1 Radon-Nikodým Process and Change of Measure

Recall in a binomial model, we have two measures: the risk-neutral measure $\tilde{P}$ and the actual measure $P$. Stock price usually overperform money market under the actual measure, however, when it comes to derivative pricing, the right measure to work with is the risk-neutral measure. One key characteristic of risk-neutral measure is that under this measure, every discounted price process is a martingale. This is not true under the actual measure. Therefore, in order to price derivatives, we need to change the measure from the actual to the risk-neutral.

The bridge that connects actual measure to risk-neutral measure is the Radon-Nikodým derivative $Z$:

$$
Z := \frac{d\tilde{P}}{dP} > 0.
$$

In order to price derivatives at any time $t \in [0,T]$, we need conditional expectation under $\tilde{P}$, and we have the Radon-Nikodým process to handle it:

$$
Z_t = E(Z|\mathcal{F}_t) = \frac{d\tilde{P}}{dP}|_{\mathcal{F}_t}.
$$

In particular, we know from binomial model that,

$$
E(Y|\mathcal{F}_s) = \frac{1}{Z_s}E(Z_t Y|\mathcal{F}_s),
$$

for any $\mathcal{F}_t$-measurable random variable $Y$.

Given a Itô process $\{Y_t\}_{t \in [0,T]}$ with non-zero drift, we know it is not a martingale under the current measure $P$. However, we can change the measure to $\tilde{P}$ so that under this new measure, $\{Y_t\}_{t \in [0,T]}$ is a martingale. This means, for any $0 \leq s < t \leq T$, we have

$$
\tilde{E}(Y_t|\mathcal{F}_s) = Y_s.
$$

In view of (3), the above equality means

$$
E(Z_t Y_t|\mathcal{F}_s) = Z_s Y_s,
$$

or equivalently, the process $\{Z_t Y_t\}$ is a martingale under the current measure $P$. If both $Z_t$ and $Y_t$ are Itô processes, then $d(Z_t Y_t)$ must have no drift term. To get this differential, we need Itô’s product rule$^1$,

$$
d(Z_t Y_t) = Z_t dY_t + Y_t dZ_t + dY_t dZ_t.
$$

2 Discounted Stock price & Measure Change

Consider a generalized geometric Brownian motion dynamics for stock price:

$$
dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \ S(0) > 0.
$$

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$^1$This is a special case of multi-dimensional Itô formula: for function $f(x,y) = xy$, where both $x$ and $y$ are spatial variables.
We also consider an adapted interest rate process $R(t)^2$, the discount factor is thus given by
\begin{equation}
D(t) = e^{-\int_0^t R(u) du}.
\end{equation}
Note that $D(t) = f(X(t))$ for $f(x) = e^{-x}$ and $X(t) = \int_0^t R(u) du$. Applying Itô’s formula we have
\begin{equation}
dD(t) = -R(t)D(t)dt.
\end{equation}

Let us study the dynamics of the discounted stock process $\{D(t)S(t)\}_{t \in [0,T]}$. Using Itô’s product rule we have
\begin{align*}
d(D(t)S(t)) &= (\alpha(t) - R(t)) D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\
&= \sigma(t)D(t)S(t)(\Theta(t)dt + dW(t)), \quad \Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.
\end{align*}

We call $\Theta(t)$ the market price of risk\(^3\). In general, $\alpha(t) \neq R(t)$, so the discounted stock price process is not a martingale under the current measure $P$. To turn it into a martingale (for pricing purpose), we need to change measure, so that under the new measure, there is no drift term. Effectively, we need to move to a new measure $\tilde{P}$ so that, under that measure, the drifted process
\begin{equation}
\tilde{W}(t) := \int_0^t \Theta(u) du + W(t),
\end{equation}
is a Brownian motion. Then under the measure $\tilde{P}$, we have a martingale.
\begin{equation}
d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t).
\end{equation}

The Radon-Nikodým process needed to fulfill the desired measure change turns out to be fairly simple. Consider the exponential martingale
\begin{equation}
Z(t) = \exp\left\{-\int_0^t \Theta(u) dW(u) - \frac{1}{2} \int_0^t \Theta^2(u) du\right\}.
\end{equation}

We use $\{Z(t)\}_{t \in [0,T]}$ to define a new measure $\tilde{P}$. Then the process in (11) is a standard Brownian motion\(^4\) under $\tilde{P}$. This new measure $\tilde{P}$ is called the risk-neutral measure.

Let us examine the discounted value process of a portfolio. We knew it is also a martingale under the risk-neutral measure in a binomial market. We show that this is still true in a geometric Brownian motion market.

