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Transformations of Gaussian random fields to Brownian sheet and nonparametric change-point tests

Ian W. McKeague [a,*,1], Yanqing Sun [b,2]

[a] Department of Statistics, Florida State University, Tallahassee, FL 32306, USA
[b] Department of Mathematics, University of North Carolina, Charlotte, NC 28223, USA

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Abstract

It is shown that three important Gaussian random fields arising in statistics can be transformed to Brownian sheet. The Kiefer process, the tied-down Brownian sheet, and the 4-sided tied-down Brownian sheet are treated in this fashion. An application to change-point analysis is developed.

Keywords: Innovation martingale; Kiefer process; Tied-down Brownian sheet; Change in distribution; Kolmogorov–Smirnov statistics

1. Introduction

Let $b(t)$ be a Wiener process and $b^{o}(t) = b(t) - tb(1)$, $0 \leqslant t \leqslant 1$, the corresponding Brownian bridge. The martingale part of the Doob–Meyer decomposition of b^{o} is

$$w(t) = b^{o}(t) + \int_0^t \frac{b^{o}(s)}{1-s}\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

which is also a Wiener process. The so-called 'innovation martingale' w plays a fundamental role in Khmaladze's (1988, 1993) theory of goodness-of-fit tests. The innovation martingale is adapted to the filtration generated by b^{o}, and the transformation $b^{o} \mapsto w$ is one-to-one (Khmaladze, 1988). The inverse transformation is given by

$$b^{o}(t) = (1-t)\int_0^t \frac{1}{1-\tau}\,w(\mathrm{d}\tau), \quad 0 \leqslant t \leqslant 1. \tag{1.2}$$

* Corresponding author.
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The above transformation can be generalized to yield the innovation martingale of any process of the form $\eta(t) = b(t) - K(t)\xi$, where ξ is any random variable, $K(t) = \int_0^t k(s)\,\mathrm{d}s$, and $k(t)$ is a nonrandom function in $L^2[0,1]$ such that $\int_t^1 k^2(s)\,\mathrm{d}s > 0$ for all $t < 1$. The innovation martingale is now

$$w(t) = \eta(t) - \int_0^t \frac{\int_s^1 k(u)\,\mathrm{d}\eta(u)}{\int_s^1 k^2(v)\,\mathrm{d}v} k(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{1.3}$$

which is again a Wiener process. Here the stochastic integral with respect to η is defined in terms of the Wiener integral with respect to b; see, e.g., Kallianpur (1980, p. 135).

McKeague et al. (1995) extended these transformations to a class of random fields defined in terms of a Brownian sheet B on $[0,1]^2$, which is a continuous Gaussian process with mean zero and covariance function $\mathrm{cov}(B(t_1,z_1),B(t_2,z_2)) = (t_1 \wedge t_2)(z_1 \wedge z_2)$. Their transformation is given as follows.

Proposition 1. *Let B be a Brownian sheet. Given any random variable ξ, and continuous function $k\colon [0,1]^2 \to \mathbb{R}$ such that $\int_u^1 k^2(s,v)\,\mathrm{d}v > 0$ for $0 \leqslant u < 1, 0 \leqslant s \leqslant 1$, let $\eta(t,z) = B(t,z) - K(t,z)\xi$, where $K(t,z) = \int_0^z\int_0^t k(s,x)\,\mathrm{d}s\,\mathrm{d}x$. Then*

$$W(t,z) = \eta(t,z) - \int_0^z \left[\int_0^t\int_x^1 \frac{k(s,u)k(s,x)}{\int_x^1 k^2(s,v)\,\mathrm{d}v}\,\mathrm{d}\eta(s,u) \right] \mathrm{d}x \tag{1.4}$$

is a Brownian sheet on $[0,1]^2$, where the stochastic integral with respect to η is defined in terms of the Wiener integral with respect to B (e.g., Wong and Zakai, 1974).

