

### 3 Vorticity, Circulation and Potential Vorticity.

#### 3.1 Definitions

- **Vorticity** is a measure of the local spin of a fluid element given by

$$\vec{\omega} = \nabla \times \vec{v} \quad (1)$$

So, if the flow is two dimensional the vorticity will be a vector in the direction perpendicular to the flow.

- **Divergence** is the divergence of the velocity field given by

$$D = \nabla \cdot \vec{v} \quad (2)$$

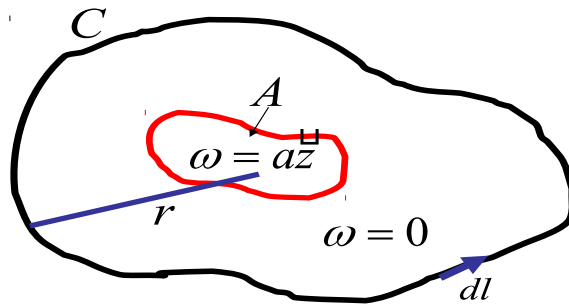
- **Circulation** around a loop is the integral of the tangential velocity around the loop

$$\Gamma = \oint_C \vec{v} \cdot d\vec{l} \quad (3)$$

For example, consider the isolated vortex patch in Fig. 1. The circulation around the closed curve C is given by

$$\Gamma = \oint_C \vec{v} \cdot d\vec{l} = \iint (\nabla \times \vec{v}) \cdot d\vec{s} = \iint \vec{\omega} \cdot d\vec{s} = \iint_A a ds = aA \quad (4)$$

where we have made use of Stokes' theorem. The circulation around the loop can also be approximated as the mean tangential velocity times the length of the loop and the length of the loop will be proportional to it's characteristic length scale  $r$  e.g. if the loop were a circle  $L = 2\pi r$ . It therefore follows that the tangential velocity around the loop is proportional to  $aA/r$  i.e. it does not decay exponentially with distance from the vortex patch. So regions of vorticity have a remote influence on the flow in analogy with electrostatics or gravitational fields. The circulation is defined to be positive for anti-clockwise integration around a loop.



**Figure 1:** An isolated vortex patch of vorticity  $a$  pointing in the direction out of the page will induce a circulation around the loop C with a tangential velocity that's proportional to the inverse of the characteristic length scale of the loop ( $r$ ).

## 3.2 Vorticity and circulation in a rotating reference frame

- **Absolute vorticity** ( $\vec{\omega}_a$ ) = vorticity as viewed in an inertial reference frame.
- **Relative vorticity** ( $\vec{\zeta}$ ) = vorticity as viewed in the rotating reference frame of the Earth.
- **Planetary vorticity** ( $\vec{\omega}_p$ ) = vorticity associated with the rotation of the Earth.

$$\vec{\omega}_a = \vec{\zeta} + \vec{\omega}_p \quad (5)$$

where  $\vec{\omega}_a = \nabla \times \vec{v}_I$ ,  $\vec{\zeta} = \nabla \times \vec{v}_R$  and  $\vec{\omega}_p = 2\vec{\Omega}$ . Often we are concerned with horizontal motion on the Earth's surface which we may consider using a tangent plane approximation or spherical coordinates. In such a situation the relative vorticity is a vector pointing in the radial direction and the component of the planetary vorticity that is important is the component pointing in the radial direction which can be shown to be equal to  $f = 2\Omega \sin\phi$ . So, when examining horizontal motion on the Earth's surface we have

$$\vec{\omega}_a = \vec{\zeta} + f \quad (6)$$

where the relative vorticity in cartesian or spherical coordinates in this situation is as follows:

$$\vec{\zeta} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \quad \text{or} \quad \vec{\zeta} = \left( \frac{1}{a} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos\phi} \frac{\partial(u \cos\phi)}{\partial \phi} \right) \hat{r}. \quad (7)$$

The absolute circulation is related to the relative circulation by

$$\Gamma_a = \Gamma_r + 2\Omega A_n \quad (8)$$

where  $A_n$  is the component of the area of the loop considered that is perpendicular to the rotation axis of the Earth.

### 3.2.1 Scalings

The relative vorticity for horizontal flow scales as  $U/L$  whereas the planetary vorticity scales as  $f$ . Therefore another way of defining the Rossby number is by the ratio of the relative to planetary vorticities.

### 3.2.2 Conventions

- **In the Northern Hemisphere**

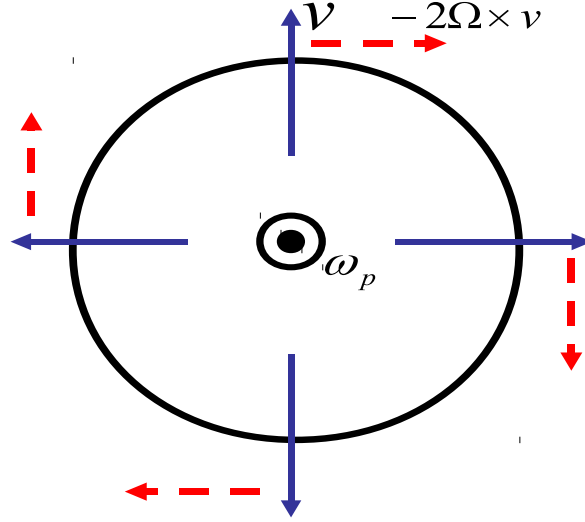
High pressure systems (anticyclones):  $\Gamma < 0$ ,  $\zeta < 0$ , Clockwise flow.

