the majority rule ($r_q = 1/2$) is implemented in the asymmetric equilibrium if and only if $E(T) = q$. The latter equality will not be satisfied generically. If in equilibrium $E(T) < q$, the turnout is more likely to be below quorum than above quorum. Hence, a super majority of reform-supporters ($r_q > 1/2$) is needed for the turnout to reach the quorum. If instead the proportion of voters in favor of the reform is a barely majority $\tilde{r} \in (1/2, r_q)$, the status quo will prevail in the referendum due to a lack of quorum. The opposite scenario occurs if in the asymmetric equilibrium we have $E(T) > q$. In this case the status quo is overturned despite being the preference of the majority.

In sum, it is easy to see that for any realized \tilde{r} the ex-post voters' revenue is maximized at $r_q = 1/2$, i.e. $w_q(\tilde{r}) \leq w_0(\tilde{r})$ which implies that

$$E(w_q(\tilde{r})) \leq E(w_0(\tilde{r})).$$

Relative to the case when $q = 0$, a participation quorum requirement never leads to an ex-ante revenue gain and, whenever quorum busting takes place, it causes generically an ex-ante revenue loss because the issue supported by the majority does not always prevail in the referendum. Moreover, the expected revenue loss due to the existence of a participation quorum requirement easily extends to more general assumptions on the distribution of \tilde{r} .

On the voting cost or expenditure side the picture that we obtain is less clear. For example, when B is high enough and $\rho(\cdot)$ belongs to the class of CARA or CRRA functions, it can be shown that overall party expenditures in the symmetric equilibrium are larger than party expenditures in the asymmetric equilibrium for any q such that an asymmetric equilibrium exists, i.e. $2R^* > \hat{R}(q)$. Indeed, if $-\rho''(x)/\rho'(x) = k$, we have that

$$\lim_{B \rightarrow \infty} \frac{\hat{R}(\bar{q})}{R^*} = \lim_{B \rightarrow \infty} \frac{4 + Bk}{2 + \frac{\bar{q}B}{\rho(\hat{R}(\bar{q}))}k} \leq 2,$$

since for B large we know that $\bar{q} > \rho(\widehat{R}(\bar{q}))/2$. The same is true if $-x\rho''(x)/\rho'(x) = k$.¹⁹

In conclusion, the overall effect on voters' welfare of the introduction of a participation quorum requirement is ambiguous. It surely decreases welfare by misrepresenting the will of the majority. However, under some assumptions, it might also reduce the total cost of voting.

A natural question is whether there is a super-majority requirement q_s that is equivalent (i.e., yields the same Nash equilibrium outcomes) to a participation quorum q . The answer is no: there is no mapping between q and q_s , as this mapping depends on the value of B . Namely, for any q , let r_q be defined as above. In the asymmetric equilibrium the threshold r_q decreases with B . Hence, for any given q and B , a quota-rule $q_s = r_q$ is indeed equivalent to a participation quorum q . However, for any given q the value of q_s depends crucially on B , which means that no quota-rule q_s is equivalent to a quorum limit q for all B . For instance, a participation quorum of 30% cannot be implemented by any fixed quota-rule q_s . In fact, the lower the value of B , the (weakly) higher the quota-rule q_s that is needed to make the quota-rule equilibrium outcomes match the participation quorum equilibrium outcomes.

6. EXTENSIONS

In this section we consider three natural generalizations of the basic model. First, we consider the case in which parties' payoffs are heterogeneous. Second we relax the assumption that the distribution of r is uniform, and allow for an asymmetric distribution. Finally, we explore the case in which there is an approval quorum requirement instead of a participation quorum.

Consider the case in which parties receive different payoffs B_s and B_r if their preferred alternative wins the referendum. In particular, we assume that $B_r = B$ and $B_s = \gamma B$ with $\gamma > 0$. Given this simple specification, the objective function of the reform party is unchanged, while the status quo party's objective function becomes $\pi_s(S, R) = \gamma B(1 - P) - S$. In this case, we can show by continuity, that for γ close to 1, there exists a unique equilibrium with positive spending such

¹⁹In this case

$$\lim_{B \rightarrow \infty} \frac{\widehat{R}(\bar{q})}{R^*} = \lim_{B \rightarrow \infty} \frac{4 + BkR^*}{2 + \frac{\bar{q}B}{\rho(\widehat{R}(\bar{q}))}k\widehat{R}(\bar{q})} \leq 2.$$

that (S^*, R^*) satisfies

$$\frac{\rho'(R^*)}{\rho(R^*)} = \gamma \frac{\rho'(S^*)}{\rho(S^*)} = \frac{1}{B},$$

and since $h(\cdot) \equiv \rho'(\cdot)/\rho(\cdot)$ is a decreasing function, it must be that $S^* > R^*$ if and only if $\gamma > 1$. Note that using the last equation we can express R^* as a function of S^* , i.e. $R^* = h^{-1}(\gamma h(S^*)) \equiv g(S^*)$, where R^* is increasing in S^* , and R^* is increasing in γ if and only if S^* is increasing in γ .²⁰ Since it can be shown that

$$\lim_{\gamma \rightarrow 1} \frac{\partial S^*}{\partial \gamma} > 0,$$

introducing a small asymmetry between parties' payoff increases parties' spending. As we have already shown before, the strategy profile (S^*, R^*) is also an equilibrium for $q > 0$ if and only if the status quo party does not have an incentive to deviate to zero, i.e. $\pi_s(S^*, R^*) \geq \pi_s(0, R^*)$. This is true if and only if $q \in [0, \underline{q}_\gamma]$, where \underline{q}_γ is increasing in γ for γ close to 1.²¹ Therefore, starting from symmetric payoffs, an increase (decrease) in the payoff of the status quo party, i.e. $\gamma > (<) 1$, enlarges (reduces) the region in which a positive spending equilibrium exists. Intuitively, a smaller payoff for the status quo party triggers the deviation to $S = 0$ for lower levels of the participation quorum requirement. Finally, note that the value of $\widehat{R}(q)$ does not depend on γ , nor the value of \bar{q} . Since \widehat{q} is such that $C(\widehat{q}, \gamma B) = 0$, and $C(q, \gamma B)$ is decreasing in q and increasing in γ , it follows that \widehat{q} is also increasing in γ . Therefore, an increase (decrease) in the payoff of the status quo party enlarges (reduces) the region in which the asymmetric equilibrium exists.