McKeague et al. (1995) used this result to derive a consistent, distribution-free test for the independence of a survival time from a covariate.

The purpose of the present paper is to give a further statistical application of this result, namely to nonparametric change-point analysis. We derive the specific forms of the transformations in Section 2. The change-point application and the results of a simulation study designed to assess the value of the approach are presented in Section 3. Proofs of some asymptotic results needed in Section 3 are given in the Appendix.

2. Transformations

In this section we obtain some interesting transformations of a few familiar Gaussian random fields to Brownian sheet. Let B denote a Brownian sheet.

Example 1. Transformation of a Kiefer process to a Brownian sheet.

A Kiefer process can be represented in terms of a Brownian sheet by $\mathscr{K}(t,z) = B(t,z) - zB(t,1)$, see Csörgő and Révész (1981, p. 80). In Proposition 1, let $k(t,z) = 1$ and $\xi = 0$. Then

$$\begin{aligned} W(t,z) &= B(t,z) - \int_0^z \frac{B(t,1) - B(t,x)}{1-x}\,\mathrm{d}x \\ &= B(t,z) - zB(t,1) + \int_0^z \frac{B(t,x) - xB(t,1)}{1-x}\,\mathrm{d}x \\ &= \mathscr{K}(t,z) + \int_0^z \frac{\mathscr{K}(t,x)}{1-x}\,\mathrm{d}x \end{aligned} \tag{2.1}$$

is a Brownian sheet. Since for each fixed $t > 0$, $t^{-1/2}W(t,\cdot)$ is a Wiener process and $t^{-1/2}\mathscr{K}(t,\cdot)$ is a Brownian bridge, it follows from the one-to-one relationship in (1.1) that the transformation (2.1) of the process $\mathscr{K}(t,z)$ is invertible. The solution to (2.1) is

$$\mathscr{K}(t,z) = (1-z)\int_0^z \frac{1}{1-x}\, W(t,\mathrm{d}x). \tag{2.2}$$

Example 2. Transformation of a tied-down Brownian sheet.

The tied-down (or pinned) Brownian sheet is defined in terms of a Brownian sheet by $B^{\circ}(t,z) = B(t,z) - tzB(1,1)$, see Gaenssler and Stute (1979, p. 225). It arises as a weak limit of empirical distribution functions in two dimensions. In Proposition 1, let $k(t,z) = 1$ and $\xi = B(1,1)$. Then

$$\begin{aligned} W(t,z) &= B^{\circ}(t,z) - \int_0^z \left[\int_0^t\int_x^1 \frac{1}{1-x}\,\mathrm{d}B^{\circ}(s,u)\right]\mathrm{d}x \\ &= B^{\circ}(t,z) - \int_0^z \frac{B^{\circ}(t,1) - B^{\circ}(t,x)}{1-x}\,\mathrm{d}x \end{aligned} \tag{2.3}$$

is a Brownian sheet. However, unlike the one-to-one transformation (1.1) between the Wiener process and the Brownian bridge, the relationship (2.3) is no longer invertible since for any random variable ξ, the process $B^{\circ}(t,z) - tz\xi$ is a solution to (2.3).

Example 3. Transformation of a 4-sided tied-down Brownian sheet.

The 4-sided tied-down Brownian sheet is defined by

$$B^{\mathrm{e}}(t,z) = B(t,z) - tB(1,z) - zB(t,1) + tzB(1,1).$$

Notice that

$$B^{\mathrm{e}}(t,z) = \mathscr{K}(t,z) - t\mathscr{K}(1,z),$$

where $\mathscr{K}(t,z) = B(t,z) - zB(t,1)$ is a Kiefer process. Let W be the Brownian sheet obtained by (2.1). It follows from Example 1 that

$$\mathscr{K}^*(t,z) = B^{\mathrm{e}}(t,z) + \int_0^z \frac{B^{\mathrm{e}}(t,x)}{1-x}\,\mathrm{d}x = W(t,z) - tW(1,z) \tag{2.4}$$

is also a Kiefer process, and the transformation (2.4) is invertible, with

$$B^{\mathrm{e}}(t,z) = (1-z)\int_0^z \frac{1}{1-x}\mathscr{K}^*(t,\mathrm{d}x). \tag{2.5}$$

Eqs. (2.4) and (2.5) give a one-to-one relationship between the Kiefer process $\mathscr{K}^*$ and the 4-sided tied-down Brownian sheet B^{e}.

