

The Simple Structure of Top Trading Cycles in School Choice*

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Abstract

Many cities determine the assignment of students to schools through a school choice mechanism. The prominent Top Trading Cycles (TTC) mechanism has attractive properties for school choice, as it is strategy-proof, Pareto efficient, and allows school boards to guide the allocation by specifying priorities. However the combinatorial description of TTC makes it non-transparent to parents and difficult for designers to analyze. We give a tractable characterization of the TTC mechanism for school choice: the TTC assignment can be simply described by n^2 admission thresholds, where n is the number of schools, and these thresholds can be easily observed after running the mechanism. To calculate these thresholds, we define TTC through trade balance equations. In a continuum model these equations correspond to a differential equation whose solution can give closed form expressions for the admission thresholds.

The model allows us to compute comparative statics, and evaluate welfare. As applications of the model we solve for optimal investment in school quality, explore the design of the priority structure, and provide comparisons between TTC and other school choice mechanisms that help explain empirical findings. To validate the continuum model we show that it gives a good approximation for strongly converging economies. Our analysis draws on an interesting connection between continuous trading procedures and continuous time Markov chains.

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1 Introduction

In recent years, many school districts have redesigned their school assignment mechanisms in order to allow students to have more choice over where they are assigned to school. Many of these reforms were inspired by the seminal paper of Abdulkadiroğlu and Sönmez (2003), which recognizes school choice as an assignment problem where students are strategic agents and school seats are objects to be assigned. Abdulkadiroğlu and Sönmez propose two mechanisms that are strategyproof for students and allow the school district to set a priority structure: the Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962); and the Top Trading Cycles (TTC) mechanism (Shapley and Scarf (1974), attributed to David Gale). The two mechanisms differ in that DA is stable with respect to priorities but not necessarily efficient, and TTC is Pareto efficient for students but not necessarily stable. In many school districts, there are no strategic constraints¹ that require sacrificing efficiency to guarantee stability, as schools do not screen students. Despite this, almost all the school districts that redesigned their school choice mechanism chose to implement the DA mechanism (Pathak and Sönmez, 2013; IIPSC, 2017), instead of the more efficient TTC.²

The limited adoption of the TTC mechanism was in part due to its lack of transparency. Based on their experiences in designing many school choice programs, Pathak (2016); Abdulkadiroğlu et al. (2017) assert that the difficulty of explaining TTC caused school districts to favor DA, as it was easier to explain.³ In particular, it is challenging to convey to parents that TTC is strategyproof, and it is hard for

¹There is no strategic concern that blocking pairs will match to each other outside the mechanism. This is because of two differences between school choice and two-sided settings like the medical match. First, in Boston and other districts schools cannot directly admit students without approval. Second, priorities are often determined by school zone, sibling status and lotteries, and are not controlled by the school. In that case, the school does not necessarily prefer students with higher priority. A notable exception is the NYC high school admission system, see Abdulkadiroğlu, Pathak, and Roth (2009).

²To the authors' knowledge, the only instances of implementation of TTC in school choice systems are in the San Francisco school district (Abdulkadiroğlu et al. 2017) and previously in the New Orleans Recovery School District (Abdulkadiroğlu et al. 2017).

³Pathak (2016) writes that:

“I believe that the difficulty of explaining TTC, together with the precedent set by New York and Boston’s choice of DA, are more likely explanations for why TTC is not used in more districts, rather than the fact that it allows for justified envy, while DA does not.”

In addition, Boston and NYC were early school redesigns that set a precedent in favor of DA. More details can be found in the discussion in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005).

administrators to explain why a student failed to receive a desired school seat.⁴ Additionally, while the priority structure is used to guide the TTC allocation, existing theory does not prescribe how to design the priority structure to achieve the school district’s goals. This is because the TTC algorithm uses the priority structure to determine the order at which students are offered seats, but after students trade seats it may be that a student is assigned to a school where they have low priority.

We provide a simpler characterization of TTC for school choice by exploiting its structure as an assignment problem with many copies of each item. We find that the TTC assignment can be concisely described by n^2 admission thresholds or cutoffs $\{p_b^c\}$. For every pair of schools b, c , the threshold p_b^c is the required school- b priority to be admitted to c . These thresholds and student priorities are a natural counterpart to prices and endowments in Walrasian markets, in that they decentralize the allocation by determining budget sets. Specifically, each student’s budget set is the set of schools the student can trade into using priority from some school (see Figures 2,3 for an illustration), and the TTC assignment is given by each student choosing their most preferred school in their budget set.

The threshold characterization elucidates how students’ priorities determine their TTC assignment. The threshold and budget set structure allows administrators to give a concise justification for why a student failed to receive a desired school seat.⁵ Additionally, explaining that TTC assigns students to the highest ranked school out of their budget set may help convey to students and parents that TTC is strategyproof.

Two simplifications allow us to calculate the thresholds $\{p_b^c\}$ directly from school capacities and the joint distribution of student preferences and priorities. First, instead of tracking trade cycles as in the discrete algorithm, we use *trade balance equations* that allow us to trace the trading process using only the number of seats offered and assigned at each school. Second, we use the continuum framework of Azevedo and Leshno (2016). Using first order conditions derived from the trade

⁴Under DA administrators could say that the school filled up with students who had higher priority at the school, but no such explanation was available for TTC.

⁵This threshold representation allows us to give the following non-combinatorial description of TTC. For each school b , each student receives b -tokens according to their priority at school b , where students with higher b -priority receive more b -tokens. The TTC algorithm publishes prices $\{p_b^c\}$. Students can purchase a single school using a single kind of token, and the required number of b -tokens to purchase school c is p_b^c . Note that $\{p_b^c\}$ can be obtained by running TTC and setting p_b^c to be the number of b -tokens of the lowest b -priority student that traded into c using priority at b . We thank Chiara Margaria, Laura Doval and Larry Samuelson for suggesting this explanation.

balance equation we define a smooth version of TTC in the continuum model that runs in continuous time. The continuous time TTC can be expressed as a solution to a differential equation, allowing us to calculate the thresholds $\{p_b^c\}$ directly from primitives. The model allows for comparative statics, tractable welfare comparisons, and closed form expressions of the TTC assignment in parameterized settings.

As an application of our model, we study optimal investment in school quality when students are assigned through TTC. First, we provide comparative statics as to how the TTC assignment changes as a school becomes more popular.⁶ We then consider the student welfare optimal investment in school quality. The marginal effect of an increase in a school’s quality on student welfare can be decomposed into two effects. The first is the positive effect of the increased utility of students assigned to the school. Under school choice there is an additional effect that arises from changes in the assignment of students to each school. This effect can be negative, for example, students sort into schools according to horizontal preferences when all schools are of equal quality, and increasing the quality difference between schools reduces sorting on the horizontal dimension. We capture this effect by looking at students’ budget sets, allowing us to produce tractable expressions for welfare and quantify the two effects. We solve for the optimal investment under TTC, and find that under certain conditions the optimal investment is equitable in that it generates equally over-demanded schools. A more equitable investment is efficient because it maximizes students’ budget sets, allowing students more choice and yielding more welfare from sorting on horizontal dimensions.

A second application of our model is to provide welfare comparisons between mechanisms. The threshold representation of TTC yields the budget set of each student, and therefore gives a tractable expression for welfare when students have unobserved taste shocks. In Example 1 we solve for TTC and DA assignments and find that when schools have equal quality TTC yields the maximal potential welfare. This is because all assigned students are given the full budget set of all schools, and efficiently sort along horizontal dimensions. DA produces lower welfare, as only a third of assigned students are allowed to efficiently sort along horizontal dimensions, because stability constrains the students who have high priority only at a single school

⁶Hatfield, Kojima, and Narita (2016) explore changes in school quality in the discrete setting, and found that it is possible for a school to be assigned lower priority students when it becomes more popular. Using the continuum framework, we are able to calculate the magnitude of the effect as well as the composition of affected students.

to be assigned to that school. Example 2 provides another welfare comparison, and shows that the optimal investment under TTC can differ from optimal investment under DA. Section 5.3 compares DA and TTC across different school choice environments and corroborates a conjecture by Pathak (2016) that the difference between the mechanisms is smaller when students have a preference for and priority at their neighborhood school. We also compare TTC to the Clinch and Trade mechanism by Morrill (2015b) and find that it is possible for TTC to produce fewer blocking pairs than the Clinch and Trade mechanism.

Our last application explores the design of priorities for TTC. We find that, under TTC, the priority structure is “bossy” in the sense that a change in the relative priority among top priority students can change the assignment of low priority students, without changing the assignment of any high priority student. Such changes to the relative priority among top students cannot be captured by supply-demand equations as in Azevedo and Leshno (2016), and therefore it is not possible to determine the TTC thresholds directly through a supply-demand equation. Another implication of this result is that the choice of tie-breaking between high priority students can affect the allocation of low priority students. We characterize the range of possible assignments generated by TTC after changes to relative priority of high-priority students, and show that a small change to the priorities will only change the allocation of a few students.

To establish the validity of our continuum framework, we provide several technical results. The continuum model is an extension of the standard discrete model in that any discrete economy can be naturally embedded into the continuum framework, and the outcome of the continuum TTC on the embedding is identical to that of discrete TTC. In particular, the threshold characterization is valid for both discrete TTC and continuum TTC. We also show that the TTC assignment changes continuously with small perturbations to the economy, and thus the continuum model can be interpreted as a limit economy.

A few technical aspects of the analysis may be of interest. First, we note that the trade balance equations circumvent many of the measure theoretic complications in defining TTC in the continuum. In particular, trading cycles in the continuum have infinitesimal mass and require schools to point to multiple students in parallel, and defining TTC through the trade balance formulation allows us to bypass these issues. Second, a connection to Markov chain theory allows us to show that a solution to

the marginal trade balance equations always exists, and to characterize the possible trades.

1.1 Related Literature

Abdulkadiroğlu and Sönmez (2003) first introduced school choice as a mechanism design problem and suggested the TTC mechanism as a solution with several desirable properties. Since then, TTC has been considered for use in a number of school choice systems. Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discusses how the city of Boston debated between using DA or TTC for their school choice systems, and ultimately chose to use DA. Abdulkadiroğlu, Pathak, and Roth (2009) compare the outcomes of DA and TTC for the NYC public school system, and shows that TTC gives higher student welfare. Kesten (2006) also study the relationship between DA and TTC, and show that they are equivalent mechanisms if and only if the priority structure is acyclic.

Threshold representations have been instrumental for empirical work on DA and variants of DA. Abdulkadiroglu, Angrist, Narita, and Pathak (Forthcoming) use admission thresholds to construct propensity score estimates. Agarwal and Somaini (2014); Kapor, Neilson, and Zimmerman (2016) structurally estimate preferences from rank lists submitted to non-strategyproof variants of DA. Both build on the threshold representation of Azevedo and Leshno (2016). We hope that our threshold representation of TTC will be similarly useful for future empirical work on TTC.

Our approach may extend to the class Pareto efficient and strategyproofness mechanisms. Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) show that TTC minimizes the number of blocking pairs subject to strategyproofness and Pareto efficiency. Additional axiomatic characterizations of TTC were given by Dur (2012) and Morrill (2013, 2015a). These characterizations explore the properties of TTC, but do not provide another method for calculating the TTC outcome or evaluating welfare. Ma (1994), Pápai (2000) and Pycia and Ünver (2015) give characterizations of more general classes of Pareto efficient and strategy-proof mechanisms. All of these mechanisms are defined through inheritance rules that determine which agent is offered which item, and the assignment is determined through clearing of trade cycles. While our analysis focuses on the TTC mechanism, we believe that our trade balance approach will be useful in analyzing these general classes of mechanisms.

Several variants of TTC have been suggested in the literature. Morrill (2015b) introduces the Clinch and Trade mechanism, which differs from TTC in that it identifies students who are guaranteed admission to their first choice and assigns them immediately without implementing a trade. Hakimov and Kesten (2014) introduce Equitable TTC, a variation on TTC that aims to reduce inequity. In Section 5.2 we use our model to analyze such variants of TTC and compare their assignments. Other variants of TTC can also arise from the choice of tie-breaking rules. Ehlers (2014) shows that any fixed tie-breaking rule satisfies weak efficiency, Alcalde-Unzu and Molis (2011); Jaramillo and Manjunath (2012) and Saban and Sethuraman (2013) give specific variants of TTC that are strategy-proof and efficient. The continuum model allows us to characterize the possible outcomes from different tie-breaking rules.

Several papers also study TTC in large markets. Che and Tercieux (2015a,b) study the properties of TTC in a large market where the number of items grows as the market gets large. Hatfield, Kojima, and Narita (2016) study the incentives for schools to improve their quality under TTC and find that a school may be assigned some less preferred students when it improves its quality.

This paper contributes to a growing literature that uses continuum models in market design (Avery and Levin, 2010; Abdulkadiroğlu, Che, and Yasuda, 2015; Ashlagi and Shi, 2015; Che, Kim, and Kojima, 2013; Azevedo and Hatfield, 2015). Our description of the continuum economy uses the setup of Azevedo and Leshno (2016), who characterize stable matchings in terms of cutoffs that satisfy a supply and demand equation. Our results from Section 5.2 imply that the TTC cutoffs depend on the entire distribution and cannot be computed from simple supply and demand equations.

Our finding that the TTC assignment can be represented in terms of cutoffs parallels the role of prices in competitive markets. Dur and Morrill (2016) show that the outcome of TTC can be expressed as the outcome of a competitive market where there is a price for each priority position at each school, and agents may buy and sell exactly one priority position. Their characterization provides a connection between TTC and competitive markets, but requires a price for each rank at each school and does not provide a method for directly calculating these prices without running TTC. He, Miralles, Pycia, Yan, et al. (2015) propose an alternative pseudo-market approach for discrete allocation problems that extends Hylland and Zeckhauser (1979) and also uses admission thresholds. Miralles and Pycia (2014) show a second welfare theorem

for discrete goods, namely that any Pareto efficient assignment of discrete goods without transfers can be decentralized through prices and endowments, but requires an arbitrary endowment structure.

1.2 Organization of the Paper

Section 2 gives a description of the TTC mechanism in the discrete model and presents our characterization of the TTC outcome in terms of cutoffs. Section 3 presents the continuum model and gives an informal description of the TTC mechanism in the continuum. Section 4 formally defines the TTC model in the continuum and presents our main results. In Section 5 we explore several applications: quantifying the effects of improving school quality and solving for optimal investment, showing the “bossiness” of the TTC priorities, and comparing the TTC assignment with the DA assignment. Omitted proofs can be found in the appendix.

2 TTC in School Choice

In this section, we describe the standard model for the TTC mechanism in the school choice literature, and outline some of the properties of TTC in this setting. In the ensuing discussion, we will index schools with the letter c .

Let \mathcal{S} be a finite set of students, and let \mathcal{C} be a finite set of schools. Each school $c \in \mathcal{C}$ has a finite capacity $q_c > 0$. Each student $s \in \mathcal{S}$ has a strict preference ordering \succ^s over schools, and we let $Ch^s(C) = \arg \max_{\succ^s} \{C\}$ denote s 's most preferred school out of the set C . Each school $c \in \mathcal{C}$ has a strict priority ordering \succ_c over students. To simplify notation, we assume that all students and schools are acceptable, and that there are more students than available seats at schools.⁷

A *feasible allocation* is $\mu : \mathcal{S} \rightarrow \mathcal{C} \cup \{\emptyset\}$ where $|\mu^{-1}(c)| \leq q_c$ for every $c \in \mathcal{C}$. If $\mu(s) = c$ we say that s is assigned to c , and we use $\mu(s) = \emptyset$ to denote that the student s is unassigned. As there is no ambiguity, we let $\mu(c)$ denote the set $\mu^{-1}(c)$ for $c \in \mathcal{C} \cup \{\emptyset\}$. A *discrete economy* is $E = (\mathcal{C}, \mathcal{S}, \succ^{\mathcal{S}}, \succ_{\mathcal{C}}, q)$ where \mathcal{C} is the set of schools, \mathcal{S} is the set of students, $q = \{q_c\}_{c \in \mathcal{C}}$ is the capacity of each school, and $\succ^{\mathcal{S}} = \{\succ^s\}_{s \in \mathcal{S}}$, $\succ_{\mathcal{C}} = \{\succ_c\}_{c \in \mathcal{C}}$.

⁷This is without loss of generality, as we can introduce auxiliary students and schools that represent being unmatched.

The Top Trading Cycles algorithm (TTC) calculates an allocation by creating a virtual exchange for priorities. The algorithm runs in discrete steps as follows.

Algorithm 1 (Top Trading Cycles). *Initialize unassigned students $S = \mathcal{S}$, available schools $C = \mathcal{C}$, capacities $\{q_c\}_{c \in C}$ and empty allocation μ .*

- While there are still unassigned students and available schools:
 - Each available school $c \in C$ tentatively offers a seat by pointing to its highest priority remaining student.
 - * Each student $s \in S$ that was tentatively offered a seat points to his most preferred remaining school.
 - * Select at least one trading cycle, that is, a list of students $s_1, \dots, s_\ell, s_{\ell+1} = s_1$ such that s_i points to the school pointing to s_{i+1} for all i , or equivalently s_{i+1} was offered a seat at s_i 's most preferred school. Assign all students in the cycles to the school they point to.⁸
 - * Remove the assigned students from S , reduce the capacity of the schools they are assigned to by 1, and remove schools with no remaining capacity from C .

TTC satisfies a number of desirable properties. An allocation μ is *Pareto efficient for students* if no group of students can improve by swapping their allocations, and no individual student can improve by swapping their allocation for an unassigned object. A mechanism is *Pareto efficient* if it always produces a Pareto efficient allocation. A mechanism is *strategy-proof for students* if reporting preferences truthfully is a dominant strategy. It is well known that the Top Trading Cycles mechanism, as used in the school choice setting, is both Pareto efficient and strategy-proof for students (Abdulkadiroğlu and Sönmez, 2003). Moreover, when type-specific quotas must be imposed, Top Trading Cycles can be easily modified to meet quotas while still maintaining constrained Pareto efficiency and strategy-proofness (Abdulkadiroğlu and Sönmez, 2003).

⁸Such a trading cycle must exist, since every vertex in the pointing graph with vertex set $S \cup C$ has out-degree 1.

