1 Addition to Last time

- Log-normal distribution is the limiting distribution for stock price when the number of periods in the binomial model is very large. This is important since even though binomial model is simple, the computational cost is high if the number of periods is very large. Last time we have seen that, under the risk-neutral measure, the stock price at time $t$ can be approximately written as

$$ S(t) = S(0) \exp \left( \sigma W(t) - \frac{1}{2} \sigma^2 t \right), $$

(1)

where $W(t)$ is a normal variable with zero mean and variance $t$. The above distribution is called log-normal distribution. If we let $t$ run from 0 to $T < \infty$, and $W(t)$ be the value of a Brownian motion at time $t$, then we obtain a process $\{S(t)\}_{t \in [0,T]}$. This process is called geometric Brownian motion. Not surprisingly, the process $\{S(t)\}_{t \in [0,T]}$ is a martingale, since it is the discounted (zero rate though) stock price under the risk-neutral measure. We will give a mathematical proof for this fact in a second.

- Lious Bachelier was the first to use Brownian motion to model stock price. That dates back to 1900, when he published his famous thesis *Théorie de la spéculation*. In quantitative finance, Brownian motion is usually referred to Bachelier model. Bachelier model captures the randomness of stock market to some extent, but it has a big drawback - the price can go negative, since normal random variable can take negative number. Nonetheless, people are still using it for different reasons.

- When a stock price process is modeled as geometric Brownian motion

$$ S(t) = S(0) \exp \left( \sigma W(t) + \left( \alpha - \frac{1}{2} \sigma^2 \right) t \right), $$

(2)

where constant $\sigma > 0$ is the volatility of the stock, and $\alpha$ is the growth rate of the stock. Note: we did not specify the probability measure here. If it is the actual measure, then $\alpha$ is the actual growth rate of stock price, e.g. 15% annually for S&P; if it is the risk-neutral measure, then $\alpha$ will be equal to the risk-free rate. Moreover, the summation used to estimate $\sigma$ last time:

$$ \frac{1}{T} \sum_{j=0}^{N-1} \left( \log \frac{S(t_{k+1})}{S(t_k)} \right)^2, $$

is called the realized variance (RV). Realized variance is useful even if the volatility $\sigma$ is not a constant, but a time-varying or even stochastic process. There are contracts on the realized variance in practice. The motivation for such contracts is to hedge volatility risk.

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1 Recursion is needed for pricing derivatives.

2 Informally, one can first sample a path for the Brownian model, then use the value on the path at time $t$ to define $S(t)$ using (1).
2 Exponential Martingale and Laplace Transform of the First Passage Time

Let \( \{W(t)\}_{t \geq 0} \) be a Brownian motion starting at zero, \( \{\mathcal{F}_t\}_{t \geq 0} \) be its filtration (information). Then for any constant \( \sigma \), the process \( \{Z(t)\}_{t \geq 0} \) defined as:

\[
Z(t) = \exp \left\{ \sigma W(t) - \frac{1}{2} \sigma^2 t \right\},
\]

is a martingale with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \).

The proof is straightforward. We make use of two fact: independent increment of Brownian motion and the moment-generating function of normal random variable. Let us consider any \( 0 \leq s < t \leq T \), then

\[
E(Z(t)|\mathcal{F}_s) = E\left(Z(s)\exp \left\{ \sigma(W(t) - W(s)) - \frac{1}{2} \sigma^2(t-s) \right\} |\mathcal{F}_s \right) = Z(s)e^{-\frac{1}{2} \sigma^2(t-s)}E(e^{\sigma(W(t)-W(s))}|\mathcal{F}_s). \tag{4}
\]

To show that \( E(Z(t)|\mathcal{F}_s) = Z(s) \), it suffices to show that

\[
e^{-\frac{1}{2} \sigma^2(t-s)}E(e^{\sigma(W(t)-W(s))}|\mathcal{F}_s) = 1. \tag{5}
\]

This can be proved. Since the increment \( W(t) - W(s) \) is independent of the past \( \mathcal{F}_s \), the conditional expectation in (5) is actually unconditional. But

\[
E(e^{\sigma(W(t)-W(s))}) = e^{\frac{1}{2} \sigma^2(t-s)}, \tag{6}
\]

since the random variable \( W(t) - W(s) \) is normally distributed with zero mean and variance \( (t-s) \).

