Calculus Section I: Single Variable Calculus I

Ivan Savic

1 Introduction to Functions:

$R^1 \Rightarrow$ set of all real numbers

1.1 Function

$\Rightarrow$ linear function: a rule that assigns a number in R^1 to each number in R^1

e.g.

$$f(x) = x + 1$$

or

$$y = x + 1$$

Where:

$f(x)/y$	-	output, dependent/endogenous variable
x	-	input, independent/exogenous variable

Note: A function assigns one and only one output y for any input x, but a given output y can correspond to more then one value of x. Thus the function $y = x^2$ assigns only one value to any x (i.e. $x = 0 \Rightarrow y = 0$; and $x = 2 \Rightarrow y = 4$, etc.) but each output y, except 0, corresponds to two inputs x (i.e. $y = 4 \Rightarrow x = 2$ & $x = -2$; and $y = 16 \Rightarrow x = 4$ & $x = -4$).

1.2 Types of Functions:

1.2.1 Monomial

$\Rightarrow \quad f(x) = \alpha x^k$

where:

k	-	the degree of the monomial
α	-	the coefficient of the monomial

1.2.2 Polynomial

⇒ function of two or more monomials

e.g. $h(x) = -x^7 + 3x^4 - 10x^2$

1.2.3 Rational Function

⇒ ration of two polynomials

e.g. $y = \frac{x^2+1}{x-1}$

1.2.4 Exponential Function

⇒ where x (i.e. the independent variable) appears as an exponent

e.g. $y = 10^x$

1.2.5 Trigonometric Function

⇒ $y = \sin x$

etc.

1.3 Increasing/decreasing functions

1.3.1 increaing:

a function y is said to be an **increasing** function of x if its output **increases** with every **increase** in the values of the input, graphically this is a function whose graph "moves" upwards from left to right.

i.e. **if** $\mathbf{x_1 > x_2}$ ⇒ $\mathbf{f(x_1) > f(x_2)}$

e.g.

$f(x) = x + 1$

$f(x) = 2x$

1.3.2 decreasing:

a function y is said to be a **decreasing** function of x if its output **decreases** with every **increase** in the values of the input, graphically this is a function whose graph "moves" downwards from left to right.

i.e. **if** $\mathbf{x}_1 > \mathbf{x}_2 \Rightarrow \mathbf{f}(\mathbf{x}_1) < \mathbf{f}(\mathbf{x}_2)$

e.g.

$$f(x) = -x^7$$
$$f(x) = -(x+4)^2$$

1.4 Maxima and Minima

1.4.1 maximum:

if at (x_0, y_0) the function changes from **increasing** to **decreasing** then the function reaches a **maximum** at this point.

if the function y always lies below the point (x_0, y_0) (i.e. $y(xi) < y_0$ for all x_i, where $i \neq 0$) then this is a **global** or **absolute maximum**.

If y does at some point lie above the point (x_0, y_0) (i.e. $\exists y(xi) < y_0$ for some x_i, where $i \neq 0$) then this is a **local** or **relative maximum**.

1.4.2 minimum:

if at (x_0, y_0) the function changes from **decreasing** to **increasing** then the function reaches a **minimum** at this point.

If the function y always lies above the point (x_0, y_0) (i.e. $y(xi) > y_0$ for all x_i, where $i \neq 0$) then this is a **global** or **absolute minimum**.

If y does at some point lie below the point (x_0, y_0) (i.e. $\exists y(xi) > y_0$ for some x_i, where $i \neq 0$) then this is a **local** or **relative minimum**.

1.5 Domain

Definition: Is the set of numbers for which a functions is defined, i.e. the set of x for which the functions can assign a set of coresponding y values,

e.g.

$y = x$ the domain is R^1

$y = \frac{1}{x}$ the domains is $R^1 - \{0\}$

$$y = \sqrt{x} \quad \text{the domain is } R^1_+$$

$$R^1_+ \equiv \left\{x \in R^1 : x \geq 0\right\} \quad \Rightarrow \quad f : D \to R^1_+$$

The domain of a function can be limited because of:

1. **Mathematics**: from the examples above you cannot divide by 0 or calculate the square root of a negative number.
2. **Application**: costs are always a positive number, a negative cost would be a benefit.

