

# Valuing a Liquidity Discount

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**T**his article is motivated by the fact that the majority of securities in the financial markets are not frequently traded and hence a discount can be applied to properly reflect such a deprived privilege. First, only a small fraction of corporate bonds are actively traded. Institutional investors hold the majority of these bonds until maturity. Similarly, term loans that assume a large part of the liabilities of a corporation are transacted extremely rarely. It is also important to note that nearly two-thirds of the \$14.2 trillion in mortgage-backed securities are highly illiquid.<sup>1</sup> Finally, the most prominent example of illiquid securities is a recent contract that allows for easy and convenient transfer of credit risk—credit default swaps, which reached over \$60 trillion in 2007 and now stay at a level of roughly \$40 trillion. Most of these contracts are illiquid.<sup>2</sup>

For such contracts that are not easily transactable, a discount must be placed on the price. In this article, we show within a simple equilibrium-modeling framework that such a discount can be exceptionally large. Putting our analysis in the perspective of a quadratic utility setting, we discover that strikingly large discounts must be placed on security prices for even very reasonable levels of risk preference.

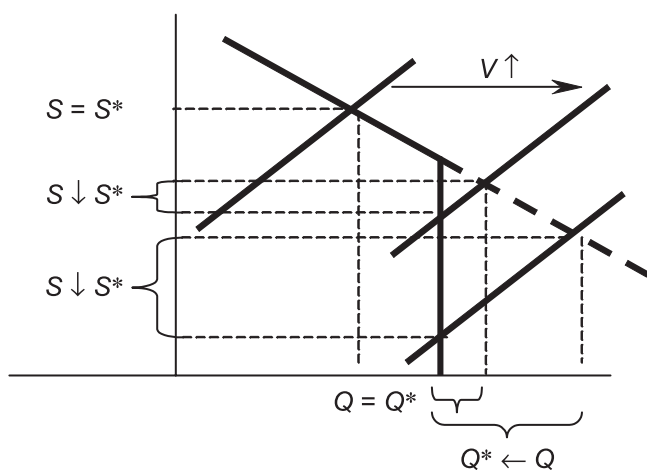
In the recent financial crisis, we have witnessed unprecedented compressions of asset prices so extreme that professionals and

regulators have started to question, and have proposed to abandon, marking to market. The advocates of abandoning marking to market argue that prices have started to deviate from economic fundamentals. Among countless other examples during the crisis, the most noticeable was Bear Stearns' stock price, which went from \$30 on March 14, 2008 (Friday), to \$10 on March 17, 2008 (Monday). Over that weekend, the stock price of Bear Stearns went down by two-thirds, even though the economic fundamentals of Bear Stearns could not have possibly changed so substantially.<sup>3</sup> To explain such a large drop in price even though no fundamentals changed, we propose a liquidity discount model that demonstrates economically how such large drops in prices, with no changes in fundamentals, are possible. Our model also helps salvage the long tradition of marking to market that Wall Street has prided itself on. With our model, price discount due to illiquidity can be computed and consequently the "fair value" of a security can be restored.

## A LIQUIDITY DISCOUNT AS A PUT OPTION

A liquidity discount occurs when supply reacts more sensitively to the economy than demand in a nonlinear fashion.<sup>4</sup> The liquidity discount reaches an extreme as demand reaches its maximum capacity. Exhibit 1 depicts this

## EXHIBIT 1 A Demand/Supply Analysis of a Liquidity Discount



Note: As the conditions of the economy change (via the state variable  $V$ ), demand and supply change accordingly. The vertical bar in the demand function (at quantity  $Q^*$ ) is where the extreme situation of the liquidity squeeze takes place. In the exhibit,  $S$  represents the liquid price and  $S^*$  represents the illiquid price.

general idea. In Exhibit 1, we assume that the demand and supply of a financial asset jointly determine the equilibrium price of the asset at any given time. The economy is represented by a single state variable (say, wealth), symbolized by  $V$ . As the economy grows, the supply curve moves to the right, as does the demand curve. To derive the liquidity discount model, we must assume that supply grows faster than demand in that less elastic demand function, which is the main cause for the price discount.

As a result, the growth of the economy results in an increased equilibrium quantity and a lower price. In the exhibit, we assume that demand is totally insensitive to economic growth and has a maximum capacity at  $Q^*$ . Clearly, our model requires that demand be less sensitive to supply, and such an exaggerated demonstration is just for the purpose of easy exposition. At the maximum capacity of the demand function, the quantity can no longer increase and equilibrium can draw the price only down.

In the exhibit, the vertical axis represents price ( $S$ ) and the horizontal axis represents quantity ( $Q$ ). We let the liquidity-constrained price be  $S^*$ . In a usual situation,  $S = S^*$ . In a liquidity-squeezed situation (represented by the situation where demand reaches its maximum capacity  $Q^*$ ),  $S > S^*$ .

The liquidity discount in this setting is defined as the impact of the slower demand reaction (than supply), which becomes dramatic as the maximum is approached. Based on the above exhibit, we can draw the following conclusion, depicted in Exhibit 2.

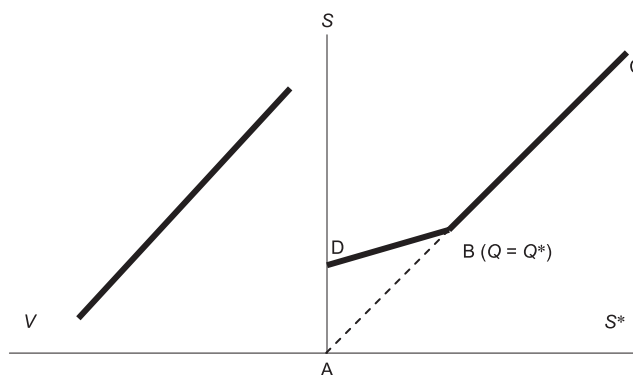
On the left side of Exhibit 2, we depict the relationship between the economy (represented by, say, wealth  $V$ ) and the perfectly liquid price  $S$ . As shown in Exhibit 1, this is a downward-sloping curve. To be shown later, the curvature of  $S$  in  $V$  must be convex in order to obtain the liquidity discount. If the relationship is linear (under which  $Q^*$  cannot exist), then there can be no liquidity discount.<sup>5,6</sup>

On the right side, we depict the relationship between the perfectly liquid price  $S$  and the liquidity-constrained price  $S^*$ . The line  $ABC$  is a 45-degree line on which the illiquid price is equal to the liquid price. At point B where the quantity reaches its maximum capacity  $Q^*$ , the illiquid price starts to decrease rapidly due to problems in liquidity and bends over toward point D. Again, the linear result (by  $BD$  line) is just a demonstration. In the next section, we derive the formal model, and the graph between points B and D is not linear and has a reflection point.

From the exhibit for which the discount is depicted linearly, it can be seen that liquidity can be explained by a put option. That is,

$$S^* = S - \text{put} \quad (1)$$

## EXHIBIT 2 A Liquidity Discount as a Put Option



Note: The left panel describes the relationship between the economic state variable and liquid price, and the right panel is a result from Exhibit 1 where the illiquid price equals the liquid price, reduced by a put option value.

In the next section, we demonstrate that this is not a simple put due to convexity requirements. But intuitively, the simple put explanation serves the purpose well and can even be used as an approximation, which has already been discussed in the industry.<sup>7</sup> What is offered here is an equilibrium model that is consistent with the Merton model widely used in modeling credit risk. Although we do not link our liquidity model yet to credit risk in this article, it would be quite straightforward to do so.

