Central limit theorems under special relativity

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Abstract

Several relativistic extensions of the Maxwell–Boltzmann distribution have been proposed, but they do not explain observed lognormal tail-behavior in the flux distribution of various astrophysical sources. Motivated by this question, extensions of classical central limit theorems are developed under the conditions of special relativity. The results are related to CLTs on locally compact Lie groups developed by Wehn, Stroock and Varadhan, but in this special case the asymptotic distribution has an explicit form that is readily seen to exhibit lognormal tail behavior.

Keywords: heavy-tailed distributions, random walks on locally compact Lie groups, κ-deformations, Maxwell–Boltzmann, Ornstein–Uhlenbeck

1. Introduction

In special relativity, the momenta of particles on parallel trajectories combine according to the κ-addition rule

$$x \overset{\kappa}{\oplus} y = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2}, \tag{1.1}$$

where $0 < \kappa < 1$ is a dimensionless parameter representing the reciprocal of the speed of light in the ambient space (with all variables expressed in dimensionless units). The classical Galilean addition law for momenta appears in the limit as $\kappa \to 0$.

The purpose of this note is to derive extensions of classical central limit theorems under κ-addition, and to relate the resulting (non-Gaussian) limit distributions to an existing parametric family, namely hyperbolic sine transformations of normal distributions. It is also shown that such distributions arise as the limit distribution of a certain type of relativistic Ornstein–Uhlenbeck process.

Background on the κ-addition rule for momenta in special relativity, and corresponding operations for velocity and energy, can be found in Kaniadakis (2002, 2006, 2009a,b, 2013), along with a comprehensive survey of the literature on various other κ-deformations, and references to applications. These include a formal extension of the Maxwell–Boltzmann distribution (used to model the energy of large systems of particles that only interact through elastic collision and are at thermal equilibrium), constructed by replacing the standard exponential by the κ-deformed exponential $\exp_\kappa(x) = \exp(\sinh^{-1}(\kappa x)/\kappa)$, and providing power-law tail behavior with exponent $-1/\kappa$.

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General formulations of central limit theorems for random walks on locally compact Lie groups, of which the real line under κ-addition is a simple example, were first established by Wehn (1962) and Stroock and Varadhan (1973). These results require indirect moment conditions on the random elements after they are logarithmically mapped into the associated Lie algebra (tangent space at the identity), and the limit distribution is described in terms of the infinitessimal generator of a semi-group of probability measures on the Lie group. Despite their great generality, such results fall short of being able to resolve whether classical CLTs extend in full to κ-addition under only standard conditions. Explicit CLTs are currently only available for random walks on certain special Lie groups, e.g., for the special orthogonal group $SO(3)$, which is of interest in the field of directional statistics. In particular, Qiu et al. (2014) recently established an explicit CLT for symmetric random elements of $SO(3)$, with an isotropic Gaussian distribution as the limit.

We establish complete relativistic extensions of the classical CLTs under standard conditions. These results are established by showing that "relativistic warping" is asymptotically equivalent to taking the hyperbolic sine of a row-sum in a transformed array, and that the standard CLT conditions on the original array suffice for applying the Lindeberg–Feller theorem to this transformed array. The proofs are elementary, avoiding the need for any background on Lie groups or infinitesimal generators. The limiting distributions have an explicit form because the exponential map from the associated Lie algebra into the Lie group in this special case takes an especially tractable form.

Our main results are presented in Section 2. Background on observed lognormal tail-behavior in astrophysical sources, and a relativistic version of the Ornstein–Uhlenbeck process that might help explain such observations, are discussed in Section 3.

2. Relativistic CLTs

In this section we first develop CLTs that apply to the relativistic averaging of momenta, and then consider extensions to the relativistic averaging of velocities and energies, along with a functional CLT.

2.1. Momenta

Our first result is a full extension of the classical CLT to κ-sums, only requiring that the iid summands have finite second moment.

