

# Financing Terms in the EOQ Model

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## 1 Introduction

This note discusses two terms that are often omitted from the standard *economic-order-quantity* (EOQ) model when the holding cost is based on financing costs. Both of these terms were first identified in Porteus (1985). One was identified explicitly, and the other was implicit in the calculation of an approximation bound. This note was written for anyone teaching the EOQ model to graduate business students. The motivation for this note is the need to reconcile the EOQ model with what students may have learned about working capital in an accounting course.

To review, total cost in the EOQ model is assumed to be the sum of purchase costs, ordering costs, and holding costs, namely

$$c\lambda + \frac{\lambda}{q}k + \frac{q}{2}h,$$

where  $\lambda$  is a constant demand per unit of time,  $c$  is a constant, per-unit purchase cost,  $k$  is a fixed cost per order, and  $h$  is a constant, per-unit holding cost. The decision variable,  $q$ , is the order size. (For an early history of the model, see Erlenkotter (1990). For a modern treatment, see, for example, Porteus (2002) or Zipkin (2000).)

When the holding cost is based on financing costs, the usual approach is to set  $h = \alpha c$ , where  $\alpha$  is the cost of money, typically an interest rate. This gives a total cost of

$$C(q) = c\lambda + \frac{\lambda}{q}k + \frac{q}{2}\alpha c$$

and an optimal order quantity of

$$q^* = \sqrt{\frac{2\lambda k}{h}} = \sqrt{\frac{2\lambda k}{\alpha c}}.$$

For students who understand working capital, it appears that the total cost is understated. (The analysis that follows is called the ‘conventional approach’ in Appendix C of Porteus (2002).) Since the cost of an order is  $k + cq$ , the cost per unit is

$$\frac{k}{q} + c.$$

This implies that the average working capital required by the cycle-stock inventory is

$$\frac{q}{2} \left( \frac{k}{q} + c \right) = \frac{1}{2}(k + cq),$$

and so the cost of this working capital is

$$\frac{\alpha}{2}(k + cq).$$

This suggests a total cost of

$$\begin{aligned} C_1(q) &= c\lambda + \frac{\lambda}{q}k + \frac{q}{2}\alpha c + \alpha\frac{k}{2} \\ &= C(q) + \alpha\frac{k}{2}. \end{aligned}$$

The extra term can be interpreted as the cost of the working capital needed for the fixed cost portion of the cycle stock inventory. The extra term is not relevant to the order size decision, so the usual approach of setting  $h = \alpha c$  yields the optimal decision. Since the EOQ model was designed to answer only the order size question, using  $C(q)$  instead of  $C_1(q)$  does no harm. But if the EOQ model is used to evaluate different vendors, the extra term in  $C_1(q)$  can be relevant. Consider the following example.

### Example

Demand	$\lambda$	32,000
Cost of Money	$\alpha$	20%

*Vendor A* charges a fixed cost of \$4,000 and a per-unit cost of \$20. *Vendor B* charges a fixed cost of \$1,000 and a per-unit cost of \$20.50. With *Vendor A*,  $q^* = 8,000$ , so let  $q_{EOQ} = 8,000$ . With *Vendor B*,  $q^* = 3,951$ , so let  $q_{EOQ} = 4,000$ . The best choice of vendor depends upon whether or not the  $\alpha k/2$  term is included in the analysis:

	<i>Vendor A</i>	<i>Vendor B</i>
$C(q_{EOQ})$	\$672,000	\$672,200
$\alpha k/2$	\$400	\$100
$C_1(q_{EOQ})$	\$672,400	\$672,300

## 2 Present Value Analysis

At the close of the previous section, we indulged in a (often employed) slip of logic: we took a model designed for determining an optimal order size and used it for a different purpose, namely evaluating different vendors. To see if this ‘slip’ can be justified, we consider the true economic cost associated with a particular choice of vendor. In this simple (and classic) version of the EOQ model, it suffices to consider the present value of the payments. (See, for example, Porteus (2002, Appendix C.1) or Zipkin (2000, Section 3.7).) The present value of the payments is

$$\begin{aligned} PV(q) &= (k + cq) + \frac{k + cq}{e^{\alpha\frac{q}{\lambda}}} + \frac{k + cq}{(e^{\alpha\frac{q}{\lambda}})^2} + \dots \\ &= \frac{k + cq}{1 - e^{-\alpha q/\lambda}}, \end{aligned}$$

where  $\alpha q/\lambda$  is the relevant interest rate, and the factor  $e^{\alpha q/\lambda}$  represents continuous discounting. The question now is whether we can find a relationship between  $PV(q)$  and  $C_1(q)$ . We can. Consider the

following expansion of  $PV(q)$ :

