Biform Games:
Online Appendix*

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These two appendices supplement Brandenburger and Stuart [2, 2006] ("Biform Games").

Appendix A: Core Calculations

Proof for Example 2.1. In Example 2.1, there are two cooperative games, resulting from the two strategies the supplier can choose. To fix notation for the games, let $v : \mathcal{P}(N) \to \mathbb{R}$ be a characteristic function. Also, given an allocation $x \in \mathbb{R}^n$, and $T \subseteq N$, write $x(T) = \sum_{j \in T} x^j$. Then the Core is the set of allocations satisfying $x(T) \geq v(T)$ for all $T \subseteq N$, and $x(N) = v(N)$. Note in particular that for each player i we must have $v(\{i\}) \leq x^i \leq v(N) - v(N \backslash \{i\})$.

It will suffice to consider two buyers in each game, so in both games the player set is $N = \{s, f_1, f_2, b_1, b_2\}$, where s is the supplier, f_1 and f_2 are the two firms, and b_1 and b_2 are the buyers. Write $F = \{f_1, f_2\}$ and $B = \{b_1, b_2\}$.

Consider first the status-quo game. For any $T \subseteq N$, if either $s \notin T$, or $F \cap T = \emptyset$, or $B \cap T = \emptyset$, then $v(T) = 0$. If $s \in T$, $F \cap T \neq \emptyset$, and $B \cap S \neq \emptyset$, then

$$v(T) = \begin{cases} 8 & \text{if } f_1 \in T, \\ 2 & \text{otherwise.} \end{cases}$$

Fix a player $i \in N \backslash \{s, f_1\}$. From the condition $x^i \leq v(N) - v(N \backslash \{i\})$ we get $x^i \leq 8 - 8 = 0$, and from the condition $x^i \geq v(\{i\})$ we get $x^i \geq 0$. This yields $x^{f2} = x^{b_1} = x^{b_2} = 0$. Thus, if the Core is nonempty, it must be a closed bounded interval on the line $x^s + x^{f_1} = 8$.

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Again using $x^i \leq v(N) - v(N\backslash\{i\})$, we get $x^s \leq 8 - 0 = 8$ and $x^{f_1} \leq 8 - 2 = 6$. Together with the above, this implies that if the Core is nonempty, it must be contained in the set

$$\mathcal{A} = \{x \in \mathbb{R}^5 : 2 \leq x^s \leq 8,\, x^{f_1} = 8 - x^s,\, x^{f2} = x^{b_1} = x^{b_2} = 0\}.$$

But it can be checked that any allocation in $\mathcal{A}$ satisfies $x(T) \geq v(T)$ for all $T \subseteq N$, and $x(N) = v(N)$. Thus, the Core is exactly $\mathcal{A}$.

Now the branded-ingredient game. For any $T \subseteq N$, if either $F \cap T = \emptyset$ or $B \cap T = \emptyset$, then

$$v(T) = \begin{cases} -1 & \text{if } s \in T, \\ 0 & \text{otherwise.} \end{cases}$$

If $s \in T$, $F \cap T \neq \emptyset$, and $B \cap T \neq \emptyset$, then

$$v(T) = \begin{cases} 7 & \text{if } f_1 \in T, \\ 5 & \text{otherwise.} \end{cases}$$

We can parallel the above argument to show that the Core is the set of allocations

$$\mathcal{B} = \{x \in \mathbb{R}^5 : 5 \leq x^s \leq 7,\, x^{f_1} = 7 - x^s,\, x^{f2} = x^{b_1} = x^{b_2} = 0\},$$

as required. ■

Proof for Example 2.2. Let $N = \{f_1, f_2, b_1, b_2, b_3\}$, where f_1, f_2 are the firms and b_1, b_2, b_3 are the buyers. Write the strategy sets of the firms as $S^{f_1} = S^{f_2} = \{\sigma, \tau\}$, where σ is the choice of the current product, and τ is the choice of the new product. Set $S = S^{f_1} \times S^{f_2}$, with typical element ρ. (We suppress the singleton strategy sets of the buyers.)

