4 The Shallow water system.

In the previous sections we have derived the equations that we require to examine the dynamics of the atmosphere. We will now examine the dynamics of particular types of motion in a simplified system: the shallow water system. Although this system is highly simplified in that it does not have the full stratification of the real atmosphere and we consider only a single layer of incompressible fluid, the motions that it supports have close analogues in the real atmosphere (and ocean).

4.1 Introduction to the shallow water approximation.

We consider the situation depicted in Fig. 1 where we have a single layer of incompressible fluid of uniform density ρ. The fluid above is taken to be of negligible density such that the surface is effectively free. There may be bottom topography of height h_B. The free surface height is $h_t(x, y, t)$ and the mean free surface height is denoted by H. At any point the free surface height perturbation (i.e. the anomaly from the mean) is denoted by $h' = h_t - H$. The system is said to be shallow as the depth H is much smaller than the horizontal scale of the fluid. A small aspect ratio of the system implies (by scale analysis of the continuity equation) that vertical velocities are much smaller than horizontal velocities. This is a condition which allows us to assume hydrostatic balance in the vertical.

4.1.1 Momentum balance in the shallow water system

In the vertical we assume that hydrostatic balance holds i.e.

$$\frac{\partial p}{\partial z} = -\rho g$$

Because density is constant this can be integrated to give

$$p = -\rho g z + C(x, y, t)$$

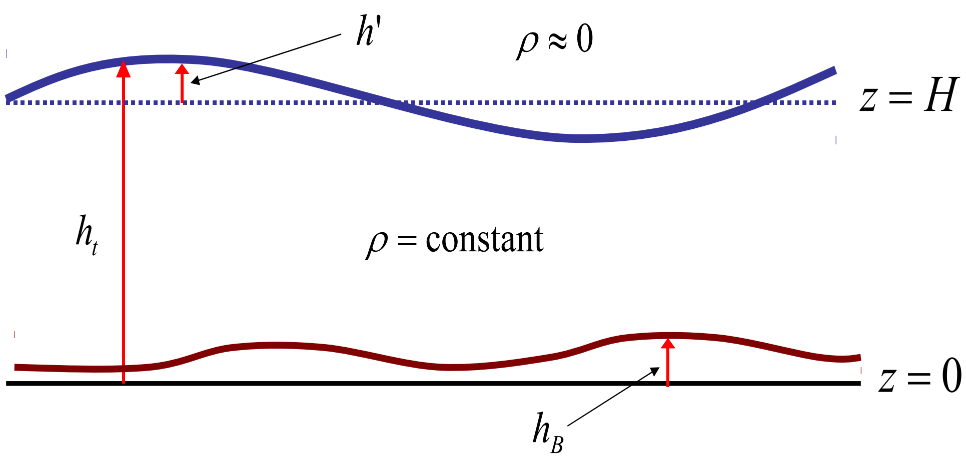

Figure 1: Schematic of the shallow water system

Assuming that $p = 0$ at $z = h_o$ gives

$$p = -\rho g(z - h_t)$$

In the horizontal we shall denote the horizontal velocity as $\vec{v}_H$. The horizontal momentum balance equation is

$$\frac{D\vec{v}_H}{Dt} + f \times \vec{v}_H = -\frac{\nabla p}{\rho}$$

where we have neglected friction or viscosity and the ∇ here is the horizontal gradient operator $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$ and $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$. But, from our momentum equation in the vertical $p = -\rho g(z - h_t)$ where $h_t = h_t(x, y, t)$. Therefore our horizontal momentum balance equation is

$$\boxed{\frac{D\vec{v}_H}{Dt} + \vec{f} \times \vec{v_H} = -g\nabla h_t.} \tag{1}$$

Since ∇h_t is independent of z then if $\vec{v}_H$ is initially independent of z then it remains so for all time. This is a consequence of assuming hydrostatic balance which, for a fluid of constant density, implies that the horizontal pressure gradients are independent of height. Note that the vertical velocity w is not zero. A vertical velocity is necessary to produce changes in the free surface height which would be associated with a convergence/divergence of the horizontal velocity field.

4.1.2 Mass continuity in the shallow water system

Consider a cylinder of thickness $h = h_t - h_B$. The mass of the cylinder is given by $\delta M = \rho h \delta A$ From mass continuity

$$\frac{D}{Dt}(\delta M) = \frac{D}{Dt}(\rho h \delta A) = \rho \frac{D}{Dt}(h\delta A) = 0$$

Therefore,

$$\rho\left(\frac{Dh}{Dt}\delta A + h\frac{D\delta A}{Dt}\right) = 0$$

But from the material derivative of line elements (Section 2.1) we have

$$\frac{D}{Dt}\delta A = \delta A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

So,

$$\rho\delta A\left(\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) = 0$$

$$\rightarrow \frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Therefore, the shallow water mass conservation equation is

$$\boxed{\frac{Dh}{Dt} + h\nabla.\vec{v}_H = 0} \tag{2}$$

So changes in the free surface height are associated with divergence/convergence of the horizontal velocity field. Since the fluid is incompressible we have in three dimensions

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3}$$

Equations 1 and 2 give us a complete set of equations for describing motion in the shallow water system.

4.1.3 Potential vorticity in the shallow water system

In section 3.5 we derived PV conservation for the fully stratified 3D system where we used Kelvin's circulation theorem on an infinitesimal fluid element bounded by isosurfaces of a materially conserved tracer. PV can also be derived algebraically from the vorticity equation. This will be done here for the shallow water system.

The vertical component of vorticity in the shallow water system ($\vec{\omega}^*$) can be defined as

$$\vec{\omega}^* = \nabla \times \vec{v}_H = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{k}\zeta$$

where $\vec{v}_H$ represents the horizontal velocity components. The horizontal momentum equation in the shallow water system is

$$\frac{D\vec{v}_H}{Dt} + f \times \vec{v}_H = -g\nabla h_t$$
$$\frac{\partial \vec{v}_H}{\partial t} + \vec{v}_H.\nabla\vec{v}_H + f \times \vec{v}_H = -g\nabla h_t$$

Use the following vector identity

$$(\vec{v}_H.\nabla)\vec{v}_H = \frac{1}{2}\nabla(\vec{v}_H.\vec{v}_H) - \vec{v}_H \times (\nabla \times \vec{v}_H)$$

to give

$$\frac{\partial \vec{v}_H}{\partial t} + (\vec{\omega}^* + f_o) \times \vec{v}_H = -\nabla(gh_t + \frac{1}{2}|v_H|^2).$$