$$\begin{aligned}
PV(q) &= \frac{k+cq}{1-e^{-\alpha q/\lambda}} \\
&= \frac{k+cq}{\frac{\alpha q}{\lambda} - \frac{\alpha^2 q^2}{2\lambda^2} + \frac{\alpha^3 q^3}{6\lambda^3} \mp \dots} = \frac{k+cq}{\frac{\alpha q}{\lambda} \left(1 - \frac{\alpha q}{2\lambda} + \frac{\alpha^2 q^2}{6\lambda^2} - \frac{\alpha^3 q^3}{24\lambda^3} \pm \dots\right)} \\
&= \frac{\lambda(k+cq)}{\alpha q} \left(1 + \frac{\alpha q}{2\lambda} + \frac{\alpha^2 q^2}{12\lambda^2} - \frac{\alpha^4 q^4}{720\lambda^4} + \dots\right) \\
&= \frac{1}{\alpha} \left(c\lambda + \frac{\lambda}{q}k + \frac{q}{2}\alpha c + \alpha \frac{k}{2}\right) + \frac{\alpha q(k+cq)}{12\lambda} - \frac{\alpha^3 q^3(k+cq)}{720\lambda^3} + \dots
\end{aligned} \tag{1}$$

From equation (1), it follows that

$$\lim_{\alpha \rightarrow 0} \left(PV(q) - \frac{C_1(q)}{\alpha}\right) = 0, \tag{2}$$

showing that the  $\alpha k/2$  term should be included in the EOQ model.

Before proceeding, it is useful to be specific about our performance criterion. Let  $k_A$  and  $c_A$  be the cost parameters associated with *Vendor A*. Define  $PV^A(q) = PV(q; k_A, c_A, \alpha)$ . Define  $PV^B$  similarly. We want to find a function  $C(q; k, c, \alpha)$  such that

$$PV^A(q) \geq PV^B(q) \Leftrightarrow C^A(q) \geq C^B(q). \tag{*}$$

Ideally, the order size  $q$  would be chosen to optimize the relevant function, but for practical reasons, we will use order sizes that are easy to compute and close to the optimal  $q$ .

Given our performance criterion, it turns out that equation (2) is misleading. (The author thanks Paul Zipkin for this observation.) Since the optimal  $q$  depends upon  $\alpha$ , we need to consider equation (1) with  $q = q^*$ .

$$\begin{aligned}
PV(q^*) &= \frac{C_1(q^*)}{\alpha} + \frac{\alpha q^*(k+cq^*)}{12\lambda} - \dots \\
&= \frac{C_1(q^*)}{\alpha} + \frac{\alpha k}{12\lambda} \sqrt{\frac{2\lambda k}{\alpha c}} + \frac{\alpha c}{12\lambda} \left(\frac{2\lambda k}{\alpha c}\right) - \dots \\
&= \frac{C_1(q^*)}{\alpha} + \sqrt{\frac{\alpha k^3}{72\lambda c}} + \frac{k}{6} - \dots
\end{aligned}$$

Thus,

$$\lim_{\alpha \rightarrow 0} \left(PV(q^*) - \frac{C_1(q^*)}{\alpha}\right) = \frac{k}{6}.$$

This suggests that if  $q$  is chosen optimally, the total cost in the EOQ model will be better represented by

$$C_2(q) = C_1(q) + \alpha \frac{k}{6}.$$

The additional  $\alpha k/6$  factor can be interpreted as incremental interest due to compounding. The first order approximation of this incremental interest is represented by the  $\alpha q(k+cq)/12\lambda$  term in equation (1). With optimal order sizes, the incremental interest on the  $k/2$  part of the cycle stock cost, namely  $\alpha q^* k/12\lambda$ , goes to zero as  $\alpha$  goes to zero. But the incremental interest on the  $cq/2$  part of the cycle stock cost does not;  $\alpha c(q^*)^2/12\lambda$  equals  $k/6$  for all  $\alpha$ .

Since

$$\lim_{\alpha \rightarrow 0} \left( PV(q^*) - \frac{C_2(q^*)}{\alpha} \right) = 0,$$

the function  $C_2$  satisfies our performance criterion (\*) in the limit.

Returning to the Example, the table below lists the costs associated with the different models. In the present value model, the optimal quantities for *Vendor A* and *Vendor B* are 7,934 and 3,935, respectively.