2.1 A Cutoff Characterization of Top Trading Cycles

Our first main contribution is that the TTC allocation can be simply characterized by n^2 cutoff students $\{s_b^c\}$, one for each pair of schools.

Theorem 1. *The TTC allocation is given by*

$$\mu(s) = \max_{\succ_s} \{c \mid s \succeq_b s_b^c \text{ for some } b\},$$

where s_b^c is the worst ranked student at school b that traded a seat at school b for a seat at school c .

Theorem 1 provides an intuitive explanation to individual students for why TTC placed them in a certain school. A student s is given a *budget set* of schools c for which there is a school b that prefers them to the cutoff student s_b^c .⁹ Each student is assigned to their favorite school in their budget set. If a student does not receive a seat at a desired school c , it is because they do not have sufficiently high priority at *any* school, and so c is not in their budget set. Moreover, the cutoff students s_b^c can be easily identified after the mechanism has been run.

However, Theorem 1 does not explain how the cutoff students $\{s_b^c\}$ change with changes in school priorities or student preferences. We therefore introduce the continuum model for TTC which allows us to directly calculate the cutoffs and do comparative statics. We omit the direct proof of Theorem 1, as it follows from Theorem 2 and Proposition 3.

3 The Continuum Model for TTC

We model the school choice problem with a continuum of students and finitely many schools, as in Azevedo and Leshno (2016). There is a finite set of schools denoted by $\mathcal{C} = \{1, \dots, n\}$, and each school $c \in \mathcal{C}$ has the capacity to admit a $q_c > 0$ mass of students. A student $\theta \in \Theta$ is given by $\theta = (\succ^\theta, r^\theta)$. The student's strict preferences over schools is \succ^θ , and we let $Ch^\theta(C) = \max_{\succ^\theta}(C)$ denote θ 's most preferred school out of the set C . The priorities of schools over students are captured by the vector

⁹We can formulate this as a condition on the percentile rank of student s at every school. Let \hat{p}_b^c be the percentile of s_b^c in the priority list of school b . A student s is assigned to their favorite school c at which there is a school b for which their percentile is higher than \hat{p}_b^c .

$r^\theta \in [0, 1]^C$. We say that r_c^θ is the rank of student θ at school c . Schools prefer students with *higher ranks*, that is $\theta \succ_c \theta'$ if and only if $r_c^\theta > r_c^{\theta'}$.

Definition 1. A *continuum economy* is given by $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ where $q = \{q_c\}_{c \in \mathcal{C}}$ is the vector of capacities of each school, and η is a probability measure over Θ .

Without loss of generality, we make the following assumptions for the sake of tractability. First, we assume that all students and schools are acceptable. Second, we normalize the mass of students to be $\eta(\Theta) = 1$. Third, we assume there is an excess of students, that is, $\sum_{c \in \mathcal{C}} q_c < 1$.

We make the following assumption for technical reasons, but it is not without loss of generality. For example, it is violated when all schools share the same priorities over students.¹⁰

Assumption 1. *The measure η admits a density ν . That is for any measurable subset of students $A \subseteq \Theta$*

$$\eta(A) = \int_A \nu(\theta) d\theta.$$

Furthermore, ν is piecewise Lipschitz continuous everywhere except on a finite grid,¹¹ bounded from above, and bounded from below away from zero on its support.¹²

An immediate consequence of this assumption is that a school's indifference curves are of η -measure 0. That is, for any $c \in \mathcal{C}$, $x \in [0, 1]$ we have that $\eta(\{\theta : r_c^\theta = x\}) = 0$. This is analogous to schools having strict preferences in the standard discrete model. Given this assumption, as r_c^θ carry only ordinal information, we can normalize a student's rank to be equal to his percentile rank in the school's preferences. That is, for any $c \in \mathcal{C}$, $x \in [0, 1]$ we have that $\eta(\{\theta : r_c^\theta \leq x\}) = x$.

In school choice, it is common for schools to have coarse priorities, and to refine these using a tie-breaking rule. Our economy \mathcal{E} captures the strict priority structure that results after applying the tie-breaking rule.

As in the discrete model, an *allocation* is a mapping $\mu : \Theta \rightarrow \mathcal{C} \cup \{\emptyset\}$ specifying the assignment of each student. An allocation μ is *feasible* if it respects capacities,

¹⁰We can incorporate an economy where two schools have perfectly aligned priorities by considering them as a combined single school in the trade balance equations. The capacity constraints still consider the capacity of each school separately.

¹¹A grid $G \subset \Theta$ is given by a finite set of grid points $D = \{d_1, \dots, d_L\} \subset [0, 1]$ as $G = \{\theta \mid \exists c \text{ s.t. } r_c^\theta \in D\}$. Equivalently, ν is Lipschitz continuous on $\Theta \setminus G$, which is a collection of hypercubes.

¹²That is, there exists $M > m > 0$ such that for every $\theta \in \Theta$ either $\nu(\theta) = 0$ or $m \leq \nu(\theta) \leq M$.

that is, for each school $c \in \mathcal{C}$ we have that $\eta(\mu(c)) \leq q_c$. To disallow two allocations that differ only on a set of zero measure we require that the assignment is right continuous, that is, for any sequence of student types $\theta^k = (\succ, r^k)$ and $\theta = (\succ, r)$, with r^k converging to r , and $r_c^k \geq r_c^{k+1} \geq r_c$ for all k, c , we can find some large K such that $\mu(\theta^k) = \mu(\theta)$ for $k > K$.

We give an informal description of the TTC algorithm here, and formally describe and characterize the algorithm in Section 4. In the continuum model the TTC algorithm runs in continuous time indexed by t , starting with $t = 0$.

Algorithm Sketch (Continuum Top Trading Cycles) *Initialize unassigned students $S = \Theta$, available schools $C = \mathcal{C}$, capacities $\{q_c\}_{c \in C}$ and empty allocation μ .*

- *At time t , if there are still unassigned students and available schools:*
 - *Each available school $c \in C$ tentatively offers a seat by pointing to the measure 0 set of remaining students with highest priority at c .*
 - * *Each student $s \in S$ that was tentatively offered a seat points to their most preferred remaining school.*
 - * *Select at least one trading cycle, that is, a list of sets of students $S_1, \dots, S_\ell, S_{\ell+1} = S_1$ such that $s_i \in S_i$ points to the school pointing to S_{i+1} for all i , or equivalently each student in S_{i+1} was offered a seat at each student in S_i 's most preferred school. Assign all students in the cycles to the school they point to.*
 - * *Remove the assigned students from S , reduce the capacity of the schools, and remove any schools with capacity $q_c = 0$.*

We remark that there are several challenges in properly defining the algorithm. Each cycle is of zero measure, but as the algorithm progresses we need to reduce school capacities appropriately. Moreover, a school will generally point to a zero measure set that includes more than one student type. This implies each school may be involved in multiple cycles at a given point, a type of multiplicity that leads to non-unique TTC allocations in the discrete setting. In the following we give a formal definition of the algorithm, and show that it has a well-defined unique outcome.

4 Main Results

In the continuum model, as in the discrete model, the TTC allocation can be characterized by n^2 cutoffs $\{p_b^c\}$.

Theorem 2. *The TTC allocation is given by*

$$\mu(\theta) = \max_{\succ_{\theta}} \{c : r_b^{\theta} \geq p_b^c \text{ for some } b\},$$

where p_b^c is the worst rank at school b that is traded for a seat at school c .

A student θ is given a *budget set* of schools c for which their score r_b^{θ} at some school b exceeds the cutoff p_b^c . Each student is assigned to their favorite school in their budget set.

In addition, in the continuum model, the TTC cutoffs can be characterized and computed using trade balance equations, which aggregate over trading cycles. The trade balance equations define a set of differential equations which can be calculated from the problem primitives.

Theorem 3. *In the continuum model, the TTC cutoffs p_b^c are given by*

$$p_b^c = \gamma_b(t^{(c)}),$$

where $t^{(c)}$ is the run-out time of school c and $\gamma(\cdot)$ is defined by the equation $\gamma'(t) = d(\gamma(t))$, and $d(x)$ is a solution to the marginal trade balance equations given the marginal distribution of students at rank $r^{\theta} = x$. The run-out times $t^{(c)}$ are calculated from γ and the capacity constraints.

The rest of this section provides the definitions required to establish these results, as well as the details for calculating the TTC cutoffs. It is structured as follows. In Section 4.1, we provide the definitions and framework. In Section 4.2 we prove Theorem 2. In Section 4.3, we prove Theorem 3. Finally in Section 4.4, we show that our definition of continuum TTC is a generalization of discrete TTC and prove a convergence result.

4.1 Defining the TTC Algorithm Through Trade Balance Equations

In this section, we show that the TTC algorithm in the continuum model can be understood and defined via a set of equations that we term *trade balance equations*, and a curve in $[0, 1]^{\mathcal{C}}$ that we call a *TTC path*. We will begin with some observations to motivate our formal definition of the continuum TTC algorithm. It is worth remarking that although our observations are used to motivate our framework for TTC in the continuum model, they are also, unless otherwise specified, valid for both the discrete and continuum models, and can be used to shed insight on both models.

The first observation is that we may track the progression of the algorithm by recording the student each school is pointing to. In the continuum model, since the set of students is given by a distribution over Θ , and since schools do not discriminate based on students' preferences, we may think of this as tracking the progression of the algorithm via a curve in $[0, 1]^{\mathcal{C}}$. Formally, denote $\gamma(t) : [0, T] \rightarrow [0, 1]^{\mathcal{C}}$ to be the *TTC path*, where $\gamma_c(t)$ is the rank of the student(s) school c points to at time t .¹³

The second observation is that it will be useful to divide the run of the algorithm into discrete *rounds* indexed by $\ell = 1, \dots, L$, corresponding to times when the set of available schools remains constant. We start the algorithm at time $t = 0$ in round $\ell = 1$, and each round ℓ ends when some school exhausts its capacity. We denote the set of schools that still available capacity in round ℓ by $\mathcal{C}^{(\ell)}$, where $\mathcal{C}^{(\ell+1)} \subsetneq \mathcal{C}^{(\ell)}$, $\mathcal{C}^{(1)} = \mathcal{C}$ and $\mathcal{C}^{(L+1)} = \phi$. Denote the time interval corresponding to round ℓ by $[t^{(\ell-1)}, t^{(\ell)}]$, with $t^{(\ell-1)} < t^{(\ell)}$, $t^{(0)} = 0$ and $t^{(L)} = T$. We refer to $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_{\ell}$ as a *run-out sequence*.

We now introduce the necessary definitions and notation to characterize the run of the TTC algorithm within a round. One way to calculate $\gamma(\cdot)$ in the discrete model is by an iterative process of identifying and implementing trade cycles. The continuum model allows a simpler characterization in terms of *trade balance equations* instead of trading cycles.

Consider an available school $c \in \mathcal{C}^{(\ell)}$. Denote the set of students who were offered a seat by school c before time t by¹⁴

¹³If $\mathcal{S}(t)$ denotes the set of students that are still unassigned at time t , then $\mathcal{S}(t) = \{\theta \mid r^{\theta} \leq \gamma(t)\}$ and $\gamma_c(t) = \sup \{r_c^{\theta} \mid \theta \in \mathcal{S}(t)\}$.

¹⁴Note that $\mathcal{T}_c(\gamma; t)$ includes students who were offered a seat in the school in previous rounds.

$$\mathcal{T}_c(\gamma; t) \stackrel{def}{=} \{\theta \in \Theta \mid \exists \tau \leq t \text{ s.t. } r_c^\theta = \gamma_c(\tau) \text{ and } r^\theta \leq \gamma(\tau)\}.$$

That is, a student θ was offered a seat at school c if at some time $\tau \leq t$ the school was pointing to students with rank r_c^θ , and student θ was still unassigned at time τ . To use the discrete terminology, we may think of the student as being pointed to by school c before step k in the algorithm. Denote the set of students who were assigned a seat at school $c \in \mathcal{C}^{(\ell)}$ before time $t \in [t^{(\ell-1)}, t^{(\ell)}]$ by

$$\mathcal{T}^c(\gamma; t) \stackrel{def}{=} \{\theta \in \Theta \mid r^\theta \not\leq \gamma(t) \text{ and } Ch_\theta(\mathcal{C}^{(\ell)}) = c\}.$$

That is, a student was assigned somewhere by time t if there exists a school c' such that $r_{c'}^\theta > \gamma_{c'}(t)$, and the student θ is assigned to c only if c is θ 's most preferred available school.

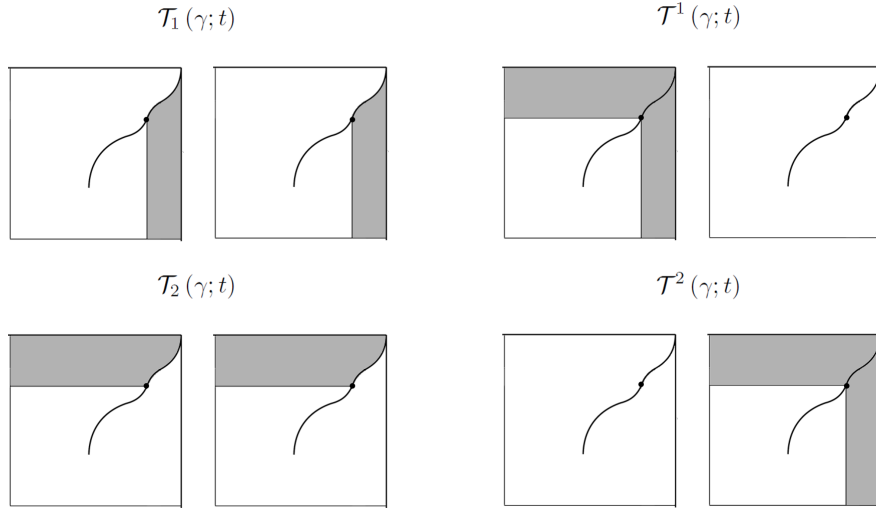


Figure 1: An illustration of the sets $\mathcal{T}^c(\gamma; t)$ and $\mathcal{T}_c(\gamma; t)$ for $c = 1, 2$. In each square the horizontal axis corresponds to the student's rank at school 1 and the vertical axis to the rank at school 2. The left square includes students who prefer school 1 and the right square includes students who prefer school 2. The curved line is γ the TTC path, and the point is at $\gamma(t)$.

Our third and main observation is that a process of cycle clearing imposes a simple condition on the sets $\mathcal{T}^c(\gamma; t)$ and $\mathcal{T}_c(\gamma; t)$. Any cycle that is cleared has the same amount of students offered a seat at a school and the amount of students assigned to the school. Therefore, at any time t and for any school c the total amount of seats offered by school c is equal to the amount of students assigned to c plus the amount

of seats that were offered but not claimed. In the continuum model the amount of seats offered but not claimed is of η -measure 0.¹⁵ It follows that if γ is a TTC path, at every time t it must satisfy the *trade balance equations* for every school c that is available at time t :

$$\eta(\mathcal{T}^c(\gamma; t)) = \eta(\mathcal{T}_c(\gamma; t)). \quad (\text{trade balance equations}) \quad (1)$$

We will show in Section 4.3 that the trade balance equations fully characterize the run of the TTC algorithm within a round.

It remains to characterize the stopping time of each round. Informally, the end of a round is determined by the first time some school fills its capacity. Formally, the stopping time $t^{(\ell)}$ and next available set $\mathcal{C}^{(\ell+1)}$ are determined by the equation $\mathcal{C}^{(\ell+1)} \subsetneq \mathcal{C}^{(\ell)}$ and the *capacity equations*:¹⁶

$$\begin{aligned} \eta(\mathcal{T}^c(\gamma; t^{(\ell)})) &= q_c \quad \forall c \notin \mathcal{C}^{(\ell+1)}, \\ \eta(\mathcal{T}^c(\gamma; t^{(\ell)})) &< q_c \quad \forall c \in \mathcal{C}^{(\ell+1)}. \end{aligned} \quad (\text{capacity equations}) \quad (2)$$

Using the trade balance equations (1) and capacity equations (2), we can formally define the TTC mechanism for a continuum economy.

Definition 2. Given a continuum economy $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ we say that a weakly decreasing function $\gamma : [0, T] \rightarrow [0, 1]^{\mathcal{C}}$ is a *valid TTC path* if it satisfies the *trade balance equations* (1) for all times t and satisfies the *capacity equations* (2) for some run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_{\ell=1..L}$.

In addition, we normalize the algorithm running time so that γ is continuous, piecewise differentiable with $\|\frac{d\gamma(t)}{dt}\|_1 = 1$ a.e., and wlog γ_c is constant on $[t^{(\ell-1)}, T]$ for $c \notin \mathcal{C}^{(\ell)}$.

If γ is a valid TTC path, then there is a unique run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_{\ell=1..L}$ that satisfies the capacity equations (1) for γ . Thus a valid TTC path γ gives a complete description of the run of the TTC algorithm, and allows us to describe the resulting TTC allocation. The set of students who are assigned to c is $\mathcal{T}^c(\gamma; t(c))$,

¹⁵A student can have a seat that is offered but not claimed in one of two ways. The first is the seat is offered at time t and not yet claimed by a trade. The second is that the student that got offered two or more seats at the same time $\tau \leq t$ (and was assigned through a trade involving only one seat). Both of these sets of students are of η -measure 0 under our assumptions.

¹⁶The trade balance equations hold even if students find some schools unacceptable.

where

$$\begin{aligned} t(c) &\stackrel{def}{=} \max \{t^{(\ell)} \mid c \in \mathcal{C}^{(\ell)}\} \\ &= \sup \{\tau \mid \eta(\mathcal{T}^c(\gamma; \tau)) < q_c\} \end{aligned}$$

denotes the time that school c fills its capacity.¹⁷ This allows us to formally define the TTC mechanism for a continuum economy.

Definition 3. Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy. The *TTC assignment* for \mathcal{E} is given by $\mu(c) = \mathcal{T}^c(\gamma; t(c))$ where γ is a valid TTC path for \mathcal{E} .

The TTC assignment is well defined and independent of the choice of the TTC path.