Take the curl of this remembering that $\nabla \times \nabla A = 0$ for any A. This gives.

$$\frac{\partial}{\partial t}(\nabla \times \vec{v}_H) + \nabla \times ((\vec{\omega}^* + f_o) \times \vec{v}_H) = 0$$
$$\frac{\partial}{\partial t}(\nabla \times \vec{v}_H) + \nabla \times ((\zeta\hat{k} + f_o\hat{k}) \times \vec{v}_H) = 0$$

Make use of the vector identity $\nabla \times (\vec{A} \times \vec{B}) = (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B} + \vec{A}(\nabla.\vec{B}) - \vec{B}(\nabla.\vec{A})$ to give

$$\begin{aligned}\nabla \times ((\zeta + f_o)\hat{k} \times \vec{v}_H) &= \vec{v}_H.\nabla(\zeta + f_o)\hat{k} - ((\zeta + f_o)\hat{k}.\nabla)\vec{v}_H + (\zeta + f_o)\hat{k}(\nabla.\vec{v}_H) - \vec{v}_H\nabla.(\zeta + f_o)\hat{k} \\ &= (\vec{v}_H.\nabla)(\zeta + f_o)\hat{k} + (\zeta + f_o)\hat{k}\nabla.\vec{v}_H\end{aligned}$$

where $((\zeta + f_o)\hat{k}.\nabla)\vec{v}_H = 0$ since the vorticity is perpendicular to the horizontal surface on which $\vec{v}_H$ is varying and $\vec{v}_H\nabla.(\zeta + f_o)\hat{k}$ is zero since f_o is constant and $\zeta = \nabla \times \vec{v}_H$ and the divergence of a curl is zero. Therefore,

$$\frac{\partial \zeta}{\partial t} + (\vec{v}_H.\nabla)(\zeta + f_o) = -(\zeta + f_o)\nabla.\vec{v}_H. \tag{4}$$

The mass conservation equation

$$\frac{Dh}{Dt} + h\nabla.\vec{v}_H = 0$$

can be re-written in the following form

$$-(\zeta + f_o)\nabla.\vec{v}_H = \frac{\zeta + f_o}{h}\frac{Dh}{Dt}. \tag{5}$$

Putting 5 into 4 gives

$$\frac{\partial \zeta}{\partial t} + (\vec{v}_H.\nabla)(\zeta + f_o) = \frac{\zeta + f_o}{h}\frac{Dh}{Dt}.$$

f_o does not vary with time so we can add that into the time derivative giving

$$\frac{\partial(\zeta + f_o)}{\partial t} + (\vec{v}_H.\nabla)(\zeta + f_o) = \frac{\zeta + f_o}{h}\frac{Dh}{Dt}$$

So,

$$\begin{aligned} &\frac{D(\zeta + f_o)}{Dt} = \frac{\zeta + f_o}{h}\frac{Dh}{Dt} \\ \rightarrow &\frac{1}{h}\frac{D(\zeta + f_o)}{Dt} - \frac{\zeta + f_o}{h^2}\frac{Dh}{Dt} = 0 \\ \rightarrow &\frac{D}{Dt}\left(\frac{\zeta + f_o}{h}\right) = 0 \end{aligned}$$

We now have an expression for a materially conserved scalar that depends on the vorticity and the height of the fluid layer. This is the shallow water expression for potential vorticity conservation

$$\boxed{\frac{Dq}{Dt} = 0, \qquad \text{where} \qquad q = \frac{\zeta + f_o}{h}} \tag{6}$$

This is rather similar to the expression of PV conservation for the fully stratified 3D system derived previously but here we are only concerned with the vertical component of vorticity and instead of the thickness term $(\partial p/\partial \theta)$ we now have the thickness of the fluid layer (h).

4.2 Waves in the shallow water system

There are various different types of small amplitude wave motions that are solutions to the shallow water equations under different circumstances. These waves in the shallow water system behave in a similar manner to those that occur in the real atmosphere or ocean so examining them in the shallow water system allows us to understand their dynamics in a simplified setting.

The four main types of wave motion that will be examined in the following sections are *Shallow water gravity waves, Poincaré (or Inertio-gravity waves), Kelvin waves* and *Rossby waves*

4.2.1 Shallow water gravity waves

The simplest case is to consider the shallow water equations in the absence of rotation. Consider small amplitude motions, linearised about a state of rest:

$$h_t(x,y,t) = H + h'(x,y,t)$$
$$\vec{v}_H(x,y,t) = \vec{v}'_h(x,y,t)$$

The linearised momentum equations in the absence of rotation ($f = 0$) therefore become

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \tag{7}$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} \tag{8}$$

and the linearised mass continuity equation becomes

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0. \tag{9}$$

Taking $\partial(7)/\partial x$ and $\partial(8)/\partial y$ and substituting into $\partial(9)/\partial t$ gives

$$\frac{\partial^2 h'}{\partial t^2} - gH(\nabla^2 h') = 0 \tag{10}$$

This is a wave equation for which we can obtain a solution of the form $h' = h_o exp(i(kx + ly - \omega t))$ where k and l represent the wavenumbers in the x and y directions respectively and ω is the wave frequency. Putting such a solution into 10 gives the following dispersion relation

$$\omega = \pm\sqrt{gH}K \qquad \text{where} \qquad K = \sqrt{k^2 + l^2} \tag{11}$$

K is the total wavenumber. The waves that obey this dispersion relation are known as shallow water gravity waves since the restoring force for the wave motion is gravity. If we were considering the 1D case of a wave propagating in the x direction we would have $\omega = \pm\sqrt{gH}k$ i.e. motion with a phase speed $\omega/k = \pm\sqrt{gH}$. The phase speed is independent of wavenuber and depends only on the depth of the fluid layer. Therefore all wavenumbers travel at the same phase speed (the waves are non-dispersive). An initial free surface height perturbation of the form $F(x)$ can be Fourier decomposed into many different wavenumber components and the general solution will be a superposition of all these different fourier components travelling in the $\pm x$ direction with the same phase speed $c = \sqrt{gH}$. Since all Fourier components travel at the same phase speed the initial perturbation will maintain the same shape over time but move to the left and right i.e. the general solution is given by

$$h'(x,t) = \frac{1}{2}\left[F(x - ct) + F(x + ct)\right] \tag{12}$$

These waves must have a wavelength that is larger than the vertical height scale in order for the hydrostatic balance appromation to hold. Therefore the above dynamics doesn't apply to short waves in a deep ocean but Tsunami's can be considered as shallow water waves due to their large length scales. For gravity waves in the atmosphere the stratification is important and they obey slightly different dynamics.