Low pressure systems (cyclones):  $\Gamma > 0$ ,  $\zeta > 0$ , Anti-clockwise flow.

- **In the Southern Hemisphere**

High pressure systems (anticyclones):  $\Gamma > 0$ ,  $\zeta > 0$ , Anticlockwise flow.

Low pressure systems (cyclones):  $\Gamma < 0$ ,  $\zeta < 0$ , Clockwise flow.



**Figure 2:** Schematic illustrating the induction of a circulation around a loop associated with the planetary vorticity.

### 3.3 Kelvin's circulation theorem

In the following section, Kelvin's circulation theorem will be derived. This theorem provides a constraint on the rate of change of circulation. Consider the circulation around a closed loop  $C$ . Consider the loop to be made up of fluid elements such that the time rate of change of the circulation around the loop is the material derivative of the circulation around the loop given by

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt} \oint_C \vec{v} \cdot d\vec{l} = \oint_C \frac{D\vec{v}}{Dt} \cdot d\vec{l} + \oint_C \vec{v} \cdot \frac{Dd\vec{l}}{Dt} \quad (9)$$

The second term can be re-written using the material derivative of line elements as follows

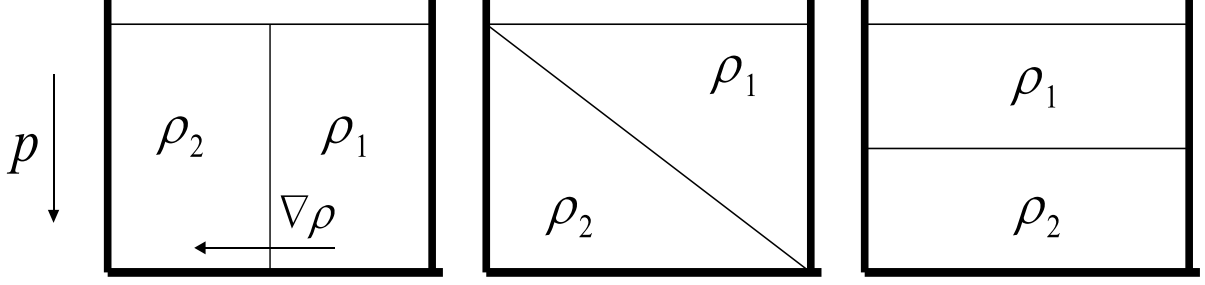
$$\oint_C \vec{v} \cdot \frac{Dd\vec{l}}{Dt} = \oint_C \vec{v} \cdot (d\vec{l} \cdot \nabla \vec{v}) = \oint_C d\vec{l} \cdot \nabla \left( \frac{1}{2} |\vec{v}|^2 \right) = 0 \quad (10)$$

This term goes to zero as it is the integral around a closed curve of the gradient of a quantity around that curve. We can then make use of our momentum equation (neglecting viscosity and including friction) to obtain an expression for the rate of change of the relative circulation

$$\boxed{\frac{D\Gamma}{Dt} = \oint_C (-2\vec{\Omega} \times \vec{v}) \cdot d\vec{l} - \oint_C \frac{\nabla p}{\rho} \cdot d\vec{l} + \oint_C \vec{F}_f \cdot d\vec{l}} \quad (11)$$

where  $\vec{F}_f$  is the frictional force per unit mass. There are therefore three terms that can act to alter the circulation, each of these will now be examined in more detail.

- **The Coriolis term:** Consider the circulation around the curve  $C$  in a divergent flow as depicted Fig. 2. It is clear that the coriolis force acting on the flow field acts to induce a circulation around the curve  $C$ .



**Figure 3:** An extremely baroclinic situation in pressure coordinates. Two fluids of different densities  $\rho_1$  and  $\rho_2$  are side by side ( $\rho_1 < \rho_2$ ). The baroclinic term generates a circulation which causes the denser fluid to slump under the lighter one until eventually an equilibrium is reached with the lighter fluid layered on top of the denser fluid and the baroclinic term =0.