We now move to consider briefly what happens when the distribution of r is not uniform, and in particular it is not symmetric. Let the distribution function of r be $F(r)$, with associated density function

²⁰In fact,

$$\frac{dR^*}{d\gamma} = \frac{\gamma h'(S^*)}{h'(\gamma h(S^*))} \frac{\partial S^*}{\partial \gamma},$$

and $\gamma h'(S^*)/h'(\gamma h(S^*)) > 0$.

²¹In particular

$$\underline{q}_\gamma = \left(\frac{\rho(S^*)}{\rho(R^*) + \rho(S^*)} - \frac{S^*}{\gamma B} \right) \rho(R^*).$$

$f(r)$. In this case,

$$\begin{aligned}\pi_r(S, R) &= B \left(1 - F \left(\frac{\rho(S)}{\rho(R) + \rho(S)} \right) \right) - R \\ \pi_s(S, R) &= BF \left(\frac{\rho(S)}{\rho(R) + \rho(S)} \right) - S,\end{aligned}$$

and, in the symmetric equilibrium, it must be that

$$f \left(\frac{1}{2} \right) \frac{\rho'(R^*)}{(\rho(R^*))^2} = \frac{1}{4B}.$$

Hence, parties' spending and expected turnout will be higher (smaller) than in the case in which r is distributed uniformly if and only if $f \left(\frac{1}{2} \right) > (<) 1$. Clearly, if $f \left(\frac{1}{2} \right) = 1$, nothing changes with respect to the uniform case. Intuitively, the higher is the mass of nearly indifferent voters, the more uncertain is the outcome of the referendum. This leads to a higher spending competition between parties, and therefore to a higher expected turnout. It is also immediate to see that the expected probability that the status quo is overturned is equal to $1 - F \left(\frac{1}{2} \right)$ and it is higher the more left-skewed is the distribution of r . Like before, the strategy profile (S^*, R^*) is an equilibrium if and only if $q \in [0, \underline{q}_r]$, where

$$\underline{q}_r = F^{-1} \left(F \left(\frac{1}{2} \right) - \frac{R^*}{B} \right) \rho(R^*).$$

Note that \underline{q}_r is larger than \underline{q} if and only if

$$F \left(\frac{1}{2} \right) - \frac{R^*}{B} > F \left(\frac{1}{2} - \frac{R^*}{B} \right).$$

In the special case of $f(r) = 2(1-r)\alpha + 2r(1-\alpha)$, where $\alpha \in (0, 1)$, the above expression is true if and only if $\alpha > 1/2$.²² Hence, in this particular example, \underline{q}_r is larger than \underline{q} if and only if $f(r)$ is right-skewed. In other words, when on average there is a majority of voters in favor of the status quo issue, the status quo party will switch later (i.e., for higher values of q) to the quorum busting strategy. Intuitively, given our mobilization technology, spending is more effective in mobilizing voters the higher the proportion of supporters a party expects to have. Therefore, if the status quo party is indifferent between $S > 0$ and $S = 0$ at \underline{q} in the case of a society split evenly, it is strictly better off mobilizing when it expects to have a majority.

²²Note that when $\alpha = 0$, $f(r) = 2r$, when $\alpha = 1$, $f(r) = 2(1-r)$, and when $\alpha = \frac{1}{2}$ we have the uniform distribution.

In the asymmetric equilibrium, it is a matter of simple algebra to show that spending $\left(\widehat{R}\right)$ and expected turnout are higher than in the case in which r is distributed uniformly if and only if $f\left(q/\rho\left(\widehat{R}\right)\right) > 1$. In the special case of $f(r) = 2(1-r)\alpha + 2r(1-\alpha)$, it follows that if $\alpha > (<) \frac{1}{2}$, $f\left(q/\rho\left(\widehat{R}\right)\right) > 1$ if and only if $q < (>) \rho\left(\widehat{R}\right)/2$. This means that when the distribution of r is left-skewed (i.e., $\alpha < \frac{1}{2}$), expected turnout is higher for low values of q such that the asymmetric equilibrium exists (since $\widehat{q} > \rho\left(\widehat{R}(\widehat{q})\right)/2$), and it is smaller for high values of q .

Finally, note that our analysis is qualitatively unchanged if we relax the assumption of $\rho(0) = 0$, as long as $\rho(0)$ is small.

We conclude this section with a comparison between an *approval* quorum requirement and the participation quorum requirement we have considered so far. Suppose that, in order to win the referendum and replace the status quo issue, the reform issue must collect more votes than the status quo issue *and* the proportion of voters in favor of the reform must be above some threshold $m \in [0, \frac{1}{2}]$.²³ Then, we can show that

$$P = \Pr((v_R > m) \cap (v_R > v_S)) = \begin{cases} (1 - K) & \text{if } M < \frac{1}{m} \\ (1 - W)^+ & \text{if } M \geq \frac{1}{m} \end{cases},$$

where

$$M = \frac{1}{\rho(R)} + \frac{1}{\rho(S)},$$

and

$$(1 - W)^+ = \begin{cases} \left(1 - \frac{m}{\rho(R)}\right) & \text{if } \rho(R) > m \\ 0 & \text{if } \rho(R) \leq m. \end{cases}$$

In particular, P takes the values shown in Figure 6 (see the appendix for the construction of the Figure). We have three probability regions. Likewise in Figure 1, if $m = 0$ the curved line collapses on the axes, and $P = 1 - K$ on the whole space. As m increases, the curved line moves northeast continuously, and below the curved line the probability that the reform issue is selected also depends on the majority requirement.

²³The best known case of approval quorum is the 40 per cent rule (or Cunningham Amendment) in Scotland. This amendemnt states that the majority in the referendum has to be at least 40 per cent of the eligible voters (see Qvortrup (2002)).