4.2.2 Poincaré (or Inertio-gravity) waves

The linearised shallow water equations in the presence of rotation on an f-plane (i.e. the coriolis parameter is constant $f_o = 2\Omega sin\phi_o$), are

$$\frac{\partial u'}{\partial t} - f_o v' = -g\frac{\partial h'}{\partial x} \tag{13}$$

$$\frac{\partial v'}{\partial t} + f_o u' = -g\frac{\partial h'}{\partial y} \tag{14}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{15}$$

These equations can be combined to give the following wave equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} + f_o^2 - c^2\nabla^2\right)h' = 0 \tag{16}$$

where $c = \sqrt{gH}$. This is again a wave equation with a plane wave solution of the form $h' = h_o exp(i(kx + ly - \omega t))$ which upon substitution into 16 gives

$$-i\omega\left(-\omega^2 + c^2(k^2 + l^2) + f_o^2\right) = 0. \tag{17}$$

which has solutions

$$\omega = 0 \qquad \text{or} \qquad \omega^2 = f_o^2 + c^2(k^2 + l^2) \tag{18}$$

The zero frequency case is the time independent flow that satisfies geostrophic balance. The waves that satisfy the other dispersion relation are known as Poincarè waves or (inertio-gravity waves). Both the effects of rotation and gravity are important for these waves. Considering the case of 1D propagation in the x direction, the phase speed is given by

$$c = \frac{\omega}{k} = \sqrt{\frac{f_o^2}{k^2} + c^2} \tag{19}$$

i.e. the phase speed is no longer independent of wavenumber. The waves are now dispersive and longer waves have larger phase speeds. The group velocity is given by

$$c_g = \frac{\partial \omega}{\partial k} = \frac{kc^2}{\sqrt{f_o^2 + c^2k^2}} \tag{20}$$

which is always less than 1 and is smallest for large wavelengths (small wavenumbers).

There are two interesting limits to consider:

- **The short wave limit** - If $(k^2 + l^2) >> f_o^2/gH$ then the dispersion relation can be approximated by

  $$\omega^2 = c^2(k^2 + l^2) \tag{21}$$

  i.e. the shallow water gravity wave solution that occurs in the absence of rotation. Stating that $(k^2 + l^2) >> f_o^2/gH$ is equivalent to stating that $\lambda << \sqrt{gH}/f_o$ where $\lambda = 2\pi/\sqrt{k^2 + l^2}$ is the horizontal wavelength. So, if the wave disturbance is of sufficiently short length scale it will not be large enough to feel the effects of the rotation and the behaviour will approximately be that of shallow water gravity waves.

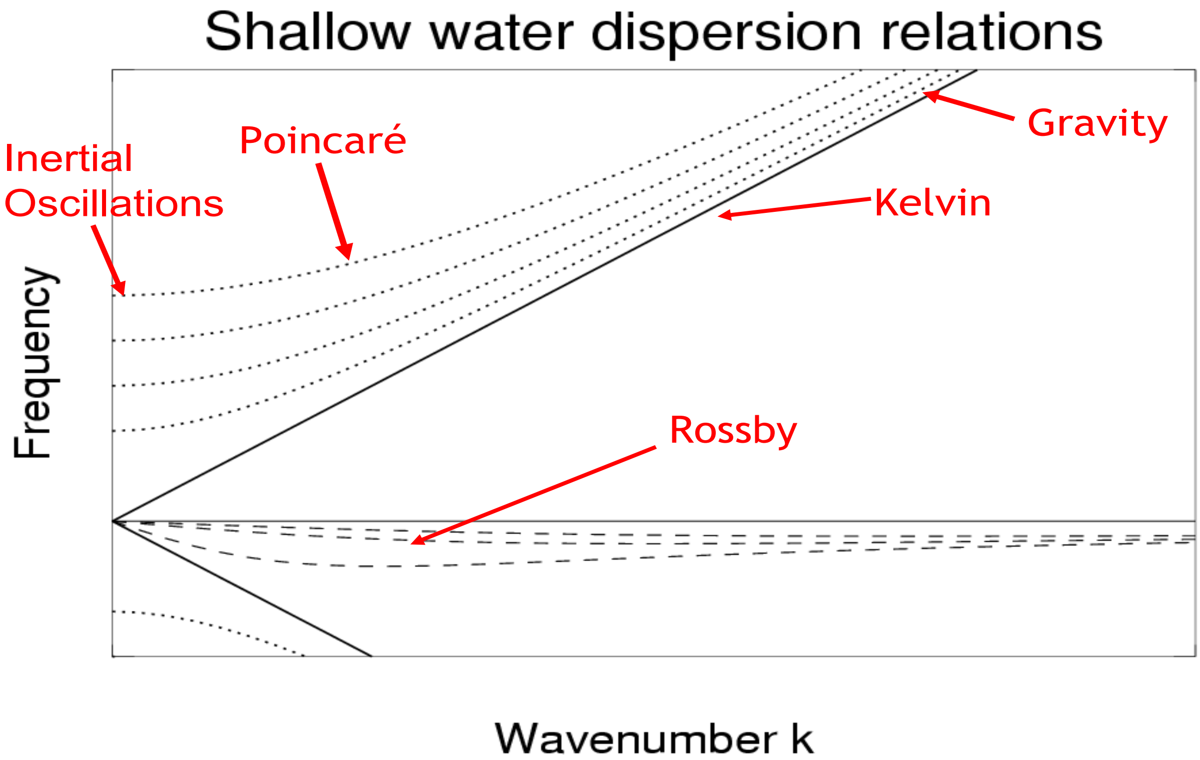

Figure 2: Dispersion relations for a rotating shallow water system. The Poincaré wave solutions are produced in the presence of a height perturbation in a rotating shallow water system. The Kelvin waves require the presence of a boundary (or the equator) and the Rossby waves require the presence of a gradient in potential vorticity.

- **The long wave limit** - If $(k^2 + l^2) << f_o^2/gH$ (or $\lambda >> \sqrt{gH}/f_o$) then the dispersion relation can be approximated by

$$\omega^2 = f_o^2 \tag{22}$$

  These are inertial oscillations: motions for which the dominant effect is rotation. The motions are of such large length scales that the effects of rotation are important and dominate over gravity effects.

The ratio $\sqrt{gH}/f_o$ that defines the boundary between motions for which gravity effects dominate and those for which rotation dominates is an important quantity. It is known as the **Rossby radius of deformation (L_D)**. For motions that have length scales much larger than L_D rotation dominates and for motions that have length scales far smaller than L_D rotation effects are not important and gravity or buoyancy effects dominate.