1.6 Interval

1.6.1 Open Inverval

An interval that does **not** contain the endpoints it is bounded by.

$$(a, b) \equiv \{x \in R^1 : a < x < b\}$$

1.6.2 Closed Interval

An interval that does contain the endpoints it is bounded by.

$$[a, b] \equiv \{x \in R^1 : a \leq x \leq b\}$$

1.6.3 Half-Opened/Closed Interval

An interval that contains **only one** of the endpoints it is bounded by.

$$(a, b] \equiv \{x \in R^1 : a < x \leq b\} \quad \textit{or} \quad [a, b) \equiv \{x \in R^1 : a \leq x < b\}$$

1.6.4 Infinite Interval

there are five kinds of infinite intervals:

1. $(a, \infty) \equiv \{x \in R^1 : x > a\}$
2. $[a, \infty) \equiv \{x \in R^1 : x \geq a\}$
3. $(-\infty, a) \equiv \{x \in R^1 : x < a\}$
4. $(-\infty, a] \equiv \{x \in R^1 : x \leq a\}$
5. $(-\infty, +\infty) \equiv \{x \in R^1\}$

2 Linear Functions

$f(x) = a + mx$

2.1 Slope of a Line

it is given by:

$$m = \frac{y-a}{x-0}$$

the slope is a constant

the slope represents the change in y given each additional change in x. Thus the slope is the rate of change in the function as can be used to indicate speed or velosity, marginal cost, marginal revenue, marginal utility, etc.

2.2 Intercept

The y intercept of a line is given by:

$$(0, a)$$

3 Slope of a Non-Linear Function

Unlike the slope of a line the slope or derivative of non-linear functions is not constant. Deriving the slope gives a function; therefore the exact value of the slope must be evaluated at a particular point.

The slope or derivative of a non-linear function at a particular point is equal to the slope of a line which is tangent to the function at that point.

Let:

$(x_0, f(x_0))$ be the point at which the deriative is computed, and
$y = f(x)$, the function for which the derivative is computed

The derivative can be expresses in various (equivalent) ways:

1. $f'(x_0)$,

2. $\frac{df}{dx}(x_0)$,or

3. $\frac{dy}{dx}(x_0)$

The slope of a tangent at x_0 is given by:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

4 Computing Derivatives

Derivatives can be used to understand functions, graph them and solve optimization problems.

4.1 Basic:

for the function:

$$f(x) = x^k$$

the derivative is:

$$f'(x) = kx^{k-1}$$

4.2 Variations:

1. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
2. $(kf')(x_0) = k(f'(x_0))$
3. $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ **Product Rule**
4. $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$ **Quotient Rule**
5. $((f(x_0))^n)' = n(f(x_0))^{n-1} \cdot f'(x_0)$ **Power Rule**
6. $(x_0^k)' = kx_0^{k-1}$

5 Characteristics of functions

5.1 Differentiable:

Differentiavle at every point x_0 in its domain D

5.2 Continuous:

1. A functions $f : D \to R^1$ is **continuous** at $x_0 \in D$ is for any sequence $\{x_n\}$ which converges to x_0 in D, $f(x_n)$ converges to $f(x_0)$.
2. A function is **continuous on a set** $U \subset D$ if it is continuous at every $x \in U$.
3. Finally we say that a function is continuous if it is continuous at every point in its domain.

5.3 Continuously Differentiable Function

If $f'(x)$ is a continuous function of x, we can that the original function $f(x)$ is continuously differentiable, or C^1 for short.