Proposition 1. *The TTC assignment for a continuum economy \mathcal{E} is well defined. That is,*

- (i) *there exists a valid TTC path γ ;*
- (ii) *if γ and γ' are both valid TTC paths for \mathcal{E} then they give the same assignment (up to a set of η -measure 0).*

In 4.3, we prove part (i) of Proposition 1 by explicitly constructing a valid TTC path γ . We prove part (ii) of Proposition 1 in Appendix B.3.

4.2 The Cutoff Description of TTC

In this section, we show that the TTC assignment can be simply and succinctly described as selection from budget sets, which are defined by n^2 cutoffs as follows.¹⁸

Let $b, c \in \mathcal{C}$ be two schools, and consider a student who wishes to use his priority for school b to get a seat in school c . For that, the student needs to receive a seat at school b at a time when c is still available. Denote the lowest b priority of a student that is offered a seat at b when school c has not filled its capacity by

$$\begin{aligned} p_b^c &\stackrel{def}{=} \min \{\gamma_b(t^{(\ell)}) \mid \ell \in \{1 \dots L\} \text{ s.t. } c \in \mathcal{C}^{(\ell)}\} \\ &= \inf \{\gamma_b(\tau) \mid \tau \in [0, T] \text{ s.t. } \eta(\mathcal{T}^c(\gamma; \tau)) < q_c \text{ and } \eta(\mathcal{T}^b(\gamma; \tau)) < q_b\}. \end{aligned}$$

¹⁷This means that for all schools c , γ_c is constant on $[t(c), T]$.

¹⁸All formal definitions in this section will be given in the continuum model, but can be analogously defined in the discrete model as well.

We refer to p_b^c as the required school- b priority to be admitted to c . Note that p_b^c depends on both b and c and can be fully calculated from the values of the TTC path at the end of the rounds $\{\gamma(t^{(\ell)})\}_{\ell=1,\dots,L}$.

We now define student budget sets. Given cutoffs $\mathbf{p} = \{p_b^c\}_{b,c \in \mathcal{C}}$, we say that the set of schools student θ can afford via their priority at school b is

$$B_b(\theta, \mathbf{p}) \stackrel{\text{def}}{=} \{c \in \mathcal{C} \mid r_b^\theta \geq p_b^c\}.$$

The budget set of student θ is the set of schools he can afford using priority at some school,

$$B(\theta, \mathbf{p}) \stackrel{\text{def}}{=} \cup_b B_b(\theta, \mathbf{p}) = \{c \in \mathcal{C} \mid \exists b \in \mathcal{C} \text{ s.t. } r_b^\theta \geq p_b^c\},$$

that is, school c is in θ 's budget set if there is some school b for which his r_b^θ priority is high enough to trade for a seat in c . We use this budget set structure to prove Theorem 2.

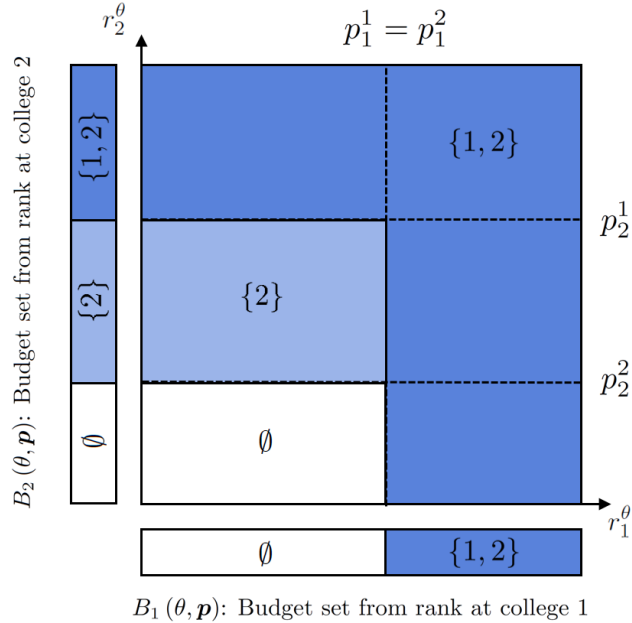


Figure 2: An example of the TTC budget sets, see Figure 3 for the corresponding assignment. Students' ranks at school 1 are given by the horizontal axis and ranks at school 2 by the vertical axis. The budget sets along the axis for school c list $B_c(\theta, \mathbf{p})$, the schools that enter the student's budget set because of their rank at school c . A student's budget set depends only on their rank, and not on their preferences. Each student is assigned to his most preferred school out of the union of the budget sets from both axes, $B(\theta, \mathbf{p}) = \cup_c B_c(\theta, \mathbf{p})$. The parameters of the economy are specified in footnote 19.

Proof of Theorem 2. Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy, and let $\mathbf{p} = \{p_c^b\}_{c,b \in \mathcal{C}}$ be cutoffs derived from a TTC path. Note that p_c^b is the worst rank at school b that is traded for a seat at school c . We show that the TTC assignment of a student θ is given by

$$\mu(\theta) = \max_{\prec_{\theta}} (B(\theta, \mathbf{p})) = \max_{\prec_{\theta}} \{c \in \mathcal{C} \mid \exists b \in \mathcal{C} \text{ s.t. } r_b^{\theta} \geq p_b^c\}.$$

$B(\theta, \mathbf{p})$ is the set of available schools when the TTC algorithm reaches student θ . To see this, observe that the TTC algorithm reaches student θ at the smallest time t such that $\gamma_b(t) = r_b^{\theta}$ for some b . If $c \notin B(\theta, \mathbf{p})$, then $r_b^{\theta} < p_b^c$ and school c has already filled its capacity. Observe that if $r_{b_1}^{\theta} = \gamma_{b_1}(t_1) \geq p_{b_1}^c$ and $r_{b_2}^{\theta} = \gamma_{b_2}(t_2) < p_{b_2}^c$ we must have that $t_1 < t_2$ because $\eta(\mathcal{T}^c(\gamma; t))$ is monotonically increasing in t . Therefore, if $c \in B(\theta, \mathbf{p})$ the algorithm reaches θ at time t such that $\gamma_b(t) = r_b^{\theta} \geq p_b^c$, and school c is still available to student θ . Finally, TTC assigns students to their most preferred available school. \square

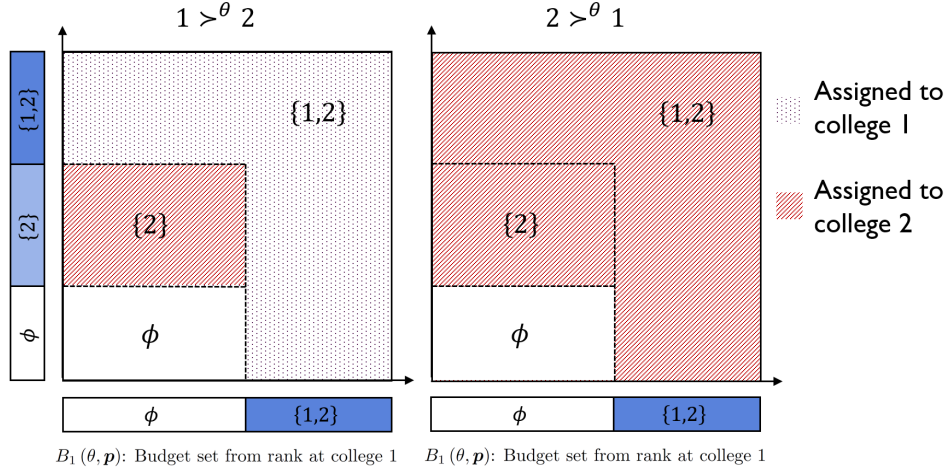


Figure 3: An example of the TTC assignment, see Figure 2 for details.

This shows that TTC assignment has a simple structure. If there are $n = |\mathcal{C}|$ schools we can describe the allocation using n^2 assignment thresholds. These n^2 thresholds determine budget sets for each student, regardless of student preferences, and students are assigned to their most preferred school in their budget set.

We remark that the need for n^2 assignment thresholds contrasts with the cutoff characterization of the deferred acceptance mechanism, which only requires n assignment thresholds Azevedo and Leshno (2016). An example for two schools is given in Figures 2,3.¹⁹ The example in Figures 2,3 shows that the TTC assignment cannot be described by only n thresholds. In Section 5.2 we show that, unlike DA, we cannot identify the TTC cutoffs solely from the demand given these cutoffs.

We further find that the budget sets derived from the cutoffs have a nested structure. If a school c is in a student's budget set because of their priority at b , then every school c' that runs out after c is also in their budget set because of their priority at b . Intuitively, the budget sets must be nested because TTC is a Pareto efficient mechanism. This is formally stated in the following proposition and illustrated in Figure 4.

Proposition 2. *Assume that schools run out in the order $1, 2, \dots, n$. For each student θ and school b , the set of schools $B_b(\theta, \mathbf{p})$ student θ can afford via their priority at school b under the TTC cutoffs \mathbf{p} is either ϕ or of the form*

$$B_b(\theta, \mathbf{p}) = \mathcal{C}^{(c)} = \{c, c + 1, \dots, n\}$$

for some $c \leq b$. Moreover $B(\theta, \mathbf{p}) = \cup_b B_b(\theta, \mathbf{p}) = \mathcal{C}^{(c)}$ for some c when it is nonempty.

Proof. We note that the TTC cutoffs $\mathbf{p} = \{p_c^b\}_{c,b \in \mathcal{C}}$ are points on the TTC path γ , and γ is weakly decreasing in every coordinate. Since $t^{(1)} \leq t^{(2)} \leq \dots \leq t^{(n)}$, it follows that $p_b^c \geq p_b^{c'}$ for all $c < c'$. Hence if $c \in B_b(\theta, \mathbf{p})$, then $r_b^\theta \geq p_b^c$ and so $r_b^\theta \geq p_b^c \geq p_b^{c'}$ and $c' \in B_b(\theta, \mathbf{p})$. This implies that if $c \leq c'$ for all $c' \in B_b(\theta, \mathbf{p})$ and $c \in B_b(\theta, \mathbf{p})$ then $B_b(\theta, \mathbf{p}) = \mathcal{C}^{(c)}$. The structure for $B(\theta, \mathbf{p})$ follows by taking unions, and also independently from the fact that $B(\theta, \mathbf{p})$ is the set of available schools when the TTC algorithm reaches student θ . \square

¹⁹In this economy 2/3 of the students prefer school 1, both schools have equal capacity. Student priorities are uniformly distributed on $[0, 1]$ independently for each school and independently of preferences.

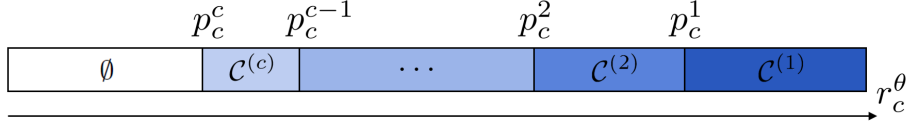


Figure 4: The schools $B_c(\theta, \mathbf{p})$ that enter a student's budget set because of their rank at school c , assuming that schools run out in the order $1, 2, \dots, n$. Students may use their rank at school c to obtain a budget set of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots, \mathcal{C}^{(c)}$. The cutoffs p_c^b are weakly decreasing in b , and equal for all $b \geq c$, $p_c^c = p_c^{c+1} = \dots = p_c^n$.

4.3 Calculating the TTC Assignment Through Differential Equations

In this section, we use the trade balance equations to show that the TTC assignment can be computed using a differential equation. We characterize the gradients of valid TTC paths as solutions to marginal trade balance equations, which are linear equations involving the marginal density of η . The TTC path within each round is a solution to the differential equation given by these gradients. We then trace the TTC path until the capacity constraints bind, calculate the cutoffs and derive the TTC assignment.

We briefly motivate each of these steps by identifying their counterparts in the discrete TTC mechanism. The valid gradients are the continuum analogue to valid trading cycles. When there are multiple valid gradients, the choice of a gradient is analogous to the selection of trading cycles to clear. The progression of the algorithm is captured through the TTC path, and increments of the algorithm and stopping conditions are governed by the marginal trade balance equations and the capacity equations. Note that the TTC allocation is unique (Proposition 1) and any choice of gradients yields the same allocation.

Discrete TTC	→	Continuum TTC	Expression	Equation
Cycle	→	Valid gradient \mathbf{d}	\mathbf{d}	$\mathbf{d} = H\mathbf{d}$
Cycle selection	→	Valid gradient selection	$\mathbf{d}(x)$	$\mathbf{d}(x) = H(x)\mathbf{d}(x)$
Algorithm progression	→	TTC Path	$\gamma(t)$	$\gamma'(t) = \mathbf{d}(\gamma(t))$

We first show how to obtain an equation for the gradient of a valid TTC path

at a given point $x \in [0, 1]^C$. Consider the incremental step of a TTC path γ from $\gamma(t) = x$ in the direction $\mathbf{d} = \gamma'(t)$. Define²⁰

$$\tilde{H}_c^b(x) \stackrel{def}{=} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^\theta \in [x - \varepsilon \cdot e^c, x] \text{ and } Ch_\theta(\mathcal{C}^\ell) = b\}),$$

that is, $\tilde{H}_c^b(x)$ is the marginal density of students who want school b that will get an offer from school c , restricting only to students who are unassigned when $\gamma(t) = x$. Then by taking the trade balance equations over an incremental step we get the *marginal trade balance equations*

$$\sum_b d_b \cdot \tilde{H}_b^c(x) = \sum_b d_c \cdot \tilde{H}_c^b(x) \quad \forall c \in \mathcal{C} \quad (3)$$

where $\gamma(t) = x$ and $d_c = \gamma'_c(t)$.²¹ The LHS is the marginal mass of students who want c and will be assigned to c , and the RHS is the marginal mass of students who will be offered a seat at c .

We can simplify the equation (3) by defining the matrix of relative marginal densities $H(x)$. Let $v_c = \sum_b \tilde{H}_b^c(x)$ be the measure of marginal students that will get an offer from school c , and let $\bar{v} = \max_b v_b$. For $\bar{v} > 0$, define the matrix $H(x)$ to have (b, c) -th entry²²

$$H_b^c(x) \stackrel{def}{=} \frac{1}{\bar{v}} \tilde{H}_b^c(x) + \mathbf{1}_{b=c} \cdot \left(1 - \frac{v_c}{\bar{v}}\right).$$

This normalization makes $H(x)$ to be a (right) stochastic matrix, and allows us to equivalently write the marginal trade balance equations in vector form.

Definition 4. For $x \in [0, 1]^C$ we say \mathbf{d} is a *valid direction from x* if \mathbf{d} has all non-positive entries and solves

$$\begin{aligned} \mathbf{d} \cdot H(x) &= \mathbf{d} \\ \|\mathbf{d}\|_1 &= 1 \end{aligned} \quad (\text{marginal trade balance}) \quad (4)$$

The first equation is equivalent to (3) and the second equation is a normalization of \mathbf{d} .

²⁰We use $e^c \in \mathbb{R}^n$ to denote the vector whose c -th coordinate is 1 and all other coordinates are 0.

²¹The formal proof is in Appendix B.6.

²²If $v_c = 0$ then the marginal students for school c were already assigned through an offer from another school. In this case $H_b^c(x) = \mathbf{1}_{b=c}$, and we can choose a direction $d = e^c$ to move school c to point to students with lower r_c^θ . It is possible that $\bar{v} = 0$, that is $v_c = 0 \forall c$. In that case $H(x) = I$ and we can choose any d .

We now show how to calculate the TTC path as a solution to a differential equation. Suppose that $\mathbf{d}(\cdot) : [0, 1]^c \rightarrow \mathbb{R}^n$ is a piecewise Lipschitz continuous function whose value at x is a solution to $d(x) \cdot H(x) = \mathbf{d}(x)$, $\|\mathbf{d}(x)\|_1 = 1$. The function $\mathbf{d}(\cdot)$ gives a possible direction $\mathbf{d}(x)$ for a TTC path γ that reaches $\gamma(t) = x$. To find a valid TTC path, we start with $\gamma(0) = \mathbf{1}$ and move in the direction given by $\mathbf{d}(\cdot)$ until we fill the capacity of some school. Once we complete a round we remove the schools that have filled their capacity and use the reduced problem to calculate the next round.

If η has full support²³ then $H(x)$ is irreducible²⁴ at every $x \in [0, 1]^c$, and there is a unique solution $\mathbf{d}(\cdot)$ to equations (4). Furthermore, $\mathbf{d}(\cdot)$ can be obtained by inverting a matrix, and $\mathbf{d}(\cdot)$ is piecewise Lipschitz continuous.

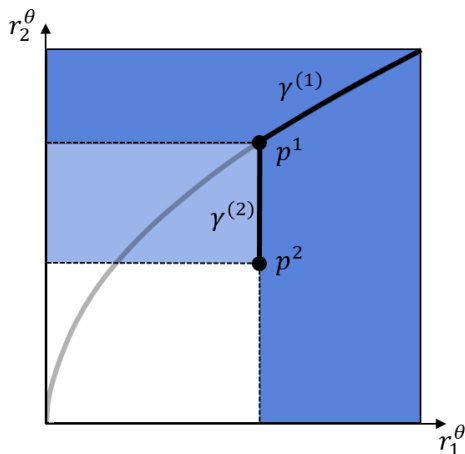


Figure 5: The TTC path used to calculate the cutoffs in Figures 2, 3. The path $\gamma^{(1)}$ indicates the order in which cycles are cleared by TTC when both schools have remaining capacity. The point p^1 shows when school 1 fills its capacity, and is calculated using the capacity equations. When school 1 fills its capacity the algorithm switches to follow the path $\gamma^{(2)}$, until the point p^2 where school 2 fills its capacity.

Theorem 4. Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy such that η has full support. Then there exists a unique valid TTC path γ . Within each round $\gamma(\cdot)$ is given by

$$\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$$

where $\mathbf{d}(x)$ is the unique valid direction from $x = \gamma(t)$ that satisfies (4).

²³We say that η has full support if for every open set $A \subset \Theta$ we have $\eta(A) > 0$.

²⁴A matrix is irreducible if it cannot be transformed by a relabeling of its rows and columns to a block upper triangular matrix.

