

Calculus Section II: Single Variable Calculus II

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1 Composite Functions

Function formed by taking the composition of two (or more) other functions.

- g & h are two functions in R^1
- function formed by first applying g to any number x
- the applying function h to the result of $g(x)$

Notation: $f(x) = h(g(x))$ or $f(x) = (h \circ g)(x)$

Note: $h \circ g \neq g \circ h$

$$g(x) = x^2 \quad \text{and} \quad h(x) = x + 4$$

$$(g \circ h)(x) = (x + 4)^2 \quad \neq \quad (h \circ g)(x) = x^2 + 4$$

Application: *dynamic models*

let:

$x = f(p)$ market demand for commodity in terms of price

$p = p(t)$ change in price over time

then:

$F(t) \equiv F(p(t))$ commodities demand over time

where:

$F(\cdot)$ outside function

$p(t)$ inside function

2 Chain Rule

$$\frac{d}{dx}(h(g(x))) = h'(g(x)) \cdot g'(x) = \frac{d}{dx}(h \circ g)(x)$$

Alternative:

given: $g(x)$ and $h(z)$

then: $\frac{d(h \circ g)}{dx}(x) = \frac{dh}{dz}(g(x)) \cdot \frac{dg}{dx}(x)$

Example 1

$$\begin{aligned}\text{let: } f(x) &= a^2 + a - 5 & \text{and } a &= x^2 \\ \frac{df}{dx} &= (2a + 1) \cdot a' \\ &= (2(x^2) + 1)(2x) & \text{substituting } a = x^2 \\ &= (2x^2 + 1)(2x) \\ &= 4x^3 + 2x\end{aligned}$$

3 Inverse Functions and Their Derivatives

$$\begin{aligned}\text{Demand function: } & x = 3 - 2p \\ \text{Inverse demand function: } & p = \frac{1}{2}(3 - x)\end{aligned}$$

any given function $f : E_1 \rightarrow R^1$
where E_1 , the domain of f , is a subset of R^1
the function $g : E_2 \rightarrow R^1$ is an **inverse** of f if:

- $g(f(x)) = x$ for all x in the domain E_1 of f , and
- $f(g(z)) = z$ for all z in the domain E_2 of g .

Function f defined on an interval E in R_1 has a well defined inverse on the interval $f(E)$ **if and only if** f is monotonically increasing on all E or monotonically decreasing on all E .

Definition 2 The function f is **monotone** or **monotonically increasing** if, whenever $x \leq y$, then $f(x) \leq f(y)$. Stated differently, a monotone function is one that **preserves the order**.

Definition 3 The function f is **monotonically decreasing** if, whenever $x \leq y$, then $f(x) \geq f(y)$, i.e., if it **reverses the order**.

A C^1 function f defined on an interval E in R^1 is one-to-one and therefore invertible on E if $f'(x) > 0$ for all $x \in E$ or $f'(x) < 0$ for all $x \in E$.

3.1 Inverse Function Theorem

Let f be a C^1 function defined on the interval I in R^1 .

If $f'(x) \neq 0$ for all $x \in I$, then:

- (a) f is invertible on I
- (b) its inverse g is a C^1 function in the interval $f(I)$, and
- (c) for all z in the domain of the inverse function g , $g'(z) = \frac{1}{f'(g(z))}$

$$\begin{aligned}\text{Assuming: } g &= f^{-1} \\ f(g(z)) &= z\end{aligned}$$

Using the chain rule:

$$\begin{aligned}f'(g(z)) \cdot g'(z) &= 1 \\ g'(z) &= \frac{1}{f'(g(z))}\end{aligned}$$

4 Exponential Functions

4.1 Basics

General form: $f(x) = a^x$

where:

a is the base of the exponential function

$a > 0$ because $\sqrt{-a}$ is undefined

e.g. $g(x) = 2^x$

1. if $x \in \mathbb{Z}^+$, i.e. is a positive integer then a^x means "multiply a by itself x times"
2. if $x = 0$ then $a^0 = 1$ by definition
3. if $x = \frac{1}{n}$ then $a^{\frac{1}{n}} = \sqrt[n]{a}$
4. if $x = \frac{m}{n}$ then $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
5. if $x \in \mathbb{Z}^-$, i.e. is a negative integer then $a^x = \frac{1}{a^{|x|}}$

4.2 Properties

1. $a^r \cdot a^s = a^{r+s}$
2. $a^{-r} = \frac{1}{a^r}$
3. $\frac{a^r}{a^s} = a^{r-s}$
4. $(a^r)^s = a^{rs}$
5. $a^0 = 1$

4.3 Derivatives

Definition 4 e is the base of the natural logarithm, it is a irrational number (cannot be expressed as a fraction or a repeating decimal) and can be defined as:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

The functions e^x has the unique property that it is its own derivative and therefore also its own antiderivative (or primitive):

1. **derivative:** $\frac{d}{dx} e^x = e^x$
2. **antiderivative:** $\int e^x dx = e^x + C$

Derivative of exponential function with a base other then e , i.e. $a \neq e$:

1. **derivative:** $\frac{d}{dx} a^x = (\ln a)a^x$, where \ln is \log of base e

2. **antiderivative:** $\int a^x dx = \frac{a^x}{\ln a} + C$

If $u(x)$ is differentiable, then:
 $(e^{u(x)})' = (e^{u(x)}) \cdot u'(x)$, and
 $(a^{u(x)})' = (u(x) \cdot \ln a)(a^{u(x)}) \cdot u'(x)$

5 Logarithmic Functions

Definition 5 A logarithmic function is the inverse of a exponential function

5.1 Basics

General form $y = \log_a(z)$ $a^{\log_a(z)} = z$ and $\log_a(a^z) = z$

Common bases:

base 10:	$y = \log z$	thus	$10^y = z$
base e (natural logarithm):	$y = \ln x$	thus	$e^y = x$
as above:	$e^{\ln x} = x$	and	$\ln e^x = x$

5.2 Properties

Each one of these properties corresponds to its equivalently numbered property for exponential functions.

1. $\log(r \cdot s) = \log r + \log s$
2. $\log(\frac{1}{s}) = \log(s^{-1}) = -\log s$
3. $\log(\frac{r}{s}) = \log r - \log s$
4. $\log r^s = s \log r$
5. $\log 1 = 0$

5.3 Derivatives of Exponents and Logs

Definition 6 \ln is the natural logarithm, that is the inverse of e

Derivative of the natural logarithm

1. **derivative:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$
2. **antiderivative:** $\int (\ln x) dx = x \ln x - x + C$

Derivative of logarithmic function with base other than e , i.e. $a \neq e$:

1. **derivative:** $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} = \frac{\log_a e}{x}$, where \ln is \log of base e

2. **antiderivative:** $\int (\log_a x) dx = x(\log_a x) - \frac{x}{\ln a} + C = x \log_a \left(\frac{x}{e}\right) + C$

If $u(x)$ is differentiable, then:

$$(\ln u(x))' = \frac{u'(x)}{u(x)}, \text{ if } u(x) > 0 \text{ and}$$

$$(\log_a u(x))' = \frac{u'(x)}{u(x) \ln a}$$