- **The baroclinic term**  $\oint_C \frac{\nabla p}{\rho} \cdot d\vec{l}$ : This term can be re-written in a more useful form using stoke's theorem and a vector identity as follows

$$-\oint_C \frac{\nabla p}{\rho} \cdot d\vec{l} = -\iint_A \nabla \times \left( \frac{\nabla p}{\rho} \right) \cdot d\vec{s} = \iint_A \frac{(\nabla \rho \times \nabla p)}{\rho^2} \cdot d\vec{s} \quad (12)$$

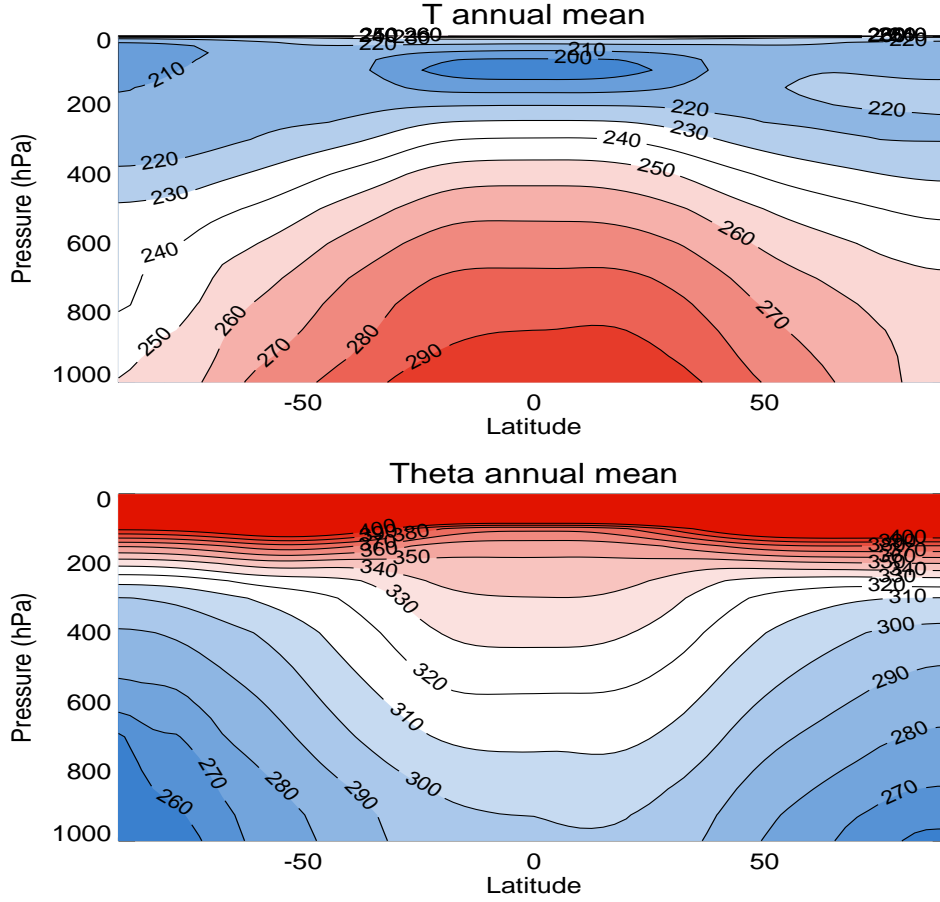
From this it can be seen that this term will be zero if the surfaces of constant pressure are also surfaces of constant density. A fluid is **Barotropic** if the density depends **only** on pressure i.e.  $\rho = \rho(p)$ . This implies that temperature does not vary on a pressure surface. In a barotropic fluid temperature does not vary on a pressure surface and therefore through thermal wind the geostrophic wind does not vary with height. In contrast a fluid is **Baroclinic** if the term  $\nabla \rho \times \nabla p \neq 0$ , for example if temperature varies on a pressure surface then  $\rho = \rho(p, T)$  and the fluid is baroclinic. In a baroclinic fluid the geostrophic wind will vary with height and there will be baroclinic generation of vorticity.

For example, consider the extremely baroclinic situation depicted in Fig. 3. We are considering here a situation with pressure decreasing with height and two fluids side by side of different densities  $\rho_1$  and  $\rho_2$  with  $\rho_1 > \rho_2$ .  $\nabla \rho \times \nabla p$  is non-zero and it can be seen that it would act to induce a positive circulation. As a result of this circulation the denser fluid slumps under the lighter fluid until eventually equilibrium is reached with the lighter fluid layered on top of the denser fluid.

This baroclinic term can also be written in terms of temperature and potential temperature

$$-\iint \nabla \times \frac{\nabla p}{\rho} ds = -\iint (\nabla \ln \theta \times \nabla T) ds \quad (13)$$

Consider Fig 4 showing the zonal mean temperature and potential temperature in the troposphere. In the tropics the temperature and potential temperature surfaces nearly coincide. In contrast in the mid-latitudes it is clear that the temperature and potential temperature surfaces do not coincide, or similarly there is a horizontal temperature gradient on pressure surfaces. Therefore, in the midlatitudes there is a large amount of baroclinic generation of circulation and vorticity responsible for the cyclonic systems always present at such latitudes.