In this section, the demonstration is intuitive but not realistic. In reality, the left and the right panels are related. In other words, how the liquidity price reacts to the economy affects the magnitude of the liquidity discount. Furthermore, it is not likely to know the exact amount of  $Q^*$ . In the next section, we endogenize the relationship amount of the economy  $V$ , the perfectly liquid price  $S$ , and the liquidity-constrained price  $S^*$ . We let the liquid price be a convex function of the economy and derive the illiquid price directly. In doing so, we arrive at an equilibrium without the specification of  $Q^*$ .

## THE MODEL

In this section, we develop a formal pricing model for “illiquidity,” which is defined as inability to transact.<sup>8</sup> Inability to transact is the exact description of Exhibit 1, in which lack of demand leads to no transactions. In a perfect world where the Black–Scholes model holds true, transactions take place at any time and investors trade securities and rebalance their portfolios continuously. When transactions are not permitted to be continuous, an extra risk is borne by the buyer, and the buyer, in return, should ask for compensation and a lower price of the security.

Once continuous trading is not allowed, the market is not (dynamically) complete in the Duffie–Huang sense [1985] and the resulting model is not preference free. As a result, one must adopt a utility function to gauge the magnitude of the risk premium. We first present a model with the quadratic utility so the standard CAPM can be used. It is straightforward to extend the model to a broader class of utility functions. Using a more complex utility function is certainly better in terms of explaining reality and providing model flexibility, but it loses the closed-form CAPM formula.

The price of an arbitrary security at current time  $t$  when no trading is allowed until a future time  $T$  can be

priced by the most fundamental discounted cash flow method:

$$X(t) = e^{-\xi(T-t)} E_t[X(T)] \quad (2)$$

where  $X(t)$  is the cash flow of an arbitrary security at time  $t$ ,  $E_t[\cdot]$  is the conditional expectation taken at time  $t$ , and  $\xi$  is the (continuously compounded) risk-adjusted return for the security. Such a discounted cash flow method requires further modeling substances in order to be operational.

To build an explicit model for Equation (2), we first assume the Black–Scholes/CAPM model in which the underlying economy (represented by a single state variable, say, wealth) obeys the following lognormal process:

$$\frac{dV}{V} = \mu dt + \sigma dW \quad (3)$$

where  $W(t)$  is the standard Wiener process and  $\mu$  and  $\sigma$  are (continuous time) mean and standard deviation of the return of the state variable  $V$ .

A perfectly liquid price,  $S$ , is a contingent claim on the state variable.<sup>9</sup> The assumption of the liquidity discount we make in this article indicates that  $S$  must be a monotonic function in wealth  $V$ . In a theorem we prove later, the function must be convex in order to arrive at the liquidity discount. There are a number of ways to construct such an explicit function. For simplicity, we choose a function that imitates a put option (as opposed to an arbitrary polynomial function) for the following reasons. First, the maturity parameter in the put option can be made equal to the liquidity discount horizon. This provides for extreme convenience in modeling the liquidity discount. Second, there is a closed-form solution for the price of a perfectly liquid asset, which allows us to compare it with the price of an equal asset constrained by liquidity. Last, the “strike price” in the put function ideally characterizes the strength of the liquidity squeeze. The higher the strike price, the stronger the liquidity discount, and the zero strike price ideally represents perfect liquidity. However, the use of the put function does suffer from one drawback. It exists at a maximum value for the liquid stock price (at the strike level), which could be unrealistic; as the economy contracts, the price of the security could become unboundedly high. Fortunately, in this situation, the liquidity discount is small and the impact would be small.

Given that  $S$  is the price of a security that can be continuously traded, it can be easily computed with the Black–Scholes model when its payoff for a fixed time horizon (time to maturity) mimics a put:

$$\begin{aligned} S(t) &= e^{-r(T-t)} \hat{E}_t[S(T)] \\ &= e^{-r(T-t)} \hat{E}_t[\max\{K - V(T), 0\}] \\ &= e^{-r(T-t)} KN(-d_-) - V(t)N(-d_+) \end{aligned} \quad (4)$$

where  $K$  is the “strike” price that reflects the strength of the liquidity squeeze,  $\hat{E}_t[\cdot]$  is the risk-neutral expectation, and

$$d_{\pm} = \frac{\ln V(t) - \ln K + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Note that the adoption of the put payoff is just a convenience to incorporate convexity. Due to the fact that the liquidity-constrained price does not have a closed-form solution (except for the extreme cases), we implement a binomial model to approximate this Black–Scholes result. For the remainder of the article, when we refer to the Black–Scholes model, it is actually the binomial approximation.

We also note that although the liquidity discount model has no closed-form solution, the binomial implementation is actually better in that the model can then be easily augmented to include credit risk, as proposed by Chen [2002], and Chen, Fabozzi, and Sverdløve [2010]. In this broader model, interactions between liquidity and credit risk can be studied.

Let  $S^*$  be the liquidity-constrained price where trading of the security is not permitted until time  $T$ . Equation (2), as a result, can be replaced with the following equation:

$$X(t) = e^{-\xi(T-t)} E_t[f(V(T), \Theta), 0] \quad (5)$$

where  $\Theta$  represents other parameters needed for the model,  $X(t)$  is either  $S$  or  $S^*$  depending on whether the equation is used for an instantaneous period or a longer time period, respectively. Note that in an instantaneous period, the physical expectation as in Equation (5) is identical to the risk-neutral expectation (details to be explored later in Theorem 1), and hence Equation (5) provides the solution to the liquid price. The expected return,  $\xi$ , must follow the capital asset pricing model (CAPM).

Note that  $S$  and  $S^*$  are identical securities with just one difference:  $S^*$  cannot be transacted (or hedged) until time  $T$ . The purpose of this article is to derive the price difference between  $S$  and  $S^*$ . We shall prove that  $S^* < S$ , and this is the model for a liquidity discount. As we shall demonstrate, such a discount can be substantial, even under very reasonable assumptions.

We begin our model for a liquidity discount with the standard CAPM. Note that the Black–Scholes model is consistent with the standard CAPM as follows:<sup>10</sup>

$$\xi = E\left[\frac{dX}{X}\right] = rdt + \eta\left[E\left[\frac{dV}{V}\right] - rdt\right] \quad (6)$$

where  $r$  is the risk-free rate and

$$\eta = \frac{\partial X}{\partial V} \frac{V}{X} \quad (7)$$

is the elasticity of the state-contingent claim with respect to the underlying economic state variable (similar to an option on its underlying stock). Note that this result is exact only in continuous time. Black and Scholes [1973] prove that the option price is a CAPM result with the underlying stock as the market. Note that  $\eta = \beta$  as shown in the following:

$$\begin{aligned} \beta &= \text{cov}\left[\frac{dX}{X}, \frac{dV}{V}\right] / \text{var}\left[\frac{dV}{V}\right] \\ &= \frac{V}{X} \frac{\text{cov}[X_v dV, dV]}{dV} \\ &= \frac{V}{X} X_v \\ &= \eta \end{aligned} \quad (8)$$

where  $X_v = \partial X / \partial V$ . Rewriting Equation (6) in discrete time for a small interval  $h$ , we have:

$$\frac{E[X(t+h)]}{X(t)} = (1+rh) + \beta\left[\frac{E[V(t+h)]}{V(t)} - (1+rh)\right] \quad (9)$$

As a result, we can derive a pricing model for the security as:

$$X(t) = \frac{1}{R(t, t+h)} (E[X(t+h)] - X_v \{E[V(t+h)] - RV(t)\}) \quad (10)$$

where  $R(t, T) = 1 + r(T - t) \approx e^{r(T-t)}$ . This result provides an alternative proof of the CAPM argument by Black and Scholes. Note that

$$\begin{aligned} X_V &= \frac{\text{cov}[X(t+h), V(t+h)]}{\text{var}[V(t+h)]} \\ &= \beta^s \end{aligned} \quad (11)$$

is the “dollar beta” (that is, delta is dollar beta). Hence, Equation (10) is also an alternative derivation to Jensen’s model [1972].