	<i>Vendor A</i>	<i>Vendor B</i>
$C(q_{EOQ})$	\$672,000	\$672,200
$C_1(q_{EOQ})$	\$672,400	\$672,300
$C_2(q_{EOQ})$	\$672,533	\$672,333
$\alpha PV(q_{EOQ})$	\$672,537	\$672,335
$\alpha PV(q_{opt})$	\$672,536	\$672,332

### 3 Discounted Average Value

Porteus (1985) provides the first formal treatment of the analysis in the preceding section by using the discounted average value approach. Porteus (2002) describes the “*discounted average value ... over a specific time interval ... [as] the cost rate that, if incurred continuously over that interval, yields the same NPV as the actual stream.*” The goal is to find a function  $DAV(q)$  such that

$$\begin{aligned} PV(q) &= DAV(q) + \frac{DAV(q)}{e^\alpha} + \frac{DAV(q)}{e^{2\alpha}} + \dots \\ &= \frac{DAV(q)}{1 - e^{-\alpha}}. \end{aligned}$$

Theorem 1 of Porteus (1985), for the parameters given in the first section above, states that

$$\begin{aligned} DAV(q) &= \frac{\lambda(k + cq)}{q} \left( 1 + \frac{\alpha q}{2\lambda} + \frac{\alpha^2 q^2}{12\lambda^2} + O(\alpha^3) \right) \\ &= C_1(q) + \frac{\alpha^2 q(k + cq)}{12\lambda} + O(\alpha^3). \end{aligned}$$

For  $q = q^*$ ,

$$\begin{aligned} DAV(q^*) &= C_1(q^*) + \frac{\alpha^2 k}{12\lambda} \sqrt{\frac{2\lambda k}{\alpha c}} + \frac{\alpha^2 c}{12\lambda} \left( \frac{2\lambda k}{\alpha c} \right) + O(\alpha^2) \\ &= C_2(q^*) + \alpha^{1.5} \sqrt{\frac{k^3}{72\lambda c}} + O(\alpha^2). \end{aligned}$$

Theorem 1 implies that

$$\lim_{\alpha \rightarrow 0} (DAV(q) - C_1(q)) = 0$$

and

$$\lim_{\alpha \rightarrow 0} (DAV(q^*) - C_2(q^*)) = 0. \quad (3)$$

As mentioned in the very beginning, the  $\alpha k/6$  term in  $C_2(q^*)$  is implicit in Porteus (1985), but it is not specifically identified. And it is discussed verbally as “interest on ... interest ... charged.”

Finally, note that since  $1/(1 - e^{-\alpha})$  does not depend on  $q$ ,  $DAV(q)$  satisfies the performance criterion (\*) in the limit.

## 4 Compound Interest Effect

We have just shown how adding two terms to the traditional EOQ model leads to a better approximation of the present value model. The intuitive explanation for the first of these terms, namely  $\alpha k/2$ , is straightforward. In fact, it was the motivation for this note. In this section, we provide some intuition for the second term, namely  $\alpha k/6$ . In the two analyses above, we have described it as interest on interest. In this section, we will try to isolate the source of this term. We start by asking where, in the present value model, is the cycle stock? In the traditional EOQ model, the cycle stock cost can actually be ‘seen.’ Each order cycle, units arrive and are drawn down. Consequently, we see the inventory value cycle in the same way. But in the present value model, where is the cycling? We have an increasing step function, in which the total costs incurred jump with each order. Further, the revenue stream does not depend on the order policy, so it can’t be used to explain the ‘missing’ cycle stock. To answer this question, consider Figure 1 below. It depicts the balance over time. ( $\lambda$  is denoted by  $l$ .)

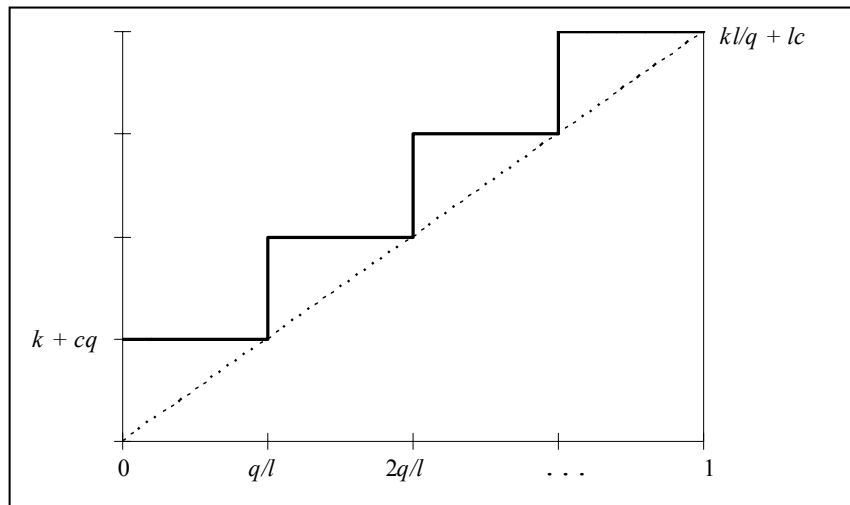


Figure 1

To find the ‘missing’ cycle stock, consider the average balance in Figure 1:

$$\frac{1}{2}(k + cq) + \frac{1}{2}\left(k\frac{\lambda}{q} + c\lambda\right).$$

The first part of the expression is the ‘missing’ cycle stock. It is the amount by which the balance exceeds the dotted, diagonal line. As the order frequency increases, this area decreases. Loosely speaking, one can think of this as money being spent on units before the money is needed to be spent. This parallels nicely with thinking of cycle stock as units on-hand before they are needed.