**Figure 4:** Annual mean era-40 climatology of (top) Temperature and (bottom) Potential temperature.

- **The friction term** - Consider friction to be a linear drag on the velocity with some timescale  $\tau$  i.e.  $\vec{F}_f = -\frac{\vec{v}}{\tau}$ . This gives

$$\frac{D\Gamma}{Dt} = \oint_C \vec{F}_f \cdot d\vec{l} = -\frac{1}{\tau} \oint_C \vec{v} \cdot d\vec{l} = \frac{\Gamma}{\tau} \quad (14)$$

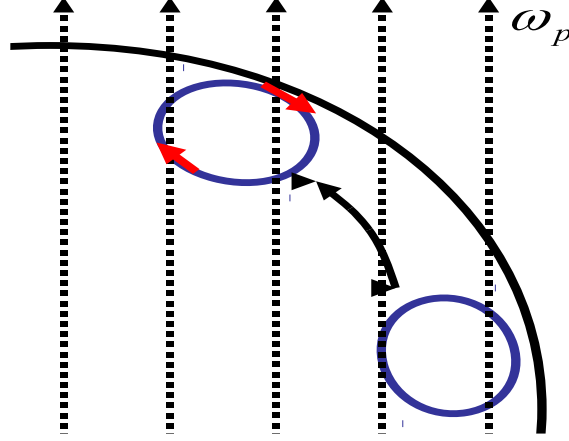
i.e. friction acts to spin-down the circulation.

Equation 11 provides us with **Kelvin's circulation theorem** which states that *if the fluid is barotropic on the material curve  $C$  and the frictional force on  $C$  is zero then absolute circulation is conserved following the motion of the fluid.* Absolute circulation being given by

$$\Gamma_a = \Gamma + 2\Omega A_n \quad (15)$$

In other words there will be a trade off between the relative and planetary vorticities. Consider Fig. 5. This shows a material curve that is shifted to higher latitude. As the curve moves to higher latitude the area normal to the Earth's rotation axis will increase and so the circulation associated with the planetary vorticity increases ( $2\Omega A_n$ ). In order to conserve the absolute circulation the relative circulation must decrease (an anti-cyclonic circulation is induced). Consider the example depicted in Fig. 3. Another

way of thinking about this is that the velocity field is acting to increase the area of the loop. The circulation associated with the planetary vorticity ( $2\Omega A_n$ ) must therefore increase and so to conserve absolute circulation the relative vorticity decreases (a clockwise circulation is induced).



**Figure 5:** Schematic illustrating the conservation of absolute vorticity in the absence of friction or baroclinicity. As the loop moves poleward the area normal to the vorticity of the Earth increases and so the planetary circulation ( $2\Omega A_n$ ) increases. In order to conserve absolute circulation the relative circulation decreases (i.e. an anticyclonic circulation is induced).

### 3.4 The Vorticity Equation

Kelvin's circulation theorem provides us with a constraint on the circulation around a material curve but it doesn't tell us what's happening to the circulation at a localised point. Another important equation is the vorticity equation which gives the rate of change of vorticity of a fluid element. Consider the momentum equation in the inertial reference frame in geometric height coordinates.

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \vec{F}_f - \nabla\Phi \quad (16)$$

Make use of the vector identity

$$\vec{v} \times (\nabla \times \vec{v}) = \nabla(\vec{v} \cdot \vec{v})/2 - (\vec{v} \cdot \nabla)\vec{v} \quad (17)$$

and the definition of absolute vorticity ( $\omega_a = \nabla \times \vec{v}$ ) and expand out the material derivative to write momentum balance in the form

$$\frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v} = -\frac{\nabla p}{\rho} + \vec{F}_f - \nabla(\Phi + \frac{1}{2}|\vec{v}|^2) \quad (18)$$

Taking the curl of this equation gives

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{v}) = \frac{(\nabla \rho \times \nabla p)}{\rho^2} + \nabla \times \vec{F}_f \quad (19)$$

Finally making use of the vector identity

$$\nabla \times (\vec{\omega} \times \vec{v}) = \vec{\omega} \nabla \cdot \vec{v} + (\vec{v} \cdot \nabla) \vec{\omega} - \vec{v} (\nabla \cdot \vec{\omega}) - (\vec{\omega} \cdot \nabla) \vec{v} \quad (20)$$

and noting that the divergence of the vorticity is zero, this gives

$$\boxed{\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{v} - \vec{\omega} (\nabla \cdot \vec{v}) + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \vec{F}_f} \quad (21)$$

This is the vorticity equation which gives the time rate of change of a fluid element moving with the flow. So, vorticity can be altered by the baroclinicity (third term) and friction (fourth term) just like in Eq. 11 for circulation. However, for vorticity there are two additional terms on the right hand side. These represent vortex stretching  $(\vec{\omega} \cdot \nabla) \vec{v}$  and vortex tilting  $((\vec{\omega} \cdot \nabla) \vec{v})$  and will be now be discussed in more detail.