It is important to note that CAPM holds only under one of the two assumptions: quadratic utility for the representative agent in the economy or normality for the returns of the risky assets. Consequently, that the above CAPM result holds in the Black–Scholes model without any utility assumption must be due to the fact that option returns and stock returns are both normally distributed. It is clear that the stock return is normally distributed, as Equation (3) postulates. The option return is normally distributed only under continuous time. This is because in continuous time, the option value is linear in the stock and must follow the same distribution of that of the stock; hence, its return is normally distributed.<sup>11</sup>

In a discrete time where  $h$  is large, the option return is no longer normally distributed and Equation (10) can no longer hold without the assumption of quadratic utility that guarantees the validity of the CAPM–quadratic utility function. The quadratic utility function is a bad assumption in that it has the wrong sign for the relative risk aversion. However, fortunately, the liquidity discount model we derive in this article suffers very little from the second-order effect of the utility function. All the model requires is that investors are risk averse.

Under quadratic utility, CAPM holds for all securities. Hence, Equation (10) holds for an  $h$  that is not infinitesimally small as follows:

$$X(t) = \frac{1}{R(t, T)} (E[X(T)] - \beta^s \{E[V(T)] - R(t, T)V(t)\}) \quad (12)$$

where  $R(t, T) = e^{r(T-t)}$  and the dollar beta is computed, for any  $T$ , as:

$$\beta^s = \frac{\text{cov}[X(T), V(T)]}{\text{var}[V(T)]} \quad (13)$$

Equation (12) is the main result of our model. It states that under quadratic utility, all assets must follow CAPM in determining their values. What we shall demonstrate is that when a liquidity discount is present, the value computed by Equation (12) is less than the perfectly liquid price computed by the Black–Scholes model.

To derive our liquidity discount model, we shall show first that if the relationship between the economy (represented by the state variable  $V$ ) and the liquid price ( $S$ ) is linear, then the liquidity discount is nil. Then, we show that a liquidity discount can exist only if the relationship between the economy and the liquid price is convex.

**Theorem 1.** *When the payoff is linear, then the liquidity discount is nil.*

**Proof.** Note that if the payoff is linear,  $X(T) = aV(T) + b$ , then the following results hold:

- i.  $E[X(T)] = aE[V(T)] + b$
- ii.  $\text{var}[X(T)] = a^2 \text{var}[V(T)]$
- iii.  $\text{cov}[V(T), X(T)] = a \text{var}[V(T)]$
- iv.  $\beta^s = a$

where the expectation, variance, and covariance operations are taken under the physical measure. As a result, following Equation (12), we have:

$$X(t) = \frac{1}{R(t, T)} \{E[X(T)] - a(E[V(T)] - R(t, T)V(t))\} \quad (14)$$

where  $R(t, T) = e^{r(T-t)}$  and

$$E[X(T)] - R(t, T)X(t) = aE[V(T)] - aR(t, T)V(t) \quad (15)$$

which gives rise to the following result:

$$X(t) = \frac{1}{R(t, T)} \{aR(t, T)V(t) + b\} \quad (16)$$

Note that this is exactly the result of risk-neutral pricing (expected value with “ $\hat{E}$ ”), that is,

$$\begin{aligned} X(t) &= \frac{1}{R(t, T)} \hat{E}[aV(T) + b] \\ &= aV(t) + b \frac{1}{R(t, T)} \end{aligned} \quad (17)$$

This is also the result described by the Martingale representation theorem, which indicates that any contingent claim under continuous trading can be replicated by the underlying asset and the risk-free asset. The result shown in Equation (17) states that Equation (12) also computes the liquid price, that is,  $X(t) = S(t)$ , if the relationship between the economy and the asset price is linear.<sup>12</sup>

### Q.E.D.

In continuous time, there are only two states in every infinitesimal time step (as described in Duffie and Huang [1985]) and hence no liquidity discount can exist. This indicates that if trading is continuous, then at each infinitesimal step the payoff is linear, and as a result, Theorem 1 holds. In other words, continuous trading breaks up a fixed time horizon into small infinitesimal time steps, each of which is a linear payoff, and hence, the liquidity discount does not exist. In the next section, we shall demonstrate this property in a numerical example, and Theorem 1 is explicitly demonstrated.

Theorem 2, in the following, proves that if the payoff is not linear and is convex, then the equilibrium price is always less than the linear price, which according to Theorem 1, is the risk-neutral price where continuous rebalancing is possible.

Although a general proof with any form of convexity is not available, a proof based on the binomial model that is consistent with our formulation of Equation (3) is provided. In particular, as any three points define the convexity, we use a two-period binomial model for the proof. It can be inferred that with more points (more periods) in the binomial model, the proof stays valid.

**Theorem 2.** *If the payoff is not linear and convex, then  $X^{cvx}(t) < X^{lnr}(t)$ , where  $X^{lnr}(t)$  is defined in [Theorem 1] and identical to  $S(t)$ , which is the perfectly liquid price, and  $X^{cvx}(t)$  is the same as  $S^*(t)$ , which is the liquidity-constrained price.*

**Proof.** We use a two-period binomial model to demonstrate the proof. We let  $V_{ij}$  represent the level of the state variable at time period  $i$  and state  $j$ , respectively. In a two-period binomial model, the three state variable values at time 2 are  $V_{20}$ ,  $V_{21}$ , and  $V_{22}$ , representing low, medium, and high prices, respectively. Without loss of generality, we also let  $V_{21} = V_0$ . Similarly, we also let  $X_{ij}$  represent the state-contingent claim price at time

period  $i$  and state  $j$ , respectively, where  $X_{20}$ ,  $X_{21}$ , and  $X_{22}$  represent low, medium, and high prices, respectively.

We let the convex payoff differ from the linear one by slightly altering the middle one as follows:  $X_{21}^{cvx} = X_{21}^{lnr} - \varepsilon$ , where  $\varepsilon$  is an arbitrary small positive amount to create the convexity of  $X$  in  $V$ . The real probability per period is  $p$ , which is time invariant. Consistent with the notation above, we symbolize compounding at the risk-free rate in two periods as  $R(0,2)$ . Our goal is to prove that:

$$\begin{aligned} X_0^{cvx} &= \frac{1}{R(0,2)} \{E[X_2^{cvx}] - \beta^s(E[V_2] - R(0,2)V_0)\} \\ &< \frac{1}{R(0,2)} \{E[X_2^{lnr}] - \beta^s(E[V_2] - R(0,2)V_0)\} \\ &= X_0^{lnr} \end{aligned} \quad (18)$$

where the symbol “2” in the subscript represents the three states in period 2, and “cvx” or “lnr” in the superscript represent convex or linear payoffs, respectively. In the binomial model, the three physical probabilities are  $p$ ,  $2p(1-p)$ , and  $(1-p)^2$  for high, medium, and low states, respectively. Hence, we have:

- i.  $E[V_2, X_2^{cvx}] = E[V_2, X_2^{lnr}] - \varepsilon V_{21} 2p(1-p)$
- ii.  $E[X_2^{cvx}] = E[X_2^{lnr}] - 2\varepsilon p(1-p)$
- iii.  $E[V_2]E[X_2^{cvx}] = E[V_2]E[X_2^{lnr}] - 2E[V_2] \varepsilon p(1-p)$
- iv.  $\text{cov}[V_2, X_2^{cvx}] = \text{cov}[V_2, X_2^{lnr}] + 2\varepsilon p(1-p)(E[V_2] - V_{21})$

and then the dollar beta under the convex function can be derived as:

$$\beta^{cvx} = \beta^{lnr} + \frac{\varepsilon 2p(1-p)(E[V_2] - V_{21})}{\text{var}[V_2]} \quad (19)$$