Let W^* be the Brownian sheet obtained by transform (2.1) of $\mathscr{K}^*$. Then

$$\begin{aligned} W^*(t,z) &= \mathscr{K}^*(t,z) + \int_0^t \frac{\mathcal{K}^*(s,z)}{1-s}\,\mathrm{d}s \\ &= B^{\mathrm{e}}(t,z) + \int_0^z \frac{B^{\mathrm{e}}(t,x)}{1-x}\,\mathrm{d}x + \int_0^t \frac{B^{\mathrm{e}}(s,z)}{1-s}\,\mathrm{d}s + \int_0^z\int_0^t \frac{B^{\mathrm{e}}(s,x)}{(1-s)(1-x)}\,\mathrm{d}s\,\mathrm{d}x \end{aligned} \tag{2.6}$$

is a Brownian sheet. This transform is one-to-one with inverse

$$B^{c}(t,z)=(1-t)(1-z)\int_0^t\int_0^z \frac{1}{(1-s)(1-x)}\,\mathrm{d}W^*(s,x). \tag{2.7}$$

It is easily seen, in view of the symmetry between t and z, how to generalize the transformations (2.6) and (2.7) to higher dimensions.

3. Nonparametric change-point problem

The stochastic processes mentioned in the previous section arise in a variety of hypothesis testing situations. In this section we explore one such application and present the results of a simulation study. The nonparametric change-point problem has been studied by Csörgő and Horváth (1987) and Hawkins (1988), among others. Let X_i, $i\geqslant 1$, be independent continuous random variables with corresponding distributions functions F_i. We wish to test the null hypothesis

$$\mathrm{H}_0\colon\ F_i=F_0,\quad i=1,\ldots,n,$$

where F_0 is a fixed distribution function (possibly specified), against the alternative that there is a change in distribution:

$$\mathrm{H}_1\colon\ F_i=F_0,\ \ i=1,\ldots,\tau-1\quad\text{and}\quad F_i=F_\tau\neq F_0,\ \ i=\tau,\ldots,n,\quad\text{where } \tau\in\{2,\ldots,n\}.$$

The null hypothesis H_0 states that there is no change in distribution from F_0 over the duration of observation. The alternative hypothesis states that a change in distribution occurs at observation τ, where τ is unspecified.

Let

$$\hat{F}_m(x)=\frac{1}{m}\sum_{i=1}^{m}I(X_i\leqslant x),$$

$$\hat{F}_{nm}(x)=\frac{1}{n-m}\sum_{i=m+1}^{n}I(X_i\leqslant x),$$

$$D^0_{nm}(x)=n^{-1/2}(n-m)(\hat{F}_{nm}(x)-F_0(x)),$$

$$D_{nm}(x)=n^{-1/2}\frac{m(n-m)}{n}(\hat{F}_{nm}(x)-F_m(x)).$$

Hawkins (1988) proposed the following test statistics. When F_0 is specified, the test statistic

$$T^0_n=\max_{0\leqslant m<n}\ \sup_x |D^0_{nm}(x)|$$

is used to measure the discrepancy between the data and the null hypothesis. When F_0 is unknown, instead use the test statistic

$$T_n=\max_{0\leqslant m<n}\ \sup_x |D_{nm}(x)|.$$

Let $F_0^{-1}(x)=\inf\{t\colon F_0(t)\geqslant x\}$, $0<x<1$, be the quantile function. Consider the 'test processes'