These two limits can be seen in the Poincaré wave dispersion relation in Fig. 2. For large scale motion (small k) the dispersion relation tends towards $\omega \simeq f_o$ and for small scale motion (large k) the dispersion relation tends toward that of shallow water gravity waves with $\omega \simeq \sqrt{gH}k$.

4.2.3 Kelvin waves

Kelvin waves are a particular type of wave solution to the shallow water equations in the presence of rotation but also in the presence of a boundary (or the equator where $f = 0$).

Figure 3: A schematic representation of the situation considered for the analysis of Kelvin waves.

Consider the case depicted in Fig. 3 where we have a boundary at $y = 0$. For Poincaré waves, a pressure (or height) gradient in the x direction would induce a velocity in the y direction due to the coriolis force. However, in the presence of a boundary we have an additional boundary condition that we cannot have a velocity perpendicular to the boundary condition. Therefore, a pressure gradient along the boundary cannot induce a velocity in the y direction at the boundary (or if our boundary is the equator then the coriolis force goes to zero) and different dynamics result.

In the case depicted in Fig. 3 we now have the additional boundary condition that $v' = 0$ at $y = 0$. It is therefore reasonable to search for solutions that have $v' = 0$ everywhere. Considering the linearised shallow water perturbation equations (13 to 15) but with $v' = 0$ the relevant equations for this system are:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \tag{23}$$

$$f_o u' = -g\frac{\partial h'}{\partial y} \tag{24}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x}\right) = 0 \tag{25}$$

i.e. in the x direction there is a balance between the acceleration and the pressure gradient and in the y direction there is geostrophic balance.

Combining 23 and 25 gives

$$\frac{\partial^2 h'}{\partial t^2} + c^2\frac{\partial^2 h'}{\partial x^2} = 0$$

where $c = \sqrt{gH}$ i.e. the shallow water gravity wave phase speed. Therefore in the x direction we have gravity wave dynamics, the same as in the non-rotating case. This has a general solution of the form

$$h' = A(y)\left[exp(i(x + ct)) + exp(i(x - ct))\right]$$

i.e. the sum of the two non-dispersive waves travelling in opposite directions. Putting this height perturbation into 23 it can be shown that the perturbation velocity in the x direction is

$$u' = -\left(\frac{g}{H}\right)^{\frac{1}{2}} A(y)\left[exp(i(x+ct)) - exp(i(x-ct))\right]$$

Turning to the remaining equation 24 and using the u' and h' above we can solve for $A(y)$. Consider the westward and eastward propagating waves seperately.

- **Westward:** $h' = A(y)exp(i(x+ct))$ and $u' = -\left(\frac{g}{H}\right)^{\frac{1}{2}} A(y)exp(i(x+ct))$ which on solving for $A(y)$ gives

  $$A(y) = exp(\frac{y}{L_D}) + C$$

  i.e. the wave grows unbounded in the y direction. This solution is unphysical - we can't have a wave that grows unbounded in the y direction.

- **Eastward:** $h' = A(y)exp(i(x-ct))$ and $u' = \left(\frac{g}{H}\right)^{\frac{1}{2}} A(y)exp(i(x-ct))$ which upon solving for y gives

  $$A(y) = exp(-\frac{y}{L_D}) + C$$

  i.e. a solution that decays exponentially away from the boundary.

So, the solutions to the shallow water perturbation equations in the presence of a boundary at $y = 0$ is

$$h' = A(y)exp\left(-\frac{y}{L_D}\right) exp(i(x-ct))$$

This is a wave that is Eastward propagating for which, at the vertical plane of the boundary and in any parallel vertical plane exhibits the behaviour of shallow water gravity waves. Where the rotation comes in is in the factor that causes the amplitude of the wave to decay exponentially away from the boundary. Over a distance of the Rossby radius away from the boundary rotation has become important for the motion and there is a balance between the pressure gradient in the y direction and the coriolis force on the velocity in the x direction (geostrophic balance in the y direction). These waves belong to a glass of waves known as boundary waves, edge waves or trapped waves.

Solving the above equations in a similar manner for any boundary in the x or y direction in the Northern or Southern hemisphere it can be demonstrated that:

- In the NH Kelvin waves always propagate with the boundary on their right hand side

- In the SH Kelvin waves always propagate with the boundary on their left hand side

Kelvin waves are particularly important in the ocean where continental boundaries are present. They will always propagate cyclonically around an ocean basin. Oceanic Kelvin waves play an important role in the dynamics of El Niño. In the atmosphere Kelvin waves occur at the equator where $f = 0$. Slightly different equations have to be solved (the β plane is considered) but the dynamics are roughly the same - they will always propagate Eastward along the equator.

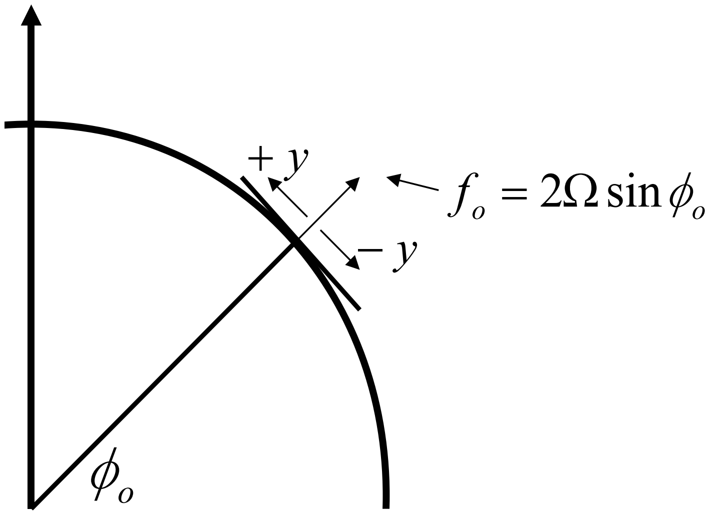

Figure 4: Schematic illustration of the β-plane

4.2.4 Rossby waves

Another important class of waves in the shallow water system are Rossby waves whose dispersion relations can be seen by the dashed curves in Fig. 2. These waves are important in both the ocean and atmosphere and rely on a background gradient of potential vorticity.