For the case where η does not have full support, we appeal to a connection to Markov chain theory, which we present briefly here, and more fully in Appendix B.3. Fix x , and note that $H(x)$ is a stochastic matrix. We reinterpret $H(x)$ as transition probabilities of a Markov chain whose states are \mathcal{C} , and \mathbf{d} as a probability distribution over \mathcal{C} . Under this interpretation, the marginal trade balance equations (4) say that the total probability flow out of state c is equal to the total probability flow into state c . In other words, \mathbf{d} is a solution to (4) if and only if it is a stationary probability distribution of the Markov chain.

We can therefore derive the set of solutions to equations (4) using well known results about Markov chains. We restate them here for completeness. Given $H(x)$, a recurrent communication class is a subset $K \subseteq \mathcal{C}$, such that $H(x)$ restricted to rows and columns with coordinates in K is an irreducible matrix, and $H_c^b(x) = 0$ for every $c \in K$ and $b \notin K$. There exists at least one recurrent communication class, and two different communication classes have an empty intersection. The restriction of the equations (4) to a subset K is given by the equations $\sum_{b \in K} d_b H_b^c(x) = d_c \forall c \in K$ and $\|\mathbf{d}\|_1 = 1$. We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor 1981) for a proof of the following result.

Lemma 1. *Fix $H(x)$ and let $\mathcal{K}(x)$ be the set of recurrent communication classes. The set of solutions to the marginal trade balance equations (4) is the set of convex combinations of $\{\mathbf{d}^K\}_{K \in \mathcal{K}(x)}$, where \mathbf{d}^K is the unique solution to equations (4) restricted to K .*

We thus find that there is always at least one solution to the marginal trade balance equations (4), but it may not be unique. However, the Markov chain and recurrent communication class structure also gives some intuition as to the proof of part (ii) of Theorem 1, which states that the TTC allocation is unique. Lemma (1) shows that having multiple possible valid directions in the continuum is parallel to having multiple possible trade cycles in the discrete model. That is, the set of possible valid directions can be decomposed into convex combinations of mutually exclusive trades. Hence the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or implement them in parallel at various relative rates. If we implement these trades in a different order we will have a different TTC path, but as in the discrete TTC, all these paths will result in the same allocation.

Therefore, we can give the following recipe for calculating the TTC path. First, we construct $\mathbf{d}(x)$ to be the unique valid direction from x whose support is minimal under some well behaved order, and we use the shortlex order.²⁵ Using \mathbf{d} we construct a valid TTC path γ that follows the direction \mathbf{d} . The path γ clears trades according to the shortlex order and results in the same allocation as any other valid TTC path.

Theorem 5. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy. Then there is a valid TTC path $\gamma(\cdot)$ such that in any round γ is given by*

$$\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$$

where $\mathbf{d}(x)$ is the valid direction from x with minimal support under the shortlex order.

4.4 Consistency with the Discrete TTC Model

In this section, we show that the continuum TTC model generalizes the standard discrete TTC model, and that the continuum TTC allocation can be used to approximate the TTC allocation on sufficiently similar economies.

To show that the continuum TTC model generalizes the discrete TTC model, we map each instance of TTC on a discrete economy into the continuum model, and show that the two produce equivalent allocations. Informally, to perform this mapping, we think of a discrete economy as a continuum economy by representing each student by a point in $\mathbb{R}^{\mathcal{C}}$ and then ‘smearing’ each of these points to obtain a finite upper bound on the density. We then run TTC on this continuum economy, and assign a student to a school if their ‘smeared’ point is fully assigned to the school.

Formally, consider a discrete economy $E = (\mathcal{C}, \mathcal{S}, \succ_c, \succ^S, q)$ with schools \mathcal{C} , students \mathcal{S} , school priorities \succ_c , student preferences \succ^S , school quotas q and $N = |\mathcal{S}|$ students. We map this to the continuum economy $\mathcal{E} = (\mathcal{C}, \Theta, \eta, \frac{q}{N})$ defined as follows. For each student $s \in \mathcal{S}$ and each school $c \in \mathcal{C}$, define $r_c^s = |\{s' \in \mathcal{S} : s' \succeq^c s\}|$ to be the rank of s at c . We identify each student $s \in \mathcal{S}$ with the N -dimensional cube $I^s = \succ^s \times \prod_{c \in \mathcal{C}} (1 - \frac{r_c}{N}, 1 - \frac{r_c - 1}{N}]$ of student types in the continuum economy.

²⁵The shortlex order over subsets of a set is a total ordering that orders subsets first by cardinality, and then by their smallest elements (Sipser, 2012). We order schools $c_i \in \mathcal{C}$ by their indices. For example, if the set of valid directions is the set of convex combinations of $d_1 = [-\frac{1}{2}, 0, \frac{1}{2}]$ and $d_2 = [0, 1, 0]$, then we select $\mathbf{d}(x) = [0, 1, 0]$.

Define η to be the measure with constant density $\frac{1}{N} \cdot N^N$ on $\cup_s I^s$, and density 0 everywhere else. Let $\mu_d : \mathcal{S} \rightarrow \mathcal{C}$ be the allocation given by discrete TTC on the discrete economy E , let $\mu : \Theta \rightarrow \mathcal{C}$ be the allocation given by continuum TTC on the continuum embedding \mathcal{E} , and let $\hat{\mu}_d : \mathcal{S} \rightarrow \mathcal{C}$ be the allocation on the discrete economy E defined in terms of the continuum allocation μ as follows:

$$\hat{\mu}_d(s) = c \quad \Leftrightarrow \quad \forall \theta \in I^s \quad \mu(\theta) = c.$$

The following proposition shows that this embedding of a discrete economy in the continuum model gives a TTC allocation that is consistent with discrete TTC.

Proposition 3. *The outcome of TTC in the continuum embedding gives the same assignment as TTC on the discrete model, that is, $\mu_d = \hat{\mu}_d$.*

This result validates the informal intuition provided in Section 4.3 that the continuum TTC process is analogous to the standard discrete TTC algorithm, and shows that it provides a strict generalization to a larger class of economies. Intuitively, we may view the continuum TTC process as performing the same assignments as the discrete TTC process, continuously and in fractional amounts instead of in discrete steps. See Appendix A for an example of an embedding of a discrete economy.

Next, we show that we can use a continuum economy to approximate sufficiently similar economies by proving that the TTC allocations for strongly convergent sequences of economies are also convergent. Specifically, in the full support setting, if a sequence of economies converges in total variation to a limit economy, then the TTC allocations also converge.

Theorem 6. *Consider two continuum economies $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ and $\tilde{\mathcal{E}} = (\mathcal{C}, \Theta, \tilde{\eta}, q)$, where the measures η and $\tilde{\eta}$ have total variation distance ε . Suppose also that both measures have full support. Then the TTC allocations in these two economies differ on a set of students of measure $O(\varepsilon|C|^2)$.*

In Section 5.2, we show that changes to the priorities of a set of high priority students can affect the final allocation of other students in a non-trivial manner. This raises the question of what the magnitude of these effects are, and whether the TTC mechanism is robust to small perturbations in student preferences or school priorities. Our convergence result implies that the magnitude of the effects of perturbations is

proportional to the total variation distance of the two economies, and suggests that the TTC mechanism is fairly robust to small perturbations in preferences.

5 Applications

5.1 Optimal Investment in School Quality

In this section, we explore how to invest in school quality when students are assigned through the TTC mechanism. School financing has been subject to major reforms, and empirical evidence suggests that increased financing has substantial impact on school quality (Hoxby, 2001; Cellini, Ferreira, and Rothstein, 2010; Jackson, Johnson, and Persico, 2016; Lafortune, Rothstein, and Schanzenbach, 2016). Under school choice, changes in school quality will affect student preferences over schools, and therefore change the assignment of students to schools. This may have a negative welfare effect, as schools that become popular will be excluded from some students' budget sets. Under heterogeneous preferences (Hastings, Kane, and Staiger, 2009; Abdulkadiroğlu, Agarwal, and Pathak, 2015) welfare depends on whether students can choose a school for which they have an idiosyncratically high preference. Observing students' budget sets allows us to track the welfare generated by student choices along horizontal dimensions.

We first provide more general comparative statics on how an increase in school quality affects the TTC assignment. We then examine the question of optimal investment in school quality under a stylized model. Omitted proofs and derivations can be found in Appendix C.1.

Model with quality dependent preferences and comparative statics

We first enrich our model from Section 3 to allow student preferences to depend on school quality investments. An economy with quality dependent preferences is given by $\mathcal{E} = (\mathcal{C}, \mathcal{S}, \eta, q)$, where $\mathcal{C} = \{1, 2, \dots, n\}$ is the set of schools and \mathcal{S} is the set of student types. A student $s \in \mathcal{S}$ is given by $s = (u^s(\cdot | \cdot), r^s)$, where $u^s(c | \delta)$ is the utility of student s for school c given the quality of each school $\delta = \{\delta_c\}$, and r_c^s is the student's rank at school c . We assume $u^s(c | \cdot)$ is differentiable, increasing in δ_c and non-increasing in δ_b for any $b \neq c$. The measure η over \mathcal{S} specifies the distribution of student types. School capacities are $q = \{q_c\}$, where $\sum q_c < 1$.

For a fixed school quality δ , we denote the induced economy by $\mathcal{E}_\delta = (\mathcal{C}, \Theta, \eta_\delta, q)$, where η_δ is the induced distribution over Θ .²⁶ We assume that for any δ the induced η_δ has a Lipschitz continuous non-negative density ν_δ that is bounded below on its support and depends smoothly on δ . We denote the TTC allocation given δ by μ_δ , and the associated cutoffs by $\{p_b^c(\delta)\}_{c \in \mathcal{C}}$. We omit the dependence on δ when it is clear from context.

When there are two schools, we can specify the direction of change of the TTC cutoffs when we slightly increase δ_ℓ for some $\ell \in \{1, 2\}$, making school ℓ more popular. We consider changes that do not change the strict order of school run-out times and without loss of generality assume that schools are numbered in order of their run-out times.

Proposition 4. *Suppose $\mathcal{E} = (\mathcal{C} = \{1, 2\}, \mathcal{S}, \eta, q)$ and δ induces an economy \mathcal{E}_δ such that the TTC cutoffs have a strict runout order $p_2^1(\delta) > p_2^2(\delta)$. Suppose $\hat{\delta}$ has higher school 2 quality $\hat{\delta}_2 \geq \delta_2$, the same quality $\delta_1 = \hat{\delta}_1$, and $\mathcal{E}_{\hat{\delta}}$ has the same runout order.²⁷ Then when we change from δ to $\hat{\delta}$ the cutoffs $p_b^c(\cdot)$ change as follows:*

- p_1^1 and p_2^1 both decrease, i.e., it becomes easier to trade into $\ell = 1$; and
- p_2^2 increases, i.e. higher 2-priority is required to get into school 2.

Figure 6 illustrates the effect of improving the quality of school $\ell = 2$ when $\mathcal{C} = \{1, 2\}$. Notice that small changes in the cutoffs can result in individual students' budget sets growing or shrinking by more than one school. In general, if the TTC cutoffs change slightly then there will be students whose budget set switches between $\mathcal{C}^{(b)}$ and $\mathcal{C}^{(c)}$ for every pair of schools $b \neq c$. As in Hatfield, Kojima, and Narita (2016), there may be low 2-rank students who will gain assignment to school 2 after the quality change because of the decrease in p_1^2 .

In general, when $n \geq 3$, increasing the quality of a school ℓ can have non-monotone effects on the cutoffs, and it is not possible to specify the direction of change of the cutoffs p_c^b . However, with additional structure we can give more descriptive comparative statics. Consider the logit economy where students' utilities for each school c are randomly distributed as a logit with mean δ_c , independently of priorities

²⁶To make student preferences strict we arbitrarily break ties in favor of school with lower index. We assume the utility of being unassigned is $-\infty$, so all students find all schools acceptable.

²⁷i.e. $p_2^1(\hat{\delta}) \geq p_2^2(\hat{\delta})$.

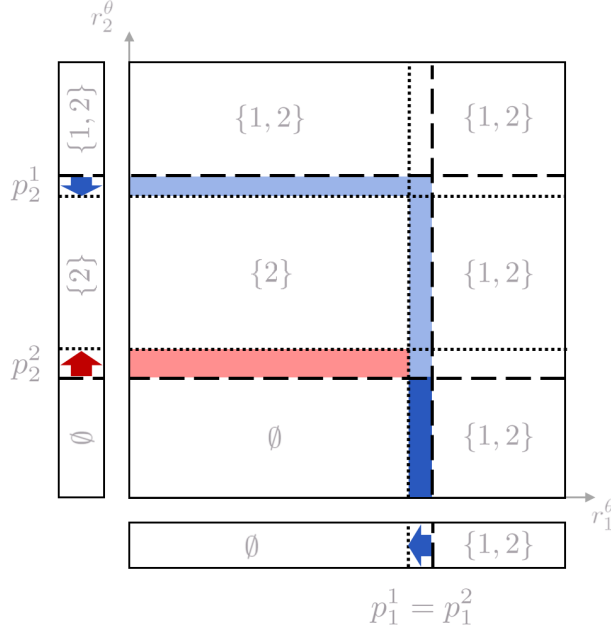


Figure 6: The effect of an increase in the quality of school 2 on TTC cutoffs and budget sets. Dashed lines indicate initial TTC cutoffs, and dotted lines indicate TTC cutoffs given increased school 2 quality. $p_1^1 = p_1^2$ and p_2^1 decrease and p_2^2 increases. Students in the colored sections receive a different budget set. Students in dark blue improve to budget set $\{1, 2\}$ instead of \emptyset , students in light blue improve to $\{1, 2\}$ from $\{2\}$, students in red go from $\{2\}$ to \emptyset .

and utilities for other schools. That is, utility for school c is given by $u^s(c | \delta) = \delta_c + \varepsilon_{cs}$ with η chosen so ε_{cs} are i.i.d. EV shifted to have mean 0 (McFadden, 1973). Schools have uncorrelated uniform priorities over the students. This model allows us to capture a fixed utility term δ_c that can be impacted by investment together with heterogeneous idiosyncratic taste shocks. Under the logit economy we have closed form expressions for the TTC cutoffs, given in Proposition 5, which allow us to describe the comparative statics.

Proposition 5. *Under the logit economy with fixed qualities δ the TTC cutoffs p_b^c for $b \geq c$ are given by²⁸*

$$p_b^c = \left(\frac{\prod_{c' < c} p_{c'}^{c-1} - \rho_c \pi_c}{\prod_{c' < c} p_{c'}^{c'}} \right)^{\pi^{b|c}} \quad (5)$$

where $\pi^{b|c}$ is the probability that a student chooses school b given budget set $\mathcal{C}^{(c)}$,

²⁸To simplify notation, we use $\prod_{c' < c} p_{c'}^{c-1} = 1$ for $c = 1$.

$\rho_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}$ is the relative residual capacity for school c , $\pi_c = \sum_{c' \geq c} e^{\delta_{c'}}$ normalizes ρ_c for when the set of available schools is $\mathcal{C}^{(c)}$, and the schools are indexed in the run-out order $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \dots \leq \frac{q_n}{e^{\delta_n}}$. Moreover, p_b^c is decreasing in δ_ℓ for $c < \ell$ and increasing in δ_ℓ for $b > c = \ell$.

Figure 7 illustrates how the TTC cutoffs change with an increase in the quality of school ℓ . Using equation (5), we derive closed form expressions for $\frac{dp_b^c}{d\delta_\ell}$, which can be found in Appendix C.1.

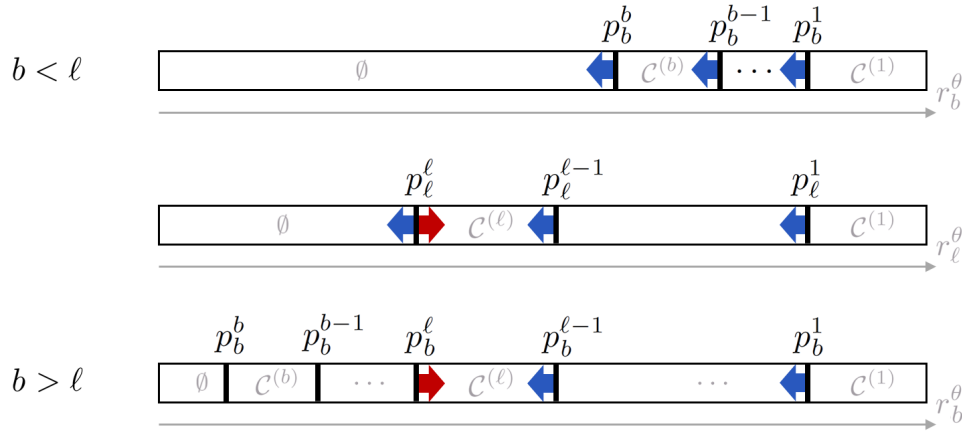


Figure 7: The effects of changing the quality δ_ℓ of school ℓ on the TTC cutoffs p_b^c under the logit economy. If $c < \ell$ then $\frac{dp_b^c}{d\delta_\ell} < 0$ for all $b \geq c$, so it becomes easier to get into the more popular schools. If $c > \ell$ then $\frac{dp_b^c}{d\delta_\ell} = 0$. If $c = \ell$ then $\frac{dp_b^c}{d\delta_\ell} = \frac{dp_b^\ell}{d\delta_\ell} > 0$ for all $b > \ell$, and p_ℓ^ℓ may increase or decrease depending on the specific problem parameters. Note that although p_b^c and p_ℓ^c look aligned in the picture, in general it does not hold that $p_b^c = p_\ell^c$ for all b .

Optimal investment in school quality

Consider a social planner that selects quality levels δ for schools in economy \mathcal{E} . The social planner wishes to maximize the social welfare of students by choosing quality levels δ . For a given assignment μ , the social welfare is given by

$$U(\delta) = \int_{s \in S} u^s(\mu(s) | \delta) d\eta.$$

First consider investment under neighborhood assignment μ_{NH} , which assigns each student to a fixed school regardless of quality and preferences. We assume this

assignment fills the capacity of each school. Social welfare for the logit economy is

$$U_{NH}(\delta) = \sum_c q_c \cdot \delta_c,$$

because $\mathbb{E}[\varepsilon_{\mu(s)s}] = 0$ under neighborhood assignment. The marginal welfare gain from increasing δ_ℓ is $\frac{dU_{NH}}{d\delta_c} = q_\ell$, as an increase in the school quality benefits each of the q_ℓ students assigned to school ℓ .