$$D^0_n(t,x)=n^{-1/2}\sum_{i=[nt]+1}^{n}\{I(X_i\leqslant F_0^{-1}(x))-x\},$$

if F_0 is specified, and

$$D_n(t,x) = n^{-1/2}\left\{\sum_{i=1}^{[nt]} I(X_i \leqslant \hat{F}_n^{-1}(x)) - \frac{[nt]}{n}\sum_{i=1}^{n} I(X_i \leqslant \hat{F}_n^{-1}(x))\right\},$$

if F_0 is unknown, where $0 < t < 1$, $0 < x < 1$, and $\hat{F}_n^{-1}(x)$ is the empirical quantile function. Csörgő and Horváth (1987) proposed a slightly different test process in the case that F_0 is unknown:

$$Y_n(t,x) = n^{-1/2}\sum_{i=1}^{[nt]} \psi_x(X_i - \hat{F}_n^{-1}(x)),$$

where

$$\psi_x(u) = \begin{cases} -(1-x) & \text{if } u \leqslant 0, \\ x & \text{if } u > 0. \end{cases}$$

As we show in the Appendix, Y_n and D_n are asymptotically equivalent in the sense that

$$|D_n(t,x) + Y_n(t,x)| \leqslant n^{-1/2} \tag{3.1}$$

for all $0 < t,x < 1$.

Hawkins (1988) and Csörgő and Horváth (1987) showed that under H_0 the test processes D_n^0 and D_n converge weakly to the time-reversed Kiefer process $\mathscr{K}'$ and 4-sided tied-down Brownian sheet B^e, respectively. Here $\mathscr{K}'(t,x) = \mathscr{K}(1-t,x)$. Csörgő and Horváth (1988) considered nine different test statistics based on Y_n and indicated where tables for the various limiting distributions can be found in the literature. For example, the distribution of the square integral of B^e has been tabulated by Blum et al. (1961) and Cotterill and Csörgő (1985).

Our approach is to transform the test processes D_n^0 and D_n by the transformations in Examples 1 and 3, so the resulting processes, denoted D_n^{0*} and D_n^*, converge in distribution to time-reversed Brownian sheet W' and Brownian sheet W, respectively. The independent increment property of Brownian sheet makes this approach especially appealing. We propose the following Kolmogorov–Smirnov-type test statistics and show that they have asymptotic null distributions of a relatively simple form:

$$T_n^{0*} \equiv \sup_{0<t<1}\sup_{0<x<1} |D_n^{0*}(t,x)| \xrightarrow{\mathscr{D}} \sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant x\leqslant 1} |W(t,x)| \tag{3.2}$$

and

$$T_n^{*} \equiv \sup_{0<t<1-n^{-1}}\sup_{0<x<1} |D_n^{*}(t,x)| \xrightarrow{\mathscr{D}} \sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant x\leqslant 1} |W(t,x)| \tag{3.3}$$

under H_0. The asymptotic null distributions of the untransformed test statistics T_n^0 and T_n have a more complicated form, being given in terms of extrema of the Kiefer process and the 4-sided tied-down Brownian sheet.

Proofs of (3.2) and (3.3) are given in the Appendix. Note that in defining T_n^* we have restricted t to the interval $(0, 1-n^{-1})$. This was done to avoid instability close to $t = 1$ in the transformed process $D_n^*(t,x)$, which is caused by $\lim_{s\nearrow 1} D_n(s,x) \neq 0$ for each n and $x < 1$. A similar restriction was used by Csörgő and Horváth (1988, p. 412).