So far we have considered shallow water waves on an $f-$plane where rotation is considered to be constant. In the real atmosphere and ocean the vertical component of the Earth's rotation varies with latitude

$$f = 2\Omega sin(\phi)$$

which has important consequences. We now consider motion on a $\beta-$plane depicted in Fig. 4. Again this is a plane that is tangential to the surface of the Earth but instead of taking f to be constant it is considered to vary linearly with latitude. A taylor expansion of the coriolis parameter around a latitude ϕ_o (at y=0) gives

$$f(y) = f(0) + \left.\frac{df}{dy}\right|_{\phi_o} y = f_o + \beta y \tag{26}$$

where $f_o = 2\Omega sin(\phi_o)$ and $\beta = 2\Omega cos\phi_o/a$ where a is the radius of the Earth.

We will consider the case without bottom topography ($h_B = 0$) and take the height of the fluid h to be a constant (H). This is known as the rigid lid approximation. There are therefore no height perturbations $h' = 0$ and no gravity wave effects.

From PV conservation we have

$$\frac{D}{Dt}\left(\frac{\zeta + f}{H}\right) = 0$$

which using 26 and expanding the material derivative of f gives

$$\frac{D\zeta}{Dt} + v\beta = 0 \tag{27}$$

In other words the relative vorticity can be altered by advection of the planetary vorticity. This makes sense because if we are considering a fluid element that exists at a low latitude and it is advected by a positive v to a higher latitude where f increases then the relative vorticity ζ must decrease in order to conserve potential vorticity. This can be seen to be exactly what is produced by equation 27.

Now, since we have constrained H to be constant the horizontal flow must be non-divergent and it can therefore be described by a stream function ψ as follows

$$\vec{v}_H = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right)$$

The PV equation 27 can therefore be re-written in the following form

$$\begin{aligned}
\frac{D\zeta}{Dt} + \beta v &= 0 \\
\frac{\partial\zeta}{\partial t} + \vec{v}_H.\nabla\zeta + \beta v &= 0 \\
\frac{\partial\zeta}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} + \beta\frac{\partial\psi}{\partial x} &= 0
\end{aligned}$$

We linearise this equation about a basic state stream function $\psi = \psi_o + \psi'$ where we take the basic state to consist of a uniform zonal flow (U) in the x direction i.e. $\psi_o = -Uy$ and note that the relative vorticity can be written in terms of the perturbation stream function $\zeta' = \nabla^2\psi'$ (and $\zeta_o = 0$) giving

$$\frac{\partial}{\partial t}(\nabla^2\psi') + U\frac{\partial(\nabla^2\psi')}{\partial x} + \beta\frac{\partial\psi'}{\partial x} = 0. \quad (28)$$

This is a non-linear wave equation for which we can find a solution of the form $\psi' = \psi_o exp(i(kx + ly - \omega t))$ with which we can obtain the following dispersion relation

$$\boxed{\omega = Uk - \frac{\beta k}{k^2 + l^2}} \quad (29)$$

where $k^2 + l^2 = K^2$ (the total wavenumber squared). This is the Rossby wave dispersion for shallow water Rossby waves in a uniform zonal flow. These are known as **Barotropic** Rossby waves since the motion is constant with height. Considering the case of one dimensional propagation in the x direction the phase velocity is given by

$$c_p = \frac{\omega}{k} = U - \frac{\beta}{k^2}$$

That is Rossby waves always have westward phase propagation with respect to the zonal flow. In the mid-latitude atmosphere there are strong Westerly winds so even though they are propagating in a Westward direction relative to the flow, if they were to be observed with respect to the surface of the Earth they are travelling in the Eastward direction. The phase velocity depends on wavenumber and therefore Rossby waves are dispersive. Furthermore due to the variation of β with latitude the westerly phase speed is larger at lower latitudes.

A stationary wave solution exists given by

$$K^2 = \frac{\beta}{U} \quad (30)$$

Considering an order of magnitude estimate for the Earth's atmosphere ($\beta \sim 10^{-11}$, $U \sim 10ms^{-2}$) gives $K \sim 10^{-12}$ m^{-2}. This is related to the total wavelength by $\lambda \sim 2\pi/\sqrt{K^2}$ giving $\lambda \sim 6000$km. So, the stationary solution only exists on a very large scale. If the waves are on a smaller scale they will not be stationary and will propagate in a westward direction relative to the zonal mean flow.

The group velocity is given by

$$c_{gx} = \frac{\partial \omega}{\partial k} = U + \frac{\beta(k^2 - l)}{(k^2 + l^2)^2}$$

so the group velocity can be positive or negative depending on the ratio of the zonal to meridional wavenumbers.

The mechanism of Rossby wave propagation

The mechanism of Rossby wave propagation can be understood by considering the situation depicted in Fig. 5. Here we are considering a resting fluid on a $\beta-$plane, so the background PV is given by

$$q = \frac{f_o + \beta y}{H}$$

Consider then the contours of constant potential vorticity ($q_1 < q_2 < q_3$). The contour q_2 is then displaced as shown by the solid red line. This contour initially consists of fluid elements and so as they are displaced they must maintain the same initial potential vorticity due to PV conservation. At position (1) the fluid elements have moved to a lower latitude where the background PV is smaller. Therefore in order to conserved PV the relative vorticity of the fluid elements (ζ) must increase and if we are considering the case in the NH then a cyclonic (anti-clockwise) flow is induced. At position (2) the

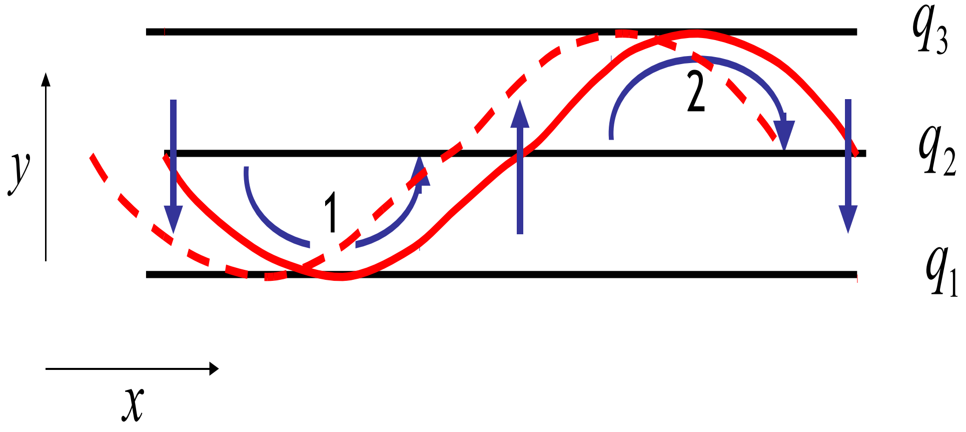

Figure 5: Schematic illustration of the mechanism of Rossby wave propagation. The PV contour q_2 is displaced to the position given by the red contour. This induces a circulation anomaly as indicated by the curved blue arrows. The straight blue arrows indicate the locations of the maximum velocity anomaly. The resulting velocity field advects the PV contour in such a way that the anomaly travels to the left (red-dashed line).

opposite occurs: an anti-cyclonic (clockwise) flow is induced. It can be seen that the resulting velocity field would act to advect the material elements of the displaced contour in such a way that the disturbance shape would move to the left (i.e. a westward phase propagation). In the SH the variation of β and the signs of vorticity are such that a westward propagation also occurs.