As a result,

$$\begin{aligned} &\beta^{cvx} \{E[V_2] - R V_0\} \\ &= \beta^{lnr} \{E[V_2] - R(0,2)V_0\} \\ &+ \frac{2p(1-p)\varepsilon(E[V_2] - V_{21})(E[V_2] - R(0,2)V_0)}{\text{var}[V_2]} \end{aligned} \quad (20)$$

and

$$\begin{aligned}
X_0^{\text{cvx}} &= E[X_2^{\text{cvx}}] - \beta^{\text{cvx}} \{E[V_2] - R(0,2)V_0\} \\
&= E[X_2^{\text{hr}}] - 2p(1-p)\epsilon - \beta^{\text{hr}} \{E[V_2] - R(0,2)V_0\} \\
&\quad - \frac{2p(1-p)\epsilon(E[V_2] - V_{21})(E[V_2] - R(0,2)V_0)}{\text{var}[V_2]} \\
&= X_0^{\text{hr}} - 2p(1-p) \\
&\quad \times \epsilon \left\{ 1 + \frac{(E[V_2] - V_{21})(E[V_2] - R(0,2)V_0)}{\text{var}[V_2]} \right\} \\
&< X_0^{\text{hr}} \tag{21}
\end{aligned}$$

This is because, clearly,  $E[V_2] > V_{21} = V_0$  (the expected payoff should be larger than today's value) and  $E[V_2] > R(0,2)V_0$  (the expected return should be more than the risk-free rate) to avoid arbitrage.

#### Q.E.D.

From Theorem 1, we know that  $X_0^{\text{hr}} = S(t)$  as the liquid price. Here,  $X_0^{\text{cvx}} = S^*(t)$  represents the illiquid price. Hence, in summary,  $S^*(t) < S(t)$  for all values of finite  $R(0,2)$  and the theorem is proved.

Note that under linearity (between wealth and liquid price), there exists no liquidity discount, which is the same result as continuous trading. As a consequence, the price under linearity  $X_0^{\text{hr}}$  is identical to the price under continuous trading  $S(t)$ . Similarly, when the relationship between wealth and liquidity price is convex, a liquidity discount exists. And the price  $X_0^{\text{cvx}}$  represents the price under the liquidity squeeze,  $S^*(t)$ .

### THE ANALYSIS

In this section, we provide a numerical example to demonstrate the enormity of the liquidity discount. The main model is Equation (12). While the analysis in this section is based on an arbitrarily chosen set of parameter values, the result holds in general. While Equation (12) is closed form, Equation (13) needs to be computed numerically as:

$$\beta^s = \frac{\sum_{j=1}^n \binom{n}{j} p^j (1-p)^{(n-j)} \{V_j(T) - \bar{V}(T)\} \{X_j(T) - \bar{X}(T)\}}{\sum_{j=1}^n \binom{n}{j} p^j (1-p)^{(n-j)} [V_j(T) - \bar{V}(T)]^2} \tag{22}$$

where

$$\begin{aligned}
\bar{V}(T) &= \sum_{j=1}^n \binom{n}{j} p^j (1-p)^{(n-j)} V_j(T) \\
\bar{X}(T) &= \sum_{j=1}^n \binom{n}{j} p^j (1-p)^{(n-j)} X_j(T)
\end{aligned}$$

are the means of the economic state variable (wealth) and the state-contingent claim, respectively. In order to demonstrate convergence to the Black-Scholes model in continuous trading, we adopt the binomial framework of Cox, Ross, and Rubinstein [1979] with  $n$  periods. Given that the binomial model will converge to the Black-Scholes model as  $n$  gets large, we use a sufficiently large  $n$  to represent the limiting Black-Scholes case. In other words, we demonstrate that under continuous trading, there is no discount for illiquid trading.

For the sake of easy exposition, we set up the following base case for the binomial model:

$n$	100
$T - t$	1 yr.
$\sigma$	0.5
$\mu$	10%
$r$	5%
$V(t)$	80

To carry out a numerical example, we need to have an explicit functional form for the relationship between the state variable and the state-contingent claim that represents either the liquid price (linear payoff) or the illiquid price (convex payoff), in the context of the Cox, Ingersoll, and Ross model [1985]. As a convenience, we choose a put payoff for the task. The put payoff is convex and negatively monotonic in the underlying economy and hence serves the purpose well. We choose an arbitrary strike of 100 to characterize convexity and note that the higher the strike, the higher the convexity.

We first compute the price of the perfect liquid contingent claim, that is,  $X(t) = S(t)$ . Note that the liquid price is such that trading takes place continuously and is represented by the Black-Scholes model. In the binomial model where  $n = 100$ , the value of the put (representing the perfectly liquid asset price) is almost identical to the Black-Scholes price of \$25.85.<sup>13</sup> We feel that for the sake of computational time, this is a minor enough difference for us to demonstrate the value of the liquidity discount.

It provides the comfort that  $n = 100$  is a good enough proxy for continuous trading. In the rest of the article, we use the binomial model with  $n = 100$  as the benchmark to examine the properties of the liquidity discount model.

Next, we turn to computing the illiquid price, that is,  $X(t) = S^*(t)$ . We start our analysis with  $n = 1$  and demonstrate that in this case, the liquid price (which is computed by the binomial model based on the risk-neutral probabilities) is identical to the illiquid price (which is computed by the CAPM based on the physical probabilities). In other words, when rebalancing is permitted at every node in the binomial model, the illiquid price is identical to the liquid price.

In a one-period model (i.e.,  $n = 1$ ), it is clear that the up and down movements in the binomial model are  $u = e^{\sigma\sqrt{\Delta t}} = e^{0.5 \times 1} = 1.6487$  and  $d = 1/u = 0.6065$ . The state variable lattice and the stock payoff are given below (left and right, respectively):

State Variable $V$	State-Contingent Claim Payoff $S$
80	28.07
131.90	0
48.52	51.48

The risk-neutral probabilities are  $\hat{p} = \frac{\exp(r\Delta t) - d}{u - d} = \frac{\exp(5\% \times 1) - 0.6065}{1.6487 - 0.6065} = 0.4267$  and  $1 - \hat{p} = 0.5733$ . The liquid price is (call):  $0.5733 \times 51.48 + e^{-5\%} = 28.07$ , as in the above binomial tree.

The physical probabilities are  $p = \frac{\exp(\mu\Delta t) - d}{u - d} = \frac{\exp(10\% \times 1) - 0.6065}{1.6487 - 0.6065} = 0.4785$  and  $1 - p = 0.5215$ . Following Equation (12), we arrive at the same exact price of \$28.07 where the expected level of the state variable is \$88.41; the dollar beta  $\beta^S$  is  $-0.6174$ ; the expected value of the illiquid price is \$26.84; and the risk-free discount is 0.9512.

The fact that the price of the illiquid asset equals the price of the liquid asset suggests that the illiquid price is independent of the physical probability  $p$  (and also independent of  $\mu$ ).<sup>14</sup> The reason is that in a single-period binomial model, the option payoff is linear in the underlying asset and Theorem 1 applies. This result is extremely crucial in our model in that once we approach continuous trading/rebalancing, the binomial model suggests that the option price within a period is linear in the underlying asset, and as a consequence, the illiquid price must equal the liquid price—the boundary condition it must satisfy by definition.