When the assignment is determined by TTC we need to use the budget set formulation of TTC to capture student social welfare.²⁹ Assume the schools are indexed according to the run-out order given by some fixed δ . A student who is offered the budget set $\mathcal{C}^{(c)} = \{c, \dots, n\}$ is assigned to the school $\ell = \arg \max_{b \in \mathcal{C}^{(c)}} \{\delta_b + \varepsilon_{bs}\}$, and the logit distribution implies that their utility is $U^c = \ln(\sum_{b \geq c} e^{\delta_b})$ (Small and Rosen, 1981). Let N^c be the mass of agents with budget set $\mathcal{C}^{(c)}$. Therefore, social welfare under the TTC assignment given quality δ simplifies to

$$U_{TTC}(\delta) = \sum_c N^c \cdot U^c.$$

This expression for welfare also allows for a simple expression for the marginal welfare gain from increasing δ_ℓ under TTC.

Proposition 6. *For the logit economy, the increase in social welfare $U_{TTC}(\delta)$ under TTC from a marginal increase in δ_ℓ is given by*

$$\frac{dU_{TTC}}{d\delta_\ell} = q_\ell + \sum_{c \leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c.$$

Under neighborhood assignment $\frac{dU_{NH}}{d\delta_\ell} = q_\ell$.

Proposition 6 shows that a marginal increase $d\delta_\ell$ in the quality of school ℓ will have two effects. It will change the utility of the q_ℓ students assigned to ℓ by $d\delta_\ell$, which is the same effect as under neighborhood assignment. In addition, the quality increase changes student preferences, and therefore changes the assignment. The second term captures the additional welfare effect of changes in the assignment by looking at the

²⁹The expected utility of student s assigned to school $\mu(s)$ depends on the student's budget $B(s, \mathbf{p})$ because of selection on taste shocks. Namely, $\mathbb{E}[u^s(\mu(s) | \delta)] = \delta_{\mu(s)} + \mathbb{E}[\varepsilon_{\mu(s)s} | \delta_{\mu(s)} + \varepsilon_{\mu(s)s} \geq \delta_c + \varepsilon_{cs} \forall c \in B(s, \mathbf{p})]$

change in the number of students offered each budget set. The additional term can be negative, and it is possible that $\frac{dU_{TTC}}{d\delta_\ell} < q_\ell = \frac{dU_{NH}}{d\delta_\ell}$ because an increase in the quality of a school can lead to less efficient sorting of students to schools.

In particular, when there are two schools $\mathcal{C} = \{1, 2\}$ with $q_1 = q_2$ and $\delta_1 \geq \delta_2$ we have that

$$\begin{aligned} \frac{dU_{TTC}}{d\delta_1} &= q_1 + \frac{dN^1}{d\delta_1} \cdot U^1 + \frac{dN^2}{d\delta_1} \cdot U^2 = q_1 + \frac{dN^1}{d\delta_1} \cdot (U^1 - U^2) \\ &= q_1 - (q_1 \cdot e^{\delta_2 - \delta_1}) (\ln(e^{\delta_1} + e^{\delta_2}) - \delta_2) < q_1, \end{aligned}$$

where we use that $N^c = \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) (\sum_{b \geq c} e^{\delta_b})$.

An increase in the quality of the higher quality school 1 gives higher utility for students assigned to 1, which is captured by the first term. Additionally, it causes some students to switch their preferences to $1 \succ 2$, making school 1 run out earlier in the TTC algorithm, and removing school 1 from the budget set of some students. Students whose budget set did not change and who switched to prefer school 1 are almost indifferent between the schools, and therefore unaffected. Students who lost school 1 from their budget set can prefer school 1 by a large margin, and incur significant loss. Thus there is a total negative effect from the changes in the assignment, which is captured by the second term, and the derivative is smaller than under neighborhood assignment. Improving the quality of school 1 when $\delta_1 \leq \delta_2$ will have the opposite effect, as it enlarges student budget sets. Specifically, $\frac{dU_{TTC}}{d\delta_1} = q_1 + q_2 \cdot e^{\delta_1 - \delta_2} (\ln(e^{\delta_1} + e^{\delta_2}) - \delta_1) > q_1$ which is larger than under neighborhood assignment. Note that holding δ_2 fixed, the function $U_{TTC}(\delta_1)$ has a kink at $\delta_1 = \delta_2$.

We now provide an illustrative example of optimal investment with quality constraints under DA, TTC and neighborhood assignment.

Example 1. Consider a logit economy with two schools and $q_1 = q_2 = \frac{3}{8}$, and let $Q = q_1 + q_2$ denote the total capacity. The planner is constrained to choose quality levels δ such that $\delta_1 + \delta_2 = 2$ and $\delta_1, \delta_2 \geq 0$.

Under neighborhood assignment $U_{NH}/Q = 1$ for any choice of δ_1, δ_2 .

Under TTC the unique optimal quality is $\delta_1 = \delta_2 = 1$, yielding $U_{TTC}/Q = 1 + \mathbb{E}[\max(\varepsilon_{1s}, \varepsilon_{2s})] = 1 + \ln(2) \approx 1.69$. This is because any assigned student has the budget set $B = \{1, 2\}$ and is assigned to the school for which he has higher

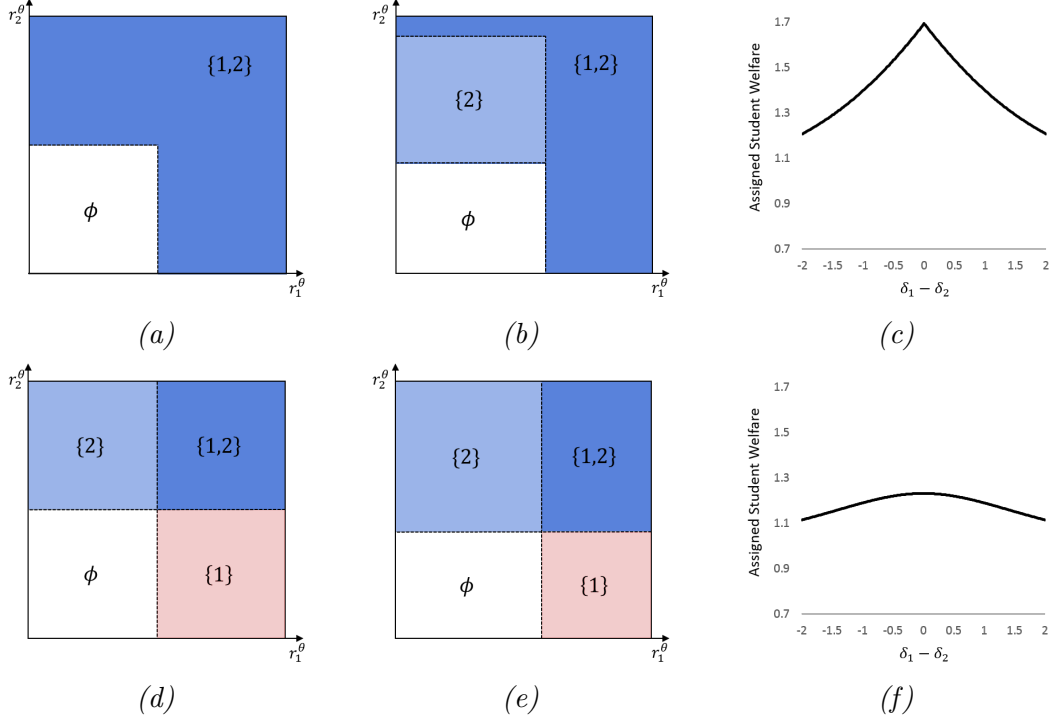


Figure 8: Illustration for Example 1. Figure (a) shows the budget sets under TTC for optimal quality levels $\delta_1 = \delta_2 = 1$, note that all assigned students are given the budget set $\{1, 2\}$. Figure (b) shows the budget sets under TTC for $\delta_1 = 2, \delta_2 = 0$. Figure (c) shows the average welfare of assigned students under TTC for quality levels $\delta_1 + \delta_2 = 2$ for different values of $\delta_1 - \delta_2$. Figures (d), (e) show the budget sets under DA for optimal quality $\delta_1 = \delta_2 = 1$ and quality $\delta_1 = 2, \delta_2 = 0$, respectively. Figure (f) shows the average welfare of assigned students under DA.

idiosyncratic taste. Welfare is lower when $\delta_1 \neq \delta_2$, because fewer students choose the school for which they have higher idiosyncratic taste. For instance, given $\delta_1 = 2, \delta_2 = 0$ welfare is $U_{TTC}/Q = \frac{1}{2} (1 + e^{-2}) \log(1 + e^2) \approx 1.20$.

Under Deferred Acceptance (DA) the unique optimal quality is also $\delta_1 = \delta_2 = 1$, yielding $U_{DA}/Q = 1 + \frac{1}{3} \ln(2) \approx 1.23$. This is strictly lower than the welfare under TTC because under DA only students that have sufficiently high priority for both schools have the budget set $B = \{1, 2\}$. The remaining assigned students have a budget set $B = \{1\}$ or $B = \{2\}$, corresponding to the single school for which they have sufficient priority. If $\delta_1 = 2, \delta_2 = 0$ welfare under DA is $U_{DA}/Q \approx 1.11$.

TTC yields higher student welfare by providing all assigned students with a full budget set, thus maximizing each assigned student's contribution to welfare from horizontal taste shocks. However, the assignment it produces is not stable. In fact, both schools admit students whom they rank at the bottom, and thus virtually all

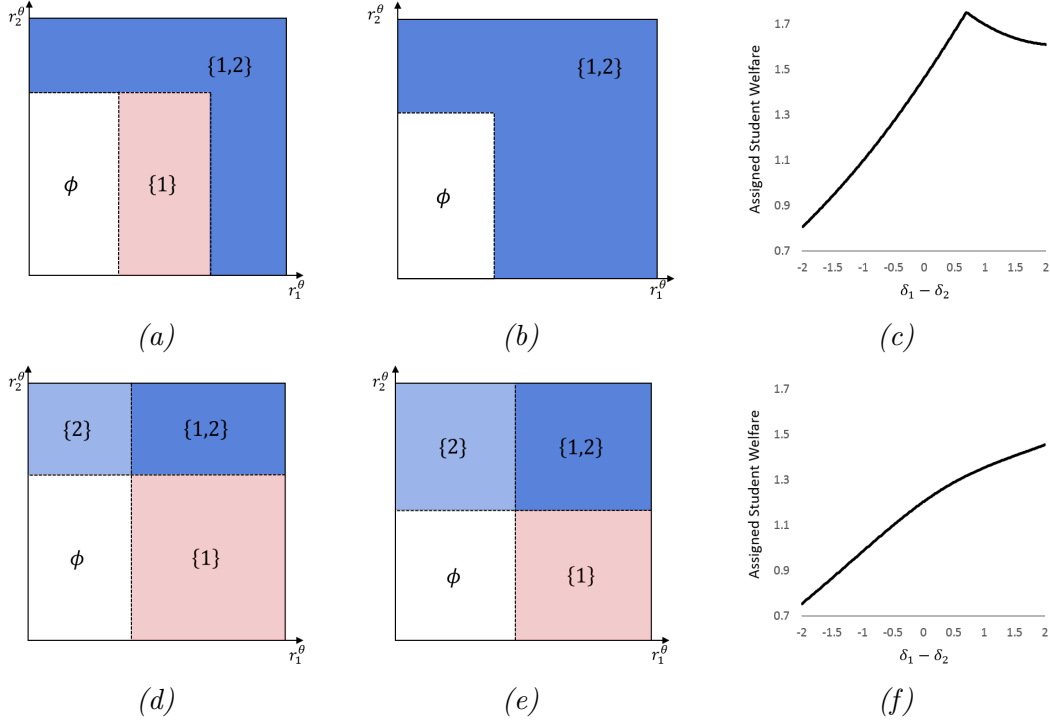


Figure 9: Illustration for Example 2. Figure (a) shows the budget sets under TTC for quality levels $\delta_1 = \delta_2 = 1$, which are not optimal. Figure (b) shows the budget sets under TTC for the optimal quality $\delta_1 = 1 + \ln(2)/2, \delta_2 = 1 - \ln(2)/2$, note that all assigned students are given the budget set $\{1, 2\}$. Figure (c) shows the average welfare of assigned students under TTC for quality levels $\delta_1 + \delta_2 = 2$ for different values of $\delta_1 - \delta_2$. Figures (d), (e) show the budget sets under DA for $\delta_1 = \delta_2 = 1$ and the optimal quality $\delta_1 = 2, \delta_2 = 0$, respectively. Figure (f) shows the average welfare of assigned students under DA.

unassigned students can potentially block with either school.³⁰ Example 1 shows that requiring a stable assignment will constrain many students from efficiently sorting on horizontal taste shocks.

We next provide an example where one school has larger capacity. Investment in the larger school yields more direct benefit as it effects more students, but balancing investments in both schools can yield larger budget sets for more students, leading to more welfare from horizontal taste shocks.

Example 2. Consider a logit economy with two schools and $q_1 = 1/2, q_2 = 1/4$, and let $Q = q_1 + q_2$ denote the total capacity. The planner is constrained to choose quality levels δ such that $\delta_1 + \delta_2 = 2$ and $\delta_1, \delta_2 \geq 0$.

³⁰Note that this may not be a concern in a school choice setting where assignments must be authorized by the department of education and blocking pairs cannot deviate and be assigned outside of the mechanism.

Under neighborhood assignment the welfare optimal quality is $\delta_1 = 2, \delta_2 = 0$, yielding $U_{NH}/Q = 4/3 \approx 1.33$.

Under TTC assignment the unique optimal quality is $\delta_1 = 1 + \frac{1}{2} \ln(2), \delta_2 = 1 - \frac{1}{2} \ln(2)$, yielding $U_{TTC}/Q = \ln\left(\frac{3e}{\sqrt{2}}\right) \approx 1.75$. Under these quality levels any assigned student has the budget set $B = \{1, 2\}$. Given $\delta_1 = 2, \delta_2 = 0$ welfare is $U_{TTC}/Q \approx 1.61$. The quality levels that are optimal in Example 1, namely $\delta_1 = 1, \delta_2 = 1$, give welfare $U_{TTC}/Q \approx 1.46$.

Under DA assignment the unique optimal quality is $\delta_1 = 2, \delta_2 = 0$, yielding $U_{DA}/Q \approx 1.45$. Given $\delta_1 = 1, \delta_2 = 1$ welfare under DA is $U_{DA}/Q \approx 1.20$.

Again we find that the optimal quality under TTC provides all assigned students with a full budget set, while the optimal quality under neighborhood assignment does not. The optimal quality levels under TTC in Example 2 imply that there is a 2/3 chance a student prefers school 1, and therefore both schools run out at the same time and all assigned students are offered a choice between both schools. Increasing δ_1 further (and decreasing δ_2) would increase welfare holding the assignment fixed, but would result in worse sorting of students to schools on the horizontal taste shocks.

Finally, consider a central school board with a fixed amount of capital K to invest in the n schools. The cost of quality δ_c is the convex function $\kappa_c(\delta_c) = e^{\delta_c}$.³¹ Using Proposition 6 we solve for optimal investment in school quality. We find that social welfare is maximized when all assigned students have a full budget set. This occurs when the amount invested in each school is proportional to the number of seats at the school.

Proposition 7. *Social welfare is uniquely maximized when the amount κ_c invested in school c is proportional to the capacity q_c , that is,*

$$\kappa_c(\delta_c) = \frac{q_c}{\sum_b q_b} K$$

and all assigned students θ receive a full budget set, i.e., $B(\theta, \mathbf{p}) = \{1, 2, \dots, n\}$ for all assigned students θ .

³¹Note that κ_c is the total school funding. This is equivalent to setting student utility of school c to be $\log(\kappa_c) = \log(\kappa_c/q_c) + \log(q_c)$, which is the log of the per-student funding plus a fixed school utility that is larger for bigger schools. Thus, schools with higher capacity also provide more efficient investment opportunities.

Under optimal investment, the resulting TTC assignment is such that every assigned student receives a full budget set and is able to attend their top choice school. More is invested in higher capacity schools, as they provide more efficient investment opportunities, but the investment is balanced across schools to prevent any school from being over-demanded. This allows the TTC mechanism to offer assigned students a choice between all schools.

5.2 Design of TTC Priorities

To better understand the role of priorities in the TTC mechanism, we examine how the TTC assignment changes with changes in the priority structure. Notice that any student θ whose favorite school is c and who is within the q_c highest ranked students at c is guaranteed admission to c . In the following example, we consider changes to the relative priority of such highly ranked students and find that these changes can have an impact on the allocation of other students, without changing the allocation of any student whose priority changed.

Example 3. The economy \mathcal{E} has two schools 1, 2 with capacities $q_1 = q_2 = q$, students are equally likely to prefer each school, and student priorities are uniformly distributed on $[0, 1]$ independently for each school and independently of preferences. The TTC algorithm ends after a single round, and the resulting allocation is given by $p_1^1 = p_1^2 = p_2^1 = p_2^2 = \sqrt{1 - 2q}$. The derivation can be found in Appendix C.2.

Consider the set of students $\{\theta \mid r_c^\theta \geq m \ \forall c\}$ for some $m > 1 - q$. Any student in this set is assigned to his top choice, regardless of his rank. Suppose we construct an economy \mathcal{E}' by arbitrarily changing the rank of students within the set, subject to the restriction that their ranks must remain in $[m, 1]^2$.³² The range of possible TTC cutoffs for \mathcal{E}' is given by $p_1^1 = p_1^2, p_2^1 = p_2^2$ where

$$\begin{aligned} p_1^1 &\in [p, \bar{p}] \\ p_2^2 &= \frac{1 - 2q}{p_1^1} \end{aligned}$$

for $p = \sqrt{(1 - 2m + 2m^2)(1 - 2q)}$ and $\bar{p} = \sqrt{\frac{1 - 2q}{1 - 2m + 2m^2}}$. Figure 10 illustrates the range of possible TTC cutoffs for \mathcal{E}' and the economy $\bar{\mathcal{E}}$ for which TTC obtains the extreme cutoffs.