3.1. Simulation results

We carried out a simulation study to compare the performance of the tests based on T_n^{0*} and T_n^* with that of T_n^0 and T_n. We considered a change in distribution from unit exponential (exp) to piecewise exponential (p-exp) having hazard rate $\lambda(x) = 1$ for $0 \leqslant x \leqslant 0.5$, and $\lambda(x) = 0.5$ for $x > 0.5$. We also considered a shift of the standard normal distribution by 0.5. Table 1 shows the observed levels and powers of the tests at an asymptotic level of 5%, for various values of n and τ. Table 2 shows the corresponding results when F_0 is unspecified. Each entry is based on 1000 samples. The 5% critical level used in the tests (i.e., the 95th percentile of the distribution of $\sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant x\leqslant 1} |W(t,x)|$) was 2.397. This was determined using a simulation of 10^4 Brownian sheets evaluated on the grid defined by 300 equally spaced points on each axis.

Inspecting the results, the power of our test is seen to be higher than that of the untransformed test if F_0 is specified or if the change in distribution takes place late in the sequence of observations. The power of the untransformed test tends to deteriorate as the change in distribution occurs later in the observation period (cf. the simulation results of Hawkins, 1988), whereas the transformed test tends to maintain its power throughout. This is not surprising in the case of T_n^* since the transformation in Example 3 involves division by the weight function $1 - t$, emphasizing changes close to $t = 1$. Similarly, the weight function $1 - x$ in Examples 1 and 3 increases the power of the tests against changes occurring in the tail of the distribution.

Csörgő and Horváth (1988) and Szyszkowicz (1994) have studied test processes of the form $D_n(t,x)/q(t)$, for various weight functions $q(t)$, with the aim of increasing the power of their tests against early or late changes in distribution. Our weight function $1 - t$ has a similar effect (for late changes), although it operates like a convolution rather than a scale change.

We have focused on statistics of Kolmogorov–Smirnov type (based on D_n^{0*} and D_n^*), but other types are possible (cf. Csörgő and Horváth, 1988, p. 411). As mentioned earlier, it is straightforward to find the required asymptotic critical values by Monte Carlo: repeatedly simulate Brownian sheet on a fine grid of points in $[0,1]^2$ and evaluate the statistic (e.g., the square integral in the case of the Cramér–von Mises statistic). FORTRAN computer code to do this can be obtained from the authors.

Table 1
Observed levels and powers of tests for a change in distribution when F_0 is specified; at an asymptotic level of 5%

n	F_0	F_τ	τ	T_n^0	T_n^{0*}
100	exp	exp	—	0.041	0.060
			25	0.65	0.981
	exp	p-exp	50	0.315	0.792
			75	0.115	0.443
150	exp	exp	—	0.07	0.066
			50	0.788	0.978
	exp	p-exp	75	0.502	0.929
			100	0.207	0.735
100	$N(0,1)$	$N(0,1)$	—	0.035	0.076
			25	0.977	0.995
	$N(0,1)$	$N(0.5,1)$	50	0.733	0.944
			75	0.285	0.456
150	$N(0,1)$	$N(0,1)$	—	0.058	0.063
			50	0.980	0.993
	$N(0,1)$	$N(0.5,1)$	75	0.902	0.906
			100	0.599	0.807

Table 2
Observed levels and powers of tests for a change in distribution when F_0 is unspecified; at an asymptotic level of 5%

n	F_0	F_τ	τ	T_n	T_n^*
100	exp	exp	—	0.058	0.051
			25	0.082	0.111
	exp	p-exp	50	0.174	0.312
			75	0.136	0.439
200	exp	exp	—	0.058	0.046
			50	0.218	0.272
	exp	p-exp	100	0.397	0.510
			150	0.132	0.570
100	$N(0,1)$	$N(0,1)$	—	0.045	0.051
			25	0.277	0.176
	$N(0,1)$	$N(0.5,1)$	50	0.511	0.445
			75	0.282	0.455
200	$N(0,1)$	$N(0,1)$	—	0.067	0.057
			50	0.569	0.385
	$N(0,1)$	$N(0.5,1)$	100	0.830	0.756
			150	0.535	0.812