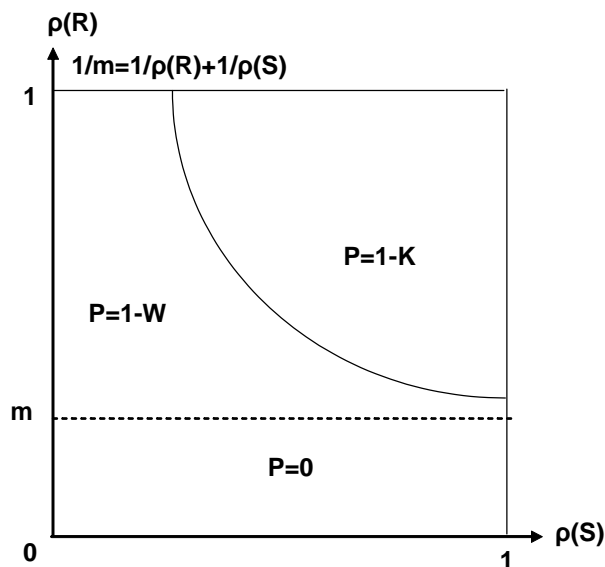


Figure 6

Note that, with the exception of the region where $P = 1 - K$, in the rest of the space P depends only on R and not on S . Hence the status quo party will choose not to mobilize voters in these regions. Recall that we defined Q as

$$Q = \frac{q - \rho(S)}{\rho(R) - \rho(S)}.$$

Hence, when $q = m$, we have that $W = Q(S = 0)$, which means that a majority quorum in the region $P = 1 - W$ is identical to a participation quorum when there is zero spending on the status-quo side ($S = 0$). Indeed, as we show in the appendix, for any (B, m) the equilibrium in the majority quorum regime is the same as the equilibrium in the participation quorum regime with $(B, q = m)$. If instead $\rho(0) = \underline{\rho} > 0$, then it is immediate to show that the pure strategies equilibria of the model with participation quorum q coincide with the pure strategies equilibria of the model with approval quorum $m - \underline{\rho}$. In conclusion, all the analysis for the participation quorum case carries over to the approval quorum case.

7. CONCLUSION

We provide an analysis of the consequences of imposing participation requirements in the context of binary elections. Turnout requirements affect the equilibrium turnout, the chance that one alternative prevails in the referendum, and the overall welfare of citizens. We show that a

participation requirement distorts drastically the incentives of parties to mobilize voters in the context of a group-based model of turnout. The result we obtain on equilibrium turnout is unambiguous: a quorum requirement can only depress turnout, sometimes even generating a “quorum paradox”. Regarding the common argument that a turnout requirement introduces a bias for the status quo, we show that, in the context of group-based models of turnout, this is not always the case. In fact, the probability that the status quo is overturned may decrease or increase in the presence of a quorum provision. The quorum provision could perhaps be an effective safeguard against so-called “false” majorities, i.e. the exploitation of voter apathy by a minority or a special interest group of committed citizens. However, the distortions that a quorum introduces suggest that more stringent requirements to call a referendum might be a better policy if the goal is to introduce a bias for status quo. The results we obtain on welfare are ambiguous, as in the presence of a quorum limit there is a welfare loss on the revenue side yet on the cost side there may be a welfare gain.

8. APPENDIX

Table 2

States	Participation Quorum	Approval Quorum
Azerbaijan, Colombia, Venezuela	25%	
Hungary		25%
Denmark	30%	
Albania, Armenia		33.3%
Uruguay*	35%	
Denmark*, Scotland		40%
Bulgaria, Italy, Lithuania, Macedonia, Malta, Poland, Portugal, Slovakia, Taiwan	50%	
Croatia, Latvia, Russia	50%	50%
Belarus, Serbia, Sweden**		50%
US States		
Massachusetts**	30%	
Mississippi**	40%	
Nebraska**	35%	
Wyoming**		50%

*Constitutional Referendum

**The percentage is with respect to voters in the general election

Construction of Figure 1.

Define M as

$$M \equiv \frac{1}{\rho(R)} + \frac{1}{\rho(S)},$$

and note that M is decreasing in $\rho(R)$ and $\rho(S)$. The curved line depicted in Figure 1 represents the case in which $M = \frac{2}{q}$. We have four cases depending on whether $M \geq \frac{2}{q}$ and whether $\rho(R) \geq \rho(S)$, where $M < (>) \frac{2}{q}$ represents the area above (below) the curved line in Figure 1 (since the probability is continuous across boundaries, we omit the boundary cases, which are self-explanatory).

(1) If $M > \frac{2}{q}$ and $\rho(S) > \rho(R)$, then

$$M > \frac{2}{q} \cap \rho(R) < \rho(S) \iff K > Q$$

$$P = \Pr(r < Q \cap r > K) = 0$$

(2) If $M > \frac{2}{q}$ and $\rho(R) > \rho(S)$, then

$$M > \frac{2}{q} \cap \rho(R) > \rho(S) \iff K < Q$$

$$P = \Pr(r > Q \cap r > K) = \begin{cases} 0 & \text{if } Q > 1 \iff \rho(R) < q \\ 1 - Q & \text{if } Q < 1 \iff \rho(R) > q \end{cases}$$

(3) If $M < \frac{2}{q}$ and $\rho(S) > \rho(R)$, then

$$M < \frac{2}{q} \cap \rho(R) < \rho(S) \iff K < Q$$

$$P = \Pr(r < Q \cap r > K) = \begin{cases} 1 - K & \text{if } Q > 1 \iff \rho(R) > q \\ Q - K & \text{if } Q < 1 \iff \rho(R) < q \end{cases}$$

(4) If $M < \frac{2}{q}$ and $\rho(R) > \rho(S)$, then

$$M < \frac{2}{q} \cap \rho(R) > \rho(S) \iff K > Q$$

$$P = \Pr(r > Q \cap r > K) = 1 - K$$

Summarizing we have 4 possible values of P which identify the 4 probability regions in Figure 1.

$$P = 0 \iff M > \frac{2}{q} \cap \rho(R) < q$$

$$P = 1 - Q \iff M > \frac{2}{q} \cap \rho(R) > q$$

$$P = 1 - K \iff M < \frac{2}{q} \cap \rho(R) > q$$

$$P = Q - K \iff M < \frac{2}{q} \cap \rho(R) < q.$$

Proof of Proposition. 1 Consider first the benchmark case of $q = 0$. For all given values of S , the profit function $\pi_r(S, R)$ is continuous for all $R \geq 0$, twice differentiable for all $R > 0$ and single peaked in R , and likewise for $\pi_s(S, R)$. For any pair of values (S^*, R^*) which jointly solve the two first order conditions it must be the case that $S^* = R^*$.