### 3.4.1 Vortex stretching and vortex tilting

To understand what the vortex stretching and tilting terms represent it is useful to think in terms of vortex filaments and vortex tubes (see Pedlosky Chapter 2). A **vortex filament** is a line in the fluid that, at each point, is parallel to the vorticity vector at that point (Fig. 6 (a)). For example, the vortex filaments associated with the Earth's rotation would be straight lines parallel to the Earth's rotation axis. A **vortex tube** is formed by the surface consisting of the vortex filaments that pass through a closed curve  $C$  (Fig. 6 (b)). The bounding curve  $C$  at any point along the tube will differ in size and orientation. If the closed curve  $C$  is taken to consist of fluid elements then according to Kelvin's circulation theorem, in the absence of baroclinicity or friction the circulation around that material curve  $C$  is constant. The vortex stretching and tilting terms arise from the fact that, although the circulation around the closed curve  $C$  is constant, the vorticity is not due to the tilting and stretching of vortex tubes given that the vorticity is related to the circulation by  $\Gamma = \iint \omega \cdot d\vec{s}$ .

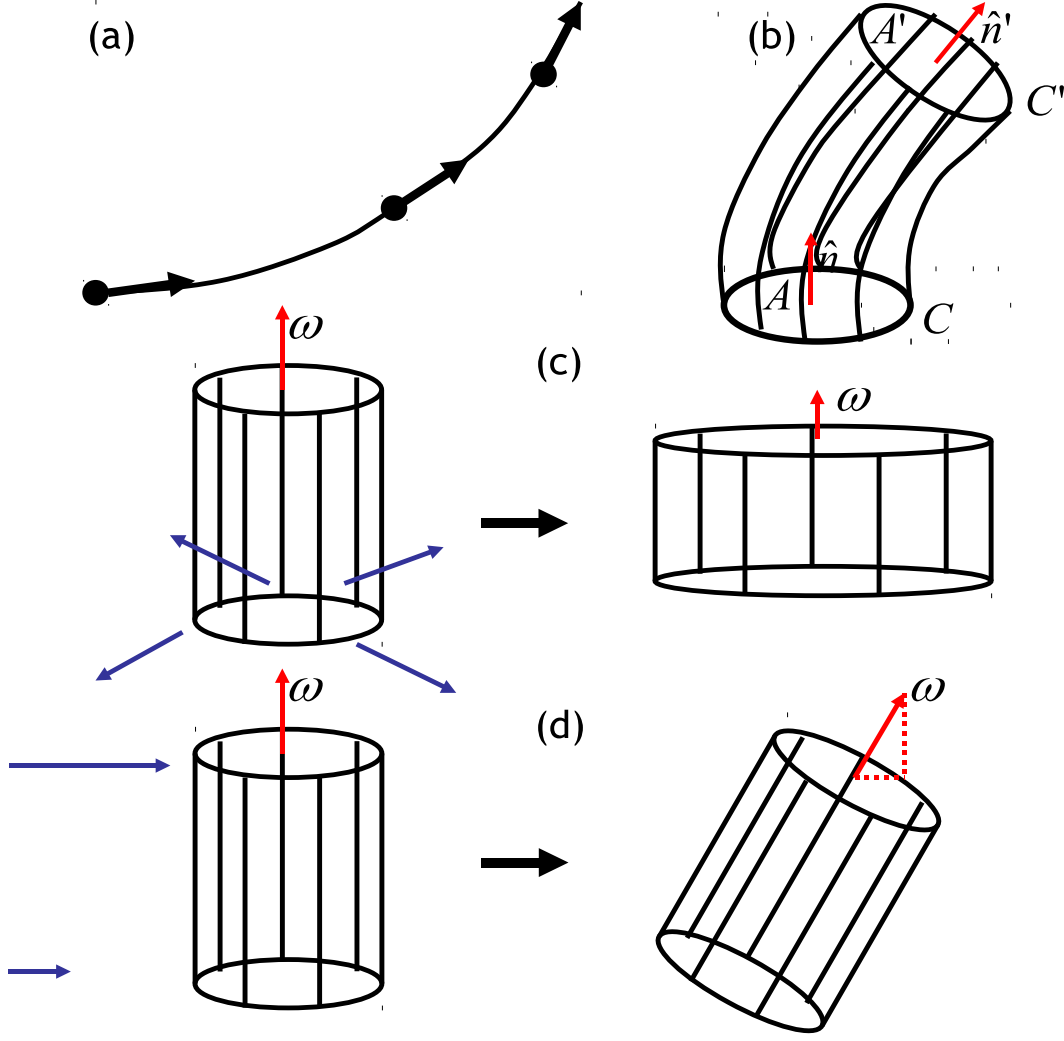
The vorticity is divergence free since  $\vec{\omega} = \nabla \times \vec{v}$  so the integral over some arbitrary volume of the divergence of vorticity is zero.