Appendix

Proof of (3.1). Rewrite Y_n as

$$Y_n(t,x) = n^{-1/2}\sum_{i=1}^{[nt]}\{-(1-x)I(X_i \leqslant \hat{F}_n^{-1}(x)) + xI(X_i > \hat{F}_n^{-1}(x))\}$$

$$= -n^{-1/2}\left\{\sum_{i=1}^{[nt]} I(X_i \leqslant \hat{F}_n^{-1}(x)) - [nt]x\right\}$$

and use the fact that $|\hat{F}_n(\hat{F}_n^{-1}(x)) - x| \leqslant n^{-1}$. □

Let $\mathbf{D}_2$ denote the extension of the usual Skorohod space to functions on $[0,1]^2$, see Neuhaus (1971), and let $\|\cdot\|$ denote the supremum norm on $\mathbf{D}_2$. For $T<1$, define the transformation $\mathrm{H}_T : \mathbf{D}_2 \to \mathbf{D}_2$ by

$$\mathrm{H}_T\phi(t,z) = \phi(t,z) + \int_0^{z\wedge T} \frac{\phi(t,x)}{1-x}\,\mathrm{d}x.$$

Let W' be the time-reversed Brownian sheet $W'(t,z) = W(1-t,z)$, and define the time-reversed Kiefer process $\mathscr{K}'$ similarly. Let H_1 be the transformation defined in Example 1, so that $\mathrm{H}_1(\mathscr{K}') = W'$ and $D_n^{0*} \equiv \mathrm{H}_1(D_n^0)$.

Proof of (3.2). This proof is essentially due to Yongyuan Li. It suffices to show that $\mathrm{H}_1(D_n^0) \xrightarrow{\mathscr{D}} \mathrm{H}_1(\mathscr{K}')$. We apply Theorem 4.2 of Billingsley (1968), for which the following three conditions need to be checked:

(i) $\mathrm{H}_T(D_n^0) \xrightarrow{\mathscr{D}} \mathrm{H}_T(\mathscr{K}')$ for each $T<1$;

(ii) $\mathrm{H}_T(\mathscr{K}') \xrightarrow{\mathscr{D}} \mathrm{H}_1(\mathscr{K}')$ as $T \nearrow 1$;
(iii) $\lim_{T \nearrow 1} \limsup_{n\to\infty} P(\|\mathrm{H}_T(D_n^0) - \mathrm{H}_1(D_n^0)\| > \varepsilon) = 0$ for every $\varepsilon > 0$.

Now (i) follows from $D_n^0 \xrightarrow{\mathscr{D}} \mathscr{K}'$ (see the proof of Theorem 1 of Hawkins, 1988) and the continuous mapping theorem (H_T is continuous for each $T < 1$). To prove (ii), notice that

$$\|\mathrm{H}_T(\mathscr{K}') - \mathrm{H}_1(\mathscr{K}')\| \leqslant \sup_{0\leqslant t\leqslant 1} \int_T^1 \frac{|\mathscr{K}'(t,x)|}{1-x}\,\mathrm{d}x$$

and for $\varepsilon > 0$,

$$P\left(\sup_{0\leqslant t\leqslant 1} \int_T^1 \frac{|\mathscr{K}'(t,x)|}{1-x}\,\mathrm{d}x > \varepsilon\right) \leqslant \frac{1}{\varepsilon}\int_T^1 \frac{E\sup_{0\leqslant t\leqslant 1}|\mathscr{K}(t,x)|}{1-x}\,\mathrm{d}x$$

$$\leqslant \frac{1}{\varepsilon}\int_T^1 \frac{2(E\mathscr{K}^2(1,x))^{1/2}}{1-x}\,\mathrm{d}x \leqslant \frac{1}{\varepsilon}\int_T^1 \frac{2(1-x)^{1/2}}{1-x}\,\mathrm{d}x = \frac{4}{\varepsilon}\sqrt{1-T} \to 0 \tag{3.4}$$