In fact, by taking the necessary and sufficient FOCs we have that

$$\frac{\rho'(R^*)\rho(S^*)}{(\rho(R^*) + \rho(S^*))^2} = \frac{1}{B} = \frac{\rho'(S^*)\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2},$$

that yields

$$\frac{\rho'(R^*)}{\rho(R^*)} = \frac{\rho'(S^*)}{\rho(S^*)}.$$

Therefore, it must be that $S^* = R^*$, where R^* solves

$$\frac{\rho'(R^*)}{4\rho(R^*)} = \frac{1}{B}.$$

Since $\frac{\rho'(R)}{\rho(R)}$ is decreasing in R , and its codomain are the positive real numbers, an equilibrium exists and it is unique for any B . Consider now the case in which $q > 0$. Note that $\pi_r(S^*, R)$ is single peaked in the $P = 1 - K$ region, it is increasing in the $P = Q - K$ region, and non-positive in the $P = 0$ region. Hence, $\pi_r(S^*, R)$ is globally single peaked at $R = R^*$. The symmetric profile $S^* = R^*$ for $q = 0$ is an equilibrium for $q > 0$ if and only if both $S^* = R^*$ lies in the $P = 1 - K$ region and s does not have an incentive to deviate to zero, i.e. $\pi_s(S^*, R^*) \geq \pi_s(0, R^*)$. This is true if and only if $q \in [0, \underline{q}(B)]$, where

$$\underline{q}(B) = \left(\frac{1}{2} - \frac{R^*}{B} \right) \rho(R^*).$$

□

In order to prove proposition (4), we first prove two preliminary Lemma.

Lemma 1 *Let \widehat{R} and \widehat{S} be defined by (2) and (3), respectively. Then*

$$\frac{d\widehat{R}}{dq} > 0, \quad \frac{d\pi_s(0, \widehat{R})}{dq} > 0, \quad \frac{d\pi_r(0, \widehat{R})}{dq} < 0, \quad \frac{d\pi_s(\widehat{S}, \widehat{R})}{dq} < 0.$$

Proof of Lemma 1. From (2), and the assumptions on $\rho(\cdot)$, it follows that \widehat{R} is the unique solution to

$$\frac{q\rho'(\widehat{R})}{\rho^2(\widehat{R})} = \frac{1}{B}.$$

Since the RHS is constant in q while the LHS is increasing in q and decreasing in \widehat{R} , then \widehat{R} is increasing in q , i.e. $\frac{d\widehat{R}}{dq} > 0$. As for the

profits, we have that

$$\pi_s(0, \widehat{R}) = B \frac{q}{\rho(\widehat{R})},$$

and

$$\pi_r(0, \widehat{R}) = B \left(1 - \frac{q}{\rho(\widehat{R})} \right) - \widehat{R},$$

and the result follows from noticing that $q/\rho(\widehat{R})$ is increasing in q . Finally,

$$\frac{d\pi_s(\widehat{S}, \widehat{R})}{dq} = -B \frac{\rho(\widehat{S}) \rho'(\widehat{R})}{(\rho(\widehat{R}) + \rho(\widehat{S}))^2} \frac{d\widehat{R}}{dq} < 0.$$

□

Lemma 2 *There exists a unique $\tilde{q} = \rho(R^*)/4 < \underline{q}$ such that $\widehat{R}(\tilde{q}) = \widehat{S}(\tilde{q}) = R^* = S^*$. Moreover, $q \neq \tilde{q}$ implies $\widehat{S} < S^*$.*

Proof of Lemma 2. \widehat{R} and R^* uniquely solve

$$q \frac{\rho'(\widehat{R})}{\rho^2(\widehat{R})} = \frac{1}{B}, \text{ and } \frac{\rho'(R^*)}{4\rho(R^*)} = \frac{1}{B}$$

respectively. It is easy to check that when $q = \rho(R^*)/4$,

$$q \frac{\rho'(\widehat{R})}{\rho^2(\widehat{R})} = \frac{\rho'(R^*)}{4\rho(R^*)}.$$

Next, from the definition of \underline{q} , we have that $\tilde{q} < \underline{q}$ if and only if $R^* < B/4$, or

$$\frac{4}{B} > \frac{\rho'(\frac{B}{4})}{\rho(\frac{B}{4})}.$$

Therefore, $\tilde{q} < \bar{q}$ if and only if

$$\Gamma(x) \equiv \frac{x\rho'(x)}{\rho(x)} < 1.$$

To prove that $\Gamma(x) < 1$, first note that $\Gamma(x)$ is differentiable hence continuous for $x > 0$. Second, $\Gamma(x) \geq 1$ implies that

$$\Gamma'(x) = \left(\frac{\rho'(x)}{\rho(x)} (1 - \Gamma(x)) + \frac{x\rho''(x)}{\rho(x)} \right) < 0.$$

Hence, $\lim_{x \rightarrow 0} \Gamma(x) \leq 1$ implies $\Gamma(x) < 1$. Since, $\lim_{x \rightarrow 0} (x\rho'(x)) = 0$, and $\lim_{x \rightarrow 0} (x\rho''(x)) = 0$, we have that

$$\lim_{x \rightarrow 0} \Gamma(x) = \begin{cases} \lim_{x \rightarrow 0} \frac{\rho'(x) + x\rho''(x)}{\rho'(x)} = 1 & \text{if } \rho(0) = 0 \\ 0 & \text{if } \rho(0) > 0. \end{cases}$$

If $q < \tilde{q}$, it follows that $\hat{R} < R^* = S^*$. Since

$$\frac{\rho'(S)\rho(R)}{(\rho(R) + \rho(S))^2}$$

is always decreasing in S , and increasing in R if and only if $S > R$, it follows that

$$\frac{\rho'(S^*)\rho(\hat{R})}{(\rho(\hat{R}) + \rho(S^*))^2} < \frac{\rho'(S^*)\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2} = \frac{1}{B},$$

and therefore $\hat{S} < S^*$. If $q > \tilde{q}$, it follows that $\hat{R} > R^* = S^*$, and

$$\frac{\rho'(S^*)\rho(\hat{R})}{(\rho(\hat{R}) + \rho(S^*))^2} < \frac{\rho'(S^*)\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2} = \frac{1}{B}.$$

Hence $q \neq \tilde{q}$ implies $\hat{S} < S^*$. □

We are now ready to prove proposition (4).