As  $n$  becomes large, the liquidity discount becomes large. As in the base case where  $n = 100$ , the discount is substantial. The binomial value is \$25.86 and the liquidity-constrained price, that is, Equation (12), is \$24.47, representing a 5% discount.

To demonstrate the binomial implementation of our model, we let  $i$  be the time index and  $j$  be the state index. At the end of the binomial lattice,  $i = n$ . At each time  $i$ , the state  $j$  is labeled as  $0 \leq j \leq i$ . The level of the state variable, the price of the liquid security, and the price of the illiquid security for the economy are then labeled as  $V_{ij}$ ,  $S_{ij}$ , and  $S_{ij}^*$ , respectively. The risk-neutral probability and the physical probability are defined as usual and given above.

In the following, we present results when the risk preference represented by the Sharpe ratio ranges from 0 (risk-free case) to 1.6. Rebalancing frequency is represented by  $k$ . And  $k = n - 1$  represents perfect rebalancing or continuous trading (i.e., for 100 periods, rebalancing 99 times in between is identical to continuous trading).

In the binomial model, the number of periods  $n$  must be divisible by one plus the rebalancing frequency, that is,  $k + 1$ , to avoid unnecessary numerical errors. In the base case where  $n = 100$ ,  $k$  can be 0, 1, 3, 4, 9, ..., 99. Take  $k = 3$  as an example; rebalancing is allowed at  $i = 25, 50, \text{ and } 75$ . At each of these times, Equation (12) is used to compute the illiquid price at every node at the given time. Specifically, at  $i = 75$ , Equation (12) is used to compute values at the nodes that are represented by  $j = 0 \sim 75$  (i.e.,  $S_{75,0}^*$  till  $S_{75,75}^*$ ). Then at  $i = 50$ , Equation (12) is again used to compute values where  $j = 0 \sim 50$  (i.e.  $S_{50,0}^*$  till  $S_{50,50}^*$ ) using the prices from  $S_{75,0}^*$  till  $S_{75,75}^*$ . This process repeats backwards until we reach today's price, which is  $S_0^*$ .

We compute a number of liquid and illiquid prices under various scenarios. Unless otherwise mentioned, the values of the input variables are taken from the base case. Note that the liquidity discount is more severe when investors are more risk averse. To measure the magnitude of risk aversion, we adopt the Sharpe ratio on the underlying state variable, which is excess return scaled by the volatility. We simulate various degrees of Sharpe ratio from 0 (risk-free case) to 1.6, with the volatility scenarios from 0.2 to 0.8. At the risk-free rate of 5%, we obtain the required rate of return ( $\mu = r + \lambda\sigma$  where  $\lambda$  is the Sharpe ratio) from 5% (risk-free case) to 133% ( $\lambda = 1.6$ ).

The results are summarized in Exhibit 3. The top panel shows binomial parameter values where  $u = \exp(\sigma\sqrt{\Delta t})$  and  $d = 1/u$  under  $n = 100$ . The middle

panel contains different expected returns under different volatility scenarios for each Sharpe ratio. The bottom panel presents physical probability values using the binomial formula  $p = [\exp(\mu \Delta t) - d] / [u - d]$ . Interestingly, each Sharpe ratio roughly corresponds to a physical probability value.

Combining the information of the physical probabilities and other input values, we can then compute liquid and illiquid prices using Equation (12). Note that when the Sharpe ratio is 0, there is no liquidity discount and the liquid value equals the illiquid value, as proved by Theorem 1. This allows us to examine the magnitude of the liquidity discount as a function of risk preference. Exhibit 4 provides all the liquid (Sharpe ratio is 0) and illiquid prices. The strike price for the result is set at 100, and the state variable is set at 80.

### EXHIBIT 3

#### Input Values

	$\sigma$ (volatility)			
	0.8	0.6	0.4	0.2
n	100	100	100	100
u	1.083287	1.061837	1.040811	1.020201
d	0.923116	0.941765	0.960789	0.980199

Sharpe ratio	$\mu$ (drift)			
	0.8	0.6	0.4	0.2
0	0.0500	0.0500	0.0500	0.0500
0.4	0.3700	0.2900	0.2100	0.1300
0.8	0.6900	0.5300	0.3700	0.2100
1.2	1.0100	0.7700	0.5300	0.2900
1.6	1.3300	1.0100	0.6900	0.3700

	p (physical probability)			
	0.8	0.6	0.4	0.2
0	0.4831	0.4892	0.4963	0.5075
0.4	0.5032	0.5092	0.5163	0.5275
0.8	0.5232	0.5293	0.5363	0.5476
1.2	0.5434	0.5494	0.5564	0.5676
1.6	0.5636	0.5695	0.5765	0.5877

Note: Numerical values reported in the article are based on p, the physical probabilities. In computing numerical values, the risk preference is represented by the Sharpe ratio, which ranges from 0 (risk-free case) to 1.6.

The top panel shows binomial parameter values, where  $u = \exp(\sigma\sqrt{\Delta t})$  and  $d = 1/u$  under  $n = 100$ . The middle panel contains different expected returns  $\mu = r + \lambda\sigma$ , where  $\lambda$  is the Sharpe ratio) under different volatility scenarios for each Sharpe ratio. The bottom panel presents physical probability values using the binomial formula  $p = [\exp(\mu\Delta t) - d] / [u - d]$ .

Reported in Exhibit 4 are simulated liquid (Sharpe ratio is 0)<sup>15</sup> and illiquid prices (Sharpe ratio is greater than 0). Note that by construction, our model for illiquid prices degenerates to liquid prices as the Sharpe

### EXHIBIT 4

#### Simulated Illiquid Prices Under Various Volatility Levels, Rebalancing Frequencies, and Risk Aversion

	$\sigma$ (volatility)			
	0.8	0.6	0.4	0.2
k = 0				
0	35.4303	29.0305	22.7066	16.9860
0.4	29.7751	25.1990	20.5670	16.2768
0.8	21.9833	19.2094	16.5282	14.3871
1.2	14.1540	12.7340	11.6111	11.4428
1.6	7.8894	7.2781	7.0476	8.0259
k = 9				
0	35.4303	29.0305	22.7066	16.9860
0.4	34.8789	28.6789	22.5240	16.9334
0.8	33.9291	28.0265	22.1478	16.8068
1.2	32.5311	27.0385	21.5570	16.5993
1.6	30.6061	25.6581	20.7160	16.2998
k = 19				
0	35.4303	29.0305	22.7066	16.9860
0.4	35.1880	28.8759	22.6264	16.9630
0.8	34.7790	28.5946	22.4642	16.9087
1.2	34.1950	28.1806	22.2166	16.8222
1.6	33.4223	27.6247	21.8784	16.7021
k = 49				
0	35.4303	29.0305	22.7066	16.9860
0.4	35.3702	28.9921	22.6867	16.9803
0.8	35.2700	28.9231	22.6469	16.9670
1.2	35.1297	28.8233	22.5871	16.9461
1.6	34.9489	28.6925	22.5073	16.9176
k = 99				
0	35.4303	29.0305	22.7066	16.9860
0.4	35.4303	29.0305	22.7066	16.9860
0.8	35.4303	29.0305	22.7066	16.9860
1.2	35.4303	29.0305	22.7066	16.9860
1.6	35.4303	29.0305	22.7066	16.9860

Note: Reported here are simulated liquid (Sharpe ratio is 0) and illiquid prices (Sharpe ratio is greater than 0) where k represents rebalancing frequency. As required by the model, when continuous trading is reached (k = 99), we obtain liquid prices and the risk preference does not matter (bottom panel). The liquidity discount is at a maximum when no trading/rebalancing is allowed (k = 0).

ratio approaches 0. Also, as required by the model, when continuous trading is reached ( $k = 99$ ), we obtain liquid prices and risk preference does not matter (bottom panel). The liquidity discount is at a maximum when no trading/rebalancing is allowed ( $k = 0$ ).