To demonstrate that  $k/6$  is due to a compound interest effect, first consider what the simple interest would be on the balances in Figure 1. This would be just the interest rate times the average balance, namely

$$\frac{\alpha}{2}(k + cq) + \frac{\alpha}{2}\left(k\frac{\lambda}{q} + c\lambda\right). \quad (4)$$

To introduce a compound interest effect, we next determine how the simple interest in equation (4) accumulates over time. This is depicted in Figure 2 below. ( $\lambda$  is denoted by  $l$ ,  $\alpha$  is denoted by  $a$ , and  $n = \lambda/q$ .) Note that the value at  $t = 1$  reduces to the value in equation (4).

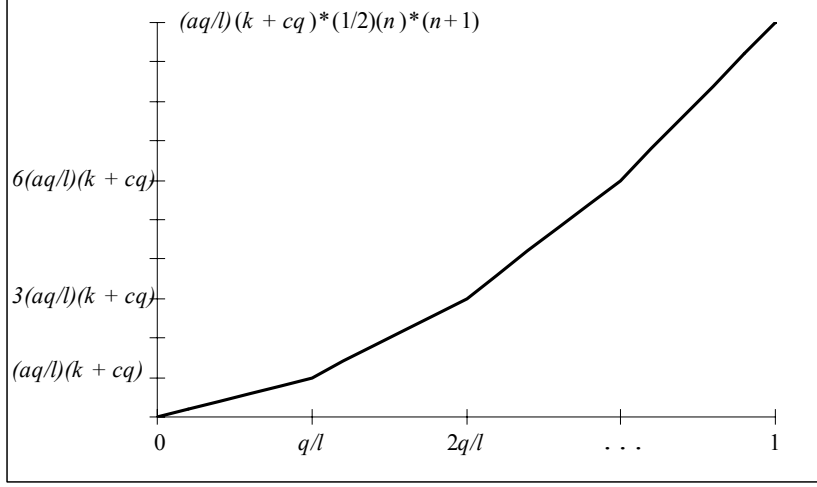


Figure 2

To approximate the effects of compound interest, we compute interest on the interest. Let  $i$  denote the order interval in the above graph, i.e.  $i$  ranges from 1 to  $n = \lambda/q$ . The average simple interest in interval  $i$  is

$$\begin{aligned} SI_i(q) &= \alpha \left( \frac{q}{\lambda} \right) (k + cq) \left( \frac{i(i-1)}{2} + \frac{i}{2} \right) \\ &= \alpha \left( \frac{q}{\lambda} \right) (k + cq) \left( \frac{i^2}{2} \right). \end{aligned}$$

The interest on the simple interest in interval  $i$  is then

$$\begin{aligned} II_i(q) &= \alpha \left( \frac{q}{\lambda} \right) SI_i(q) \\ &= \alpha^2 \left( \frac{q}{\lambda} \right)^2 (k + cq) \left( \frac{i^2}{2} \right). \end{aligned}$$

Summing over the intervals yields

$$\begin{aligned} II(q) &= \sum_{i=1}^n II_i(q) = \alpha^2 \left( \frac{q}{\lambda} \right)^2 (k + cq) \sum_{i=1}^n \frac{i^2}{2} \\ &= \alpha^2 \left( \frac{q}{\lambda} \right)^2 (k + cq) \left( \frac{n^3}{6} + \frac{n^2}{4} + \frac{n}{12} \right) \\ &= \alpha^2 (k + cq) \left( \frac{\lambda}{6q} + \frac{1}{4} + \frac{q}{12\lambda} \right). \end{aligned}$$

At  $q = q^*$ ,

$$\lim_{\alpha \rightarrow 0} \frac{II(q^*)}{\alpha} = \frac{\alpha c (q^*)^2}{12\lambda} = \frac{k}{6}.$$

Thus, we see why  $c(q^*)^2/12\lambda$  can be thought of as a compound interest effect on the  $cq$  part of the cycle stock.

## References

- [1] Erlenkotter, Donald, "Ford Whitman Harris and the Economic Order Quantity Model," *Operations Research*, 1990, **38:6**, 937-946.
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- [4] Zipkin, Paul H., *Foundations of Inventory Management*. New York: McGraw-Hill, 2000.