Proof of Proposition. 4 First, we show that

$$\underline{q} < \hat{q} < \frac{1}{2},$$

and that the thresholds \underline{q} and \hat{q} are well defined. Define

$$C(q) = \pi_s(\hat{S}, \hat{R}) - \pi_s(0, \hat{R}),$$

and

$$D(q) = \pi_s(S^*, R^*) - \pi_s(0, R^*).$$

Hence \underline{q} and \hat{q} are implicitly defined by

$$C(\hat{q}) = 0, \quad D(\underline{q}) = 0.$$

Clearly $D'(q) < 0$, and from Lemma (1) $C'(q) < 0$. So the thresholds \underline{q} and \hat{q} are uniquely defined. From Lemma (2) $\tilde{q} < \underline{q}$. Hence, $D(\tilde{q}) =$

$C(\tilde{q}) > 0$, and $\tilde{q} < \hat{q}$. To show that $\underline{q} < \hat{q}$, it suffices to show that for $q \geq \tilde{q}$, it is true that $D'(q) < C'(q)$, that is

$$\frac{1}{\rho(R^*)} > \frac{1}{\rho(\hat{R})} + \frac{d\hat{R}}{dq} \left(\frac{\rho(\hat{S}) \rho'(\hat{R})}{(\rho(\hat{R}) + \rho(\hat{S}))^2} - \frac{1}{B} \right).$$

But since, for $q \geq \tilde{q}$ we have $\hat{R} \geq R^* \geq \hat{S}$, and since $\frac{d\hat{R}}{dq} > 0$, it follows that the term in brackets in the above inequality is non positive and therefore $D'(q) < C'(q)$. Next, we show that $\underline{q} < \bar{q}$ and that \bar{q} is well defined. Recall that when $q = \bar{q}$, $\pi_r(0, \hat{R}) = 0$ and, by the envelope theorem, we have that

$$\frac{d\pi_r(0, \hat{R})}{dq} = \frac{\partial \pi_r(0, \hat{R})}{\partial q} = -\frac{B}{\rho(\hat{R})} < 0.$$

Hence \bar{q} is uniquely defined. To show that $\underline{q} < \bar{q}$, note that when $q \geq \tilde{q}$, we have that

$$\begin{aligned} 0 < D(\tilde{q}) &= B \left(\frac{1}{2} - \frac{\tilde{q}}{\rho(R^*)} \right) - R^* < \\ &< B \left(1 - \frac{\tilde{q}}{\rho(R^*)} \right) - R^* = \pi_r(0, \hat{R}(\tilde{q})), \end{aligned}$$

and

$$\frac{dD(q)}{dq} = -\frac{B}{\rho(R^*)} < -\frac{B}{\rho(\hat{R})} = \frac{d\pi_r(0, \hat{R})}{dq} < 0.$$

Hence, since $D(q)$ is smaller and decreases faster than $\pi_r(0, \hat{R})$, the desired inequality follows. \square

Construction of Figure 3.

As for \underline{q} , note that

$$\frac{d\underline{q}}{dB} = \rho'(R^*) \left(\frac{\partial R^*}{\partial B} \left(\frac{1}{4} - \frac{R^*}{B} \right) + \frac{1}{4} \frac{R^*}{B} \right).$$

Since

$$\frac{\partial R^*}{\partial B} = \frac{1}{4 - B \frac{\rho''(R^*)}{\rho'(R^*)}} > 0,$$

if $R^*/B < 1/4$, it follows that $\frac{dq}{dB} > 0$. Finally, $R^*/B < 1/4$ if and only if $\Gamma(x) < 1$ for $x > 0$, which is true by the proof of Lemma (2). Since

$$\lim_{B \rightarrow 0} R^* = 0, \quad \lim_{B \rightarrow \infty} R^* = \infty, \quad \lim_{B \rightarrow \infty} \frac{R^*}{B} = \lim_{B \rightarrow \infty} \frac{\partial R^*}{\partial B} \leq \frac{1}{4},$$

it follows that

$$\lim_{B \rightarrow 0} \underline{q} = 0, \quad \lim_{B \rightarrow \infty} \frac{dq}{dB} = 0, \quad \lim_{B \rightarrow \infty} \underline{q} \in \left[\frac{1}{4}, \frac{1}{2} \right].$$

In particular, a sufficient condition for $\lim_{B \rightarrow \infty} \underline{q} = \frac{1}{2}$ is $\lim_{x \rightarrow \infty} \frac{\rho''(x)}{\rho'(x)} = c < 0$ (this is true for example in the case of $\rho(x) = 1 - e^{-\alpha x}$, and $\alpha > 0$). Recall that if $\rho(0) > 0$ we can have that $\lim_{B \rightarrow \infty} \underline{q} > \frac{1}{2}$.

As for $\hat{q}(B)$, recall that \hat{R} is a function of q and B , and we have that

$$\begin{aligned} \frac{\partial \hat{R}}{\partial q} &= \frac{1}{q} \frac{1}{\frac{2\rho(\hat{R})}{qB} - \frac{\rho'(\hat{R})}{\rho'(\hat{R})}} \in \left(0, \frac{B}{2\rho(\hat{R})} \right) \\ \frac{\partial \hat{R}}{\partial B} &= \frac{1}{B} \frac{1}{\frac{2\rho(\hat{R})}{qB} - \frac{\rho'(\hat{R})}{\rho'(\hat{R})}} = \frac{q}{B} \frac{\partial \hat{R}}{\partial q} \in \left(0, \frac{q}{2\rho(\hat{R})} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d\hat{q}}{dB} &= \frac{\rho(\hat{R}(\hat{q})) \left(\frac{\hat{S}(\hat{q})}{B^2} + \left(B^{-1} - \frac{\rho(\hat{S}(\hat{q}))\rho'(\hat{R}(\hat{q}))}{(\rho(\hat{R}(\hat{q})) + \rho(\hat{S}(\hat{q})))^2} \right) \frac{\partial \hat{R}(\hat{q})}{\partial B} \right)}{1 - \rho(\hat{R}(\hat{q})) \left(B^{-1} - \frac{\rho(\hat{S}(\hat{q}))\rho'(\hat{R}(\hat{q}))}{(\rho(\hat{R}(\hat{q})) + \rho(\hat{S}(\hat{q})))^2} \right) \frac{\partial \hat{R}(\hat{q})}{\partial \hat{q}}} > 0 \\ \lim_{B \rightarrow 0} \hat{q} &= 0, \quad \lim_{B \rightarrow \infty} \hat{q} \in \left[\lim_{B \rightarrow \infty} \underline{q}, \frac{1}{2} \right], \end{aligned}$$