To visualize the effect, we translate the values in Exhibit 4 from dollar terms to percentage terms. In each case, the liquid price serves as the benchmark (named Black–Scholes value). This is the value consistent with continuous trading. Various comparisons are provided in Exhibit 5 and Exhibit 6 as follows.

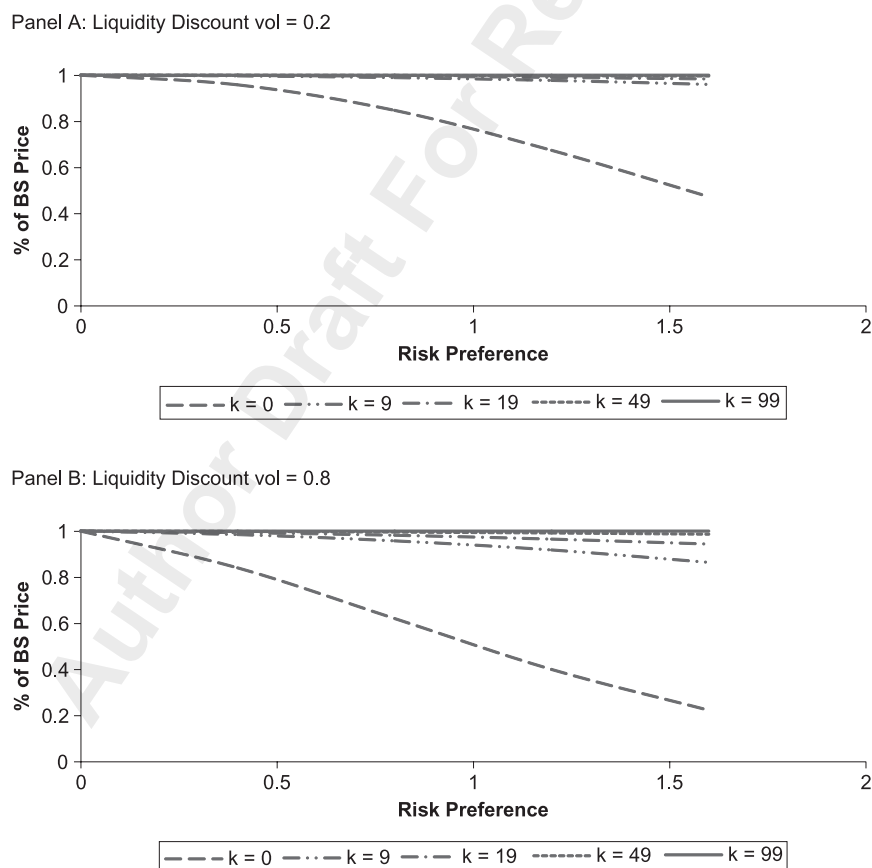
In Panel A of Exhibit 5, we present the result of the liquidity discount under various trading frequencies. The binomial model of  $n = 100$  is used as the benchmark and regarded as the perfectly liquid price, which in the

limiting case, converges to the Black–Scholes model. As a result, in the perfectly liquid case where  $n = 100$ , the number of rebalancing times is  $k = 99$ . Under no rebalancing,  $k = 0$ , which represents the case of extreme illiquidity when investors hold their securities to maturity. Panel A of Exhibit 5 plots the result using  $\sigma = 0.2$ . The horizontal axis is the expected rate of return of the stock, used to represent risk preference.

It is clear that as the model allows for “continuous” rebalancing (represented by  $k = 99$  in the case of  $n = 100$ ), the price should be the same as the “Black–Scholes price” where continuous rebalancing is part of the assumption. In Exhibit 5, we do see that the price ratio (of equilibrium over Black–Scholes) is 1 throughout the whole range of risk preference. When  $k = 0$ , the discount

## EXHIBIT 5

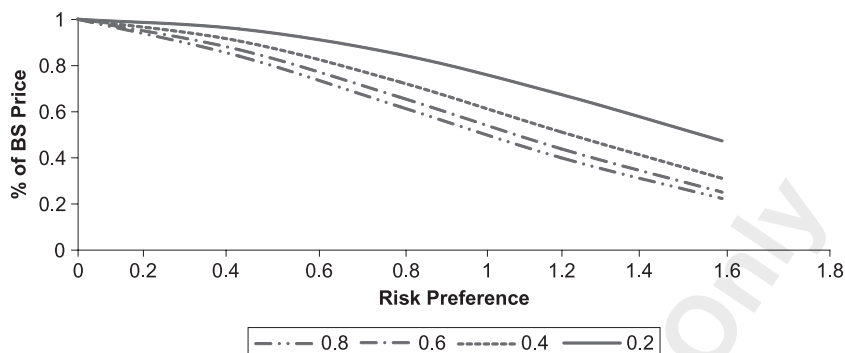
### A Liquidity Discount Under Various Rebalancing Frequencies



Note: Both panels depict how the liquidity discount becomes more severe as risk aversion increases. Risk aversion plotted on the horizontal axis is represented by the Sharpe ratio. The Black–Scholes price is the  $n = 100$  binomial price. The parameter  $k$  is the number of rebalancing frequencies where  $k = 0$  represents no rebalancing (most severe liquidity discount) and  $k = 99$  represents continuous trading (no liquidity discount).

## EXHIBIT 6

### A Liquidity Discount Under Various Volatility Levels at $k = 0$



Note: This Exhibit is the same as Exhibit 5 but plots a liquidity discount for various volatility scenarios. The x-axis is the risk preference that is represented by the Sharpe ratio.

because of illiquidity can be severe. We can see from Exhibit 5 that the discount is as bad as 40% as the Sharpe ratio reaches 1.6 and no rebalancing is permitted.

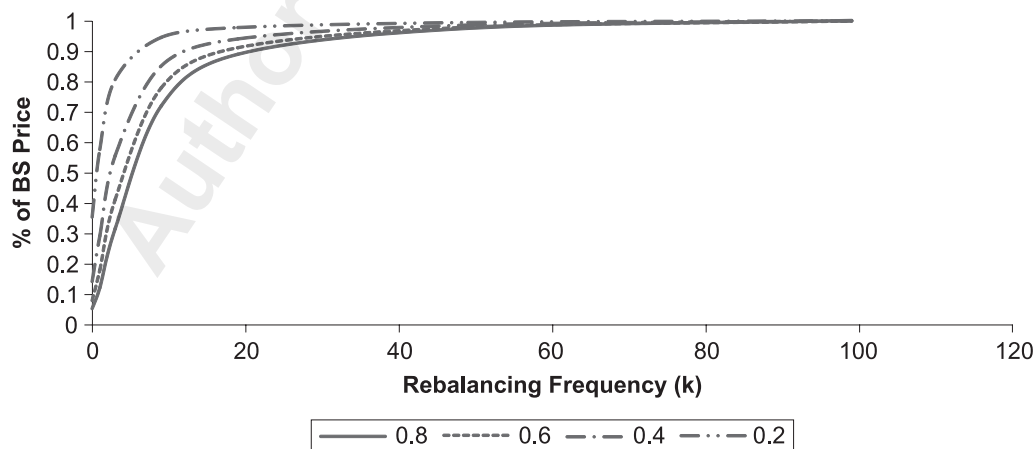
Panel B of Exhibit 5 is similar to Panel A with a higher volatility value (0.8). As we can see, the liquidity discount is more severe as volatility increases. We recall that during the 2007–2008 crisis, volatility was high. For example, in the case of Lehman Brothers (see Chen, Chidambaran, Imerman, and Soprazetti [2011]), volatility in many months of 2008 exceeded 100%.

