1 Limiting Distribution of the Scaled Random Walk

Recall that we defined a scaled simple random walk last time, \( \{W_t^{(N)}\}_{t \in [0,T]} \): Fix a \( T > 0 \) and large positive integer \( N \), define \( X_0^{(N)} = 0 \) and

\[
W_t^{(N)} = \sqrt{\frac{T}{N}} \sum_{j=1}^{N} X_j, \quad t = 0, \frac{T}{N}, \frac{2T}{N}, \ldots, T,
\]

(1)

where \( \{X_j\}_{1 \leq j \leq N} \) are i.i.d. random variables that take \( \pm 1 \) with probability 0.5. The value of \( X_t \) for general \( t \) is defined using linear interpolation.

Obviously, the path of \( W_t^{(N)} \) is continuous. Moreover, we have seen that \( \{W_t^{(N)}\} \) is a martingale for time points \( t = \frac{kT}{N}, 0 \leq k \leq N \). In fact, the increment \( W_t^{(N)} - W_s^{(N)} \) is independent of \( W_u^{(N)} \) for all \( T \geq t \geq s \geq u + \frac{T}{N} \). As \( N \to \infty \), the first-order variation of the path \( \{W_t^{(N)}\}_{t \in [0,T]} \) tends to \( \infty \), but the quadratic variation tends to \( T \).

It is interesting to know the limiting distribution of \( W_T^{(N)} \) as \( N \to \infty \). The impact of this is that, when we choose a finer partition of the interval \([0,T]\) with a larger \( N \), we would like to know what we will obtain in the end. A good tool to study limiting probability distribution is the moment-generating function. Recall that, for a random variable \( Z \), its moment-generating function is defined as

\[
\phi(u) = E(e^{uZ}).
\]

(2)

Since moment-generating function is the signature of a probability distribution\(^1\), to study the limiting distribution of \( W_T^{(N)} \), it suffices to know its limiting moment-generating function. Thus, we proceed to define:

\[
\phi_N(u) = E(e^{uW_T^{(N)}}) = E \exp \left( u \sqrt{\frac{T}{N}} \sum_{j=1}^{N} X_j \right) X_j \text{’s are i.i.d.} = \left[ E \exp \left( u \sqrt{\frac{T}{N}} X_1 \right) \right]^N.
\]

(3)

To see the limit \( \lim_{N \to \infty} \phi_N(u) \), it suffices to know \( \lim_{N \to \infty} \log \phi_N(u) \). Let \( x = \frac{1}{\sqrt{N}} \), then

\[
\lim_{N \to \infty} \log \phi_N(u) = \lim_{x \to 0^+} \log \left( \frac{1}{2} e^{x} + \frac{1}{2} e^{-x} \right) x^2 = \frac{1}{2} u^2 T,
\]

(4)

where the last equality is a result of L’Hôpital’s rule. As a result, the limiting distribution of \( W_T^{(N)} \) has moment-generating function

\[
\phi(u) = e^{\frac{1}{2} u^2 T}.
\]

\(^1\)Probability distribution can be uniquely determined by its moment-generating function.
Recall that the moment-generating function of a normal random variable with mean \( \mu \) and variance \( \sigma^2 \) has moment-generating function \( e^{u\mu + \frac{1}{2}u^2\sigma^2} \). (5) implies that, the limiting distribution of \( W_N^{NT} \), as \( N \to \infty \), is normal with mean 0 and variance \( T \). In other words, the limiting distribution has density
\[
f(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}.
\]

**Remark 1.** The above result is a special case of central limit law. In fact, the result still holds if we replace \( \{X_j\} \) by any other i.i.d. random variable with zero mean and unit variance (e.g. standard normals, which is quite useful for simulations).

### 2 Log-Normal Distribution as the Limit of the Binomial Model

We play the scaling game for binomial model in this section. Fix a time horizon \( T > 0 \), choose a large positive integer \( N \) to partition \([0, T]\) with equal length, so that we have a \( N \)-period binomial model. We consider zero rate for simplicity: \( r = 0 \). Let us fix a positive number \( \sigma > 0 \), we then take the up factor to be \( u_N = 1 + \sigma \sqrt{\frac{T}{N}} \) and the down factor to be \( d_N = 1 + \sigma \sqrt{\frac{T}{N}} \). The risk-neutral probabilities are then
\[
\tilde{p} = \frac{1 + r - d_N}{u_N - d_N} = \frac{\sigma \sqrt{\frac{T}{N}}}{2\sigma \sqrt{\frac{T}{N}}} = \frac{1}{2} = \tilde{q}.
\]