Theorem 1. *Let $\{X_i\}$ be a sequence of iid zero-mean r.v.s with variance $\sigma^2 < \infty$, and let $X_{ni} = X_i/\sqrt{n}$. Then*

$$X_{n1} \overset{\kappa}{\oplus} X_{n2} \overset{\kappa}{\oplus} \dots \overset{\kappa}{\oplus} X_{nn} \xrightarrow{d} \frac{1}{\kappa}\sinh(\kappa Z), \tag{2.1}$$

where $Z \sim N(0, \sigma^2)$.

Proof. The Lie group $(\mathbb{R}, \overset{\kappa}{\oplus})$ is group isomorphic to its Lie algebra $(\mathbb{R}, +)$, with the logarithmic map $x \mapsto \sinh^{-1}(\kappa x)$ providing the isomorphism. That is,

$$\sinh^{-1}[\kappa(x \overset{\kappa}{\oplus} y)] = \sinh^{-1}(\kappa x) + \sinh^{-1}(\kappa y), \tag{2.2}$$

which can be directly checked (without recourse to the theory of Lie groups) using the expression $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$. This provides the following decomposition of the κ-sum:

$$X_{n1} \overset{\kappa}{\oplus} X_{n2} \overset{\kappa}{\oplus} \ldots \overset{\kappa}{\oplus} X_{nn} = \frac{1}{\kappa} \sinh(\kappa T_n + \mu_n), \tag{2.3}$$

where $T_n = \frac{1}{\kappa} \sum_{i=1}^n (Y_{ni} - EY_{ni})$, $Y_{ni} = \sinh^{-1}(\kappa X_i/\sqrt{n})$, and $\mu_n = nEY_{n1}$.

Since $\sqrt{n}Y_{n1} \to \kappa X_1$ a.s. and $|\sqrt{n}Y_{n1}| \leq \kappa |X_1|$, by dominated convergence we have $\mu_n/\sqrt{n} = \sqrt{n}EY_{n1} \to \kappa EX_1$, and

$$\mathrm{Var}(T_n) = \frac{1}{\kappa^2} \sum_{i=1}^n \mathrm{Var}(Y_{ni}) = \frac{1}{\kappa^2} \left\{ E(\sqrt{n}Y_{n1})^2 - [E(\sqrt{n}Y_{n1})]^2 \right\} \to \mathrm{Var}(X_1) = \sigma^2.$$

Also, if t is a continuity point of the distribution of $\kappa|X_1 - EX_1|$, then by dominated convergence

$$\begin{aligned} & nE([Y_{n1} - EY_{n1}]^2 1\{\sqrt{n}|Y_{n1} - EY_{n1}| > t\}) \\ = \; & E[\sqrt{n}Y_{n1} - \mu_n/\sqrt{n}]^2 - E([\sqrt{n}Y_{n1} - \mu_n/\sqrt{n}]^2 1\{|\sqrt{n}Y_{n1} - \mu_n/\sqrt{n}| \leq t\}) \\ \to \; & \kappa^2 E([X_1 - EX_1]^2 1\{\kappa|X_1 - EX_1| > t\}). \end{aligned}$$

Thus, noting that the last term above tends to zero as $t \to \infty$, it follows that

$$\sum_{i=1}^n E([Y_{n1} - EY_{n1}]^2 1\{|Y_{n1} - EY_{n1}| > \epsilon\}) = nE([Y_{n1} - EY_{n1}]^2 1\{\sqrt{n}|Y_{n1} - EY_{n1}| > \epsilon\sqrt{n}\}) \to 0$$

for every $\epsilon > 0$, so the Lindeberg condition holds and $T_n \xrightarrow{d} N(0, \sigma^2)$ by the Lindeberg–Feller theorem.