$$\iiint_V dV \nabla \cdot \vec{\omega} = \iint_A (\vec{\omega} \cdot \hat{n}) dA = 0, \quad (22)$$

using the divergence theorem, where  $\hat{n}$  is the outward normal over the surface of the volume. By definition the component of vorticity perpendicular to the sides of a vortex tube is zero. Therefore, considering the vortex tube in Fig. 6 (b) we have

$$0 = \iint_A \vec{\omega} \cdot (-\hat{n}_A) dA + \iint_{A'} \vec{\omega} \cdot \hat{n}_{A'} dA \rightarrow \iint_A (\vec{\omega} \cdot \hat{n}_A) dA = \iint_{A'} (\vec{\omega} \cdot \hat{n}_{A'}) dA \quad (23)$$

In other words, the vortex strength, or circulation is constant along the length of the tube. We can now understand the terms  $\vec{\omega} (\nabla \cdot \vec{v})$  and  $(\vec{\omega} \cdot \nabla) \vec{v}$  as the stretching and tilting of vortex tubes. For example consider the situation depicted in Fig. 6 (c) depicting a vortex tube where the vorticity is oriented completely in the vertical ( $\hat{k}$ ). The vortex



**Figure 6:** Illustrations of (a) a vortex filament, (b) a vortex tube, (c) vortex stretching and (d) vortex tilting.

stretching and tilting terms give

$$\begin{aligned}
 \frac{D\vec{\omega}_a}{Dt} &= \vec{\omega} \cdot \nabla \vec{v} - \vec{\omega} \nabla \cdot \vec{v} \\
 &= \omega \frac{\partial}{\partial z} (u\hat{i} + v\hat{j} + w\hat{k}) - \omega \hat{k} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\
 &= \hat{i}\omega \frac{\partial u}{\partial z} + \hat{j}\omega \frac{\partial v}{\partial z} - \hat{k}\omega \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
 \end{aligned} \tag{24}$$

Consider first the example depicted in Fig. 6 (c) where a horizontal divergent velocity field is present. This gives

$$\frac{D\omega}{Dt} \hat{k} = -\omega \hat{k} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{25}$$

i.e. the divergent velocity field would act to decrease the vertical component of vorticity. This makes sense if the curve  $C$  consists of fluid elements moving with the velocity field



and the circulation around the curve  $C$  is conserved. The divergent velocity field would act to increase the area enclosed by the curve  $C$  which must be counteracted by a decrease in the vorticity to keep the circulation constant. In contrast if the velocity field was convergent it would result in an increase in vorticity. The term is given the name vortex stretching because if an incompressible fluid was being considered, an increase in the vorticity associated with a decrease in the area enclosed by the material curves  $C$  can only be achieved if the vortex tube is stretched.

Consider the second case depicted in Fig. 6 (d) where a vertical shear in wind in the  $x$  ( $\hat{i}$ ) direction is present. From 24 this gives  $D\omega/Dt = \hat{i}\omega\frac{\partial u}{\partial z}$  i.e. it would act to increase the component of vorticity in the  $x$  direction. This can be understood by the wind shear acting to tilt the vortex tube such that the closed curves  $C$  and  $C'$  have a larger component of their normal vector pointing in the  $x$  direction and therefore the component of vorticity in the  $x$  direction is increased.

### 3.5 Potential Vorticity (PV)

So far we have derived Kelvin's circulation theorem which demonstrated that in the absence of friction or baroclinicity the absolute circulation is conserved. This provides us with a constraint on the circulation around a closed curve but it's non-local. It doesn't tell us what's happening to an individual fluid element. We would need to know how the material curve  $C$  evolves. The vorticity equation tells us how the vorticity of a localised point may change but there's no constraint - the terms on the right hand side of 21 could be anything.

What we really need to describe the flow is a scalar field that is related to the velocities that is materially conserved. The theory of Potential Vorticity due to Ertel (1942) provides us with such a quantity. It provides us with a quantity that is related to vorticity that is materially conserved. The theorem really combines the vorticity equation and Kelvin's circulation theorem.

#### 3.5.1 Case 1: The barotropic case

Kelvin's circulation theorem was valid for barotropic fluids in the absence of friction

$$\frac{D\Gamma_a}{Dt} = 0 \rightarrow \frac{D}{Dt} \oint_C \vec{u} \cdot d\vec{l} = \frac{D}{Dt} \iint_A \vec{\omega} \cdot d\vec{s} = 0 \quad (26)$$

Consider now an infinitesimal volume element that is bounded by two isosurfaces of a materially conserved tracer ( $\chi$ ) as depicted in Fig. 7. Since  $\chi$  is materially conserved  $D\chi/Dt = 0$ . If we consider Kelvin's circulation theorem around the infinitesimal fluid element then we have