At time \( T \), which is the \( N \)-th step of the \( N \)-period binomial tree, the stock price is
\[
S_N(T) = S_0 u_N^{H_N} d_N^{T_N} = S_0 \left(1 + \sigma \sqrt{\frac{T}{N}}\right)^{H_N} \left(1 - \sigma \sqrt{\frac{T}{N}}\right)^{T_N}, \quad (6)
\]
where \( H_N/T_N \) is the number of heads/tails observed in the \( N \) coin tosses. Suppose we have a simple random walk starting from zero, whose increment is 1 if a head shows up and \(-1 \) if a tail shows up, and let us denote the value of the simple random walk at the \( N \)-th step by \( M_N \), then
\[
M_N = H_N - T_N. \quad (7)
\]

On the other hand,
\[
N = H_N + T_N. \quad (8)
\]

Thus, we have
\[
H_N = \frac{1}{2}(N + M_N), \quad T_N = \frac{1}{2}(N - M_N). \quad (9)
\]

Combining (6) with (9) and taking log we obtain that
\[
\log S_N(T) = \log S_0 + \frac{1}{2}(N + M_N) \log \left(1 + \sigma \sqrt{\frac{T}{N}}\right) + \frac{1}{2}(N - M_N) \log \left(1 - \sigma \sqrt{\frac{T}{N}}\right). \quad (10)
\]

Moreover, using the fact that
\[
\log(1 + x) = x - \frac{1}{2}x^2 + O(x^3),
\]
we obtain that
\[
\log S_N(T) = \log S_0 + \frac{1}{2}(N + M_N) \left(\sigma \sqrt{\frac{T}{N}} - \sigma^2 \frac{T}{2N} + O(N^{-\frac{3}{2}})\right) + \frac{1}{2}(N - M_N) \left(-\sigma \sqrt{\frac{T}{N}} - \sigma^2 \frac{T}{2N} + O(N^{-\frac{3}{2}})\right)
\]
\[
= \log S_0 - \frac{1}{2}\sigma^2 T + O(N^{-\frac{3}{2}}) + \sigma \sqrt{\frac{T}{N}} M_N \left(1 + O(n^{-1})\right). \quad (11)
\]
Recall that in last section, the scaled random walk

\[ W_T^{(N)} = \sqrt{\frac{T}{N}} M_N = \sqrt{\frac{T}{N}} \sum_{j=1}^{N} X_j \rightarrow W(T), \text{ as } N \rightarrow \infty \]

where \( W(T) \) is a normal with zero mean and variance \( T \). In conclusion, as \( N \rightarrow \infty \), the stock price at time \( N \) has distribution

\[ S(T) = S_0 \exp \left( \sigma W(T) - \frac{1}{2} \sigma^2 T \right). \tag{12} \]

That is, the log price is normally distributed.

### 3 Brownian Motion

We studied the limiting marginal distribution of scaled random walk and \( N \)-period binomial model. If we look at the whole path, we obtain Brownian motion. Brownian motion has a (random) continuous path \( W(t) \) for all \( t \geq 0 \) that satisfies \( W(0) = 0 \). For all \( 0 = t_0 < t_1 < t_2 < \ldots t_k \) the increments

\[ W(t_i) = W(t_i) - W(t_0), \quad W(t_2) - W(t_1), \ldots, \quad W(t_k) - W(t_{k-1}) \tag{13} \]

are independent and each of these increments is normally distributed with

\[ E(W(t_{i+1}) - W(t_i)) = 0, \quad \text{Var}(W(t_{i+1}) - W(t_i)) = t_{i+1} - t_i. \tag{14, 15} \]

In other words, increments of Brownian motion are independent and stationary\(^2\).

The joint distribution of the values of Brownian motion at different times can be easily obtained. When the marginal distributions are known, two jointly normal variables are uniquely determined by their covariance. In the case of Brownian motion, \( W(t) \) and \( W(s) \) are zero mean normal variables with variance \( t \) and \( s \), respectively. Their covariance is

\[ \text{Cov}(W(t), W(s)) = E(W(t)W(s)) = \min\{t, s\}. \tag{16} \]

Alternatively, one can also easily obtain the joint moment-generating function of \( (W(t), W(s)) \) using the independence of increments. Without loss of generality, assume \( t > s \), then for \( u, v \in \mathbb{R} \),

\[
\phi(u, v) = E(e^{uW(t)+vW(s)}) = E(e^{u(W(s)+W(t)-W(s))+vW(s)}) \\
= E(e^{uW(s)+vW(t)-W(s)}) \text{ independence} = E(e^{uW(s)})E(e^{vW(t)-W(s)}) \\
= e^{\frac{1}{2}u^2s} e^{\frac{1}{2}v^2(t-s)} \tag{17}
\]

Let us denote by \( \mathcal{F}_t \) the information available at time \( t \) for the Brownian motion. That is, \( \mathcal{F}_t \) contains all the history of \( W(s) \) for any \( 0 \leq s \leq t \). Then Brownian motion \( \{W(t)\}_{t \geq 0} \) is a martingale with respect to the filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \). This is a simple consequence of the fact that increments of Brownian motion have zero mean and are independent of the past.