³²The remaining students still have ranks distributed uniformly on the complement of $[1 - r, 1]^2$.

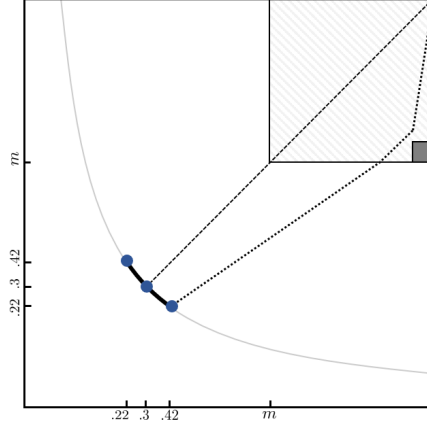


Figure 10: The range of possible TTC cutoffs in example 3 with $q = 0.455$ and $m = 0.6$. The points depict the TTC cutoffs for the original economy and the extremal cutoffs for the set of possible economies \mathcal{E}' , with the range of possible TTC cutoffs for \mathcal{E}' given by the bold curve. The dashed line is the TTC path for the original economy. The shaded squares depict the changes to priorities that generate the economy $\bar{\mathcal{E}}$ which has the extremal cutoffs. In $\bar{\mathcal{E}}$ the priority of all top ranked students is uniformly distributed within the smaller square. The dotted line depicts the TTC path for $\bar{\mathcal{E}}$, which results in cutoffs $p_1^1 = \sqrt{\frac{1-2q}{1-2m+2m^2}} \approx 0.42$ and $p_2^2 = \sqrt{(1-2q)(1-2m+2m^2)} \approx 0.22$.

Example 3 has several implications. First, it shows that it is not possible to directly compute TTC cutoffs from student demand. The set of cutoffs such that student demand is equal to school capacity (depicted by the grey curve in Figure 10) are the cutoffs that satisfy $p_1^1 = p_1^2$, $p_2^1 = p_2^2$ and $p_1^1 p_2^2 = 1 - 2q$. Under any of these cutoffs the students in $\{\theta \mid r_c^\theta \geq m \forall c\}$ have the same demand, but the resulting TTC outcomes are different. It follows that the mechanism requires more information to determine the allocation. However, Theorem 6 implies that the changes in TTC outcomes are small if $1 - m$ is small.

A second implication is that the TTC priorities can be ‘bossy’ in the sense that changes in the relative priority of high priority students can affect the assignment of other students, even when all high priority students receive the same assignment. Notice that in all the economies considered in Example 3, we only changed the relative priority within the set $\{\theta \mid r_c^\theta \geq m \exists c\}$, and all these students were always assigned to their top choice. However, these changes resulted in a different allocation for low priority students. For example, if $q = 0.455$ and $m = 0.4$, a student θ with priority $r_1^\theta = 0.35, r_2^\theta = 0.1$ could possibly receive his first choice or be unassigned depending on the choice of \mathcal{E}' . Such changes to priorities may naturally arise when there are many indifferences in student priorities, and tie-breaking is used. Since priorities

are bossy, the choice of tie-breaking between high-priority students can have indirect effects on the allocation of low priority students.

Example 3 can also be used to compare TTC with the Clinch and Trade (C&T) mechanism, introduced by Morrill (2015b). The C&T mechanism identifies students whose favorite school is c and have priority $r_c^\theta \geq 1 - q$ and allows them to immediately “clinch” and be assigned to c without doing a trade. Morrill (2015b) gives an example where the C&T allocation has fewer blocking pairs than the TTC allocation. The fact that allowing students to clinch can change the allocation can be interpreted as another example for the bossiness of priorities under TTC: we can equivalently implement C&T by running TTC on a changed priority structure where students who clinched at school c have higher rank at c than any other student.³³ The following proposition builds on Example 3 and shows that C&T may produce more blocking pairs than TTC.

Proposition 8. *The Clinch and Trade mechanism can produce more, fewer or an equal number of blocking pairs compared to TTC.*

Proof. Morrill (2015b) provides an example where C&T produces fewer blocking pairs than TTC. Both mechanism give the same assignment for the symmetric economy in the beginning of Example 3. It remains to construct an economy \mathcal{E}_1 for which C&T produces more blocking pairs than TTC. Economy \mathcal{E}_1 is the same as $\bar{\mathcal{E}}$, except that school 2 rank is redistributed among students with $r_2^\theta \leq \underline{p}$ so that students with $r_1^\theta \geq \bar{p}$ have higher school 2 rank.³⁴ The C&T allocation for \mathcal{E}_1 is given by $p_1^1 = p_2^2 = 0.3$, while TTC gives $p_1^1 = \bar{p}$ and $p_2^2 = \underline{p}$ (and under both $p_1^1 = p_1^2$, $p_2^1 = p_2^2$). Under TTC unmatched students will form blocking pairs only with school 2, while under C&T all unmatched students will form a blocking pair with either school. See Figure 13 for an illustration. \square

³³For brevity, we abstract away from certain details of C&T mechanism that are important when not all schools run out at the same round.

³⁴Specifically, select $\ell_1 < \ell_2$. Among students with $r_2^\theta \leq \underline{p}$ and $r_1^\theta \geq \bar{p}$ the school 2 rank is distributed uniformly in the range $[\ell_2, \underline{p}]$. Among students with $r_2^\theta \leq \underline{p}$ and $r_1^\theta < \bar{p}$ the school 2 rank is distributed uniformly in the range $[0, \ell_1]$. Within each range r_1^θ and r_2^θ are still independent. See Figure 13 for an illustration.

5.3 Comparing Top Trading Cycles and Deferred Acceptance

Both TTC and Deferred Acceptance (DA) (Gale and Shapley, 1962) are strategyproof, but differ in that TTC is efficient whereas DA is stable. In theory, the choice between the mechanisms requires a trade-off between efficiency and stability (this trade-off is evident in Example 1). Kesten (2006); Ehlers and Erdil (2010) show the two mechanisms are equivalent only under strong conditions that are unlikely to hold in practice. However, Pathak (2016) evaluates the two mechanisms on application data from school choice in New Orleans and Boston, and reports that the two mechanisms produced similar outcomes. Pathak (2016) conjectures that neighborhood priority leads to correlation between student preferences and school priorities that may explain the similarity between the TTC and DA allocations.

We consider a simple model with neighborhood priority to evaluate the effect of the resulting correlation between student preferences and school priorities.³⁵ There are n neighborhoods, each with one school and a mass q of students. Schools have capacities $q_1 \leq \dots \leq q_n = q$, and each school gives priority to students in their neighborhood. For each student, the neighborhood school is their top ranked choice with probability α . With probability $1 - \alpha$ the students ranks the neighborhood school in position k drawn uniformly at random from $\{2, 3, \dots, n\}$. Student preference ordering over non-neighborhood schools are drawn uniformly at random.

This model supports the conjecture of Pathak (2016), as the proportion of students whose assignments are the same under both mechanisms scales linearly with the probability of preference for the neighborhood school α .

Proposition 9. *The proportion of students who have the same assignments under TTC and DA is given by*

$$\alpha \frac{\sum_i q_i}{nq}.$$

Proof. We use the methodologies developed in Section 4 and in Azevedo and Leshno (2016) to find the TTC and DA allocations respectively. Students with priority are given a lottery number uniformly at random in $[\frac{1}{2}, 1]$, and students without priority

³⁵Che and Tercieux (2015b) also show that when there are a large number of schools with a single seat per school and preferences are random both DA and TTC are asymptotically efficient and stable and give asymptotically equivalent allocations. As Example 1 shows, these results do not hold when there are many students and a few large schools.

are given a lottery number uniformly at random in $[0, \frac{1}{2}]$, where lottery numbers at different schools are independent. For all values of α , the TTC cutoffs are given by $p_j^i = p_i^j = 1 - \frac{q_i}{2q}$ for all $i \leq j$, and the DA cutoffs are given by $p_i = 1 - \frac{q_i}{2q}$. The derivations of the cutoffs can be found in Appendix C.3.

The students who have the same assignments under TTC and DA are precisely the students at neighborhood i whose ranks at school i are above $1 - \frac{q_i}{2q}$, and whose first choice school is their neighborhood school. This set of students comprises an $\alpha \frac{\sum_i q_i}{nq}$ fraction of the entire student population, which scales proportionally with the correlation between student preferences and school priorities. \square

6 Discussion

We can simplify how the TTC outcome is communicated to students and their families by using the cutoff characterization. The cutoffs $\{p_b^c\}$ are calculated in the course of running the TTC algorithm. The cutoffs can be published to allow parents to verify their allocation, or the budget set structure can be communicated using the language of tokens (see footnote 5). We hope that these methods of communicating TTC will make the mechanism more palatable to students and their parents, and facilitate a more informed comparison with the Deferred Acceptance mechanism, which also has a cutoff structure.

The model assumes for simplicity that all students and schools are acceptable. It can be naturally extended to allow for unacceptable students or schools by erasing from student preferences any school that they find unacceptable or that finds them unacceptable. Type-specific quotas can be incorporated, as in Abdulkadiroğlu and Sönmez (2003), by adding type-specific capacity equations and erasing from the preference list of each type all the schools which do not have remaining capacity for their type.

In many school choice systems, indifferences in school priority are broken using tie-breaking lotteries. Our model can be used to calculate the TTC outcome given a tie-breaking rule. In Section 5.2 we characterize all the possible TTC outcomes for a class of tie-breaking rules, and find that the choice of tie-breaking rule can have significant effect on the allocation. We leave the problem of determining the optimal choice of tie-breaking lottery for future research.

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A Example: Embedding a discrete economy in the continuum model

Consider the discrete economy $E = (\mathcal{C}, \mathcal{S}, \succ^{\mathcal{S}}, \succ^{\mathcal{C}}, q)$ with two schools and six students, $\mathcal{C} = \{1, 2\}$, $\mathcal{S} = \{a, b, c, u, v, w\}$. School 1 has capacity $q_1 = 4$ and 2 has capacity $q_2 = 2$. The school priorities and student preferences are given by

$$\begin{aligned} 1 & : a \succ u \succ b \succ c \succ v \succ w, \\ 2 & : a \succ b \succ u \succ v \succ c \succ w, \\ a, b, c & : 1 \succ 2, \\ u, v, w & : 2 \succ 1. \end{aligned}$$

In Figure 11, we display three TTC paths for the continuum embedding \mathcal{E} of the discrete economy E . The first path γ_{all} corresponds to clearing all students in recurrent communication classes, that is, all students in the maximal union of cycles in the pointing graph. The second path γ_1 corresponds to taking $K = \{1\}$ whenever possible. The third path γ_2 corresponds to taking $K = \{2\}$ whenever possible. We remark that the third path gives a different first round cutoff point p^1 , but all three paths give the same allocation.

A.1 Calculating the TTC paths

We first calculate the TTC path in the regions where the TTC paths are the same.

In what follows, we will let \tilde{H} be the matrix with (i, j) th entry \tilde{H}_i^j , the marginal density of students who want school j and get an offer from school i . Let H be the matrix with (i, j) th entry $\frac{1}{\bar{v}}\tilde{H}_i^j + \mathbf{1}_{i=j}(1 - \frac{v_i}{\bar{v}})$, where $v_i = \sum_j \tilde{H}_i^j$ is the i th row sum of \tilde{H} , and $\bar{v} = \max_i v_i$ is the largest row sum of \tilde{H} , as defined in Section 4.3.

At every point (x_1, x_2) with $\frac{5}{6} < x_1 \leq x_2 \leq 1$ the \tilde{H} matrix is $\begin{bmatrix} x_2 - \frac{5}{6} & 0 \\ x_1 - \frac{5}{6} & 0 \end{bmatrix}$,

so $v_1 = x_2 - \frac{5}{6} > v_2$ and $H = \begin{bmatrix} 1 & 0 \\ \frac{6x_1-5}{6x_2-5} & \frac{6x_2-6x_1}{6x_2-5} \end{bmatrix}$. Hence $d = [-1, 0]$ is the unique (non-positive) gradient satisfying $dH = d$ and $d\mathbf{1} = 1$, and the TTC path is defined uniquely for $t \in [0, \frac{1}{6}]$ by $\gamma(t) = (1 - t, 1)$. This section of the TTC path starts at $(1, 1)$ and ends at $(\frac{5}{6}, 1)$.

At every point $(\frac{5}{6}, x_2)$ with $\frac{5}{6} < x_2 \leq 1$ the \tilde{H} matrix is $\begin{bmatrix} 0 & \frac{1}{6} \\ 0 & 0 \end{bmatrix}$, so $\bar{v} = v_1 = \frac{1}{6}, v_2 = 0$ and hence $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $d = [0, -1]$ is the unique (non-positive) gradient, and the TTC path is defined uniquely for $t \in [\frac{1}{6}, \frac{1}{3}]$ by $\gamma(t) = (\frac{5}{6}, \frac{7}{6} - t)$. This section of the TTC path starts at $(\frac{5}{6}, 1)$ and ends at $(\frac{5}{6}, \frac{5}{6})$.

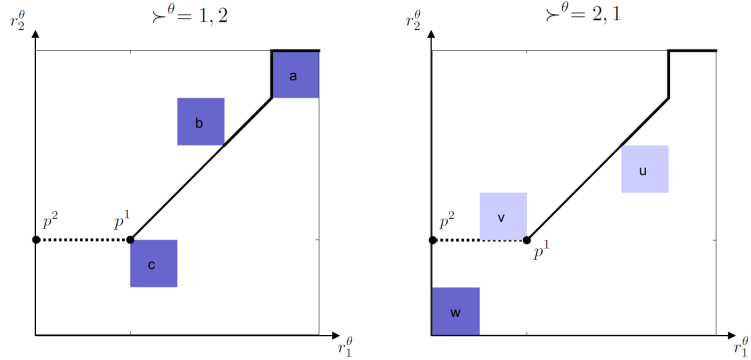
At every point (x_1, x_2) with $\frac{2}{3} < x_1, x_2 \leq \frac{5}{6}$ the \tilde{H} matrix is $\begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix}$, and so $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $d = [-\frac{1}{2}, -\frac{1}{2}]$ is the unique gradient, the TTC path is defined uniquely to lie on the diagonal $\gamma_1(t) = \gamma_2(t)$, and this section of the TTC path starts at $(\frac{5}{6}, \frac{5}{6})$ and ends at $(\frac{2}{3}, \frac{2}{3})$.

At every point $x = (\frac{1}{3}, x_2)$ with $\frac{1}{3} < x_2 \leq \frac{2}{3}$ the \tilde{H} matrix is $\begin{bmatrix} 0 & 6x_2 - 2 \\ 0 & 0 \end{bmatrix}$, and so $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $d = [0, -1]$ is the unique (non-positive) gradient, and the TTC path is parallel to the y axis.

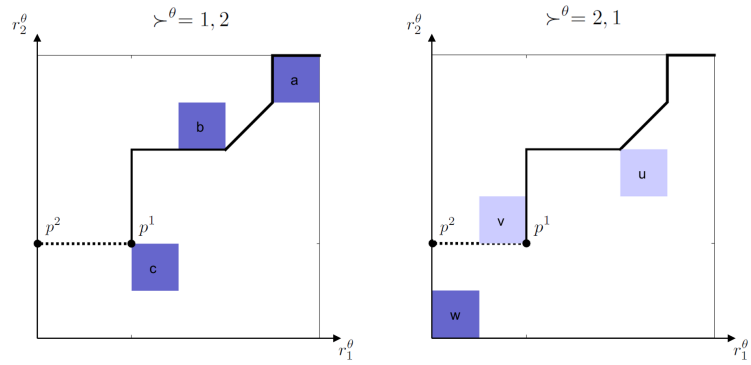
Finally, at every point $(x_1, \frac{1}{3})$ with $0 < x_1 \leq \frac{2}{3}$, the measure of students assigned to school c_1 is at most 3, and the measure of students assigned to school c_2 is 2, so c_2 is unavailable. Hence, from any point $(x_1, \frac{1}{3})$ the TTC path moves parallel to the x_1 axis.

We now calculate the various TTC paths where they diverge.

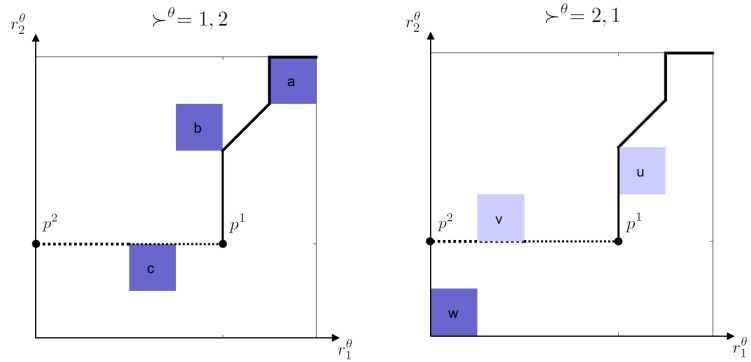
At every point $x = (x_1, x_2)$ with $\frac{1}{2} < x_1, x_2 \leq \frac{2}{3}$ the \tilde{H} matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (i.e. there are no marginal students), and so $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, at every point $x = (x_1, x_2)$ with $\frac{1}{3} < x_1, x_2 \leq \frac{1}{2}$ the \tilde{H} matrix is $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$, and so $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, at every point $x = (x_1, x_2)$ with $\frac{1}{3} < x_1 \leq \frac{1}{2}$ and $\frac{1}{2} < x_2 \leq \frac{2}{3}$, the \tilde{H} matrix is $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix}$ so again $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The same argument with the coordinates swapped gives that $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when $\frac{1}{2} < x_1 \leq \frac{2}{3}$ and $\frac{1}{3} < x_2 \leq \frac{1}{2}$. Hence in all these regions, both



TTC path γ_{all} clears all students in recurrent communication classes.



TTC path γ_1 clears all students who want school 1 before students who want school 2.



TTC path γ_2 clears all students who want school 2 before students who want school 1.