where we used

$$\begin{aligned} \frac{d\hat{R}(\hat{q})}{dB} &= \frac{\partial \hat{R}(\hat{q})}{\partial \hat{q}} \frac{d\hat{q}}{dB} + \frac{\partial \hat{R}(\hat{q})}{\partial B} > 0 \\ \frac{\hat{S}(\hat{q})}{B} &\in \left(0, \frac{1}{2} \right), \quad \frac{\rho(\hat{S}(\hat{q}))}{\rho(\hat{R}(\hat{q})) + \rho(\hat{S}(\hat{q}))} \in \left(0, \frac{1}{2} \right) \\ \frac{1}{2} &\geq \lim_{B \rightarrow \infty} \hat{q} \geq \lim_{B \rightarrow \infty} \underline{q} \in \left[\frac{1}{4}, \frac{1}{2} \right]. \end{aligned}$$

As for $\bar{q}(B)$,

$$\begin{aligned} \frac{d\bar{q}}{dB} &= \frac{1}{B^2} \left(- \left(\left(\frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}} \frac{d\bar{q}}{dB} + \frac{\partial \hat{R}(\bar{q})}{\partial B} \right) B - \hat{R}(\bar{q}) \right) \rho \left(\hat{R}(\bar{q}) \right) + \right. \\ &\quad \left. + B^2 \left(1 - \frac{\hat{R}(\bar{q})}{B} \right) \rho' \left(\hat{R}(\bar{q}) \right) \left(\frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}} \frac{d\bar{q}}{dB} + \frac{\partial \hat{R}(\bar{q})}{\partial B} \right) \right) \\ &= \frac{-\frac{\partial \hat{R}(\bar{q})}{\partial B} \rho \left(\hat{R}(\bar{q}) \right) + \frac{\hat{R}(\bar{q})}{B^2} \rho \left(\hat{R}(\bar{q}) \right) + \left(1 - \frac{\hat{R}(\bar{q})}{B} \right) \rho' \left(\hat{R}(\bar{q}) \right) \frac{\partial \hat{R}(\bar{q})}{\partial B}}{1 + \frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}} \rho \left(\hat{R}(\bar{q}) \right) - \left(1 - \frac{\hat{R}(\bar{q})}{B} \right) \rho' \left(\hat{R}(\bar{q}) \right) \frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}}} \\ &= \frac{\hat{R}(\bar{q})}{B^2} \rho \left(\hat{R}(\bar{q}) \right) > 0 \end{aligned}$$

where the last equality is obtained by substituting back the equation for $\bar{q}(B)$. Moreover,

$$\lim_{B \rightarrow 0} \bar{q} = 0, \quad \lim_{B \rightarrow \infty} \frac{d\bar{q}}{dB} = 0, \quad \lim_{B \rightarrow \infty} \bar{q} \geq \frac{1}{2},$$

where we used

$$\frac{\hat{R}(\bar{q})}{B} \in (0, 1)$$

and

$$\begin{aligned} \frac{d\hat{R}(\bar{q})}{dB} &= \frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}} \frac{d\bar{q}}{dB} + \frac{\partial \hat{R}(\bar{q})}{\partial B} = \frac{\partial \hat{R}(\bar{q})}{\partial \bar{q}} \left(\frac{d\bar{q}}{dB} + \frac{\bar{q}}{B} \right) \\ &= \frac{1}{2 - \rho \left(\hat{R}(\bar{q}) \right) \frac{\rho''(\hat{R}(\bar{q}))}{(\rho'(\hat{R}(\bar{q})))^2}} > 0 \end{aligned}$$

$$\lim_{B \rightarrow 0} \hat{R}(\bar{q}) = 0, \quad \lim_{B \rightarrow \infty} \hat{R}(\bar{q}) = \infty, \quad \lim_{B \rightarrow \infty} \frac{\hat{R}(\bar{q})}{B} = \lim_{B \rightarrow \infty} \frac{d\hat{R}(\bar{q})}{dB} \leq \frac{1}{2}$$

In particular, if $\lim_{x \rightarrow \infty} \frac{\rho''(x)}{(\rho'(x))^2} = -\infty$, then $\lim_{B \rightarrow \infty} \bar{q} = 1$ (this is true for example in the case of $\rho(x) = 1 - e^{-\alpha x}$, and $\alpha > 0$). Summarizing, we have that

$$\begin{aligned} \frac{dq}{dB} &> 0, \quad \lim_{B \rightarrow 0} q = 0, \quad \lim_{B \rightarrow \infty} q \in \left[\frac{1}{4}, \frac{1}{2} \right] \\ \frac{d\hat{q}}{dB} &> 0, \quad \lim_{B \rightarrow 0} \hat{q} = 0, \quad \lim_{B \rightarrow \infty} \hat{q} \in \left[\lim_{B \rightarrow \infty} q, \frac{1}{2} \right] \\ \frac{d\bar{q}}{dB} &> 0, \quad \lim_{B \rightarrow 0} \bar{q} = 0, \quad \lim_{B \rightarrow \infty} \bar{q} \in \left[\frac{1}{2}, 1 \right]. \end{aligned}$$

In order to prove proposition (5), we need the following Lemma.