The martingale in (3) is called the exponential martingale. A direct application of it is to study the first passage time of Brownian motion. For any real number \( m \), the first passage time to \( m \) is the first time at which the Brownian motion reaches \( m \). We denote this time by \( \tau_m \):

\[
\tau_m = \inf \{ s \geq 0 : W(s) = m \}. \tag{7}
\]

The first passage time \( \tau_m \) is a stopping time, since the information \( \mathcal{F}_t \) can tell us if \( \tau_m \) has past \( (\tau_m < t) \). If we “stop” the exponential martingale \( \{Z(t)\}_{t \geq 0} \) at time \( \tau_m \), then its expectation will be the same as \( Z(0) \), simply because the stopped martingale is still a martingale. Thus\(^3\)

\[
1 = Z(0) = E(e^{\sigma m - \frac{1}{2} \sigma^2 \tau_m}) = e^{\sigma m} E(e^{-\frac{1}{2} \sigma^2 \tau_m}). \tag{8}
\]

If we let \( \lambda = \frac{1}{2} \sigma^2 > 0 \), then we obtain that

\[
E(e^{-\lambda \tau_m}) = e^{-|m|\sqrt{2\lambda}}. \tag{9}
\]

This is the Laplace transform\(^4\) of random variable \( \tau_m \). Laplace transform is similar as moment-generating function. We can invert the Laplace transform (similar as Fourier inversion) to get the density function of \( \tau_m \). The existence of moment-generating function requires that all moments of the random variable \( Z \) are finite:

\[
E(|Z|^k) < \infty, \ k = 1, 2, \ldots.
\]

The first passage time \( \tau_m \), however, does not even have a finite first moment:

\[
E(\tau_m) = \lim_{\lambda \to 0^+} \frac{\partial}{\partial \lambda} E(e^{-\lambda \tau_m}) = \lim_{\lambda \to 0^+} \frac{|m|}{\sqrt{2\lambda}} e^{-|m|\sqrt{2\lambda}} = \infty. \tag{10}
\]

Why first passage time is relevant? Consider a Bachelier model with zero rate, with \( S(0) = 100 \), and \( S(t) = 100 + 0.2W(t) \), for \( t \in [0, T] \). Now an option that pays one dollar at time \( T \) if the stock price ever drops to 30 before \( T \). How are you going to price it at time zero?

\(^3\)\( \{Z(t)\} \) is a martingale no matter if \( \sigma \) is positive or not. Here we choose the \( \sigma \) that has the same sign as \( m \), this is a technical reason.

\(^4\)Laplace transform of stopping times can also be interpreted as the price of a perpetual option (maturity= \( \infty \)) that pays one dollar at this stopping time (continuously compound interest rate= \( \lambda \)). Laplace transform is also very useful for pricing European options under geometric Brownian motion dynamics with stochastic volatility. This is called (Peter) Carr’s randomization.
3 Reflection Principle and Lookback Options

Let \( \{W(t)\}_{t \in [0, T]} \) be a Brownian motion starting at zero. We define its running maximum process \( \{M(t)\}_{t \in [0, T]} \) as:

\[
M(t) = \sup_{s \in [0, t]} W(s).
\]  

(11)

It is known that \( W(t) \) has probability density function:

\[
f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.
\]  

(12)

If we want to evaluate an option\(^5\) whose payoff depends on both the stock price at time \( T \) and its running maximum at that time, then the joint distribution of \( (W(T), M(T)) \) is needed. Suppose \( f_{W(T), M(T)}(x, m) \) is the unknown joint probability density function for \( (W(T), M(T)) \), then for any \( m > 0 \) and \( w \leq m \), we have

\[
P(S(T) \leq w, M(T) \geq m) = \int_{0}^{w} du \int_{m}^{\infty} dv f_{W(T), M(T)}(u, v).
\]  

(13)

If the probability on the left hand side of (13) can be obtained in some smart way, then the density on the right hand side is available to us.

Usually it is difficult to get the probability in (13) without using the joint density, but because Brownian motion has the reflection principle: when the path hits a level \( m \), the future part of the path has equal chances to go up or down. Thus, hitting \( m \) and then staying below \( w \) at time \( T \) has the same probability as hitting \( m \) and then staying above \( 2m - w \) at time \( T \):

\[
P(S(T) \leq w, \tau_m \leq T) = P(S(T) \geq 2m - w, \tau_m \leq T).
\]  

(14)

Moreover, notice that \( 2m - w \geq m \geq w \) so

\[
\{S(T) \leq w, \tau_m \leq T\} = \{S(T) \leq w, M(T) \geq m\},
\]

\[
\{S(T) \geq 2m - w, \tau_m \leq T\} = \{S(T) \geq 2m - w\}.
\]

As a result, we obtain that

\[
P(S(T) \leq w, M(T) \geq m) = P(S(T) \geq 2m - w) = \int_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2t}} dx.
\]  

(15)

Taking derivatives with respect to \( w \) and \( m \) in (14), we immediately obtain that

\[
f_{W(T), M(T)}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2t}}, \quad w \leq m, m > 0.
\]  

(16)

From (16), we can further obtain the marginal density of \( M(T) \), as well as the conditional density of \( M(T) \) given \( W(T) \). The detailed computation can be found in the textbook.

\(^5\)For example, a lookback option.