Write ψ^{f_1} (resp. ψ^{f_2}) for the indicator function $\chi_{\{\tau\}\times S^{f_2}}$ (resp. $\chi_{S^{f_1}\times\{\tau\}}$) on S. Also, for $T \subseteq N$, let $r_T = \min\{2 \times |\{f_1, f_2\} \cap T|, |\{b_1, b_2, b_3\} \cap T|\}$, where $|X|$ denotes the cardinality of X. Then the characteristic functions are given by

$$V(\rho)(T) = 4r_T + 3\min\{2[\psi^{f_1}(\rho)\chi_T(f_1) + \psi^{f_2}(\rho)\chi_T(f_2)], r_T\} - 5\psi^{f_1}(\rho)\chi_T(f_1) - 5\psi^{f_2}(\rho)\chi_T(f_2).$$

Note that

$$\begin{aligned} V(\rho)(N) &= 12 + 3\min\{2[\psi^{f_1}(\rho) + \psi^{f_2}(\rho)], 3\} - 5\psi^{f_1}(\rho) - 5\psi^{f_2}(\rho), \\ V(\rho)(\{f_1, b_1, b_2\}) &= 8 + 6\psi^{f_1}(\rho) - 5\psi^{f_1}(\rho) = 8 + \psi^{f_1}(\rho), \quad \text{(A1)} \\ V(\rho)(\{f_2, b_1, b_3\}) &= 8 + \psi^{f_2}(\rho), \quad \text{(A2)} \end{aligned}$$

and, for $i = b_1, b_2, b_3$,

$$V(\rho)(N\backslash\{i\}) = 8 + 6\max\{\psi^{f_1}(\rho), \psi^{f_2}(\rho)\} - 5\psi^{f_1}(\rho) - 5\psi^{f_2}(\rho).$$

Also, if $\rho \neq (\tau, \tau)$, then

$$V(\rho)(N) = 12 + \psi^{f_1}(\rho) + \psi^{f_2}(\rho), \tag{A3}$$

and, for $i = b_1, b_2, b_3$,

$$V(\rho)(N) - V(\rho)(N\backslash\{i\}) = 4. \tag{A4}$$

Consider the allocation $x(\rho) = (\psi^{f_1}(\rho), \psi^{f_2}(\rho), 4, 4, 4)$. It is straightforward to verify that $x(\rho)(T) \geq V(\rho)(T)$ for all $T \subseteq N$. We now show that this is the only Core allocation. Adding A1 and A2 gives

$$x^{b_1}(\rho) + x(\rho)(N) = x(\rho)(\{f_1, b_1, b_2\}) + x(\rho)(\{f_2, b_1, b_3\}) \geq \\ V(\rho)(\{f_1, b_1, b_2\}) + V(\rho)(\{f_2, b_1, b_3\}) = 8 + \psi^{f_1}(\rho) + 8 + \psi^{f_2}(\rho) = 4 + V(\rho)(N),$$

using A3. Thus $x^{b_1}(\rho) \geq 4$, so that $x^{b_1}(\rho) = 4$, using A4. A similar argument applies to $x^{b_2}(\rho)$. The condition $x(\rho)(\{f_1, b_1, b_2\}) \geq V(\rho)(\{f_1, b_1, b_2\})$, together with A1, then implies $x^{f_1}(\rho) \geq \psi^{f_1}(\rho)$. A similar argument yields $x^{f_2}(\rho) \geq \psi^{f_2}(\rho)$. Thus $x^{f_1}(\rho) = \psi^{f_1}(\rho)$ and $x^{f_2}(\rho) = \psi^{f_2}(\rho)$, using A3 again.

The remaining case uses

$$V(\tau, \tau)(N) = 11, \tag{A5}$$

and, for $i = b_1, b_2, b_3$,

$$V(\tau, \tau)(N) - V(\tau, \tau)(N\backslash\{i\}) = 7. \tag{A6}$$

Consider the allocation $x(\tau, \tau) = (-5, -5, 7, 7, 7)$. As before, it is straightforward to verify that $x(\tau, \tau)(T) \geq V(\tau, \tau)(T)$ for all $T \subseteq N$. This is also the only Core allocation. Adding A1 and A2 gives