6 Higher-Order Derivatives

Let $f(x)$ be a C^1 function on $R^1$and since its derivative $f'(x)$ is continuous on R^1, then $f'(x)$ is differentiable at x_0

6.1 second-order derivative:

Thus if a function is C^1 then its second derivative is expressed as:

$$f''(x_0), \qquad f^2(x_0), \qquad \text{or} \qquad \frac{d}{dx}(\frac{df}{dx}(x_0)) = \frac{d^2f}{dx^2}(x_0)$$

6.2 third-order derivative:

The third derivative of a C^2 function is expressed as:

$$f'''(x_0), \qquad f^3(x_0), \qquad \text{or} \qquad \frac{d^3f}{dx^3}(x_0)$$

6.3 kth-order derivative

For a function that is C^k :

$$f^k(x_0), \qquad \text{or} \qquad \frac{d^kf}{dx^k}(x_0)$$

If f is C^k for every positive integer k, then we say that f is C^∞ or "infinitely differentiable." All polynomials are C^∞ functions.

7 Approximation by Differentials or linear approximation

A differential is an infinitesimal change in the value of a function. Differentials help to constitute **derivatives** and **integrals**.

From the definition of a derivative above we can say that:

$$\frac{f(x_0+h)-f(x_0)}{h} \approx f'(x_0)$$

Setting $h = 1$ gives us:

$$f(x_0 + h) - f(x_0) \approx f'(x_0)$$

So $f'(x_0)$ is a good approximation of the marginal change of f to x_0.

So a change in Δy can be approximated by:

$$\Delta y \equiv f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$$

or

$$\Delta y \approx dy = f'(x_0)\Delta x$$

8 Convexity

8.1 Convex:

A function f is **concave up** or simply **convex** on an interval I if:

$$f''(x) \geq 0$$

or if and only if:

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$

8.2 Concave:

A function f is **concave down** or simply **concave** on an interval I if:

$$f''(x) \leq 0$$

or if and only if:

$$f((1-t)a + tb) \geq (1-t)f(a) + tf(b)$$

9 Maxima and Minima Revisited

9.1 Determining Max/Min

A function's maxima and minima are found at its critical value. For a point to be a max/min it is necessary but not sufficient for it to be a critical value. A critical value of a function is:

1. where its derivative is equal to zero ($f'(x) = 0$),
2. where the derivative is not defined (e.g. for $f(x) = |x|$ $f'(x)$ is not defined for $x = 0$, and this is also the global minimum of the function), or
3. at the boundary points or points of discontinuity of a function (i.e. given the function $g(x) = \{ \begin{array}{cc} x+1 & 0 \leq x \leq 1 \\ -x^2 - 1 & -1 \leq x < 0 \end{array}$ where the domain of the function is $D \rightarrow [-1, 1]$, then $(-1, -2)$, $(0, 1)$, and $(1, 2)$ are all critical points and a global min, local min and global min respectively)

We can use the second derivative of a function to determine its maxima and minima. Thus:

1. if $f'(x_0) = 0$ & $f"(x_0) < 0$, then x_0 is a **maximum** of f
2. if $f'(x_0) = 0$ & $f"(x_0) > 0$, then x_0 is a **minimum** of f
3. if $f'(x_0) = 0$ & $f"(x_0) = 0$, then x_0 can be a max, min or neither

In the case where $f'(x_0) = 0$ & $f''(x_0) = 0$, there are four possible outcomes:

1. e.g. $f_1(x) = x^4$, the function is **concave up** and has a **minimum** at $x = 0$
2. e.g. $f_2(x) = -x^4$, the function is **concave down** and has a **maximum** at $x = 0$
3. e.g. $f_3(x) = x^3$, the function is **increasing** and has an **infection point** at $x = 0$, where the function turns from being **concave down** to being **concave up**
4. e.g. $f_4(x) = -x^3$, the function is **decreasing** and has an **inflection point** at $x = 0$, where the function turns from being **concave up** to being **concave down**

An **inflection point** is where a function changes its concavity, that is it change from being convex to being concave or vice versa.