The first derivative of $\sinh^{-1}(x)$, namely $1/\sqrt{1+x^2}$, is bounded between $1 - |x|^\alpha$ and 1 for any $0 < \alpha \leq 2$. By taking $\alpha = 1$ when $|x| \geq 1$ and $\alpha = 2$ when $|x| < 1$, then integrating, we obtain $|\sinh^{-1}(x) - x| \leq x^2 \min(|x|, 1)$. Using this inequality and the condition $EX_1 = 0$

(not used up to now), we have

$$\begin{aligned}|\mu_n| &= |nE[Y_{n1} - \kappa X_1/\sqrt{n}]| \le nE|\sinh^{-1}(\kappa X_1/\sqrt{n}) - \kappa X_1/\sqrt{n}| \\ &\le \kappa^2 E[X_1^2 \min(\kappa|X_1|/\sqrt{n}, 1)],\end{aligned}$$

which tends to zero by dominated convergence. The result then follows by (2.3) and the continuous mapping theorem. □

Example 1. Let the X_i be iid copies of a discrete r.v. X having distribution formed by standardizing (to have zero-mean and unit variance) the probability mass function

$$p_k = \frac{C}{k^3(\log k)(\log\log k)^2}, \quad k = 3, 4, \dots$$

where C is a normalizing constant. Note that $EX^2 < \infty$ and Theorem 1 is applicable, but $E[X^2 \log(1 + |X|)] = \infty$, so it is a borderline case. Also, X is not symmetric and the bias μ_n in (2.3) does not vanish. As proved in Theorem 1, $\mu_n \to 0$, but its rate of convergence to zero is very slow, as seen in Figure 1, indicating that the distribution of the κ-sum may be substantially skewed, even for large samples. Curiously, the bias does not tend monotonically to zero, but initially becomes more severe (for $n \le 64$).

In contrast, μ_n tends very rapidly to zero in the following example, see Figure 1.

Example 2. Let $X = \sqrt{2}$ with probability 1/3, and $-1/\sqrt{2}$ with probability 2/3. This X is not symmetric but has finite moments of all orders (as well as zero-mean and unit variance).

Note that the asymmetry in Example 1, rather than the borderline moment condition, is the prime cause of the slow rate of convergence of the bias to zero. For CLTs on general Lie groups, a suitable symmetry assumption can have the convenient effect of removing bias. For example, in connection with the explicit CLT on the (compact) Lie group $SO(3)$ mentioned in the Introduction, the main result of Qiu et al. (2014) assumes that the angular distribution of the random rotation is symmetric on $(-\pi, \pi]$; the possibility of an extension to the asymmetric case was not discussed.

Our second result extends the classical Lindeberg–Feller theorem and contains Theorem 1 as a special case, although the proof is less revealing in the sense that the bias is no longer made explicit.

Theorem 2. *If $\{X_{ni}, i = 1, \dots, n\}$ is a triangular array of independent zero-mean r.v.s such that $\sum_{i=1}^n E[X_{ni}^2 1\{|X_{ni}| > \epsilon\}] \to 0$ for all $\epsilon > 0$, and $\sum_{i=1}^n \mathrm{Var}(X_{ni}) \to \sigma^2 < \infty$, then (2.1) holds.*

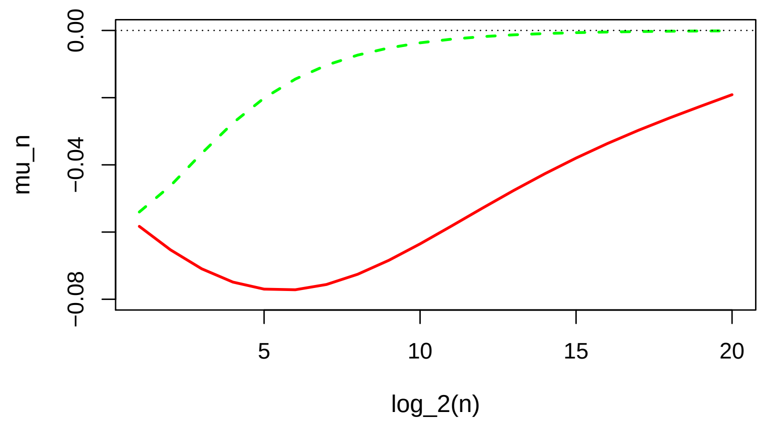

Figure 1: The behavior of the bias μ_n as a function of $\log_2 n$ in Example 1 (red solid line) and Example 2 (green dashed line), when $\kappa = \sigma = 1$.