$$x^{b_1}(\tau, \tau) + x(\tau, \tau)(N) = x(\tau, \tau)(\{f_1, b_1, b_2\}) + x(\tau, \tau)(\{f_2, b_1, b_3\}) \geq \\ V(\tau, \tau)(\{f_1, b_1, b_2\}) + V(\tau, \tau)(\{f_2, b_1, b_3\}) = 18 = 7 + V(\tau, \tau)(N),$$

using A5. Thus $x^{b_1}(\tau, \tau) \geq 7$, so that $x^{b_1}(\tau, \tau) = 7$, using A6. A similar argument applies to $x^{b_2}(\tau, \tau)$. The condition $x(\tau, \tau)(\{f_1, b_1, b_2\}) \geq V(\tau, \tau)(\{f_1, b_1, b_2\})$, together with A1, implies $x^{f_1}(\tau, \tau) \geq -5$. A similar argument yields $x^{f_2}(\tau, \tau) \geq -5$. Thus $x^{f_1}(\tau, \tau) = -5$ and $x^{f_2}(\rho) = -5$, using A5 again. ■

Proof for Example 5.1. Let $N = \{f_1, f_2, f_3, b_1, b_2\}$, where f_1, f_2, f_3 are the firms and b_1, b_2 are the buyers. Write the strategy set of f_1 as $S = \{\sigma, \tau\}$, where σ is the status-quo strategy and τ is the negative-advertising strategy. (We suppress the singleton strategy sets of the other players.)

Fix the indicator function $\chi_{\{\tau\}}$ on S. For $T \subseteq N$, let $r_T = \min\{|\{f_1, f_2, f_3\} \cap T|, |\{b_1, b_2\} \cap T|\}$. Then the characteristic functions are given by

$$V(\rho)(T) = \begin{cases} r_T(2 - \chi_{\{\tau\}}(\rho)) & \text{if } f_1 \notin T, \\ 2 + (r_T - 1)(2 - \chi_{\{\tau\}}(\rho)) & \text{if } f_1 \in T \text{ and } r_T \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} V(\rho)(N) &= 4 - \chi_{\{\tau\}}(\rho), \\ V(\rho)(N \backslash \{f_1\}) &= 4 - 2\chi_{\{\tau\}}(\rho), \end{aligned}$$

and, for $i = f_2, f_3$, and $j = b_1, b_2$,

$$\begin{aligned} V(\rho)(N \backslash \{i\}) &= 4 - \chi_{\{\tau\}}(\rho), \\ V(\rho)(N \backslash \{j\}) &= 2. \end{aligned}$$

Thus

$$V(\rho)(N) - V(\rho)(N \backslash \{f_1\}) = \chi_{\{\tau\}}(\rho),$$

and, for $i = f_2, f_3$, and $j = b_1, b_2$,

$$\begin{aligned} V(\rho)(N) - V(\rho)(N \backslash \{i\}) &= 0, \\ V(\rho)(N) - V(\rho)(N \backslash \{j\}) &= 2 - \chi_{\{\tau\}}(\rho), \end{aligned}$$

from which AU is satisfied. By Lemma 5.1 in the text, if the Core is nonempty, each player $k \in N$ gets exactly $x^k(\rho) = V(\rho)(N) - V(\rho)(N \backslash \{k\})$. But it is straightforward to verify that $x(\rho)(T) \geq V(\rho)(T)$ for all $T \subseteq N$. ■

Proof for Example 5.2. Let $N = \{1, 2, 3\}$, and let ρ be a strategy profile. AU is satisfied in each cooperative game. So, by Lemma 5.1, if the Core is nonempty, each player $k \in N$ gets exactly $x^k(\rho) = V(\rho)(N) - V(\rho)(N \backslash \{k\})$. Now when $|N| = 3$, AU gives that for any $i, j \in N$, with $i \neq j$,

$$V(\rho)(\{i, j\}) = [V(\rho)(N) - V(\rho)(N \backslash \{i\})] + [V(\rho)(N) - V(\rho)(N \backslash \{j\})].$$

It follows that $x(\rho)(T) = V(\rho)(T)$ if $|T| = 2$. Also, since $V(\rho)(\{k\}) = 0$, we certainly have $x(\rho)(T) > V(\rho)(T)$ if $|T| = 1$. Finally, $x(\rho)(N) = V(\rho)(N)$, by AU again. ■

