Lecture 17: Exotic Options

November 10, 2010

We assume zero rate Black-Scholes model without further mention. That is
\[ dS_t = \sigma S_t dW_t, \quad S_0 > 0. \]

1 Put-Call Symmetry

For European puts and calls we have put-call parity. That property is inherited from the structure of the payoff functions:
\[ (x - K)^+ - (K - x)^+ = x - K. \] (1)

Put-call parity can be used to link a call with a put, via a forward contract. However, this is not the only linkage between puts and calls. In B-S model, we actually have a put-call symmetry: a call has the same value as certain number of puts struck at appropriate price. Let us show this below.

Recall that in a B-S model, the call price at time \( t \) is given by B-S formula
\[ C(t, S_t, K) = S_t N(d_+(T - t, S_t, K)) - K N(d_-(T - t, S_t, K)), \] (2)
and
\[ d_\pm(\tau, S_t, K) = \frac{1}{\sigma \sqrt{\tau}} \left( \log \frac{S_t}{K} \pm \frac{1}{2} \sigma^2 \tau \right). \] (3)

Let us do simple math from the right hand side of (2),
\[ S_t N(d_+(T - t, S_t, K)) - K N(d_-(T - t, S_t, K)) = \frac{K}{S_t} \left\{ \frac{S_t^2}{K} N(d_+(T - t, S_t, K)) - S_t N(d_-(T - t, S_t, K)) \right\}. \] (4)

Note that
\[ d_\pm(\tau, S_t, K) = - \frac{1}{\sigma \sqrt{\tau}} \left\{ \log \frac{K}{S_t} \mp \frac{1}{2} \sigma^2 \tau \right\} = - \frac{1}{\sigma \sqrt{\tau}} \left\{ \log \frac{S_t}{(S_t^2/K)} \mp \frac{1}{2} \sigma^2 \tau \right\} \]
\[ = -d_\mp(\tau, S_t, S_t^2/K). \] (5)

Thus, from (2), (4) and (5) we have that
\[ C(t, S_t, K) = \frac{K}{S_t} \left\{ \frac{S_t^2}{K} N(-d_-(T - t, S_t, S_t^2/K)) - S_t N(-d_+(T - t, S_t, S_t^2/K)) \right\}. \] (6)

On the other hand, using B-S formula and put-call parity we know that the time-\( t \) price of a put struck at \( K_1 \) is given by
\[ P(t, S_t, K_1) = C(t, S_t, K_1) - S_t + K \]
\[ = S_t N(d_+(T - t, S_t, K_1)) - K N(d_-(T - t, S_t, K_1)) - S_t + K \]
\[ = K N(-d_-(T - t, S_t, K_1)) - S_t N(-d_+(T - t, S_t, K_1)). \] (7)
since \(1 - N(x) = N(-x)\) for any real number \(x\). We compare (6) with (7) we immediately obtain that

\[
C(t, S_t, K) = \frac{K}{S_t} P(t, S_t, S_t^2 / K).
\]  

(8)

From (8) we see that, at time \(t\), a call struck at \(K\) has the same value as \(K / S_t\) shares of puts struck at \(S_t^2 / K\). Similarly, we can also show that

\[
P(t, S_t, K) = \frac{K}{S_t} C(t, S_t, S_t^2 / K).
\]  

(9)

These formulas can be used to value some barrier options.

2 Up-and-Out Call, etc

We want to first evaluate an up-and-out call. Specifically, there is a barrier \(B > K > 0\), where \(K\) is the strike of the call. The call pays \((S_T - K)^+\) at maturity \(T\) if and only if the stock price stay below the barrier \(B\) during the time interval \([0, T]\).

Instead of using tedious computation listed in the textbook, we evaluate the up-and-out call with semi-static replication\(^1\). The idea is, to value this exotic option, we just need to construct a portfolio of vanilla puts and calls to replicate its payoff. By no arbitrage the price of this exotic option is the same the cost of the portfolio\(^2\).

How we are going to replicate it? First just consider a regular call struck at \(K\), whose time-\(t\) price is \(C(t, S_t, K)\). Of course this call will be more expensive than the price of the one we want to price. What is the difference? We know that at the first time the stock hits the knockout barrier \(B\), the up-and-out call is knocked out, which means that the value should be equal to zero at that time. We use (8) to get that zero difference? We know that at the first time the stock hits the knockout barrier \(B\), the up-and-out call is knocked out, which means that the value should be equal to zero at that time. We use (8) to get that zero.

\[
\text{Using (8), the call left has the same value as the put in (11), so we can liquidate all the options and receive zero. Thus, the portfolio of puts and calls fully replicates the payoff of the up-and-out call (UOC henceforth). By no arbitrage, the price of this exotic option is given by}
\]

\[
\text{UOC(t, S_t, K, B) = C(t, S_t, K) - \frac{K}{B} C(t, S_t, B^2 / K) - (B - K) OT(t, S_t, B), t \in [0, \min \{\tau_B, T\}].} \tag{12}
\]

(12)

Since we have B-S formula for European calls, we just need to price the one-touch. That is not difficult, since

\[
\text{OT(t, S_t, B) = } \tilde{E}\{\mathbb{1}_{\tau_B \leq T} | \mathcal{F}_t\} = \tilde{E}\{\mathbb{1}_{\tau_B \geq B} | \mathcal{F}_t\}, \tag{13}
\]

(13)

where \(M_T = \sup_{t \in [0, T]} S_t\) is the running maximum of the stock price. We know that \(S_t = S_0 \exp(-\frac{1}{2} \sigma^2 t + \sigma W_t)\), so we have

\[
\tilde{E}\{\mathbb{1}_{M_T \geq B} | \mathcal{F}_t\} = \tilde{P}\left(\sup_{s \in [T-t]} \{ -\frac{1}{2} \sigma^2 s + \sigma W_s \} \geq \log(B / S_t) \right). \tag{14}
\]

(14)

\(^{\text{1}}\)It is called semi-static because the replication strategy does not require continuous hedging.