Lemma 3 For all $q \in (\underline{q}, \hat{q})$ there exists a unique $\tilde{R}(q) \in (R^*, \hat{R})$ such that the best response of party s is $S \in \{0, \tilde{S} > 0\}$. Moreover $\tilde{R}(\underline{q}) = R^*$, $\tilde{R}(\hat{q}) = \hat{R}$, and

$$\frac{\partial \tilde{R}}{\partial q} > 0, \quad \frac{\partial \tilde{S}}{\partial q} < 0.$$

Proof of Lemma 3. Denote $S = S(R)$ as the best response of party s to R and let

$$\begin{aligned} C(R, q) &= \pi_s(S, R) - \pi_s(0, R) \\ &= B \left(\frac{\rho(S)}{\rho(R) + \rho(S)} - \frac{q}{\rho(R)} \right) - S. \end{aligned}$$

The indifference condition that defines $\tilde{R}(q)$ is $C(R, q) = 0$. Since $S^* = S(R^*)$ and $\hat{S} = S(\hat{R})$, we have that $\tilde{R}(\underline{q}) = R^*$, and $\tilde{R}(\hat{q}) = \hat{R}$. Since $\frac{\partial C}{\partial q} < 0$, for $q \in (\underline{q}, \hat{q})$ we have that $C(R^*, q) < C(R^*, \underline{q}) = 0$, and $C(\hat{R}, q) > C(\hat{R}, \hat{q}) = 0$. If $\frac{\partial C}{\partial R} > 0$ for all $R \in [R^*, \hat{R}]$ and $q \in (\underline{q}, \hat{q})$, then for any $q \in (\underline{q}, \hat{q})$ there exists a unique $\tilde{R} \in (R^*, \hat{R})$ such that $C(\tilde{R}, q) = 0$. What is left to show is that $\frac{\partial C}{\partial R} > 0$ when $R \in [R^*, \hat{R}]$. Using the fact that

$$B \frac{\rho'(S(R)) \rho(R)}{(\rho(R) + \rho(S(R)))^2} = 1,$$

we have that

$$\begin{aligned} \frac{\partial C}{\partial R} &= B \frac{q\rho'(R)}{(\rho(R))^2} - B \frac{\rho'(R)\rho(S(R))}{(\rho(R) + \rho(S(R)))^2} + \\ &\quad + \left(B \frac{\rho'(S(R))\rho(R)}{(\rho(R) + \rho(S(R)))^2} - 1 \right) \frac{\partial S(R)}{\partial R} \\ &= B \left(\frac{q\rho'(R)}{(\rho(R))^2} - \frac{\rho'(R)\rho(S(R))}{(\rho(R) + \rho(S(R)))^2} \right), \end{aligned}$$

and by using the definition of \hat{R} and R^* we have that for $R \in [R^*, \hat{R}]$,

$$\frac{q\rho'(R)}{(\rho(R))^2} > \frac{1}{B} > \frac{\rho'(R)\rho(S(R))}{(\rho(R) + \rho(S(R)))^2}.$$

Finally, since $C(R, q)$ is differentiable in both arguments, the implicit function theorem implies that $\tilde{R}(q)$ is differentiable and

$$\frac{\partial \tilde{R}}{\partial q} = -\frac{\frac{\partial C}{\partial \tilde{R}}}{\frac{\partial C}{\partial q}} > 0.$$

Since $\tilde{S} > 0$ is the best response to $\tilde{R} > R^*$, then by the proof of Lemma (2) $\frac{\partial \tilde{S}}{\partial \tilde{R}} < 0$ and therefore

$$\frac{\partial \tilde{S}}{\partial q} = \frac{\partial \tilde{S}}{\partial \tilde{R}} \frac{\partial \tilde{R}}{\partial q} < 0.$$

□

We are now ready to prove proposition (5).

Proof of Proposition. 5 By construction, \tilde{R} makes party s indifferent between playing 0 and $S(\tilde{R})$. We have an equilibrium if s chooses the mix $(\alpha, 1 - \alpha)$ (with α on $S = 0$) such that the best response of party r is \tilde{R} . Let

$$R(\alpha) \equiv \arg \max_R \left(\alpha \pi_r(0, R) + (1 - \alpha) \pi_r(S(\tilde{R}), R) \right),$$

be the best response of party r to party s mixing between 0 and $S(\tilde{R})$.

We want to find an α such that $R(\alpha) = \tilde{R}$. Note that it must be the case that $R(\alpha) \in (R(0), R(1))$, where

$$R(1) \equiv \arg \max_R \left(B \left(1 - \frac{q}{\rho(R)} \right) - R \right) = \hat{R}$$

$$R(0) \equiv \arg \max_R \left(B \left(\frac{\rho(R)}{\rho(R) + \rho(S(\tilde{R}))} \right) - R \right) = R'',$$

where $R'' < R^* < \tilde{R} < \hat{R}$. Since the objective

$$\left(\alpha \pi_r(0, R) + (1 - \alpha) \pi_r(S(\tilde{R}), R) \right)$$

is concave in R for all α , the FOC delivers uniquely our target, namely

$$\alpha = \frac{\frac{1}{B} - \frac{\rho'(\tilde{R})\rho(S(\tilde{R}))}{(\rho(\tilde{R}) + \rho(S(\tilde{R})))^2}}{\frac{q\rho'(\tilde{R})}{(\rho(\tilde{R}))^2} - \frac{\rho'(\tilde{R})\rho(S(\tilde{R}))}{(\rho(\tilde{R}) + \rho(S(\tilde{R})))^2}}.$$

Finally, note that $\alpha(\underline{q}) = \alpha(\tilde{R} = R^*) = 0$, $\alpha(\hat{q}) = \alpha(\tilde{R} = \hat{R}(\hat{q})) = 1$. \square

Proof of Claim 1. To show that $E(T) > q$ when $q \in (\underline{q}, \hat{q})$, note that in this region

$$B \frac{q}{\rho(\tilde{R}(q))} = B \frac{\rho(S(\tilde{R}(q)))}{\rho(\tilde{R}(q)) + \rho(S(\tilde{R}(q)))} - S(\tilde{R}(q)),$$

and $\hat{R}(\hat{q}) > \tilde{R}(q) > R^* > S(\tilde{R}(q))$. Hence, it must be the case that

$$\frac{1}{2} > \frac{\rho(S(\tilde{R}(q)))}{\rho(\tilde{R}(q)) + \rho(S(\tilde{R}(q)))} > \frac{q}{\rho(\tilde{R}(q))},$$

that implies

$$\frac{\rho(\tilde{R})}{2} > q,$$

and therefore

$$E(T) = \frac{\rho(\tilde{R})}{2} + (1 - \alpha(q)) \frac{\rho(\tilde{S})}{2} > q.$$

Continuity of the expected turnout for all $q \neq \bar{q}$ implies that

$$\lim_{q \rightarrow \underline{q}} E(T) = \rho(R^*), \text{ and } \lim_{q \rightarrow \hat{q}} E(T) = \frac{\rho(\hat{R}(\hat{q}))}{2}.$$