9.2 Global Max/Min

- when f has only one critical point in its domain
- when $f'' > 0$ or $f'' < 0$ throughout the domain of f

$$\left|\begin{array}{llll} \text{if} & f & \text{is} & C^2 \\ & D \to I & & \\ \& & f'' \neq 0 & on & I \end{array}\right\} \Rightarrow \left\{\begin{array}{llll} f & \text{has at} & \text{most} & \text{one critical point} \\ \text{if} & f'' > 0 & \Rightarrow & \text{global min} \\ \text{if} & f'' < 0 & \Rightarrow & \text{global max} \end{array}\right|$$

- $D \to$ closed finite interval $[a, b]$
  1. find critical points
  2. evaluate f at critical points and at a and b
  3. chose point that gives largest/smallest value of f

10 Graphing a Function

When graphing a function you should do the following:

1. **Domain**: Determine the function's domain D.
2. **Intercepts**: Find at which points the graph of the function intercepts the x and y-axis. For the y intercepts this is determined by setting $x = 0$ and solving the function (**remember**: by definition a graph of a function will have only one y intercept). To find the x intercepts you set the set $y = 0$ and solve for all values of x for which $y = 0$ (**remember**: a function can have more then one x intercept).
3. **Critical values**: Find the functions critical values by finding the first order derivative and solving for $f' = 0$. Other important values are also where the derivative f' is not defined, If the function's domain is defined over a bounded interval, e.g. $D \to [a, b]$, then the boundary points are also critical values, and finally if a function is discontinuous points of discontinuity are also critical values.
4. **Max/Min**: if at a point x_0, $f'(x_0) = 0$ & $f"(x_0) < 0$, then x_0 is a **maximum,** if $f'(x_0) = 0$ & $f"(x_0) > 0$, then x_0 is a **minimum**. and if $f'(x_0) = 0$ & $f"(x_0) = 0$ then x_0 is a **max**, **min** or **inflection point**. In the last case you have to look at the behavior of a function in the immediate vicinity of x_0 to determine whether it is a max, min, or inflection point. If a function is defined by a closed interval $D \to [a, b]$, then you should also look at the behavior of the function as it approaches its boundary points to determine if they are maxima, minima or neither.

5. **Absolute Max/Min**: In order to determine whether the function has a **global maximum** or **minimum** you must compute the value of the function at its critical values as well as the values it approaches at its boundaries. The highest value is the global max and the lowers in the global min all other mas or min are local max/min. **Note** if a function approaches $+\infty$ at any point then it does not have a global max and if it approaches $-\infty$ it does not have a global min.

6. **Inflection points**: Are points where the function changes its concavity. Inflection points are those critical values for which $f"(x_0) = 0$ but which are not a max or min.

7. **Tails and Asymptotes**: It is useful to see how the function behaves at its tails, i.e. when $|x| \rightarrow \infty$. If a function is undefined at a point, e.g. $\{x = a\}$, or has an open ended domain, e.g. $D \rightarrow [a, b)$, $D \rightarrow [a, +\infty)$ or $D \rightarrow (-\infty, +\infty)$, then you must determine what value the function approaches as it gets closer to the value(s) it is not defined for. To do this you simply take the limit of the function as it approaches this/these point(s). The function will either approach a specific point or will approach $\pm\infty$. If the function is not defined at a particular point a, but is defined on either side of the point, then you must take the limit of the function as it approaches both from the left and the right. The function may approach the same value from either side (e.g. in the function $y = x$, $D \rightarrow (R - \{0\})$ where the function approach $y = 0$ on either side of $x = 0$) or it can approach different values (e.g. $y = 1/x$, $\lim_{x\rightarrow 0^+} y \rightarrow +\infty$ and $\lim_{x\rightarrow 0^-} y \rightarrow -\infty$).

8. **Increasing/Decreasing**: Determine whether the function is increasing or decreasing in x. The function may be increasing/decreasing through its entire domain (e.g. for $y = x$ the function is increasing throughout) or it may be decreasing/decreasing over only certain interval in its domain (e.g. $y = x^2$ in decreasing for $x < 0$ and increasing for $x > 0$). These intervals are bounded by a function's domain, its intercepts, critical values and asymptotes.

9. **Concavity**: Use the second order derivative to determine concavity. If on a particular interval $f"(x) \geq 0$ then the function is **concave up/convex** if $f"(x) \leq 0$ then the function is **concave down/concave**. As in point 8, a function can have the same concavity throughout its domain or it may change from one interval to another. The intervals are defined by the function's domain, inflection points, its intercepts, and asymptotes