\(^{\text{2}}\)The no-arbitrage argument is the same as before. The only difference is that, we do not use stocks and cash to replicate, but use European puts and calls.

\(^{\text{3}}\)One touches are single barrier options that pay one dollar at maturity if the strike is hit by maturity.
Using Corollary 7.2.2 you can finish the computation in (14).

Once we have the price for UOC, we also have the price for up-and-in call. This is simply because of the fact

$$\text{up-and-out} + \text{up-and-in} = \text{vanilla}.$$  

So the up-and-in call (UIC henceforth) at time \(t\) is worth

$$UIC(t, S_t, K, B) = C(t, S_t, K) - UOC(t, S_t, K) = \frac{K}{B} C(t, S_t, B^2/K) + (B - K) OT(t, S_t, B), \ t \in [0, \min\{\tau_B, T\}].$$  

(15)

One can similarly get the price of an up-and-in put (UIP henceforth). Let us assume the knock-in barrier \(B > K > 0\). Then using “put-call parity” for barrier option we have

$$UIC(t, S_t, K, B) - UIP(t, S_t, K, B) = (B - K) OT(t, S_t, B), \ t \in [0, \min\{\tau_B, T\}].$$  

(16)

This is easy to verify: if \(B\) is hit by time \(T\), then both UIC and UIP turn into regular call and put, we can use put-call parity at that time; if \(B\) is not hit by time \(T\), then both sides expire worthless. By no arbitrage, both sides should always be equal to each other.

From (15) and (16) we have that

$$UIP(t, S_t, K, B) = \frac{K}{B} C(t, S_t, B^2/K), \ t \in [0, \min\{\tau_B, T\}].$$  

(17)

In fact, if \(B\) is hit before \(T\), then at that time, the calls struck at \(B^2/K\) can be sold to buy one put struck at \(K\) (see formula (9)).

If we take differential of an UIP with respect to the strike \(K\), then we obtain a up-and-in digital put: it pays one dollar at expire if and only if barrier \(B\) is hit by time \(T\), and \(S_T \leq K\). Its price is given by

$$UIDP(t, S_t, K, B) = \frac{\partial}{\partial K} UIP(t, S_t, K, B), \ t \in [0, \min\{\tau_B, T\}].$$  

(18)

In practice, we can construct a UIDP by a vertical spread:

$$UIDP(t, S_t, K, B) \approx \frac{1}{\epsilon} [UIP(t, S_t, K + \epsilon, B) - UIP(t, S_t, K, B)], \ t \in [0, \min\{\tau_B, T\}].$$  

(19)

3 Lookbacks

We continue to price a lookback using static replication. Recall that a lookback pays \(M_T - S_T\) at maturity \(T\), where \(M_t = \sup_{s \in [0, t]} S_s\). At time \(t\), we can construct a portfolio as follows:

- buy one put struck at \(M_t\)
- buy \(dK\) shares of UIDP with strike \(K\) and knockout barrier \(B, K > M_t\).

The value of the portfolio at time \(t\) is

$$V(t, S_t, M_t) = P(t, S_t, M_t) + \int_{M_t}^\infty UIDP(t, S_t, K, K) dK.$$  

(20)

If the maximum increase from \(M_t\) to \(M_u\), \(t < u\), then all the UIDP struck at \(K \in (M_t, M_u)\) knocks into regular digital puts, they will helps us to move up the strike of the first put: the total payoff at time \(T\) of the standalone put and those knocked-in digital puts is

$$(M_t - S_T)^+ + \int_{M_t}^{M_u} \mathbb{1}_{S_T \leq K} dK = (M_u - S_T)^+.$$  

(21)

So the above portfolio will automatically shift up the strike of the standalone put with the running maximum. At time \(T\), the total value of the portfolio is

$$(M_T - S_T)^+ + \int_{M_T}^\infty UIDP(T, S_T, K, K) dK.$$  

(22)
We claim that all the remaining UIDP expire worthless, since their knock-in barriers have not been reached by time $T$. So finally the portfolio replicates the payoff of a lookback. By no arbitrage, the price of the lookback at any time $t \in [0, T]$ must be the same as the value of the replicating portfolio:

$$LB(t, S_t, M_t) = P(t, S_t, M_t) + \int_{M_t}^{\infty} UIDP(t, S_t, K, K)dK.$$  \hspace{0.5cm} (23)

On the right hand side of (23), put price is known from B-S formula, UIDP’s prices can be derived from B-S formula (see (18) and (17)). What is more important is that, the pricing formula (23) also suggests a static replicating portfolio. In view of (17), (19) and (23), you just need to buy a lot of European calls and puts, and trade appropriately when new maximum is reached, then at time $T$, you can receive the “drawdown” payment $M_T - S_T$. 

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