Proof. Consider the following alternative decomposition of the κ-sum as

$$X_{n1} \overset{\kappa}{\oplus} X_{n2} \overset{\kappa}{\oplus} \dots \overset{\kappa}{\oplus} X_{nn} = \frac{1}{\kappa} \sinh(\kappa S_n + R_n),$$

where $S_n = \sum_{i=1}^n X_{ni}$ and $R_n = \sum_{i=1}^n \left\{\sinh^{-1}(\kappa X_{ni}) - \kappa X_{ni}\right\}$. By the Lindeberg–Feller theorem, $S_n \xrightarrow{d} Z \sim N(0, \sigma^2)$. The result then follows using the continuous mapping theorem and Slutsky's lemma if the remainder $R_n \to_p 0$. By the same bound on $|\sinh^{-1}(x) - x|$ used to handle the bias term μ_n in the proof of Theorem 1,

$$E|R_n| \le \kappa^2 \sum_{i=1}^n E[X_{ni}^2 \min(\kappa |X_{ni}|, 1)] = \kappa^2 (A_n + B_n),$$

where for any given $\epsilon > 0$,

$$A_n = \sum_{i=1}^n E[X_{ni}^2 \min(\kappa |X_{ni}|, 1) 1\{|X_{ni}| > \epsilon\}] \le \sum_{i=1}^n E[X_{ni}^2 1\{|X_{ni}| > \epsilon\}] \to 0$$

by the Lindeberg condition, and

$$B_n = \sum_{i=1}^n E[X_{ni}^2 \min(\kappa |X_{ni}|, 1) 1\{|X_{ni}| \le \epsilon\}] \le \kappa\epsilon \sum_{i=1}^n E X_{ni}^2$$

if $\epsilon < 1/\kappa$. Thus $\limsup B_n \leq \kappa\epsilon\sigma^2$, and since ϵ can be arbitrarily small, we also have $B_n \to 0$, completing the proof. $\square$

Our approach has involved showing that a Lindeberg–Feller type result holds for the transformed array $\sinh^{-1}(\kappa X_{ni})$ under standard conditions on the underlying array $\{X_{ni}\}$. In contrast, for the Lie group CLTs mentioned in the Introduction, moment conditions are needed on the transformed array (logarithmically mapped into the Lie algebra); in our special setting, those conditions are a consequence of the simpler conditions on the underlying array.

We conclude this section by noting that a full extension of the classical strong law of large numbers also holds. If X_i are iid with finite mean and $X_{ni} = X_i/n$, then

$$X_{n1} \overset{\kappa}{\oplus} X_{n2} \overset{\kappa}{\oplus} \ldots \overset{\kappa}{\oplus} X_{nn} \to \frac{1}{\kappa}\sinh(\kappa E X_1) \quad \text{a.s.} \tag{2.4}$$

The proof uses a similar argument to what we have already seen, except the relevant remainder term is handled using the inequality $|\sinh^{-1}(x) - x| \leq |x|\min(|x|, 1)$.

2.2. Arcsinh-normal distributions

We refer to a r.v. of the form $a \sinh X$, where X is normally distributed and a is a constant, as *arcsinh-normal*, in parallel with the term lognormal. Figure 2 shows the pdf of $\frac{1}{\kappa}\sinh(\kappa Z + \mu)$, where $Z \sim N(0, 1)$, for various choices of κ and μ. The distribution is close to normal for small κ, but has lognormal tail behavior; the lognormal tails become especially apparent as μ increases.