The game of Example 5.2 can be derived from a game which explicitly includes buyers as well as firms, as follows. Let $N = F \cup T$, with $|F| = 3$ and $|T| = 4$, and let $S^i = \{s_0, s_1\}$ for each $i \in F$. For each $i \in F$, for any $s \in S$, define $I^i(s) = 1$ if $s^i = s_1$, 0 otherwise. For any $A \subseteq N$, define

$r_A = \min\{|A \cap F|, |A \cap T|\}$, and define $F_A = \{B \subseteq |A \cap F| : |B| = r_A\}$. Then for all $A \subseteq N$, let

$$V(s)(A) = \begin{cases} 0 & \text{if } r_A < 2, \\ 2w + \max_{B \in F_A} \left(2p \prod_{i \in B} I^i(s) - \sum_{i \in B} I^i(s) \right) & \text{if } r_A = 2, \\ \max \left\{ 3w + 3p \prod_{i \in F} I^i(s) - \sum_{i \in F} I^i(s), \max_{i \in F} \left(2w + 2p \prod_{k \in F \setminus \{i\}} I^k(s) - \sum_{k \in F \setminus \{i\}} I^k(s) \right) \right\} & \text{if } r_A = 3, \end{cases}$$

where $w = p = 2$.

To interpret this game, suppose there are three firms, each with unit capacity, and four buyers. Each firm decides whether or not to adopt a new technology. (The buyers have singleton strategy sets, which we suppress.) The new technology is compatible with the old, but not vice versa. There is also a network effect. The quantity w is what a buyer would be willing to pay, provided there is another buyer 'present.' (If the new technology is being used, a buyer is willing to pay $w + p$.) Thus, in a given subset, there must be at least two firms and two buyers for value to be created. Further, in such a subset, the players choose the new technology if possible. Finally, if a firm adopts the new technology, it incurs a \$1 fixed operating cost (even if it ends up operating with the old technology).

To determine the Core for this game, first note that $r_{N \setminus \{j\}} = 3$ for any buyer j. That is, each buyer has zero added value. Next, define $c(s) = \sum_{i \in F} I^i(s)$ for all $s \in S$. It suffices to consider three cases, namely $c(s) \leq 1$, $c(s) = 2$, and $c(s) = 3$. If $c(s) \leq 1$,

$$V(s)(N) = 3w - \sum_{i \in F} I^i(s) = 6 - \sum_{i \in F} I^i(s)$$

and for any $i \in F$,

$$V(s)(N) - V(s)(N \setminus \{i\}) = w - I^i(s) = 2 - I^i(s) \tag{A7}$$

If $c(s) \geq 2$,

$$\begin{aligned} V(s)(N) &= 2(w+p) + (w+p) \prod_{i \in F} I^i(s) - \sum_{i \in F} I^i(s) \\ &= 8 + 4 \prod_{i \in F} I^i(s) - \sum_{i \in F} I^i(s). \end{aligned}$$

If $c(s) = 2$, for any $i \in F$,

$$V(s)(N) - V(s)(N \setminus \{i\}) = 2pI^i(s) - I^i(s) = 3I^i(s) \tag{A8}$$

If $c(s) = 3$, for any $i \in F$,

$$V(s)(N) - V(s)(N \backslash \{i\}) = w + p - 1 = 3. \tag{A9}$$

In each of the three cases, equations A7-A9 imply that AU is satisfied. Further, note that the added values of the firms correspond to the payoffs in Figure 5.3 in the text. It is straightforward to show that the allocation in which each player receives its added value is in the Core. By Lemma 5.1, the Core consists of this allocation alone. Finally, since the Core is a singleton, there is a unique reduced game among the firms alone. The characteristic function of Example 5.2 is this reduced game.

Proof for Example 5.3. Let $N = \{f_1, f_2, f_3, b_1, b_2\}$, where f_1, f_2, f_3 are the firms and b_1, b_2 are the buyers. Write the strategy set of f_2 as $S = \{\sigma, \tau\}$, where σ is the status-quo strategy and τ is the repositioning strategy. (We suppress the singleton strategy sets of the other players.)