Exhibit 6 presents the same result as Exhibit 5 but examines how various volatility levels affect the liquidity discount. Exhibit 6 sets  $k$  to be 0 for the maximum amount of the liquidity discount. As we can see from the exhibit, the deterioration of the asset price is rather fast. As volatility rises, the deterioration is faster.

The next necessary step is to test the model against liquid prices—in other words, when we permit perfect liquidity, that is,  $k = n - 1$ . In this situation, the equilibrium price must equal the Black–Scholes (or binomial) price. Exhibit 7 presents the result of convergence under

## EXHIBIT 7

### The Convergence Result



Note: This exhibit provides a pictorial representation of the convergence test of the liquidity discount model. As the rebalancing frequency approaches  $n - 1$  (in our example, it is  $k = n - 1 = 100 - 1 = 99$ ), the illiquid price should converge to the liquid price.

various volatility levels. As we can see, convergence is faster when volatility is smaller.

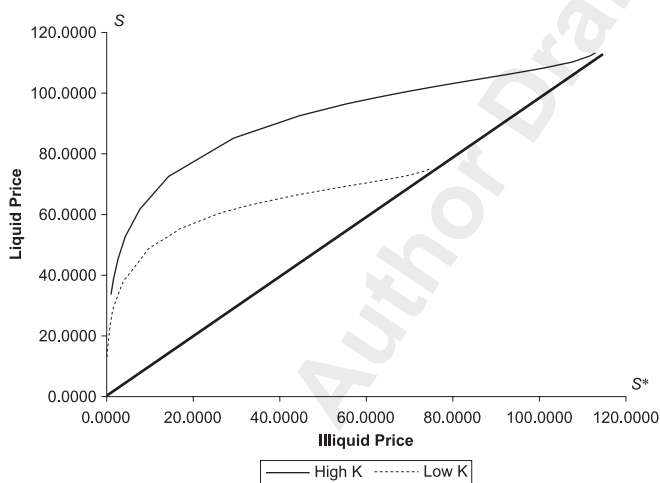
In Exhibit 8, we provide the result of our model and the relationship between liquid and illiquid prices. This is the main result of our model, which describes the illiquid price ( $S^*$ ) as a function of the liquid price  $S$ . Exhibit 8 is similar to Exhibit 2 but is presented with our model. The physical probability is set at 0.6 to exaggerate the result for the visual presentation.

We see in Exhibit 8 that as the liquid price decreases, the illiquid price decreases but at a much faster rate. This is consistent with the description in Exhibit 2 where the B–D line bends over to touch the vertical axis.

When the liquid price is high, the liquidity discount is small and the two prices are equal to each other. In the numerical example plotted in Exhibit 8, toward the right where the prices are both high, the liquidity discount disappears and the curve approaches the 45-degree line asymptotically.

Note that in our model, there is no explicit put option as in Exhibit 2. Our liquidity discount model is derived by limiting trading/rebalancing in a binomial model. Our liquidity discount is computed by assuming a quadratic utility function so that pricing can be achieved

**EXHIBIT 8**  
Liquid vs. Illiquid Prices



Note: This exhibit is similar to the right panel of Exhibit 2. The inputs used to produce this exhibit are the same as those used in Exhibit 5 and Exhibit 6 but with two different “strike prices”: 100 and 120. The high strike ( $K = 120$ ) is the solid line, and the low strike ( $K = 100$ ) is the dotted line.

via the CAPM. Nevertheless, the result of our model describes the liquidity discount as a put option.

We also note that the liquidity discount model derived in this article is closely connected with the relationship between the economy (represented by a single state variable: wealth) and the perfectly liquid price. As a result, the convexity of the relationship determines the severity of the liquidity discount.

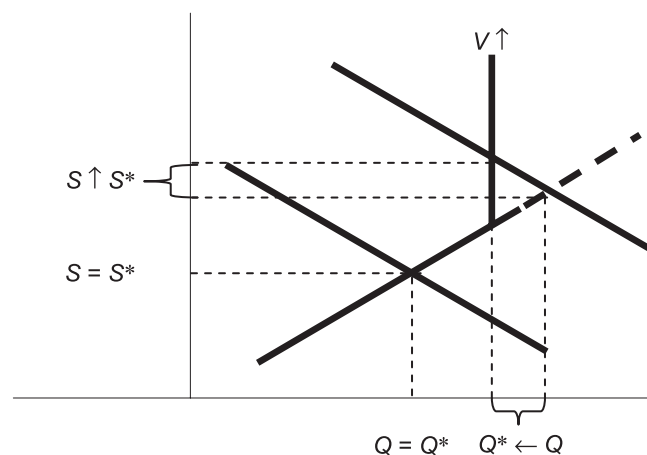
## A LIQUIDITY PREMIUM

One immediate extension to our model is that we can analyze assets with liquidity premiums. While there are a number of financial assets that suffer from a liquidity discount (i.e., lower prices due to limited trading), other assets, not necessarily financial, enjoy a liquidity premium. Gold, oil, and even real estate are good examples of such assets.

Symmetrical to the cause of the liquidity discount, the cause of the liquidity premium is the limited capacity of supply. Parallel to the model of the liquidity discount, when an agent is risk averse and the payoff of a security is concave, limited supply causes such an asset to enjoy the liquidity premium. The analysis is straightforward, as follows.

Similar to Exhibit 1, Exhibit 9 depicts a situation in which demand is more sensitive to economic changes

**EXHIBIT 9**  
A Demand/Supply Analysis of a Liquidity Premium

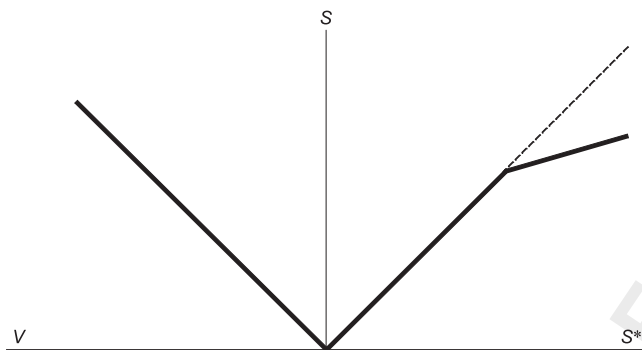


Note: This exhibit is symmetrical to Exhibit 1 but is formulated to price a liquidity premium. In this case, the supply function is constrained (at  $Q^*$ ).  $V$  represents economy,  $S$  represents the liquid price, and  $S^*$  represents the illiquid price.  $Q$  is quantity.

than supply and supply is bounded by a fixed quantity  $Q^*$ . As the economy grows, it approaches maximum capacity and a liquidity squeeze (supply-driven) takes place. Contrary to the demand-driven squeeze, now the price under the liquidity squeeze is higher with the squeeze than without the squeeze. In this situation, the equilibrium price rises but quantity falls.

A counterpart of Exhibit 2 is shown in Exhibit 10. The left panel of Exhibit 10 describes the relationship between the economy  $V$  and the perfectly liquid price  $S$ . The right panel describes the relationship between the liquid price  $S$  and the illiquid price  $S^*$ .

### EXHIBIT 10 A Liquidity Premium as a Call Option



Note: This exhibit is symmetrical to Exhibit 2, but the supply-driven liquidity squeeze results in a liquidity premium.  $V$  represents economy,  $S$  represents the liquid price, and  $S^*$  represents the illiquid price.