Figure 11: Three TTC paths and their cutoffs and allocations for the discrete economy in example A. In each set of two squares, students in the left square prefer school 1 and students in the right square prefer school 2. The first round TTC paths are solid, and the second round TTC paths are dotted. The cutoff points p^1 and p^2 are marked by filled circles. Students shaded light blue are assigned to school 1 and students shaded dark blue are assigned to school 2.

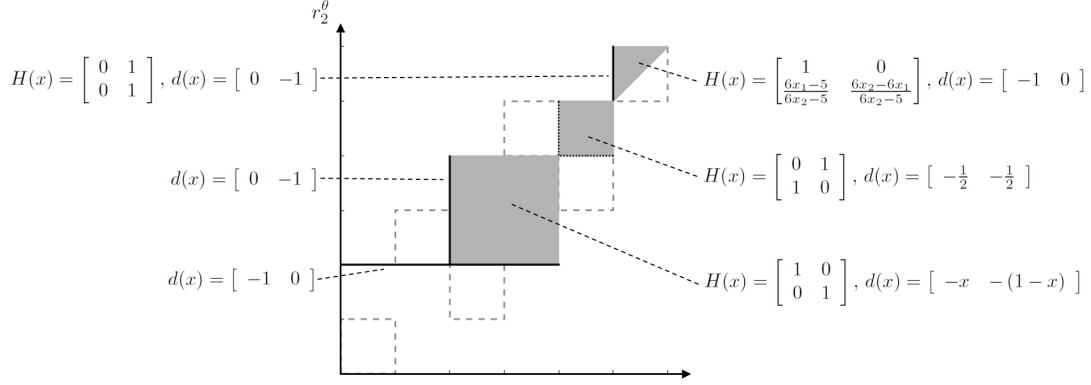


Figure 12: The matrices $H(x)$ and valid directions $d(x)$ for the continuum embedding \mathcal{E} . The borders of the squares corresponding to the students are drawn using dashed grey lines.

schools are in their own recurrent communication class, and any vector d satisfies $dH = d$.

The first path corresponds to taking $d = [-\frac{1}{2}, -\frac{1}{2}]$, the second path corresponds to taking $d = [-1, 0]$ and the third path corresponds to taking $d = [0, -1]$. The first path starts at $(\frac{2}{3}, \frac{2}{3})$ and ends at $(\frac{1}{3}, \frac{1}{3})$ where school 2 fills. The third path starts at $(\frac{2}{3}, \frac{2}{3})$ and ends at $(\frac{2}{3}, \frac{1}{3})$ where school 2 fills. Finally, when $x = (\frac{1}{3}, x_2)$ with $\frac{1}{3} < x_2 \leq \frac{1}{2}$, the \tilde{H} matrix is $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and so $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $d = [0, -1]$ is the unique gradient, and the second TTC path starts at $(\frac{1}{3}, \frac{1}{2})$ and ends at $(\frac{1}{3}, \frac{1}{3})$ where school 2 fills. All three paths continue until $(0, \frac{1}{3})$, where school 1 fills.

Note that all three paths result in the same TTC allocation, which assigns students a, b, c, w to school 1 and u, v to school 2. All three paths assign the students assigned before p^1 (students a, u, b, c for paths 1 and 2 and a, u, b for path 3) to their top choice school. All three paths assign all remaining students to school 1.

B Proofs for Main Results

Definitions and Notation

In our proofs, we will be comparing allocations TTC paths across rounds. Therefore we need to expand all our previous definitions that include the notion of a top choice school so that they also specify the set of schools available to a student. We do so

below.

Let \underline{x}, \bar{x} be vectors. We let $(\underline{x}, \bar{x}] = \{x : x \not\leq \underline{x} \text{ and } x \leq \bar{x}\}$ denote the set of vectors that are weakly smaller than \bar{x} along every coordinate, and strictly larger than \underline{x} along some coordinate. Let $K \subseteq \mathcal{C}$ be a set of schools. For all vectors x , we let $\pi_K(x)$ denote the projection of x to the coordinates indexed by schools in K .

Recall that a *valid TTC path* γ is a weakly decreasing function $\gamma : [0, T] \rightarrow [0, 1]^{\mathcal{C}}$ that satisfies the *trade balance equations* for all times t , and satisfies the *capacity equations* for some run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_{\ell=1..L}$. For brevity, for each t we will let $\ell(t)$ denote the round being run at $\gamma(t)$, that is, $\ell(t)$ is the unique round ℓ that satisfies $t^{(\ell-1)} \leq t < t^{(\ell)}$. We will also relax the assumption that $\|\frac{d\gamma(t)}{dt}\|_1 = 1$ a.e, and allow for rescaling of the time parameter t .

Let us now incorporate information about the set of available schools. We denote by

$$\Theta^{c|C} = \{\theta \in \Theta \mid Ch_{\theta}(C) = c\}$$

the set of students whose top choice in C is c , and denote by $\eta^{c|C}$ the measure of these students. That is, for $S \subseteq \Theta$, let $\eta^{c|C}(S) := \eta(S \cap \Theta^{c|C})$. In an abuse of notation, for a set $A \subseteq [0, 1]^{\mathcal{C}}$, we will often let $\eta(A)$ denote $\eta(\{\theta \in \Theta \mid r^{\theta} \in A\})$, the measure of students with ranks in A , and let $\eta^{c|C}(A)$ denote $\eta(\{\theta \in \Theta^{c|C} \mid r^{\theta} \in A\})$, the measure of students with ranks in A whose top choice school in C is c .

For a set of schools C and individual schools $b, c \in C$, let

$$\begin{aligned} \tilde{H}_b^{c|C}(x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^{\theta} \in [x - \varepsilon \cdot e^b, x] \text{ and } Ch_{\theta}(C) = c\}) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta^{c|C} \mid r^{\theta} \in [x - \varepsilon \cdot e^b, x]\}) \end{aligned}$$

be the marginal density of students pointed to by school b at the point x whose top choice school in C is c .

Let $H^C(x)$ be the $|C| \times |C|$ matrix with (b, c) th entry

$$H^C(x)_{b,c} = \frac{1}{\bar{v}} \tilde{H}_b^{c|C}(x) + \mathbf{1}_{b=c} \left(1 - \frac{v_c}{\bar{v}}\right),$$

where $v_c = \sum_{d \in C} \tilde{H}_c^{d|C}(x)$ is the row sum of $\tilde{H}(x)$, and $\bar{v} = \max_c v_c$ is the maximum row sum.

Let $M^C(x)$ be the Markov chain with state space C , and transition probability

from state b to state c equal to

$$H^C(x)_{b,c}.$$

We remark that such a Markov chain exists, since $H^C(x)$ is a (right) stochastic matrix for each pair C, x .

We will also need the following definitions. For a matrix H and sets of indices I, J we let $H_{I,J}$ denote the submatrix of H with rows indexed by elements of I and columns indexed by elements of J . Recall that, by Assumption 1, the measure η is defined by a probability density ν that is right-continuous and piecewise Lipschitz continuous with points of discontinuity on a finite grid. Let the finite grid be the set of points $\{x \mid x_i \in D_i \forall i\}$, where the D_i are finite subsets of $[0, 1]$. Then there exists a partition \mathcal{R} of $[0, 1]^c$ into hyperrectangles such that for each $R \in \mathcal{R}$ and each face of R , there exists an index i and $y_i \in D_i$ such that the face is contained in $\{x \mid x_i = y_i\}$.

The following notion of continuity will be useful, given this grid-partition. We say that a multivariate function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *right-continuous* if $f(x) = \lim_{y \geq x} f(y)$, where x, y are vectors in \mathbb{R}^n and the inequalities hold coordinate-wise. For an $m \times n$ matrix A , let $\mathbf{1}(A)$ be the $m \times n$ matrix with entries

$$\mathbf{1}(A)_{ij} = \begin{cases} 1 & \text{if } A_{ij} \neq 0, \\ 0 & \text{if } A_{ij} = 0. \end{cases}$$

We will also frequently make use of the following lemmas.

Lemma 2. *Let γ be a TTC path. Then γ is Lipschitz continuous.*

Proof. By assumption, γ is normalized so that $\|\frac{d\gamma(t)}{dt}\|_1 = 1$ a.e., and so since $\gamma(\cdot)$ is monotonically decreasing, for all c it holds that $\gamma_c(\cdot)$ has bounded derivative and is Lipschitz with Lipschitz constant L_c . It follows that $\gamma(\cdot)$ is Lipschitz with Lipschitz constant $\max_c L_c$. \square

Lemma 3. *Let $C \subseteq \mathcal{C}$ be a set of schools, and let D be a region on which $H^C(x)$ is irreducible for all $x \in D$. For each x let $A(x)$ be given by replacing the n th column of $H^C(x) - I_C$ with the all ones vector $\mathbf{1}$.³⁶ Then the function $f(x) = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}$ is piecewise Lipschitz continuous in x .*

Proof. It suffices to show that the function which, for each x , outputs the matrix $A(x)^{-1}$ is piecewise Lipschitz continuous in x .

³⁶ I_C is the identity matrix with rows and columns indexed by the elements in C .

Now

$$\tilde{H}_b^{clC}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\theta: r^\theta \geq x, r^\theta \not\geq x_b + \varepsilon \cdot e_b, c \succ^\theta C} \nu(\theta) d\theta,$$

where $\nu(\cdot)$ is bounded below on its support and piecewise Lipschitz continuous, and the points of discontinuity lie on the grid. Hence $\tilde{H}_b^{clC}(x)$ is Lipschitz continuous in x for all b, c , and $\sum_d \tilde{H}_c^{d|C}(x)$ nonzero and hence bounded below. Further, since $H^C(x)_{b,c} = \frac{1}{\bar{v}(x)} \tilde{H}_b^{clC}(x) + \mathbf{1}_{b=c} \left(1 - \frac{v_c(x)}{\bar{v}(x)}\right)$, where $v_c(x) = \sum_d \tilde{H}_c^{d|C}(x)$ and $\bar{v}(x) = \max_d v_d(x)$, this implies that $H^C(x)_{b,c}$ is bounded above and piecewise Lipschitz continuous in x , and therefore so is $A(x)$. Finally, since $H^C(x)$ is an irreducible row stochastic matrix for each $x \in D$, it follows that $A(x)$ is full rank and continuous. This is because when $H^C(x)$ is irreducible, $H^C(x) - I_C$ has strictly negative diagonal entries and weakly positive off-diagonal entries, and every choice of $n - 1$ columns of $H^C(x) - I_C$ gives an independent set whose span does not contain the all ones vector $\mathbf{1}_C$. Therefore if we let $A(x)$ be given by replacing the n th column in $H^C(x) - I_C$ with $\mathbf{1}_C$, then $A(x)$ has full rank.

Since $A(x)$ is full rank and continuous, in each piece $\det(A(x))$ is bounded away from 0, and so $A(x)^{-1}$ is piecewise Lipschitz continuous, as required. \square

B.1 Proof of Theorem 4

We prove the following slightly more general theorem.

Theorem 7. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy such that $H(x)$ is irreducible for all x . Then there exists a unique valid TTC path γ . Within each round $\gamma(\cdot)$ is given by*

$$\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$$

where $d(x)$ is the unique valid direction from $x = \gamma(t)$ that satisfies equations (4).

Moreover, if we let $A(x)$ be obtained from $H(x) - I$ by replacing the n th column with the all ones vector $\mathbf{1}$, then

$$d(x) = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}.$$

Proof. The main steps of the proof are as follows. We first show that $d(\cdot)$ is unique by solving equations (4) explicitly. This gives a closed form expression for $d(\cdot)$ which is Lipschitz continuous. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$

follows by invoking Picard-Lindelöf. Finally, we verify that the solution $\gamma(\cdot)$ is a valid TTC path.

Consider the equations (4),

$$\begin{aligned} d(x) H(x) &= d(x) \\ \|d(x)\|_1 &= 1. \end{aligned}$$

When $H(x)$ is irreducible, $H(x) - I$ has strictly negative diagonal entries and weakly positive off-diagonal entries, and every choice of $n - 1$ columns of $H(x) - I$ gives an independent set whose span does not contain $\mathbf{1}$. Therefore if we let $A(x)$ be given by replacing the n th column in $H(x) - I$ with $\mathbf{1}$, then $A(x)$ has full rank, and the equations (4) are equivalent to

$$\begin{aligned} d(x) A(x) &= \begin{bmatrix} \mathbf{0}^T & \mathbf{1} \end{bmatrix}, \\ \text{i.e. } d(x) &= \begin{bmatrix} \mathbf{0}^T & \mathbf{1} \end{bmatrix} A(x)^{-1}. \end{aligned}$$

Next, we invoke Lemma 3 to show that $d(x)$ is Lipschitz continuous. It follows from the Picard-Lindelöf theorem that there exists a unique function $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = d(\gamma(t))$. Since all valid TTC paths must satisfy the differential equation, it suffices to show that the unique solution $\gamma(\cdot)$ is a valid TTC path, that is, it satisfies the trade balance equations (1) and capacity equations (2). This is easily shown to be true by integrating over the marginal trade balance equations (4). \square

B.2 Proof of Theorem 5

Proof of Theorem 5. The main steps of the proof are as follows. We first show that if we take $\mathbf{d}(x)$ to be the valid direction from x with minimal support under the shortlex order, then $\mathbf{d}(\cdot)$ is piecewise Lipschitz continuous. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$ follows by invoking Picard-Lindelöf. Finally, we verify that the solution $\gamma(\cdot)$ is a valid TTC path.

Let C be the set of available schools. Fix a point x , and consider the set of vectors \mathbf{d} such that $\mathbf{d} \cdot H^C(x) = \mathbf{d}$. We invoke the following theorem, whose proof we defer to Section B.3.

Theorem 8. *Let C be the set of available schools, and let $\mathcal{K}(x)$ be the set of subsets $K \subseteq C$ for which $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C \setminus K}$ is the zero matrix. Then*

the equation $\mathbf{d} = \mathbf{d} \cdot H^C(x)$ has a unique solution \mathbf{d}^K that satisfies $\|\mathbf{d}^K\| = 1$ and $\text{supp}(\mathbf{d}^K) \subseteq K$. Moreover, if $\|\mathbf{d}\| = 1$ and \mathbf{d} is a solution to the equation $\mathbf{d} = \mathbf{d} \cdot H^C(x)$, then \mathbf{d} is a convex combination of the vectors in $\{\mathbf{d}^K\}_{K \in \mathcal{K}(x)}$.

It follows that if $\mathbf{d}(x)$ is the valid direction from x with minimal support under the shortlex order, then $\mathbf{d}(x) = \mathbf{d}^{K(x)}$ for the element $K(x) \in \mathcal{K}(x)$ that is the smallest under the shortlex ordering. As the density $\nu(\cdot)$ defining $\eta(\cdot)$ is Lipschitz continuous, it follows that $\mathcal{K}(\cdot)$ and $K(\cdot)$ are piecewise constant. Hence we may invoke Lemma 3 to conclude that $\mathbf{d}(\cdot)$ is piecewise Lipschitz within each piece, and hence piecewise Lipschitz in $[0, 1]^C$.

Since $\mathbf{d}(\cdot)$ is piecewise Lipschitz, it follows from the Picard-Lindelöf theorem that there exists a unique function $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$. Since all valid TTC paths must satisfy the differential equation, it suffices to show that the unique solution $\gamma(\cdot)$ is a valid TTC path, that is, it satisfies the trade balance equations (1) and capacity equations (2). This is easily shown to be true by integrating over the marginal trade balance equations (4), which are equivalent to $\mathbf{d}(x) \cdot H^C(x) = \mathbf{d}(x)$ for all x . \square

B.3 Proof of Proposition 1

Connection to Continuous Time Markov Chains & General Solutions for the Gradient

In Section 4.3, we showed that if a TTC path has gradient $\mathbf{d}(x)$ at x , then $\mathbf{d}(x)$ satisfies the equation $\mathbf{d}(x) \cdot H(x) = \mathbf{d}(x)$. We also appealed to a connection with Markov chain theory to provide a method for solving for all the possible values of $\mathbf{d}(x)$. Specifically, we showed, in Lemma 1, that if $\mathcal{K}(x)$ is the set of recurrent communication classes of $H(x)$, then the set of valid directions $\mathbf{d}(x)$ is identical to the set of convex combinations of $\{\mathbf{d}^K\}_{K \in \mathcal{K}(x)}$, where \mathbf{d}^K is the unique solution to equations (4) restricted to K . We present the relevant definitions, results and proofs here in full.

Let us first present some definitions from Markov chain theory.³⁷ A square matrix P is a *right-stochastic matrix* if all the entries are non-negative and each row sums to 1. A *probability vector* is a vector with non-negative entries that add up to 1. Given a

³⁷See standard texts such as Karlin and Taylor (1975) for a more complete treatment.

right-stochastic matrix P , the *Markov chain with transition matrix P* is the Markov chain with state space equal to the column/row indices of P , and a probability P_{ij} of moving to state j in one time step, given that we start in state i . Given two states i, j of a Markov chain with transition matrix P , we say that states i and j *communicate* if there is a positive probability of moving to state i to state j in finite time, and vice versa.

For each Markov chain, there exists a unique decomposition of the state space into a sequence of disjoint subsets C_1, C_2, \dots such that for all i, j , states i and j communicate if and only if they are in the same subset C_k for some k . Each subset C_k is called a *communication class* of the Markov chain. A Markov chain is *irreducible* if it only has one communication class. A state i is *recurrent* if, starting at i and following the transition matrix P , the probability of returning to state i is 1. A communication class is recurrent if it contains a recurrent state.

The following proposition gives a characterization of the stationary distributions of a Markov chain.

Proposition 10. *Suppose that P is the transition matrix of a Markov chain. Let \mathcal{K} be the set of recurrent communication classes of the Markov chain with transition matrix P . Then for each recurrent communication class $K \in \mathcal{K}$, the equation $\pi = \pi P$ has a unique solution π^K such that $\|\pi^K\| = 1$ and $\text{supp}(\pi^K) \subseteq K$. Moreover, the support of π^K is equal to K . In addition, if $\|\pi\| = 1$ and π is a solution to the equation $\pi = \pi P$, then π is a convex combination of the vectors in $\{\pi^K\}_{K \in \mathcal{K}}$.*

We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor (1975)) for a proof of this result.