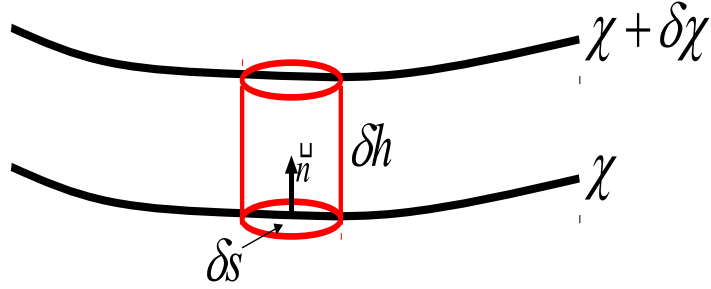
$$\frac{D}{Dt} \vec{\omega}_a \cdot d\vec{s} = \frac{D}{Dt} (\vec{\omega}_a \cdot \hat{n}) ds = 0.$$

The unit vector  $\hat{n}$  normal to the isosurface  $\chi$  is given by

$$\hat{n} = \frac{\nabla\chi}{|\nabla\chi|}$$

and the infinitesimal volume  $\delta V$  bounded by the two isosurfaces is  $\delta V = \delta h \delta S$ , where  $\delta h$  is the separation between the isosurfaces. Therefore,

$$(\vec{\omega}_a \cdot \hat{n}) ds = \vec{\omega}_a \cdot \frac{\nabla\chi}{|\nabla\chi|} \frac{\delta V}{\delta h}. \quad (27)$$



**Figure 7:** A fluid element that is bounded by two isosurfaces of a materially conserved tracer ( $\chi$ )

We can then write  $\delta h$  in terms of  $\delta\chi$  since  $\delta\chi = \delta h|\nabla\chi|$ . Putting this into 27 gives

$$(\vec{\omega}_a \cdot \hat{n}) ds = \vec{\omega}_a \cdot \frac{\nabla\chi}{\delta\chi} \delta V.$$

So, from Kelvin's circulation theorem we have

$$\frac{D}{Dt} \left[ \frac{(\vec{\omega}_a \cdot \nabla\chi) \delta V}{\delta\chi} \right] = \frac{1}{\delta\chi} \frac{D}{Dt} [(\vec{\omega}_a \cdot \nabla\chi) \delta V] = \frac{\delta M}{\delta\chi} \frac{D}{Dt} \left[ \frac{(\vec{\omega}_a \cdot \nabla\chi)}{\rho} \right] = 0,$$

since  $\chi$  is materially conserved therefore  $\delta\chi$  is also materially conserved. Also the mass of the fluid element ( $\delta M$ ) is materially conserved. Therefore we have the result that

$$\frac{Dq}{Dt} = 0, \quad \text{where} \quad q = \frac{\vec{\omega}_a \cdot \nabla\chi}{\rho}. \quad (28)$$

This is a statement of potential vorticity conservation where  $q$  is the potential vorticity.  $\chi$  may be any materially conserved quantity e.g.  $\theta$  for adiabatic motion of an ideal gas.

### 3.5.2 Case 2: The baroclinic case

Kelvin's circulation theorem only holds for a barotropic atmosphere. But, throughout a large proportion of the atmosphere (particularly the mid-latitudes) the baroclinic term

$$\iint_a \left( \frac{\nabla\rho \times \nabla p}{\rho^2} \right) \cdot d\vec{s} = - \iint_A (\nabla \ln\theta \times \nabla T) \cdot d\vec{s}$$

is non-zero. However, we can make it zero by choosing the correct tracer  $\chi$ . If we considered our volume to be between isosurfaces of constant  $\rho$ ,  $p$ ,  $\theta$  or  $T$  then the baroclinic term would go to zero. But, we also require that  $\chi$  be materially conserved and the only one of these quantities that is conserved following the motion of an ideal gas is potential temperature ( $\theta$ ). So, if we choose  $\theta$  as our materially conserved tracer then we have the result that

$$\frac{Dq}{Dt} = 0 \quad \text{where} \quad q = \left[ \frac{\vec{\omega}_a \cdot \nabla\theta}{\rho} \right] = 0 \quad (29)$$

This is a general statement of PV conservation which holds even in a baroclinic atmosphere. **The potential vorticity is a quantity that is related to the vorticity**

( $\vec{\omega}_a$ ) and the stratification ( $\nabla\theta$ ) that is materially conserved in the absence of friction or diabatic heating. If friction or diabatic heating were present then they would be source or sink terms on the right hand side of 29.

Note that in order to derive this we assumed that we were dealing with the motion of an ideal gas which allowed us to write the baroclinic term in terms of  $\theta$  and  $T$  and allowed us to assume that  $\theta$  was a materially conserved tracer. For most purposes in the atmosphere this is true. But, if for some reason this was not true then potential vorticity conservation would take on a different form.

### 3.5.3 Interpretation of PV

Potential vorticity conservation (29) is the foundation of our theories of atmospheric dynamics. It allows for the prediction of the time evolution of flows that are in near geostrophic balance and allows us to understand the propagation and generation of various different types of atmospheric waves (as we shall see in the following sections). We will examine PV in different systems: the shallow water model, two-layer quasi-geostrophic theory and the fully stratified 3D equations. In each of these systems potential vorticity takes on a slightly different form but the concept is the same. It is a materially conserved tracer that is related to both the velocities and the stratification. We can therefore assume it is advected by the mean flow. Therefore if the potential vorticity at a point in time is known, and the velocity field is also known then we can work out how the PV field will evolve allowing us to calculate the PV at a point a later time from which an inversion can give us the new velocity field.