Arcsinh-normal distributions form a subclass of the translation system S_U introduced by Johnson (1949). Various classical characterization results for normal distributions can be translated immediately into arcsinh-normal versions. For example, if X and Y are independent and their κ-sum is arcsinh-normal, then both X and Y must be arcsinh-normal; this is a consequence of (2.2) and Cramér's theorem. Likelihood-based methods of inference for such distributions are available (Jones and Pewsey 2009), but, as far as we know, their appearance as limit laws in special relativity has not previously been noted.

2.3. Velocity and energy

Under special relativity, the velocity $u(q)$ and energy $\mathcal{E}(q)$ of a particle with momentum q are given (in dimensionless units) by

$$u(q) = \frac{q}{\sqrt{1+\kappa^2 q^2}}, \qquad \mathcal{E}(q) = \frac{1}{\kappa^2}\sqrt{1+\kappa^2 q^2}, \tag{2.5}$$

and the corresponding κ-sum rules are

$$u_1 \overset{\kappa}{\oplus}_{\mathrm{v}} u_2 = \frac{u_1 + u_2}{1 + \kappa^2 u_1 u_2}$$

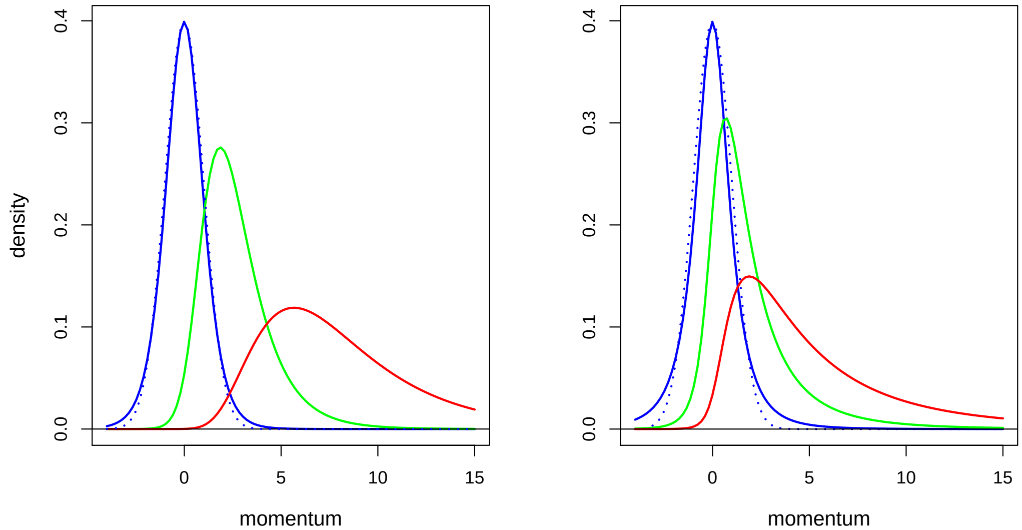

Figure 2: The pdf of $\frac{1}{\kappa}\sinh(\kappa Z+\mu)$ compared with the pdf of $Z \sim N(0,1)$ (dashed), for $\mu = 0$ (blue), $\mu = 1$ (green) and $\mu = 2$ (red), for $\kappa = .5$ (left panel) and $\kappa = .9$ (right panel).

and

$$\mathcal{E}_1 \overset{\kappa}{\oplus}_{\mathrm{e}} \mathcal{E}_2 = \kappa^2 \mathcal{E}_1 \mathcal{E}_2 + \frac{1}{\kappa^2}\sqrt{(\kappa^4 \mathcal{E}_1^2 - 1)(\kappa^4 \mathcal{E}_2^2 - 1)},$$

respectively, see Kaniadakis (2006). The previous CLTs for momenta can be translated into CLTs for velocity and energy as follows.