So, what we have is an initial displacement of a PV contour induces a relative vorticity anomaly due to the gradient of background planetary vorticity with latitude. This relative vorticity anomaly is associated with a circulation change that acts to advect the PV contour in such a way that the initial displacement propagates to the west. Here the background PV gradient is associated with the variation of the coriolis parameter with latitude. More generally the background PV gradient could be due to both the variation of the coriolis parameter with latitude and the PV associated with the background flow. We could have equally examined Rossby waves in a situation where the coriolis parameter is constant with latitude but there is a variation of H: all that is required is a background gradient of potential vorticity. But, for the atmosphere, the troposphere could be considered to be a layer where the high static stability of the tropopause effectively acts as a rigid lid and it is really the variations in the coriolis parameter with latitude that are key to the mechanism of Rossby wave propagation.

4.3 Summary of shallow water waves

- **Gravity waves**: Occur in the situation without rotation. Non-dispersive with Eastward or Westward phase propagation at speed $c = \sqrt{gH}$

- **Poincaré waves (Inertio-gravity waves)**: Eastward or Westward moving waves that are influenced by both gravity and rotation. In the long-wave limit they are dominated by rotation and behave as inertial oscillations. In the short-wave limit they are too short to feel the effects of rotation and so behave as gravity waves

- **Kelvin waves**: Non-dispersive waves that occur in the situation with rotation and in the presence of a boundary (or a region where $f = 0$). At the boundary and in all vertical plains parallel to the boundary they behave as shallow water gravity waves always propagating with the boundary on the right in the NH and on the left in the SH. Rotation influences the amplitude of the wave away from the boundary.

- **Rossby waves**: Westward propagating (relative to the zonal flow) dispersive waves that exist in the presence of a background gradient of potential vorticity. The group velocity (i.e. the direction of propagation of energy) may be either Eastward or westward. The restoring force for these waves is the meridional gradient of potential vorticity.

The relative phase speeds of the waves in a rotating shallow water system are as follows:

$$\text{Poincaré} > \text{Kelvin} > \text{Rossby}$$

It turns out that the oscillations that can exist in a hydrostatic gravitationally stable atmosphere are very similar. They are continually excited by various forces acting on the atmosphere.

4.4 Geostrophic adjustment

Synoptic and larger scale motions in the mid-latitudes are in approximate geotrophic balance. Departures from geostrophic balance can lead to the excitation of inertia-gravity waves which can act to adjust the mass and momentum distributions so that the flow returns toward geostrophic balance. The questions that this section aims to address are Why is the atmosphere close to geostrophic balance and how is that balance achieved?

In order to understand the process of adjustment toward geostrophic balance we will consider the case of a shallow layer of fluid that is in an initially unbalanced state, depicted in Fig. 6. A partition is inserted in the fluid at the origin $x = 0$ and the depth of the fluid on the left hand side of the partition made to be $h = H + h_o$ and on the right hand side $h = H - h_o$ where H is the mean depth of the fluid. So, there is a discontuity in the height of the fluid. We consider a state that is initially at rest and examine what happens if the partition is remove instantaneously. To what state does the fluid adjust to at equilibrium and how does it get there? The initial conditions are

$$\begin{aligned} u' &= 0 \\ h'(x) &= h_o \text{ for } x < 0 \\ &= -h_o \text{ for } x > 0 \\ &\rightarrow h'(x) = -h_o sgn(x) \end{aligned} \tag{31}$$

The height perturbation h_o is assumed to be much smaller than the mean layer depth so that the equations that are relevant to this system are the linearised shallow water equations

Two cases will now be examined 1) without rotation and 2) with constant rotation f_o i.e. on an $f-$plane.

Case 1: No rotation

In the absence of rotation the linearised shallow water equations are

$$\begin{aligned} \frac{\partial u'}{\partial t} &= -g\frac{\partial h'}{\partial x} \\ \frac{\partial v'}{\partial t} &= -g\frac{\partial h'}{\partial y} \\ \frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) &= 0 \end{aligned}$$

The height perturbation h' is the solution of the equation

$$\frac{\partial^2 h'}{\partial t^2} = c^2\frac{\partial^2 h'}{\partial x^2} \tag{32}$$

(see section 4.2.1). In Section 4.2.1 it was found that for the non-rotating shallow water system the solutions to this equation were a pair of counter-propagating solutions that preserve the shape of the initial conditions and move with a speed c where $c = \sqrt{gH}$. At a time t the height field h' will therefore be given by

$$h'(x,t) = \frac{1}{2}\left(-h_o sgn(x + ct) - h_o sgn(x - ct)\right) \tag{33}$$

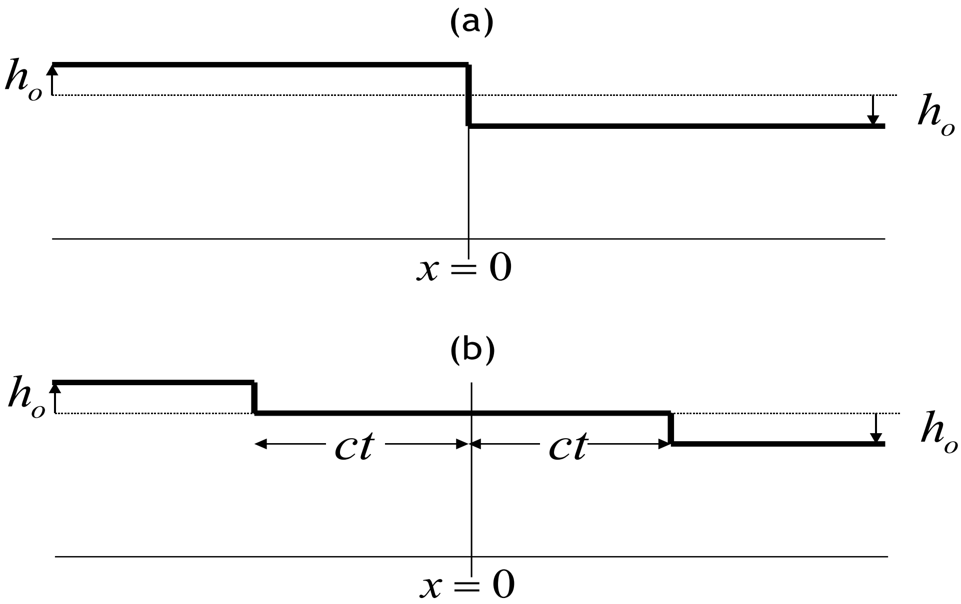

Figure 6: Illustration of the free surface height perturbation (a) before adjustment and (b) at a time t after the adjustment process has begun.