For many purposes in the atmosphere it is the vertical component of vorticity that dominates.

$$q = \frac{\omega_{a,z} \frac{\partial\theta}{\partial z}}{\rho}$$

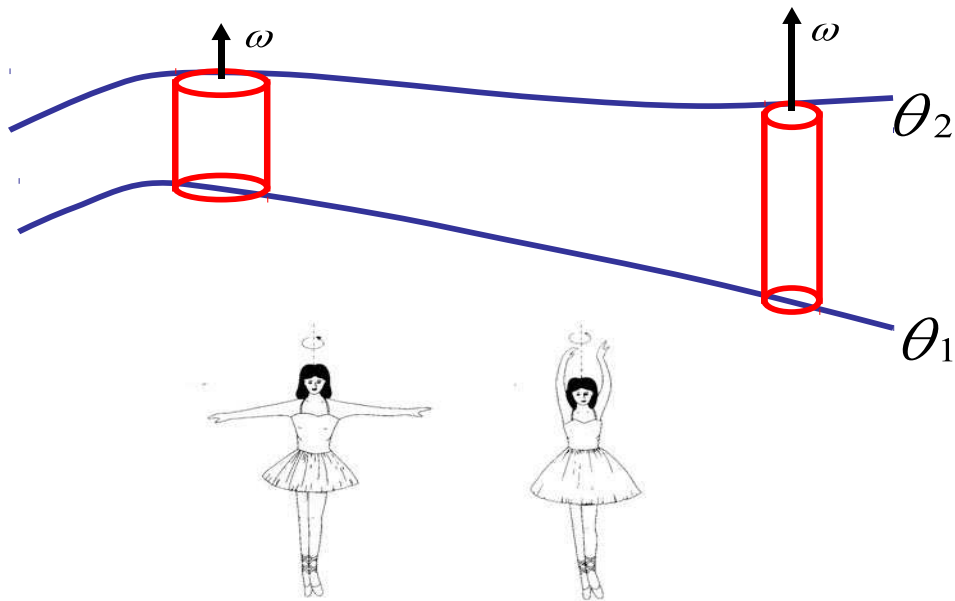
From hydrostatic balance we can re-write this as

$$\frac{Dq}{Dt} = 0, \quad q = \frac{(f + \zeta)}{\frac{\partial p}{\partial \theta}} \quad \text{where} \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

from which it is clear that PV is given by the absolute vorticity multiplied by a term that is a measure of the stratification.  $x$  and  $y$  are the local cartesian coordinates on a tangent plane. Strictly speaking the component of vorticity here is not actually the component in the vertical but it is the component that is perpendicular to isentropic surfaces (surfaces of constant  $\theta$ ) i.e. the derivatives with respect to  $x$  and  $y$  would be carried out with  $\theta$  held constant. If there are strong horizontal gradients of  $\theta$  (which by thermal wind balance indicates strong vertical wind shears) then this will differ significantly from the vertical component of vorticity (vortex tilting is important).

But, if the isentropic slope is small and tilting effects are neglected then the component of vorticity here is approximately the vertical one. If  $\partial\theta/\partial p$  was constant then temperature isn't varying on a pressure surface and we have a barotropic atmosphere. In that case, following the motion, air parcels would conserve the sum of their relative and planetary vorticities.

The quantity  $\partial p/\partial\theta$  is known as the **thickness**. It is the thickness between isentropic surfaces in pressure coordinates. It can be seen by considering Fig. 8 that potential vorticity conservation takes into account the effects of vortex stretching. We are considering a



**Figure 8:** Illustration of potential vorticity conservation. A fluid column is bounded by two surfaces of potential temperature. As potential temperature is conserved following the motion of the fluid column it stretches or compresses as the thickness between the potential temperature surfaces varies. Since the mass of the fluid column is conserved the stretching reduces the surface area of the column and vice versa. Therefore, via the circulation theorem the vorticity must increase/decrease if the thickness increases/decreases. Therefore, it is the ratio of the vorticity to the thickness that is conserved.

cylindrical column bounded by the isentropic surfaces  $\theta_1$  and  $\theta_2$ . The mass of the column is materially conserved and since the  $\theta$  is also materially conserved, as the column moves from the thin to the thick region it is stretched. As a result the area of the column on the isentropic surfaces is decreased and therefore the vorticity must increase to conserve the circulation. It can be seen that PV conservation takes this into account:  $|\partial p / \partial \theta|$  increases and so  $f + \zeta$  increases to conserve PV. This is really conservation of angular momentum for fluids. It can be seen to be analogous to the effect that when a ballerina or ice skater goes from a position with their arms out horizontally to their arms stretched vertically their vorticity (spin) increases.