as $T \nearrow 1$, where the second inequality in (3.4) follows by Doob's inequality since $\mathscr{K}(t,x)$ is a martingale in t for each fixed x. The proof of (iii) is parallel to that of (ii), with $\mathscr{K}'$ replaced by D_n^0. Notice that

$$D_n^0(t,x) \stackrel{\mathscr{D}}{=} n^{-1/2}\sum_{i=1}^{n-[nt]} \{I(X_i \leqslant F_0^{-1}(x)) - x\} \equiv M_n(n-[nt],x),$$

and that $M_n(k,x)$ is a martingale in k for each fixed n and x, so we can rewrite (3.4) with $\mathscr{K}'(t,x)$ replaced by $M_n(n-[nt],x)$. □

Proof of (3.3). First note that (t,x) can be restricted to the rectangle $R_n \equiv (0,1-n^{-1})^2$ without affecting T_n^*, since $D_n(t,x) = 0$ for $x \geqslant 1-n^{-1}$. Next, from Theorem 2.1 of Csörgő and Horváth (1987) and (3.1), one can define a sequence of 4-sided tied-down Brownian sheets $\{B_n^e(t,x), 0\leqslant t,x\leqslant 1\}$, $n \geqslant 1$, such that

$$P\left\{\sup_{0<t,x<1} |D_n(t,x) - B_n^e(t,x)| > A_1 n^{-1/4}(\log n)^{3/4}\right\} \leqslant B_1 n^{-\varepsilon}$$

for all $\varepsilon > 0$, where $A_1 = A(\varepsilon)$ and B_1 are constants. This leads to

$$\sup_{(t,x)\in R_n} |D_n(t,x) - B_n^e(t,x)|(1-t)^{-v_1}(1-x)^{-v_2} \xrightarrow{\mathrm{P}} 0 \tag{3.5}$$

as $n \to \infty$, for any $v_1 > 0$ and $v_2 > 0$ such that $v_1 + v_2 < 1/4$. Denoting the transformation in Example 3 by J, we have that $J(B_n^e)$, $n \geqslant 1$, is a sequence of Brownian sheets,

$$\sup_{(t,x)\in R_n} |J(B_n^e)(t,x)| \xrightarrow{\mathscr{D}} \sup_{0\leqslant t,x\leqslant 1} |W(t,x)| \tag{3.6}$$

and

$$D_n^* \equiv J(D_n) = J(B_n^e) + J(D_n - B_n^e) \quad \text{on } R_n. \tag{3.7}$$

The last term above is asymptotically negligible on R_n since

$$\sup_{(t,x)\in R_n} |J(D_n - B_n^{\rm e})(t,x)|$$

$$\leqslant \sup_{(t,x)\in R_n} |D_n(t,x) - B_n^{\rm e}(t,x)| + \sup_{(t,x)\in R_n} \left| \int_0^x \frac{D_n(t,u) - B_n^{\rm e}(t,u)}{1-u}\,{\rm d}u \right|$$

$$+ \sup_{(t,x)\in R_n} \left| \int_0^t \frac{D_n(s,x) - B_n^{\rm e}(s,x)}{1-s}\,{\rm d}s \right| + \sup_{(t,x)\in R_n} \left| \int_0^x \int_0^t \frac{D_n(s,u) - B_n^{\rm e}(s,u)}{(1-s)(1-u)}\,{\rm d}s\,{\rm d}u \right|,$$

which tends to zero in probability by (3.5). We conclude from (3.6) and (3.7) that

$$T_n^* \equiv \sup_{(t,x)\in R_n} |J(D_n)(t,x)| \xrightarrow{\mathscr{D}} \sup_{0\leqslant t,x\leqslant 1} |W(t,x)|,$$

as required. □

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