Moreover,

$$\frac{\rho(\hat{R}(\hat{q}))}{2} = \frac{\rho'(\hat{R}(\hat{q})) \hat{q} B}{2\rho(\hat{R}(\hat{q}))} < \frac{\rho'(R^*) B}{4} = \rho(R^*),$$

since

$$\rho'(\hat{R}(\hat{q})) \hat{q} < \rho'(R^*) \frac{\rho(\hat{R}(\hat{q}))}{2}.$$

Finally, $\lim_{q \rightarrow \underline{q}} \frac{\partial E(T)}{\partial q} < 0$ follows from

$$\begin{aligned} \lim_{q \rightarrow \underline{q}} \frac{\partial E(T)}{\partial q} &= \frac{1}{2} \left(\rho'(R^*) - \lim_{q \rightarrow \underline{q}} \frac{\partial \alpha}{\partial \tilde{R}} \rho(R^*) \right) \lim_{q \rightarrow \underline{q}} \frac{\partial \tilde{R}}{\partial q} = \\ &= \frac{1}{2} \left(\rho'(R^*) - \frac{4\rho'(R^*) - B\rho''(R^*)}{16\underline{q} - B\rho'(R^*)} \rho(R^*) \right) \lim_{q \rightarrow \underline{q}} \frac{\partial \tilde{R}}{\partial q} < \\ &= \frac{1}{2} \left(\rho'(R^*) - \frac{4\rho'(R^*)}{16\underline{q} - B\rho'(R^*)} \rho(R^*) \right) \lim_{q \rightarrow \underline{q}} \frac{\partial \tilde{R}}{\partial q} = \\ &= \frac{\rho'(R^*)}{2} \left(\frac{16\rho(R^*) \left(\frac{1}{2} - \frac{R^*}{B} \right) - B\rho'(R^*) - 4\rho(R^*)}{16\rho(R^*) \left(\frac{1}{2} - \frac{R^*}{B} \right) - B\rho'(R^*)} \right) \lim_{q \rightarrow \underline{q}} \frac{\partial \tilde{R}}{\partial q} < 0, \end{aligned}$$

where we used Lemma 3,

$$\lim_{q \rightarrow \underline{q}} \frac{\partial \alpha}{\partial \tilde{R}} = \frac{4\rho'(R^*) - B\rho''(R^*)}{16\underline{q} - B\rho'(R^*)} > 0,$$

and

$$16\rho(R^*) \left(\frac{1}{2} - \frac{R^*}{B} \right) - B\rho'(R^*) - 4\rho(R^*) < 4\rho(R^*) - B\rho'(R^*) = 0.$$

□

Proof of Claim 2. That $E(T)|_{q=\hat{q}} > \hat{q}$ follows directly from Claim 1, by continuity of $E(T)$. Next, since \bar{q} is increasing in B , and $\lim_{B \rightarrow \infty} \bar{q} \geq \frac{1}{2} > \rho(\hat{R}(\bar{q}))/2$, there exists a \bar{B} such that for $B > \bar{B}$ we have that

$$\frac{\rho(\hat{R}(\bar{q}))}{2} < \bar{q}.$$

Since,

$$\frac{\partial \hat{R}(q)}{\partial q} < \frac{B}{2\rho(\hat{R}(q))},$$

it follows that

$$\frac{\partial E(T)}{\partial q} = \frac{\rho'(\hat{R}(q))}{2} \frac{\partial \hat{R}(q)}{\partial q} < 1.$$

Hence, when $B > \bar{B}$, there exists a unique q' such that $E(T) > q$ if and only if $q < q'$. □

Majority Quorum

If the majority quorum requirement is $m \in [0, \frac{1}{2}]$, then

$$\begin{aligned} P &= \Pr((v_R > m) \cap (v_R > v_S)) \\ &= \Pr\left(\left(\tilde{r} > \frac{m}{\rho(R)}\right) \cap \left(\tilde{r} > \frac{\rho(S)}{\rho(R) + \rho(S)}\right)\right) \\ &= 1 - \max\left(\min\left(1, \frac{m}{\rho(R)}\right), \frac{\rho(S)}{\rho(R) + \rho(S)}\right) \\ &= \min((1 - W)^+, 1 - K), \end{aligned}$$

where we define

$$\begin{aligned} M &= \frac{1}{\rho(R)} + \frac{1}{\rho(S)}, & W &= \frac{m}{\rho(R)}, & K &= \frac{\rho(S)}{\rho(R) + \rho(S)} \\ (1 - W)^+ &= \begin{cases} \left(1 - \frac{m}{\rho(R)}\right) & \text{if } \rho(R) > m \\ 0 & \text{if } \rho(R) \leq m \end{cases} \end{aligned}$$

Given that $(1 - W) < (1 - K)$ if and only if $M > 1/m$, we have

$$P = \begin{cases} (1 - K) & \text{if } \frac{1}{M} < \frac{1}{m} \\ (1 - W)^+ & \text{if } \frac{1}{M} \geq \frac{1}{m}. \end{cases}$$

Note that, the symmetric spending profile (S^*, R^*) is an equilibrium if and only if s does not deviate to zero, that is $m \in [0, \underline{m}]$, where

$$\begin{aligned} \underline{m} &= \min\left(\rho(R^*), \frac{\rho(S^*)\rho(R^*)}{\rho(R^*) + \rho(S^*)}, \frac{\rho(S^*)\rho(R^*)}{\rho(R^*) + \rho(S^*)} - S^* \frac{\rho(R^*)}{B}\right) \\ &= \left(\frac{1}{2} - S^* B^{-1}\right) \rho(R^*) = \underline{q}. \end{aligned}$$

Hence, when $m = q$ the symmetric equilibrium existence conditions are the same. In addition, it is immediate to see that the value of \hat{R} that satisfies the FOC for r in the region $P = 1 - W$ is given by

$$m \frac{\rho'(\hat{R}(m))}{\rho^2(\hat{R}(m))} = \frac{1}{mB}.$$

Hence, $\hat{R}(m)$ is the best response to $S = 0$ as long as it gives to the reform party a non-negative payoff, i.e.

$$B \left(1 - \frac{m}{\rho(\hat{R}(m))}\right) - \hat{R}(m) \geq 0.$$

If the above condition is violated, the best response is $R = 0$. It is easy to see by just substituting q with m that, for any B , all the boundary

conditions for existence of equilibria in the participation quorum case coincide with the boundary conditions for existence in the majority quorum case, and that the equilibria are the same.

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