from which it can be seen that for $|x| > ct$, $h' = -h_o sgn(x)$ and for $-ct < x < ct$, $h' = 0$. What happens here is that the fluid starts to adjust in the form of two counter propagating gravity waves. The locations which have been reached by this wave have equilibrated to a state that is undisturbed with a zero height perturbation. Regions that are too far from the initial perturbation to have yet experienced the effect of the wave front will remain undisturbed until the front reaches that point (See Fig. 6). The velocity field is obtained from

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x}$$

to give

$$u' = -\frac{g}{2c}\left[-h_o sgn(x+ct) + h_o sgn(x-ct)\right]$$

So, for locations which have not yet been reached by the wave front $u' = 0$ whereas locations that have been reached have a velocity $h_o g/c$. There is a complete conversion from potential energy to kinetic energy and in the case of boundaries of the system at $\pm\infty$ there is an infinite amount of potential energy.

Case 2: with constant rotation f_o

With rotation the linearised shallow water equations become

$$\frac{\partial u'}{\partial t} - f_o v' = -g\frac{\partial h'}{\partial x}$$
$$\frac{\partial v'}{\partial t} + f_o u' = -g\frac{\partial h'}{\partial y}$$
$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

It can imediately be seen by examining these equations that the presence of rotation makes a massive difference to the problem. It is now possible to have a steady state solution that has non-zero height perturbation (h'). Such a solution was not possible in the case without rotation and it therefore equilibrated to a situation where h' was 0.

If we wanted to find the steady state solution we would want to solve the following equations:

$$-f_o v' = -g\frac{\partial h'}{\partial x} \tag{34}$$

$$f_o u' = -g\frac{\partial h'}{\partial y} \tag{35}$$

$$H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{36}$$

However, a difficulty with this becomes apparent if we take $\partial(34)/\partial y$ and $\partial 35/\partial x$ and try to plug that into 36. The three equations are not independent. Any height field that satisfies 34 and 35 i.e. one that is in geostrophic balance with the velocity fields, automatically satisfies 36. We therefore have 3 unknowns and 2 equations. This problem cannot be solved in this way - we can think up any number of solutions of height fields and velocity fields that will satisfy these equations but we don't know which one is the right one. This is known as **Geostrophic Degeneracy**. Geostrophic balance provides us with a powerful diagnostics relationship between the height fields and velocity fields but we cannot use it to predict the equilibrium solution of a system that is initially out of balance or examine the time evolution until that balance is achieved. In order to look at the time evolution of systems that are out of balance we must consider the small departures from geostrophy.

For the particular case of the geostrophic adjustment problem, if we wanted to find the steady state solution we need some additional constraint to allow us to solve the equations. That constraint comes from the conservation of potential vorticity.

Consider the PV equation

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \vec{v_H}'.\nabla q' = 0 \tag{37}$$

When linearising this equation the $\vec{v_H}'.\nabla q'$ term can be omitted and we are left with $\partial q'/\partial t = 0$ i.e. the PV field is fixed in space. Linearising the PV then gives

$$\begin{aligned}
\frac{\partial q'}{\partial t} &= \frac{\partial}{\partial t}\frac{\zeta' + f_o}{H + h'} \\
&= \frac{\partial}{\partial t}\left(\frac{1}{H}\left(\zeta' + f_o - \frac{f_o h'}{H}\right)\right) \\
&= \frac{1}{H}\frac{\partial}{\partial t}\left(\zeta' - \frac{f_o h'}{H}\right) \\
\rightarrow &\frac{\partial}{\partial t}\left(\zeta' - \frac{f_o h'}{H}\right) = 0
\end{aligned} \tag{38}$$

The initial PV is given by

$$q'(x,y) = \begin{cases} -f_o h'/H & x < 0 \\ f_o h'/H & x > 0 \end{cases} \tag{39}$$

We therefore need to search for a final state that is a solution of the following equations

$$f_o v' = g\frac{\partial h'}{\partial x} \tag{40}$$

$$f_o u' = -g\frac{\partial h'}{\partial y} \tag{41}$$

$$\zeta' - f_o\frac{h'}{H} = q'(x,y) \tag{42}$$

where $\zeta' = \partial v'/\partial x - \partial u'/\partial y$. Since f_o is constant the velocity field is horizontally non-divergent and so can be written in terms of a stream function $\psi' = gh'/f_o$ where $u' = -\partial\psi'/\partial y$ and $v' = \partial\psi'/\partial x$. Equation 42 can therefore be re-written as

$$\begin{aligned} \nabla^2\psi' - \frac{f_o h'}{H} &= q'(x,y) \\ \nabla^2\psi' - \frac{f_o^2}{gH}\psi' &= q'(x,y) \\ \left(\nabla^2 - \frac{1}{L_D^2}\right)\psi' &= q'(x,y) \end{aligned} \tag{43}$$

where $L_D = \sqrt{gH}/f_o$ is the Rossby radius of deformation. The initial conditions are that h' is initially independent of y and so it'll remain that way for all time and we get

$$\begin{aligned} \frac{\partial^2\psi'}{\partial x^2} - \frac{1}{L_D^2}\psi' &= q'(x) \\ \frac{\partial^2\psi'}{\partial x^2} - \frac{1}{L_D^2}\psi' &= \frac{f_o h_o}{H}sgn(x) \end{aligned}$$

This is then solved seperately for $x < 0$ and $x > 0$ and the solutions and their first derivative are matched at $x = 0$

- For $x > 0$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{L_D^2}\right)\psi' = \frac{f_o h_o}{H}$$