Fix the indicator function $\chi_{\{\tau\}}$ on S and the indicator χ_T on N. For $T \subseteq N$, let $r_T = \min\{|\{f_1, f_2, f_3\} \cap T|, |\{b_1, b_2\} \cap T|\}$. Then the characteristic functions are given by

$$V(\rho)(T) = \begin{cases} 7r_T & \text{if } f_2 \notin T, \\ 7r_T + \chi_{\{\tau\}}(\rho) & \text{if } f_2 \in T \text{ and } r_T \geq 1, \\ -\chi_{\{\tau\}}(\rho)\chi_T(f_2) & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} V(\rho)(N) &= 14 + \chi_{\{\tau\}}(\rho), \\ V(\rho)(N \backslash \{f_2\}) &= 14, \end{aligned}$$

and, for $i = f_1, f_3$, and $j = b_1, b_2$,

$$\begin{aligned} V(\rho)(N \backslash \{i\}) &= 14 + \chi_{\{\tau\}}(\rho), \\ V(\rho)(N \backslash \{j\}) &= 7 + \chi_{\{\tau\}}(\rho). \end{aligned}$$

Thus

$$V(\rho)(N) - V(\rho)(N \backslash \{f_2\}) = \chi_{\{\tau\}}(\rho),$$

and, for $i = f_1, f_3$, and $j = b_1, b_2$,

$$\begin{aligned} V(\rho)(N) - V(\rho)(N \backslash \{i\}) &= 0, \\ V(\rho)(N) - V(\rho)(N \backslash \{j\}) &= 7, \end{aligned}$$

from which AU is satisfied. By Lemma 5.1, if the Core is nonempty, each player $k \in N$ gets exactly $x^k(\rho) = V(\rho)(N) - V(\rho)(N \backslash \{k\})$. But it is straightforward to verify that $x(\rho)(T) \geq V(\rho)(T)$ for all $T \subseteq N$. ■

Appendix B: Axiomatization of the Confidence Index

Let the choice set X consist of the closed bounded intervals of the real line, i.e.

$$X = \{[p, q] : p, q \in \mathbb{R} \text{ with } p \leq q\},$$

and let $\succsim$ be a preference relation on X. (For our application, fix a player i. The intervals are then the projections onto the ith coordinate axis of the Cores of cooperative games. The assumption is that player i evaluates these intervals according to the preference relation $\succsim$.) Consider the following axioms on $\succsim$:

- A1 (Order): The relation $\succsim$ is complete and transitive.
- A2 (Dominance): If $p > s$, then $[p, q] \succ [r, s]$.
- A3 (Continuity): If $[p_m, q_m] \succ [r_m, s_m]$ for all m, where $[p_m, q_m] \to [p, q]$ and $[r_m, s_m] \to [r, s]$, then $[p, q] \succsim [r, s]$.
- A4 (Positive affinity): If $[p, q] \succ [r, s]$, then $[\lambda p + \mu, \lambda q + \mu] \succ [\lambda r + \mu, \lambda s + \mu]$ for any strictly positive number λ and any number μ.

Proposition B1 *A preference relation $\succsim$ on X satisfies Axioms A1 through A4 if and only if there is a number α, with $0 \leq \alpha \leq 1$, such that*

$$[p, q] \succsim [r, s] \text{ if and only if } \alpha q + (1 - \alpha)p \geq \alpha s + (1 - \alpha)r.$$

Furthermore, the number α is unique.

Proof. Sufficiency and uniqueness are readily checked, so let us establish necessity.

Step 0: Let

$$A = \{\alpha' : \alpha' \in [0, 1] \text{ and } [\alpha', \alpha'] \precsim [0, 1]\}.$$

The set A is well-defined due to Order.

Step 1: The set A contains the point 0, and so is nonempty. To see this, note that Dominance implies that $[0 - 1/n, 0 - 1/n] \prec [0, 1]$ for every integer n. Thus $[0, 0] \precsim [0, 1]$ by Continuity.