From Exhibit 10, it is clear that the liquidity premium can be described as a call option:

$$S^* = S + \text{call} \quad (23)$$

As in the liquidity discount case, in our model, there is no need to explicitly model the call option. As long as the liquid price is concave in the state variable of the economy, the call-option-like result will be naturally derived.

The following theorem shows that in such a case, the liquidity-squeezed price  $S^*$  is higher than the perfectly liquid price  $S$  when the liquid price is a concave function of the economy.

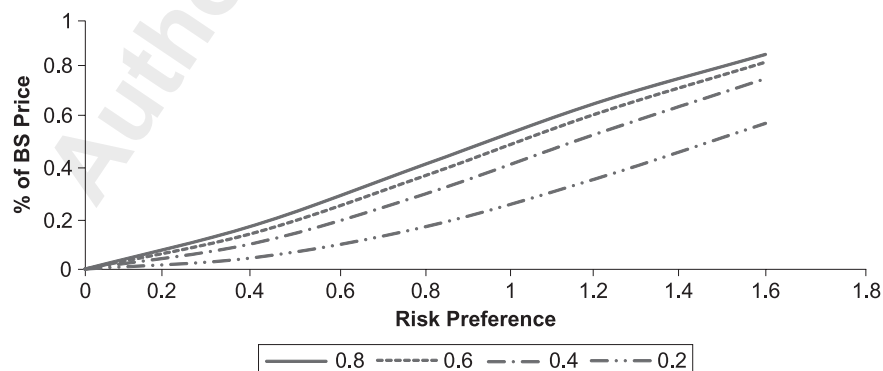
**Theorem 3.** *If the payoff is concave, then  $S^{lr}(t) < S^{av}(t)$ , where  $S^{lr}(t)$  is defined in [Theorem 1] as  $S(t)$ , which is the perfectly liquid price, and  $S^{av}(t)$  is the same as  $S^*(t)$ , which is the liquidity-constrained price.*

**Proof.** Repeat the same procedure as for the proof of Theorem 2 and the result follows.

**Q.E.D.**

The model for the liquidity premium is a result that represents the case where supply of the asset is less sensitive than demand to the underlying economy. This is useful in explaining prices of several commodities in the current situation, such as gold and oil. These commodities are assets with very limited supply. As the demand for such assets grows stronger, prices rise

### EXHIBIT 11 A Liquidity Premium Under Various Volatility Levels at $k = 0$



Note: This exhibit is symmetrical to Exhibit 6 but describes the case of a liquidity premium. The x-axis is the risk preference that is represented by the Sharpe ratio.

disproportionally to the rest of the economy, resulting in liquidity premiums. Our model argues that if the liquid price is linear in the state variable, then the price of the asset will rise, but there is no liquidity premium. If the liquid price is concave in the state variable, then there is a liquidity premium. Similar to the liquidity discount, such a liquidity premium can be substantial even when the fundamental economy does not change materially.

A counterpart of Exhibit 6 is plotted in Exhibit 11. As the expected return becomes lower, the impact of the liquidity premium is more profound. This exhibit is generated with the same inputs as the base case, with the relationship between the liquid price and the economic state variable as  $S = \min\{V, K\}$ .

## SUMMARY AND FUTURE RESEARCH

In this article, we provide a liquidity discount (premium) model that shows that severe discounts in prices are possible in a normal economic environment. The model is derived in an equilibrium framework and hence can be combined with (and extended to include) existing models for credit risk, such as Merton's [1974] or Geske's [1977].

We discover that a liquidity discount is associated with a convex reaction of the security price to the economy (and symmetrically, a liquidity premium is associated with a concave reaction). Given that securities with convex payoffs are regarded as risky assets and securities with concave payoffs are regarded as insurance assets, we conclude that a liquidity discount is more likely associated with risky assets and a liquidity premium is more likely associated with insurance assets.

Such an extension of our work is necessary because liquidity risk and credit risk are highly intertwined. In the last economic crisis, the collapse of Bear Stearns and Lehman Brothers, because of the credit crisis, triggered a liquidity crisis for the entire industry. The liquidity model derived in this article has the potential to be extended to include credit risk in the structural modeling framework. Such extensions are important in that with such integrated modeling work, we can understand individual liquidity and credit risks as well as their interaction. This remains to be the future work.

Another possible extension of the model is to adopt a more reasonable utility function. Note that the convenience of the quadratic utility is the closed-form CAPM. Yet the price to be paid is its incorrect relative

risk aversion. With its more reasonable utility function, the liquidity discount model can be more realistic.

## ENDNOTES

I thank Bill Filonuk, Danny Tan, An Yan, and Sris Chatterjee for their valuable comments on this article.

<sup>1</sup>According to Wikipedia ([http://en.wikipedia.org/wiki/Mortgage-backed\\_security](http://en.wikipedia.org/wiki/Mortgage-backed_security)), which does not reveal a proper reference, only \$5 trillion out of the \$14.2 trillion that is guaranteed by a government-sponsored enterprise (GSE) can be considered liquid.

<sup>2</sup>See Chen, Fabozzi, and Sverdlow [2010].

<sup>3</sup>On Sunday of that weekend (March 16, 2008), JP Morgan offered \$2 per share for the stock. Because the Fed would have offered a discount window had there been any liquidity problems, the price went up to \$10.

<sup>4</sup>Or equivalently, as the economy contracts, supply decreases faster than demand. In other words, a liquidity discount exists when supply is more sensitive to the economy than demand.

<sup>5</sup>The converse statement is that a liquidity discount can still exist without the existence of  $Q^*$ , as long as the relationship between wealth and the liquid price is convex. Hence, our model is much more general than the exhibit depicts. The use of  $Q^*$  is for the convenience of exposition.

<sup>6</sup>To put our model in the context of the Cox, Ingersoll, Ross model [1985], the security price  $S$  is a state-contingent claim that needs to be risky in order to generate the liquidity discount. If the state-contingent claim is insurance, then it should be concave in wealth, and in this case, our model predicts a liquidity premium. We shall explore this theory later.

<sup>7</sup>For example, see the Kshitij Consultancy's website: <http://www.kshitij.com/moneymkt/icdrates.shtml> and wikipedia: [http://en.wikipedia.org/wiki/Liquidity\\_risk](http://en.wikipedia.org/wiki/Liquidity_risk)

<sup>8</sup>While our analysis of liquidity is focused on large compressions of prices, we find that it is not inconsistent with microstructure literature, for example, Hodrick and Moulton [2009] who define perfect liquidity as the "ability to trade at quantity desired at time desired and at a price not worse than the uninformed expected value."

<sup>9</sup>See Cox, Ingersoll, and Ross [1985] for the general equilibrium theory of the pricing of financial securities.

<sup>10</sup>Black and Scholes [1973] derived their option-pricing model with the CAPM; see Section 3 of their article.

<sup>11</sup>This is obvious as by Ito's lemma, the drift and diffusion of the dynamics of the option (i.e., innovations of the option price) are both deterministic.

<sup>12</sup>Clearly, it can be seen that the result is independent of risk preference (to be represented by the actual probability  $p$ ).

<sup>13</sup>The binomial price with  $n = 100$  is \$25.84, which differs from the Black–Scholes price by \$0.01. As  $n = 160$ , there is no difference in the price of the two models in two decimal places. The difference is \$0.0013.

<sup>14</sup>It is easy to show algebraically that the physical probability  $p$  disappears from the pricing formula when  $n = 1$ .

<sup>15</sup>It is straightforward to see that when the Sharpe ratio is 0,  $\hat{p} = p$  and the illiquid price equals the liquid price.

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