Calculus Section III: Multiple Variable and Integral Calculus

Ivan Savic

1 Functions of Several Variables

$f : A \rightarrow B$

where:

A: set of elements for which f is defined (its domain)

B: set in which f takes its values (target or target space) image or range

often this is expressed as:

$f : R^n \rightarrow R^m$

where:

R^n: n dimensional set of real numbers, where n is the number of **independent** variables

R^m: m dimensional set of real numbers, where m is the number of **dependent** variables

Example 1 *production function:*

input bundle: x_1, x_2, x_3
output bundle: q_1, q_2

the output function is given by the following notations:
$q = (q_1, q_2) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \equiv F(x_1, x_2, x_3)$
$F = (f_1, f_2)$
$F : R^3 \rightarrow R^2$

Example 2 *investment function:*

$z = A(1 + \frac{r}{n})^{nt}$

dependant variable:
z - return on investment

independent variables:
A - initial investment
r - rate or return
n - compounded n times a year
t - number or years till maturity

Thus this function can be expressed as:

$F : R^4 \to R^1$

Example 3 *voter utility mapping:*

let there be:
m - voters, and
k - candidates

each voter has a preference set over the candidates, $x^i = (x_1, x_2, ..., x_k)$ so that:

$x = (x_1^1, x_2^1, ..., x_k^1; x_1^2, x_2^2, ..., x_k^2; ...; x_1^m, x_2^m, ..., x_k^m) \in R^{km}$

$U : R^{km} \to R^m$

2 Types of Functions

2.1 Linear functions (transformations)

$f : R^K \to R^M$

where:

$f(x + y) = f(x) + f(y)$, and

$f(rx) = rf(x)$

Let $f : R^K \to R^1$ be a linear function, then there exists a vector $a \in R^K$ such that $f(x) = ax$ for all $x \in R^K$

i.e.

$$f(x) = a \cdot x = a_1 x_1 + a_2 x_2 + ... + a_k x_k = \begin{pmatrix} a_1 & ... & a_k \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

Let $f : R^K \to R^M$ be a linear function, then there exists a $m\ x\ k$ matrix A s.t. $f(x) = Ax$ for all $x \in R^K$

$$f(x) = A \cdot x = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

2.2 Quadratic forms

A quadratic form on R^k is a real-valued function of the form:

$$q(x_1, ..., x_k) = \sum_{i,j=1}^{k} a_{ij} x_i x_j = \begin{pmatrix} x_1 & ... & x_k \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

Example 4

$$Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$
$$Q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2$$

2.3 Polynomials

A function $f : R^K \to R^1$ is a **monomial** if it can be written as

$$f(x_1, ..., x_k) = c \cdot x_1^{a_1} \cdot x_2^{a_2} \cdot \cdots \cdot x_k^{ak}$$

A function $f : R^K \to R^1$ is a **polynomial** if f is a finiter sum of monomials on R^k. The higest degree which occurs among these monomials is called the degree of the plynomial. A function $f : R^K \to R^M$ is called a **polynomial** if each of its component functions is a rea-valued polynomial.

$$f(x_1, x_2) = -4x_1^2x_2$$
$$f(x_1, x_2, x_3) = 3x_1^2x_2 + 4x_2x_3^3$$

3 Partial Derivatives

When taking a partial derivative with respect to one independent variable you follow the same rules as taking a linear derivative, you simply treat all other independent variables in the function as if they were constants:

Example 5 $\frac{\partial}{\partial x}(3x^2y^3) = 2x \cdot 3y^3 = 6xy^3$

3.1 Notation:

3.1.1 First-order partial derivatives:

$$\frac{\partial f}{\partial x_i} = f_i = f_{x_i} = D_i f = \partial_{x_i} f$$

3.1.2 Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x_i^2} = f_{ii} = f_{x_i x_i} = D_{ii} f = \partial_{x_i x_i} f$$

3.1.3 Second-order mixed derivatives:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{ij} = f_{x_i x_j} = D_{ij} f = \partial_{x_i x j} f$$