### 4 Quadratic Variation of Brownian Motion and Volatility Estimation

In Lecture 7 we have shown that, as \( N \rightarrow \infty \), the scaled random walk \( \{W_T^{(N)}\}_{0 \leq t \leq N} \) has infinite first-order variation and finite quadratic variation \( T \). Brownian motion has the same property.

For a given time-horizon \( T > 0 \), we consider an arbitrage partition \( \Pi = \{t_0, t_1, \ldots, t_N\} \) such that

\[ 0 = t_0 < t_1 < \ldots < t_N = T. \]

\(^2\)The history does not matter. What matters is the time lapse of the increment.
The diameter of the partition Π is defined as the maximum time lapse:

\[ \|\Pi\| := \max_{0 \leq i \leq N-1} \{t_{i+1} - t_i\}. \]  

Then the quadratic variation of Brownian motion \{W(t)\}_{t \geq 0} at time \(T\) is defined as\(^3\)

\[ \langle W \rangle(T) := \lim_{\|\Pi\| \to 0^+} \sum_{j=1}^{N-1} (W(t_{j+1}) - W(t_j))^2 \]  

(19)

The above limit has a surprisingly simple result. One may get some flavor by taking the expectation of both sides and interchanging\(^4\) the order of limit and expectation:

\[ E(\langle W \rangle(T)) = \lim_{\|\Pi\| \to 0^+} \sum_{j=1}^{N-1} E(W(t_{j+1}) - W(t_j))^2 = \lim_{\|\Pi\| \to 0^+} \sum_{j=1}^{N-1} (t_{j+1} - t_j) = T. \]  

(20)

To completely show that \(\langle W \rangle(T) = T\) one just need to show that

\[ \text{Var}(\langle W \rangle(T)) = 0. \]  

(21)

This is not a problem because, using independence of increments,

\[
\text{Var} \left[ \sum_{j=1}^{N-1} (W(t_{j+1}) - W(t_j))^2 \right] = \sum_{j=1}^{N-1} \text{Var} \left[ (W(t_{j+1}) - W(t_j))^2 \right] = 2 \sum_{j=1}^{N-1} (t_{j+1} - t_j)^2 \\
\leq 2\|\Pi\| \sum_{j=0}^{N-1} (t_{j+1} - t_j) = 2T\|\Pi\| \to 0^+, \]

(22)

as \(\|\Pi\| \to 0^+\). The above result is the fundamental of stochastic calculus. If we use \(dW(t)\) to express the infinitesimal increment of Brownian motion, then we can informally write

\[ T = \langle W \rangle(T) = \int_0^T (dW(t))^2. \]  

(23)

In other words,

\[ (dW(t))^2 = dt. \]  

(24)

This is a striking result, as we know that \((dt)^2 = 0\). Moreover, one can also show that \(dW(t)dt = 0\).

The fact that the quadratic variation of Brownian motion is simply the time past can be used to estimate the volatility of stock price. Let us consider the classical geometric Brownian motion model for stock price

\[ S(t) = S(0) \exp \left\{ \sigma W(t) + \left( \alpha - \frac{1}{2} \sigma^2 \right) t \right\}. \]  

(25)

where \(\alpha\) and \(\sigma > 0\) are constants. We observe the price process \(\{S_t\}_{t \geq 0}\) and we want to estimate the volatility \(\sigma\).

Fix a time-horizon \(T > 0\) (say, a day), consider a partition \(0 = t_0 < t_1 < \ldots < t_N = T\). Note that for any \(0 \leq k \leq N - 1\),

\[ \log \frac{S(t_{k+1})}{S(t_k)} = \sigma (W(t_{k+1}) - W(t_k)) + \left( \alpha - \frac{1}{2} \sigma^2 \right) (t_{k+1} - t_k). \]  

(26)

\(^3\)Note that the limit here is more complicated than that in random walk case: not only \(N \to \infty\), but also \(\|\Pi\| \to 0^+\).

\(^4\)The interchange is legal because, the value of the summation is increasing as we choose more points and finer partitions. In probability this is called the monotone converge theorem.
If the partition is fine (relatively high-frequency data), then

\[
\sum_{k=0}^{N-1} \left( \log \frac{S(t_{k+1})}{S(t_k)} \right)^2
\]

\[=\sigma^2 \sum_{k=0}^{N-1} (W(t_{k+1}) - W(t_k))^2 + \text{terms proportional to } (t_{k+1} - t_k)^2 \text{ or } (W(t_{k+1}) - W(t_k))(t_{k+1} - t_k)
\]

\[\approx \sigma^2 \int_0^T (dW(t))^2 = \sigma^2 T. \tag{27}
\]

Hence we have the estimation

\[
\hat{\sigma}^2 = \frac{1}{T} \sum_{j=0}^{N-1} \left( \log \frac{S(t_{k+1})}{S(t_k)} \right)^2.
\] \tag{28}