Let $U_{ni} = u(X_{ni})$ and $\mathcal{E}_{ni} = \mathcal{E}(X_{ni})$, where $\{X_{ni}, i = 1, \ldots, n\}$ is a triangular array of the form considered earlier. Using the identity (cf. Kaniadakis 2006)

$$U_{n1} \overset{\kappa}{\oplus}_{\mathrm{v}} U_{n2} \overset{\kappa}{\oplus}_{\mathrm{v}} \ldots \overset{\kappa}{\oplus}_{\mathrm{v}} U_{nn} = u(X_{n1} \overset{\kappa}{\oplus} X_{n2} \overset{\kappa}{\oplus} \ldots \overset{\kappa}{\oplus} X_{nn}),$$

applying the above theorems (and noting that $u(\cdot)$ is continuous) we have

$$U_{n1} \overset{\kappa}{\oplus}_{\mathrm{v}} U_{n2} \overset{\kappa}{\oplus}_{\mathrm{v}} \ldots \overset{\kappa}{\oplus}_{\mathrm{v}} U_{nn} \xrightarrow{d} \frac{\sinh(\kappa Z)}{\kappa\sqrt{1+\sinh^2(\kappa Z)}} = \frac{1}{\kappa}\tanh(\kappa Z)$$

and similarly

$$\mathcal{E}_{n1} \overset{\kappa}{\oplus}_{\mathrm{e}} \mathcal{E}_{n2} \overset{\kappa}{\oplus}_{\mathrm{e}} \ldots \overset{\kappa}{\oplus}_{\mathrm{e}} \mathcal{E}_{nn} \xrightarrow{d} \frac{1}{\kappa^2}\sqrt{1+\sinh^2(\kappa Z)} = \frac{1}{\kappa^2}\cosh(\kappa Z).$$

The pdfs of these limiting distributions are illustrated in Figs. 3 and 4.

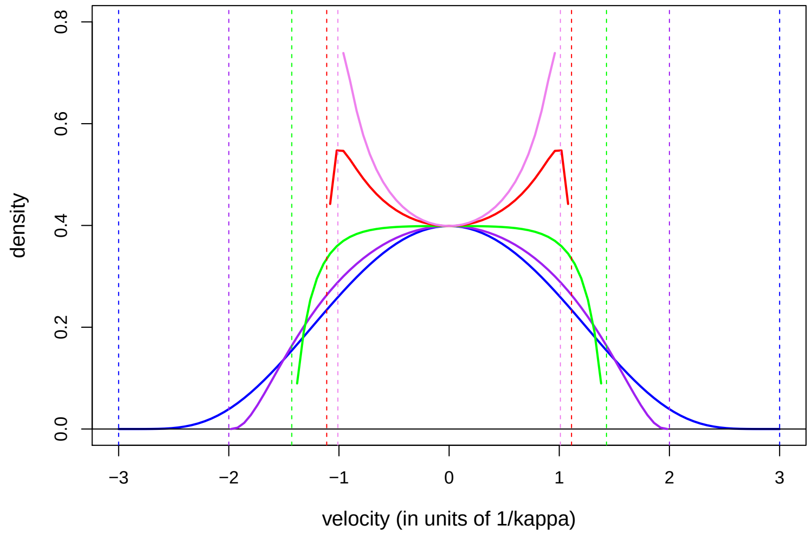

Figure 3: The pdf of the limiting distribution for velocity: $\kappa = .33$ (blue), $\kappa = .5$ (purple), $\kappa = .7$ (green), $\kappa = .9$ (red) and $\kappa = .99$ (violet). The vertical dashed lines indicate the lower and upper bounds ($\pm 1/\kappa$) on the velocity.