  The general solution is the solution of

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{L_D^2}\right)\psi' = 0$$

  which gives $\psi' = Ae^{(1/L_D)x} + Be^{-(1/L_D)x}$. We can see that a particular solution is of the form $\psi' = -\frac{f_o^2 h_o L_D^2}{H}$. So the solution is

$$\psi' = Aexp\left(\frac{x}{L_D}\right) + Bexp\left(-\frac{x}{L_D}\right) - \frac{f_o h_o L_D^2}{H} \tag{44}$$

- For $x < 0$ - Following the same processes as above we have

$$\psi' = Aexp\left(\frac{x}{L_D}\right) + Bexp\left(-\frac{x}{L_D}\right) + \frac{f_o h_o L_D^2}{H} \tag{45}$$

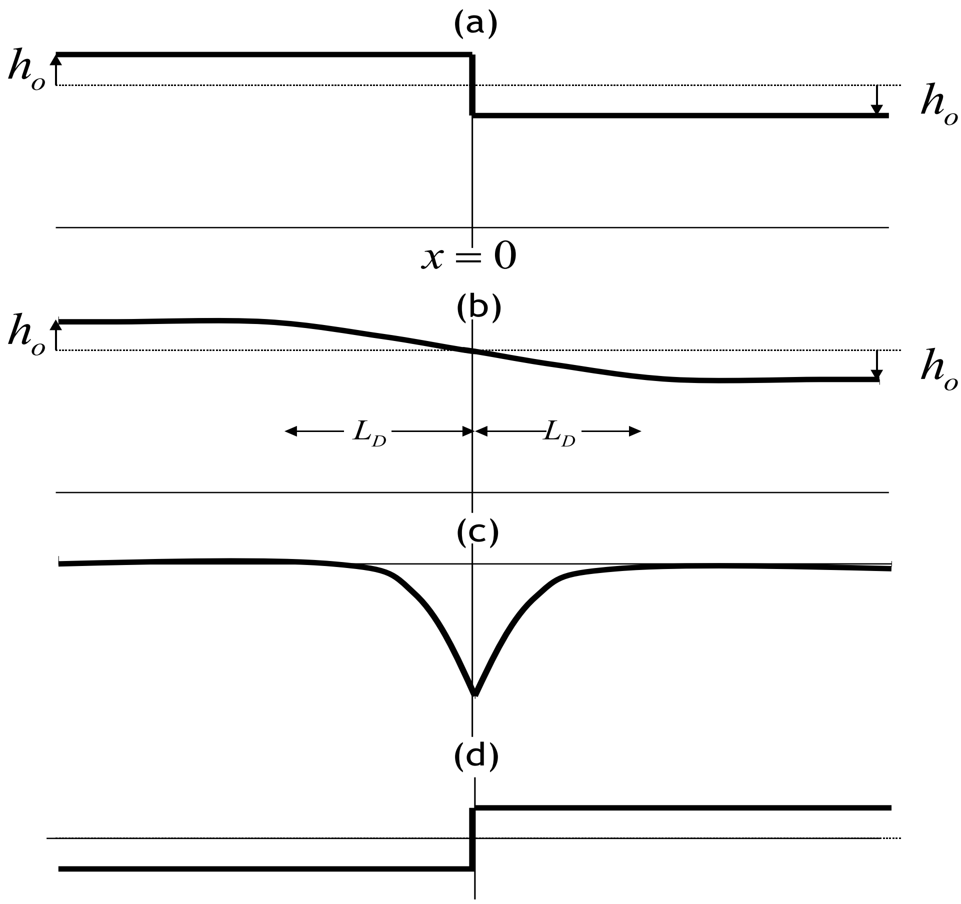

Figure 7: Illustration of various fields in the process of geostrophic adjustment in the presence of rotation. (a) the initial free surface height perturbation, (b) the equilibrated free surface height perturbation, (c) the equilibrated velocity field in the y direction and (d) the potential vorticity both before and after the adjustment process.

Imposing the condition that $\psi' \to 0$ at $x = \pm\infty$ and matching the solutions and the first derivative at $x = 0$ gives

$$\psi' = \begin{cases} -\frac{gh_o}{f_o}\left(1 - exp\left(-\frac{x}{L_D}\right)\right) & x > 0 \\ \frac{gh_o}{f_o}\left(1 - exp\left(\frac{x}{L_D}\right)\right) & x < 0 \end{cases} \tag{46}$$

From the relationship between h' and ψ' it can be seen that this gives the height structure shown in Fig. 7 (b). As $x \to \infty$, $h' \sim -h_o$ and as $x \to -\infty$, $h' \sim h_o$. At $x = 0$, $h' = 0$ and the variation in the height perturbation away from the origin takes the form of an exponential with e-folding length scale equal to the Rossby radius of deformation. Unlike the case without rotation, the variations in the height field are not radiated away to infinity. The conservation of PV and the presence of rotation constrains the influence of the adjustment to within a deformation radius of the initial disturbance.

The velocity field can be calculated according to 35 and is shown in Fig. 7 (c). In the case with rotation the equilibrium velocity is perpendicular to the slope of the free surface. This is the velocity that is in geostrophic balance with the pressure gradient that develops.

The process by which adjustment to this equilibrium occurs is as follows: The initial height perturbation produces poincaré waves that are radiated away. Given the relationship between group velocity and wavenumber for poincaré waves (Equation 20), the shorter waves propagate away faster than the longer ones. The longest waves (close to inertial oscillations) travel away more slowly but even they too propagate away to leave in the end the other solution to the time evolving shallow water equations (the $\omega = 0$ solution). That is the velocity field in geostrophic balance with the height perturbation.

Over the length scale of the Rossby radius of deformation this adjustment process acts to smooth out the pressure gradient until an equilibrium situation is achieved in which the pressure gradient is balanced by the coriolis force on the velocity perturbation.

Quite unlike the case without rotation there is not a complete conversion of potential to kinetic energy. The presence of rotation restricts the amount of potential energy that is converted to kinetic energy. A calculation of the energetics of the adjustment process demonstrates that some energy is also lost and this is lost in the form of the Poincaré waves that are radiated away to infinity.

Over length scales that are small compared with L_D the rotation effects are small and the fluid is allowed to adjust. Over larger scales the pressure gradient is balanced by the coriolis force on the resulting meridional velocity.

So, when a height perturbation occurs, the result is an instability which produces inertio-gravity waves that propagate away allowing the fluid to adjust toward a new geostrophic equilibrium. Since these modes propagate quickly, after a certain amount of time only the large scale modes that are close to geostrophic balance and heavily influenced by the rotation will remain. Then eventually if everything is allowed to adjust to equilibrium we will be left with a flow that is in geostrophic balance.