Step 2: Set $\alpha = \sup A$. Then $\alpha \in A$. To prove this, it suffices to show that $\alpha \leq 1$. First, note that Dominance implies that $[1 + 1/n, 1 + 1/n] \succ [0, 1]$ for all n. Thus by Continuity,

$$[1, 1] \succsim [0, 1]\,. \tag{B1}$$

Second, note that by definition of α,

$$[\alpha - 1/n, \alpha - 1/n] \precsim [0,1]\,. \tag{B2}$$

Now suppose $\alpha > 1$. Then there is an n^* such that $\alpha - 1/n > 1$ for $n > n^*$. Hence by Dominance,

$$[\alpha - 1/n, \alpha - 1/n] \succ [1,1]\,. \tag{B3}$$

Combining equations B1-B3, and using Order, yields

$$[\alpha - 1/n, \alpha - 1/n] \succ [1,1] \succsim [0,1] \succsim [\alpha - 1/n, \alpha - 1/n]$$

for $n > n^*$, a contradiction. Thus $\alpha \leq 1$, as was to be shown.

Step 3: The number α satisfies $[\alpha, \alpha] \sim [0,1]$. First suppose that $\alpha = 1$. Then $[1,1] \precsim [0,1]$ since $\alpha \in A$. Using equation B1 and Order gives $[1,1] \precsim [0,1] \precsim [1,1]$, from which $[\alpha, \alpha] \sim [0,1]$. Next suppose that $\alpha < 1$. Note that $[\alpha, \alpha] \precsim [0,1]$ since $\alpha \in A$. Suppose, contra hypothesis, that $[\alpha, \alpha] \prec [0,1]$. By the definition of α, it must be that $[\alpha + 1/n, \alpha + 1/n] \succ [0,1]$ for all n. Using Continuity and Order then yields $[\alpha, \alpha] \succsim [0,1] \succ [\alpha, \alpha]$, a contradiction.

Step 4: Using Positive Affinity,

$$[\alpha(q-p)+p, \alpha(q-p)+p] \sim [0(q-p)+p, 1(q-p)+p] = [p,q]\,,$$

or

$$[p,q] \sim [\alpha q + (1-\alpha)p, \alpha q + (1-\alpha)p]\,,$$

as required. ■

Some comments on this result:

i. Discussion of the Axioms Axioms A1 through A3 are standard, and don't require an independent justification in the present context. Axiom A4 is crucial and accounts for the specific form that the representation of preferences takes. In fact, Axiom A4 is immediately implied by the context. It ensures that a player's preferences are invariant over strategically-equivalent games.[1] To see this, consider two cooperative games Γ_1 and Γ_2. Fix a player i, and numbers $\lambda > 0$ and μ. Let Γ_3 be derived from Γ_1 by multiplying the value of every coalition in Γ_1 by λ and, if the coalition contains player i, also adding μ. (If you like, we change the 'currency' in which the game is played and give player i some money from outside the game.) Let Γ_4 be derived from Γ_2 in similar fashion. In cooperative theory, the games Γ_1 and Γ_3 are strategically equivalent, as are the games Γ_2 and Γ_4. Now let player i's Core projection in Γ_1 be $[p,q]$, and that in Γ_2 be $[r,s]$ Then player i's Core

[1] See Owen [8, 1995, pp.215-216], where the (general) concept is called S-equivalence. Theorem X.3.4 there establishes that if two games are S-equivalent, then there is an isomorphism between their imputation sets that preserves the domination relation. This is the basis for treating the two games as equivalent.

projection in Γ_3 will be $[\lambda p + \mu, \lambda q + \mu]$, and that in Γ_4 will be $[\lambda r + \mu, \lambda s + \mu]$. If player i prefers the first interval to the second, then, using strategic equivalence, player i should prefer the third to the fourth. This is precisely Axiom A4.

ii. Application to Biform Games To apply our axiomatization to a biform game, we have to make two choices. The first is that a player doesn't distinguish between two Cores that yield the same projection for that player. This says that what matters to player i is 'what' competition implies for i's payoffs, not 'how' competition implies this range of payoffs. This seems a sensible assumption, but one could certainly imagine the alternative where, to capture the 'how,' players have preferences over (entire) polytopes rather than intervals. This could be an interesting extension of our approach.