2.4. Relativistic invariance principle

Next we discuss a κ-sum version of the functional CLT (invariance principle) of Donsker (1951). As in Theorem 1, let $\{X_i\}$ be a sequence of iid zero-mean r.v.s with variance $\sigma^2 < \infty$, and set $X_{n,i} = X_i/\sqrt{n}$. Then define the *relativistic random walk for momentum* as the process

$$B_n(t) = X_{n,1} \overset{\kappa}{\oplus} X_{n,2} \overset{\kappa}{\oplus} \dots \overset{\kappa}{\oplus} X_{n,\lceil nt \rceil} \quad \text{for} \quad 0 < t \leq 1,$$

where $\lceil \cdot \rceil$ is the ceiling function, and set $B_n(0) = 0$. Viewing B_n as a random element in the space of bounded functions on $[0, 1]$ endowed with the uniform norm, under the conditions of Theorem 1 we have

$$B_n(t) \xrightarrow{d} B(t) = \frac{1}{\kappa} \sinh(\kappa W(t)),$$

where $W(t)$ is a Wiener process with infinitesimal variance σ^2. This follows from the classical Donsker theorem using a similar expansion to (2.3) and noting that the drift term μ_n, now a function of t, converges uniformly to zero.

3. Relativistic Maxwell–Boltzmann distributions

The first relativistic extension of the Maxwell–Boltzmann distribution was due to Jüttner (1911), but in recent years various authors have questioned whether Maxwell–Boltzmann–

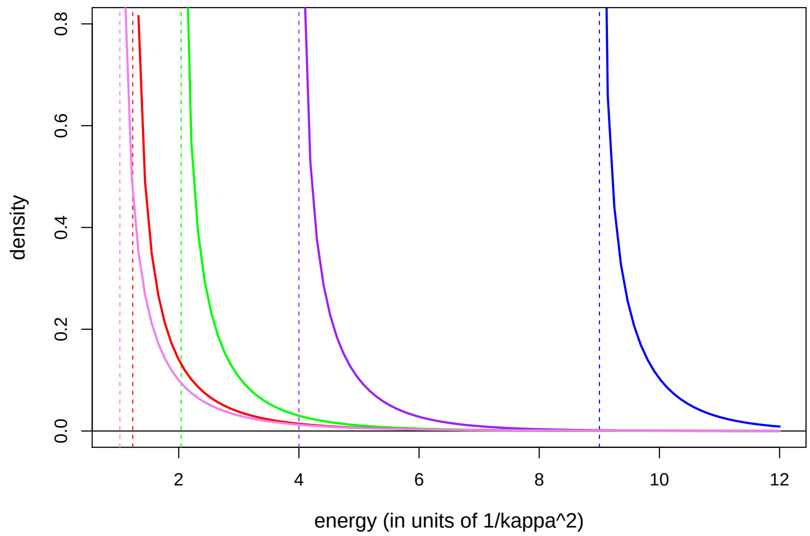

Figure 4: The pdf of the limiting distribution for energy, with the same color coding as in Figure 3. The vertical dashed lines indicate the lower bounds ($1/\kappa^2$) on the energy.

Jüttner theory is adequate to explain the flux distributions of relativistic sources, especially those that expand into an infinite surrounding space (e.g., cosmic rays, quasars, gamma ray bursts, and X-rays from black hole objects), rather than being confined to a closed vessel. Indeed, it has been observed that lognormal distributions may provide better approximations in such systems, see, e.g., Ioka and Nakamura (2002); Gaskell (2004); Gandhi (2009); Kunjaya et al. (2011); Gladders et al. (2013).

Multiplicative processes have typically been used to explain lognormal distributions. Yet existing approaches (e.g., based on self-organized criticality) used to model the X-ray flux of black hole objects rely on additive processes, and produce power-law tail behavior rather than lognormal (see Kunjaya et al. (2011), who develop a more sophisticated multiplicative model to address this issue). By extending classical central limit theory to allow the type of addition that is relevant to special relativity, and that is relevant to an open system, our results may provide an alternative explanation for the lognormal behavior that is a common feature of relativistic particle systems.