Let $X(t)$ be the value of a portfolio, and $\Delta(t)$ be the number of shares of stocks held at time $t$. Then we have
\begin{align*}
x(t) &= \Delta(t)ds(t) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= \Delta(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)], \quad \Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.
\end{align*}

Using Itô’s product rule, from (9), (12) and (14) we obtain that
\begin{equation}
d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] = \Delta(t)d(D(t)S(t)).
\end{equation}

Since under the risk-neutral measure $\tilde{P}$, $\{D(t)S(t)\}_{t \in [0,T]}$ is a martingale, the process $\{D(t)X(t)\}$ must also be a martingale.

\(^2\)Interest rate may change over time, we choose the most general setting here. But when it comes to computation of stock option price, we choose a constant interest rate $r$.

\(^3\)There is a similar notion in portfolio management: Sharpe ratio.

\(^4\)To prove this fact, one need to use Le\'vy’s characterization of Brownian motion: a martingale whose quadratic variation is equal to $t$. It is easy to check the second part: $(d\tilde{W}(t))^2 = dt$. To show $\{W(t)\}_{t \in [0,T]}$ is a martingale under $\tilde{P}$, it is equivalently to show that $\{Z(t)\tilde{W}(t)\}$ is a martingale under the old measure $P$, so checking the drift of $d(Z(t)\tilde{W}(t))$ will be enough.
3 Risk-neutral Pricing

Recall in a binomial market, we drive a portfolio to replicate the payoff of an European option at maturity. Then by arbitrage argument, we proved that the value of the replicating portfolio at time \( t \) is the price of the option at the same time. Suppose we are able to find such a portfolio\(^5\) so that an European option that pays \( V(T) \) at its maturity \( T \) is fully replicated:

\[
X(T) = V(T). \tag{16}
\]

Then since under the risk-neutral measure \( \tilde{P} \), \( \{D(t)X(t)\}_{t \in [0,T]} \) is a martingale. We have that

\[
D(t)X(t) = \tilde{E}(D(T)V(T)|\mathcal{F}_t) = \tilde{E}(D(T)V(T)|\mathcal{F}_t). \tag{17}
\]

By no arbitrage, the price of the option at time \( t \), \( V(t) \) is given by

\[
V(t) = X(t) = \frac{1}{D(t)}\tilde{E}(D(T)V(T)|\mathcal{F}_t) = \tilde{E}\left(e^{-r\int_t^TR(s)ds}V(T)|\mathcal{F}_t\right). \tag{18}
\]

Again, the price at time \( t \) is the risk-neutral expectation of the discounted payoff at maturity.

Example 1. For a geometric Brownian motion market with constant parameters: \( r, \sigma \), the time \( t \) price of a call with strike \( K \) and maturity \( T \) is given by

\[
V(t) = e^{-r(T-t)}\tilde{E}\left((S(T) - K)^+|\mathcal{F}_t\right) = e^{-r(T-t)}\tilde{E}\left(S(t)\exp\left\{\sigma(\bar{W}(T) - \bar{W}(t)) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right\} - K\right)^+, \tag{19}
\]

where the last equality is because the Brownian increment \( \bar{W}(T) - \bar{W}(t) \) is independent of \( \mathcal{F}_t \).

4 How to Hedge

The risk-neutral pricing formula in (18) is based on the assumption that we can find a portfolio to replicate the payoff of that European option. We need to find out the right replicating strategy to do so. First note that the discounted option price process \( D(t)V(t) \) is a martingale:

\[
D(t)V(t) = \tilde{E}(D(T)V(T)|\mathcal{F}_t). \tag{20}
\]

Probability theory tells us, if \( \{M(t)\}_{t \in [0,T]} \) is a martingale with respect to the filtration of Brownian motion \( \{\bar{W}(t)\}_{t \in [0,T]} \) under \( \tilde{P} \), then there exists an adapted process \( \{\Gamma(t)\}_{t \in [0,T]} \) such that,

\[
M(t) = M(0) + \int_0^t \Gamma(u)d\bar{W}(u). \tag{21}
\]

Now use this theorem to martingale \( D(t)V(t) \), we can also find a process \( \{\Gamma(t)\} \) such that

\[
d(D(t)V(t)) = \Gamma(t)d\bar{W}(t). \tag{22}
\]

Recall that, on the other hand,

\[
d(D(t)X(t)) = \Delta(t)\sigma(t)D(t)S(t)d\bar{W}(t). \tag{23}
\]

Because the portfolio is replicating the option, we always have \( X(t) = V(t) \). By comparing (22) with (23), we obtain that

\[
\Delta(t) = \frac{\Gamma(t)}{\sigma(t)D(t)S(t)}, \ 0 \leq t \leq T. \tag{24}
\]

Formula (24) is not applicable now, since we do not know what \( \Gamma(t) \) is. But we will figure it out in the future.

\(^5\)An initial investment \( X(0) \) and a trading strategy \( \{\Delta(t)\}_{t \in [0,T]} \).