Second is whether a player has one preference relation over all of the second-stage intervals, or a potentially different relation for each first-stage strategy profile $s \in S$. That is, do we let the players' preferences depend on their strategic choices, or not? In the definition in the text (Definition 4.1), we assumed not, in part simply to keep things simple. But here too, the alternative case can be considered.

	No	Yes
No	–	–
Yes	–	$v(1, 2) = v(N) = w$

No

	No	Yes
No	–	$v(2, 3) = v(N) = w$
Yes	$v(1, 3) = v(N) = w$	$v(N) = w$

Yes

Figure B1

The second issue is relevant to mutual consistency question raised in Section 6d. Consider the biform game in Figure B1. There are three players, each with two strategies *No* and *Yes*. Player 1 chooses the row, player 2 the column, and player 3 the matrix. Figure B1 depicts the cooperative game associated with each strategy profile, where $w > 0$ and the values of all subsets not shown are 0. First suppose that each player has one confidence index for all three of the second-stage games– denote these α^1, α^2, and α^3 respectively. Then, considering the second-stage games following (*Yes*, *Yes*, *No*), (*No*, *Yes*, *Yes*), (*Yes*, *No*, *Yes*), and (*Yes*, *Yes*, *Yes*) respectively, mutual consistency requires

$$\begin{aligned} \alpha^1 + \alpha^2 &= 1, \\ \alpha^2 + \alpha^3 &= 1, \\ \alpha^3 + \alpha^1 &= 1, \\ \alpha^1 + \alpha^2 + \alpha^3 &= 1, \end{aligned}$$

a contradiction.

Nevertheless, if we allow a player different confidence indices for different second-stage games, then mutual consistency can always be satisfied: For each second-stage game, take an arbitrary point in the Core, project it onto the players' axes, and treat each projected point as a weighted average of the upper and lower endpoints of the projection of the whole Core onto that axis. We repeat what we said in Section 6d–mutual consistency is not conceptually necessary. But if it is wanted, we have shown how it can be achieved.

iii. The Literature Proposition B1 is closely related to Milnor's [6, 1954] derivation of the Hurwicz optimism-pessimism index [4, 1951]. (See also Arrow [1, 1953].[2]) But the difference in contexts is significant. Milnor was concerned only with one-person decision problems, whereas our context is multi-person. More important, Milnor adopted a states-consequences formulation, whereas we consider intervals of possible (monetary) consequences.

Luce and Raiffa [5, 1957, pp.282-298] list various criticisms of the Hurwicz decision criterion. On examination, however, it turns out that these criticisms have force only to the extent that the decision maker faces a problem with well-defined states. In our present, state-free context, they lose their bite. In particular, the Hurwicz criterion cannot be made to satisfy admissibility without, at the same time, losing continuity (Milnor [6, 1954, p.55]). But in our set-up, admissibility and continuity do not conflict. To see this, consider the following extra axiom and proposition.

A5 (Admissibility): If $p > r$, then $[p, q] \succ [r, q]$; if $q > r$, then $[p, q] \succ [p, r]$.

Proposition B2 *A preference relation $\succsim$ on X satisfies Axioms A1 through A5 if and only if there is a number α, with $0 < \alpha < 1$, such that*

$$[p, q] \succsim [r, s] \text{ if and only if } \alpha q + (1 - \alpha)p \geq s + (1 - \alpha)r.$$

Furthermore, the number α is unique.

Proof. Again, sufficiency and uniqueness are immediate, so we establish necessity. Using Proposition B1, we have only to show that $0 < \alpha < 1$. We have $[\alpha, \alpha] \sim [0, 1]$. Admissibility implies $[1, 1] \succ [0, 1]$ and $[0, 1] \succ [0, 0]$. Using Order, we find $[1, 1] \succ [\alpha, \alpha] \succ [0, 0]$. Dominance yields $0 < \alpha < 1$. ■

[2] Ghirardato [3, 2001] obtains a similar representation. He starts with a state-space formulation, but then coarsens the decision maker's perception of the state space. Nehring and Puppe [7, 1996] present conditions under which preferences over sets depend on the maximal and minimal elements of the sets. The treatment in this paper is more specific than theirs in two ways. The choice set consists of closed bounded intervals of the real line, and the representation involves a convex combination of the maximal and minimal elements.

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