Dunkel et al. (2007) showed that the Jüttner (and modified-Jüttner) distribution can be obtained from the maximum entropy principle under the constraint that the average (relativistic) energy at equilibrium is fixed; this is done by defining the entropy with respect to the Haar measure on the relevant state space (a locally compact group, on which the Haar measure is preserved under the group operation). The Haar measure for the real line under κ-addition is $dq/\sqrt{1+\kappa^2 q^2}$, and the arcsinh-normal distribution maximizes entropy with respect to this Haar measure when a constraint is placed on the second moment of *rapidity*: $\varphi = \sinh^{-1}(\kappa q) = \tanh^{-1}(\kappa u)$, where q is the momentum and $u \in (-1/\kappa, 1/\kappa)$ is the velocity. The addition rule for rapidities is the usual addition (for parallel trajectories), so this result can be obtained directly from the entropy-maximizing property of the normal distribution. However, it is not natural from the physical point of view to place a constraint on the second moment of the rapidity, in contrast to a constraint on the expected energy say (used to derive Jüttner type distributions).

A physically more compelling approach to deriving a relativistic Maxwell–Boltzmann distribution is to construct a relativistic Langevin equation driven by a Wiener process, and determine its limiting distribution. Many types of relativistic Ornstein–Uhlenbeck (OU) processes directly driven by a standard Wiener process have been proposed, and shown to converge to either Jüttner or modified-Jüttner stationary distributions, see Dunkel and Hänggi (2009) and Angst (2011) for references to this extensive literature.

We now describe a different type of relativistic OU process and show that it has an arcsinh-normal stationary distribution. This involves using the relativistic Wiener process discussed in the previous section to drive a κ-deformed version of the Langevin equation. Since the exponential map $[x] \equiv \sinh(\kappa x)/\kappa$ is a field isomorphism from $(\mathbb{R}, +, \cdot)$ to its κ-deformed version, we have

$$[x] \overset{\kappa}{\otimes} [y] = [x \cdot y] \quad \text{and} \quad [x] \overset{\kappa}{\oplus} [y] = [x + y],$$

for κ-multiplication and κ-addition, respectively. Let X_t be an OU process on the real line satisfying the stochastic differential equation $dX_t = \alpha X_t \, dt + dW_t$, where $\alpha < 0$ and W_t is a standard Wiener process. It is then easily seen that the transformed process $Y_t = [X_t]$ satisfies the following κ-deformed Langevin equation driven by the relativistic Wiener process $B_t = [W_t]$:

$$d_\kappa Y_t = \beta \overset{\kappa}{\otimes} Y_t \overset{\kappa}{\otimes} d_\kappa t \overset{\kappa}{\oplus} d_\kappa B_t,$$

where $\beta = [\alpha]$, and we have used the κ-differential $d_\kappa Y_t \equiv Y_{t+dt} \overset{\kappa}{\ominus} Y_t$ as defined in Kaniadakis (2013). Formally, this means that Y_t satisfies the stochastic integral equation

$$Y_t = Y_0 \overset{\kappa}{\oplus} \int_0^t \frac{\beta \overset{\kappa}{\otimes} Y_s}{\sqrt{1 + \kappa^2 s^2}} \, ds \overset{\kappa}{\oplus} B_t, \quad t \geq 0, \tag{3.1}$$

where the Haar measure for the real line under κ-addition now plays a role because Y_t is being treated as a stochastic process on this Lie group. The stationary distribution of X_t is normal, so the stationary distribution of Y_t is arcsinh-normal. In related work, OU processes on Lie groups that are driven by a Wiener process on the associated Lie algebra have been studied by Baudoin et al. (2008), who showed the existence of "natural" OU processes with stationary distribution induced by the exponential map applied to the Wiener process at time 1.

To conclude, we have provided in (3.1) a natural and explicit construction of a relativistic OU process having a stationary distribution with lognormal tails, which may help explain the observed flux distributions of various astrophysical sources. Extensions of classical CLTs and OU processes to other Lie groups on the real, such as the Lie group associated with Tsallis entropy (see Tempesta (2011) for background) can also be developed using our approach. This will be the topic of a future paper.

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