3.1.4 Higher-order partial and mixed derivatives:

$$\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} = f^{(i,j,k)}$$

4 Antiderivatives and Integration

Antiderivative: $F : F' = f$

Indefinite integral: $F(x) = \int f dx$

4.1 Some examples and properties:

1. $\int af(x)dx = a \int f(x)dx$ constant factor rule of integration
2. $\int (f + g)dx = \int f dx + \int g dx$ sum rule of integration
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)$ counterpart to basic derivative
4. $\int \frac{1}{x} dx = \ln x + C$
5. $\int e^x dx = e^x + C$
6. $\int e^{f(x)} f'(x)dx = e^{f(x)} + C$
7. $\int (f(x))^n f'(x)dx = \frac{1}{n+1}(f(x))^{n+1} + C. \quad (n \neq -1)$
8. $\int \frac{1}{f(x)} f'(x)dx = \ln f(x) + C$

$$\int (4x^2 + x^{\frac{1}{2}} - \frac{3}{x})dx = \frac{4x^3}{3} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3\ln x + C = \frac{4}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} - 3\ln x + C$$

4.2 Techniques

4.2.1 Linearity of integration:

linearity is a fundamental property of the integral that follows from the sum rule in integration and the constant factor rule in integration

$$\int af(x) + bg(x)dx = \int af(x)dx + \int bg(x)dx = a \int f(x)dx + b \int g(x)dx$$

4.2.2 Integration by substitution

This is the counterpart of the **chain rule**.

4.2.3 Integration by parts

This is the counterpart of the **product rule:** $(u \cdot v)' = u'v + uv'$

$\int udv = uv - \int vdu$

Example 6 $\int \ln(x)dx$

Let:

$$u = ln(x);$$
$$du = \frac{1}{x}dx;$$
$$v = x;$$
$$dv = 1{\cdot}dx$$

Then:

$$\begin{aligned}\int \ln(x)dx &= x\ln(x) - \int x(\tfrac{1}{x})dx \\ &= x\ln(x) - \int 1dx \\ &= x\ln(x) - x + C\end{aligned}$$

Example 7 $\int xe^{2x}dx$

Let:

$$u = x;$$
$$du = dx;$$
$$v = \frac{1}{2}e^{2x};$$
$$dv = e^{2x}dx$$

Then:

$$\begin{aligned}\int xe^{2x}dx &= x \cdot \tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}dx \\ &= \tfrac{xe^{2x}}{2} - \tfrac{1}{2}\int e^{2x}dx \\ &= \tfrac{xe^{2x}}{2} - \tfrac{1}{2}(\tfrac{1}{2}e^{2x}) + C \\ &= \tfrac{xe^{2x}}{2} - \tfrac{e^{2x}}{4} + C \\ &= \tfrac{2xe^{2x} - e^{2x}}{4} + C \\ &= \tfrac{e^{2x}(2x-1)}{4} + C\end{aligned}$$

4.3 Fundamental theorem of Calculus

definite integral:

$\int_a^b f(x)dx = F(b) - F(a), \qquad \text{where} \qquad F' = f$

To calculate an are under a cure in the interval $[a, b]$ divide the interval into N equal subintervals

each $\qquad \Delta = \frac{(b-a)}{N}$

with endpoints: $x_0, x_1, x_2, ..., x_n$
$x_0 = a$
$x_1 = a + \Delta$
$x_2 = a + 2\Delta$
$\vdots$
$x_n = a + N\Delta = b$

summing up these segments we get the **Riemann sum**:

$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + ... + f(x_n)(x_n - x_{n-1}) = \sum_{i-1}^{N} f(x_i)\Delta$

Definition 8 *The **fundamental theorem** states that interating this process using smaller and smaller subintervals, in the limit we obtain the definite integral* $\int_a^b f(x)dx$ *:*

$\lim_{\Delta \to 0} \sum_{i-1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$