To make use of this proposition, define at each point x and for each set of schools C a Markov chain $M^C(x)$ with transition matrix $H^C(x)$. We will relate the valid directions $\mathbf{d}(x)$ to the recurrent communication classes of $M^C(x)$, where C is the set of available schools. We will need the following notation and definitions. Given a vector v indexed by \mathcal{C} , a matrix Q with rows and columns indexed by C and subsets $K, K' \subseteq C$ of the indices, we let v_K denote the restriction of v to the coordinates in K , and we let $Q_{K,K'}$ denote the restriction of Q to rows indexed by K and columns indexed by K' .

The following lemma characterizes the recurrent communication classes of the Markov chain $M^C(x)$ using the properties of the matrix $H^C(x)$, and can be found in any standard stochastic processes text.

Lemma 4. *Let C be the set of available school at a point x . Then a set $K \subseteq C$ is a recurrent communication class of the Markov chain $M^C(x)$ if and only if $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C \setminus K}$ is the zero matrix.*

Proposition 10 and Lemma 4 allow us to characterize the valid directions $d(x)$.

Theorem 9. *Let C be the set of available schools, and let $\mathcal{K}(x)$ be the set of subsets $K \subseteq C$ for which $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C \setminus K}$ is the zero matrix. Then the equation $\mathbf{d} = \mathbf{d} \cdot H^C(x)$ has a unique solution \mathbf{d}^K that satisfies $\|\mathbf{d}^K\| = 1$ and $\text{supp}(\mathbf{d}^K) \subseteq K$, and its projection onto its support K has the form*

$$(\mathbf{d}^K)_K = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A_K^C(x)^{-1},$$

where $A_K^C(x)$ is the matrix obtained by replacing the $(|K| - 1)$ th column of $H^C(x)_{K,K} - I_K$ with the all ones vector $\mathbf{1}_K$.

Moreover, if $\|\mathbf{d}\| = 1$ and d is a solution to the equation $\mathbf{d} = \mathbf{d} \cdot H^C(x)$, then \mathbf{d} is a convex combination of the vectors in $\{\mathbf{d}^K\}_{K \in \mathcal{K}(x)}$.

Proof. Proposition 4 shows that the sets K are precisely the recurrent sets of the Markov chain with transition matrix $H(x)$. Hence uniqueness of the d^K and the fact that d is a convex combination of d^K follow directly from Proposition 10. The form of the solution d^K follows from Theorem 7. \square

This has the following interpretation. Suppose that there is a unique recurrent communication class K , such as when η has full support. Then there is a unique infinitesimal continuum trading cycle of students, specified by the unique direction \mathbf{d} satisfying $\mathbf{d} = \mathbf{d} \cdot H(x)$. Moreover, students in the cycle trade seats from every school in K . Any school not in K is blocked from participating, since there is not enough demand to fill the seats they are offering. When there are multiple recurrent communication classes, each of the \mathbf{d}^K gives a unique infinitesimal trading cycle of students, corresponding to those who trade seats in K . Moreover, these trading cycles are disjoint. Hence the only multiplicity that remains is to decide the order, or the relative rate, at which to clear these cycles. We will show in the next section that, as in the discrete setting, the order in which cycles are cleared does not affect the final allocation.

Proof of Uniqueness

In this section, we prove part (ii) of Proposition 1, that any two valid TTC paths give the same allocation. In other words, we show that the TTC allocation is unique.

The intuition for the result is the following. The connection to Markov chains shows that having multiple possible valid direction in the continuum is parallel to having multiple possible trade cycles in the discrete model. Hence the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or to implement them in parallel at various relative rates. We can show that the set of cycles is independent of the order in which cycles are selected, or equivalently that the sets of students who trade with each other is independent of the order in which possible trades are executed. It follows that any pair of valid TTC paths give the same final allocation.

We remark that the crux of the argument is similar to the argument used to show that discrete TTC gives a unique allocation. However, the lack of discrete cycles and the ability to implement sets of trades in parallel both complicate the argument and lead to a rather technical proof.

The formal proof proceeds in a number of steps. We first produce a rectangular subdivision \mathcal{R}' of the space $[0, 1]^C$ such that the sets of schools that are involved in trading cycles is constant on each rectangle $R \in \mathcal{R}'$. We then formally define *cycles* in the continuum setting, and define a partial order over the cycles corresponding to the order in which cycles can be cleared under TTC. We then define the set of cycles $\Sigma(\gamma)$ associated with a valid TTC path γ . Finally, we show that the sets of cycles associated with two valid TTC paths γ and γ' are the same, $\Sigma(\gamma) = \Sigma(\gamma')$. This last step is the most involved, and hence will be presented in a number of steps.

Recall that

$$\tilde{H}_b^{c|C}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^\theta \in [x - \varepsilon \cdot e^b, x) \text{ and } Ch_\theta(C) = c\})$$

is the marginal density of students pointed to by school b at the point x whose top choice school in C is c , and $H^C(x)$ is the $|C| \times |C|$ matrix with (b, c) th entry

$$H^C(x)_{b,c} = \frac{1}{\bar{v}} \tilde{H}_b^{c|C}(x) - \mathbf{1}_{b=c} \left(1 - \frac{v_c}{\bar{v}}\right),$$

where $v_c = \sum_{b \in C} \tilde{H}_c^{b|C}(x)$ is the row sum of $\tilde{H}(x)$, and $\bar{v} = \max_c v_c$ is the maximum

row sum.

We begin with some observations about $\tilde{H}_c^{b|C}(\cdot)$ and $H^C(\cdot)_{bc}$. For all $b, c \in C$ the function $\tilde{H}_c^{b|C}(\cdot)$ is right-continuous on $[0, 1]^C$, Lipschitz continuous on R for all $R \in \mathcal{R}$ and uniformly bounded away from zero on its support. Hence $\mathbf{1}(\tilde{H}_c^{b|C}(\cdot))$ is constant on R for all $R \in \mathcal{R}$. It follows that $H^C(\cdot)_{bc}$ is also right-continuous, and Lipschitz continuous on R for all $R \in \mathcal{R}$. Moreover, there exists some finite rectangular subpartition \mathcal{R}' of \mathcal{R} such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}(H^C(\cdot))$ is constant on R for all $R \in \mathcal{R}'$.

Definition 5. The partition \mathcal{R}' is the minimal rectangular subpartition of \mathcal{R} such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}(H^C(\cdot))$ is constant on R for all $R \in \mathcal{R}'$.

We now translate this to a result about recurrent communication classes of a matrix. Recall that for a square matrix Q with rows and columns indexed by C , and subsets $D, D' \subseteq C$ of the index set, $Q_{D, D'}$ denotes the restriction of Q to the rows indexed by D and columns indexed by D' .

For each $x \in [0, 1]^C$ and $C \subseteq \mathcal{C}$, let $\mathcal{K}^C(x)$ be the set of recurrent communication classes of the Markov chain $M^C(x)$. The following result is an immediate corollary of Proposition 4, since $\mathbf{1}(H^C(\cdot))$ is constant on R for every $R \in \mathcal{R}'$, and irreducibility of a matrix Q and whether it is a zero matrix depend only on $\mathbf{1}(Q)$.

Lemma 5. $\mathcal{K}^C(\cdot)$ is constant on R for every $R \in \mathcal{R}'$.

For each $K \in \mathcal{K}^C(x)$, let $d^K(x)$ be the unique vector satisfying $d = d\tilde{H}^C(x)$, which exists by Theorem 9.

We now move to defining formally the notion of a (non-infinitesimal) cycle in the continuum setting.

Definition 6. A (continuum) cycle $\sigma = (K, \underline{x}, \bar{x})$ is a set $K \subseteq \mathcal{C}$ and a pair of vectors $\underline{x} \leq \bar{x}$ in $[0, 1]^C$. A continuum cycle is *valid* for sets of available schools $\{C(x)\}_{x \in [0, 1]^C}$ if $K \in \mathcal{K}^{C(x)}(x)$ for all $\underline{x} < x \leq \bar{x}$.

Intuitively, a cycle is defined by two time points in a run of TTC, which gives a set of students,³⁸ and the set of schools they most desire. A cycle is valid if the set of schools involved is a recurrent communication class of the associated Markov chains.

³⁸The set of students is given by taking the difference between two nested hyperrectangles, one with upper coordinate \bar{x} and the other with upper coordinate \underline{x} .

We remark that a cycle is valid only with respect to sets of available schools, where we specify a set of available schools at each point x satisfying $\underline{x} < x \leq \bar{x}$.

We say that a cycle $\sigma = (K, \underline{x}, \bar{x})$ appears at time t in a run of TTC with path γ if $K \in \mathcal{K}^{C(\gamma(t))}(\gamma(t))$ and $\gamma_c(t) = \bar{x}_c$ for all $c \in K$, that is, if the set of schools in K is a recurrent communication class of the Markov chain at $\gamma(t)$, and the same students are at the top of the remaining priority list of each school in K under σ and under $TTC(\gamma)$ at time t .

We define a partial order over continuum cycles that specifies when one cycle must clear before another. To develop some intuition for why such an ordering exists, consider an instance of TTC in the discrete setting with one school c with capacity 2 and two students s_1, s_2 . Consider the cycles $\sigma_i = c \rightarrow s_i \rightarrow c$ for $i = 1, 2$. If school c prefers student s_1 to student s_2 , and both cycles σ_1, σ_2 clear, then σ_1 must clear before σ_2 . In general, some cycles must clear before other cycles, since they involve higher priority students or schools. We extend this idea to the continuum setting as follows.

Definition 7. Let θ be a student type and $\sigma = (K, \underline{x}, \bar{x})$, $\sigma' = (K', \underline{x}', \bar{x}')$ be cycles. We say that a student θ is in cycle σ if $r^\theta \in (\underline{x}, \bar{x}]$ ³⁹, and a school c is in cycle σ if $c \in K$.

The cycle σ blocks the cycle σ' , denoted by $\sigma \triangleright \sigma'$, if at least one of the following hold:

(Blocking student) There exists a student θ in σ' who prefers a school in K to all those in K' , that is, there exists θ and $c \in K \setminus K'$ such that $c \succ^\theta c'$ for all $c' \in K'$.

(Blocking school) There exists a school in σ' that prefers a positive measure of students in σ to all those in σ' , that is, there exists $c \in K'$ such that $\eta(\theta | \theta \text{ in } \sigma, r_c^\theta > \bar{x}'_c) > 0$.⁴⁰

Let us now define the set of cycles associated with a run of TTC. Intuitively, an infinitesimal cycle is a minimal set of students that trades their seats at a given time, and a cycle is given by aggregating these infinitesimal cycles over some period of time. We make this formal below.

Let γ be a TTC path with run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_\ell$. We first define the set of times that we aggregate over to form cycles, and then formally define the cycles.

³⁹Recall that since r^θ, \underline{x} and \bar{x} are vectors, this is equivalent to saying that $r^\theta \not\leq \underline{x}$ and $r^\theta \leq \bar{x}$.

⁴⁰We note that it is necessary but not sufficient that $\bar{x}_c > \bar{x}'_c$.

For each set of schools $K \subseteq \mathcal{C}$ and each round ℓ , let $\tau^{(\ell)}(K, \gamma)$ be the set of times in $[t^{(\ell-1)}, t^{(\ell)})$ when K is a recurrent communication class for $\tilde{H}^{\mathcal{C}^{(\ell)}}(\gamma(t))$. Since γ is continuous and weakly decreasing, it follows that $\tau^{(\ell)}(K, \gamma)$ is the finite disjoint union of intervals of the form $[\underline{t}, \bar{t}]$. Let $\mathcal{I}(\tau^{(\ell)}(K, \gamma))$ denote the set of intervals in this disjoint union. We may assume that for each interval τ , $\gamma(\tau)$ is fully contained in some hyperrectangle $R \in \mathcal{R}'$.⁴¹ Intuitively, each cycle in the TTC path γ will correspond to some time interval $\tau \in \mathcal{I}(\tau^{(\ell)}(K, \gamma))$, and will be the set of students that trade their seats in K .

Consider a time interval $\tau = [\underline{t}, \bar{t}] \in \mathcal{I}(\tau^{(\ell)}(K, \gamma))$. We define the cycle $\sigma(\tau) = (K, \underline{x}(\tau), \bar{x}(\tau))$ as follows. Intuitively, we want to define it simply as $\sigma(\tau) = (K, \gamma(\underline{t}), \gamma(\bar{t}))$, but in order to minimize the dependence on γ , we define the endpoints $\underline{x}(\tau)$ and $\bar{x}(\tau)$ of the interval of ranks to be as close together as possible, while still describing the same set of students (up to a set of η -measure 0). Formally, consider the set

$$\cup_{c \in K} \mathcal{T}^c(\gamma; \bar{t}) \setminus \mathcal{T}^c(\gamma; \underline{t})$$

of students who are assigned in round ℓ during the time interval τ and whose top choice available school is in K . Define

$$\begin{aligned} \underline{x}(\tau) &= \max \{x : \gamma(\underline{t}) \leq x \leq \gamma(\bar{t}), \eta(\theta : Ch_\theta(\mathcal{C}^{(\ell)}) \in K, r^\theta \in (\underline{x}(\tau), \gamma(\bar{t}))) = 0\}, \\ \bar{x}(\tau) &= \min \{x : \gamma(\underline{t}) \leq x \leq \gamma(\bar{t}) : \eta(\theta : Ch_\theta(\mathcal{C}^{(\ell)}) \in K, r^\theta \in (\gamma(\underline{t}), \underline{x}(\tau))) = 0\}, \end{aligned}$$

to be the points chosen to be maximal and minimal respectively such that the set of students allocated by γ during the time interval τ has the same η -measure as if $\gamma(\underline{t}) = \underline{x}(\tau)$ and $\gamma(\bar{t}) = \bar{x}(\tau)$. In other words, the set

$$(\cup_{c \in K} \mathcal{T}^c(\gamma; \bar{t}) \setminus \mathcal{T}^c(\gamma; \underline{t})) \setminus \{\theta : Ch_\theta(\mathcal{C}^{(\ell)}) \in K, r^\theta \in (\underline{x}(\tau), \bar{x}(\tau))\}$$

has η -measure 0.

In a slight abuse of notation, if $\sigma = \sigma(\tau)$ we will let $\underline{x}(\sigma)$ denote $\underline{x}(\tau)$ and $\bar{x}(\sigma)$ denote $\bar{x}(\tau)$.

Definition 8. The set of cycles cleared by TTC (γ) in round ℓ , denoted by $\Sigma^{(\ell)}(\gamma)$,

⁴¹This is without loss of generality, since if $\gamma(\tau)$ is not contained we can simply partition τ into a finite number of intervals $\cup_{R \in \mathcal{R}'} \gamma^{-1}(\gamma(\tau) \cap R)$, each contained in a hyperrectangle in \mathcal{R}' .

is given by

$$\Sigma^{(\ell)}(\gamma) := \bigcup_{K \subseteq \mathcal{C}^{(\ell)}} \bigcup_{\tau \in \mathcal{I}(\tau^{(\ell)}(K, \gamma))} \sigma(\tau).$$

The set of cycles cleared by $TTC(\gamma)$, denoted by $\Sigma(\gamma)$, is the set of cycles cleared by $TTC(\gamma)$ in some round ℓ ,

$$\Sigma(\gamma) := \bigcup_{\ell} \Sigma^{(\ell)}(\gamma).$$

For any cycle $\sigma \in \Sigma(\gamma)$ and time t we say that the cycle σ is *clearing at time t* if $\gamma(t) \not\leq \underline{x}(\sigma)$ and $\gamma(t) \not\geq \bar{x}(\sigma)$. We say that the cycle σ is *cleared at time t* or *finishes clearing at time t* if $\gamma^{(\ell)}(t) \leq \underline{x}(\sigma)$ with at least one equality. We remark that for any TTC path γ there may be multiple cycles clearing at a time t , each corresponding to a different recurrent set. For any TTC path γ the set $\Sigma(\gamma)$ is finite.

We first show that, given a TTC path γ , we can define available sets $C(x)$ at each point $x \in [0, 1]^{\mathcal{C}}$ that are consistent with the available sets along the TTC path γ , such that every cycle $\sigma \in \Sigma(\gamma)$ is a valid cycle with respect to these available sets. We note that this is a non-trivial exercise, since a run of the TTC algorithm gives available sets only at points x on the TTC path γ , and we are defining available sets for all $x \in [0, 1]^{\mathcal{C}}$.

Lemma 6. *Let γ be a TTC path with run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_{\ell}$, and let $\Sigma(\gamma)$ be the set of cycles cleared by $TTC(\gamma)$. For each $x \in [0, 1]^{\mathcal{C}}$, let $t(x) = \max\{t : \gamma(t) \geq x\}$, and let $C(x) = C^{(\ell)}$ if and only if $t(x) \in [t^{(\ell-1)}, t^{(\ell)}]$. Then*

- $C(\gamma(t))$ is the set of available schools at time t for all $t \in [0, t^{(|\mathcal{C}|)}]$, that is, $\forall \ell, t$ s.t. $t \in [t^{(\ell-1)}, t^{(\ell)}]$, $C(\gamma(t)) = \mathcal{C}^{(\ell)}$; and
- every $\sigma \in TTC(\gamma)$ is a valid cycle for the sets of available schools $C(x)$.

Proof. The first claim holds almost trivially, since if $x = \gamma(t)$ with $t \in [t^{(\ell-1)}, t^{(\ell)}]$, then $t(x) = t$ and so $C(\gamma(t)) = \mathcal{C}^{(\ell)}$ by definition. It remains to show that every $\sigma \in TTC(\gamma)$ is a valid cycle for the sets of available schools $C(x)$.

Fix x such that $\underline{x} \leq x \leq \bar{x}$, and let $y \in \gamma([0, 1])$ be the point such that $y = \gamma(t(x))$. Then, by definition, $C(x) = C(y) = C$. Moreover, y is in the image of γ , and since $\underline{x} \leq x, y \leq \bar{x}$ and the cycle σ was clearing at time t it holds that K is a

recurrent communication class of $M^C(y)$. By Lemma 4 it follows that $H^C(y)_{K,K}$ is irreducible and $H^C(y)_{K,C \setminus K}$ is the zero matrix.

We want to prove that K is a recurrent communication class of $M^C(x)$. By Lemma 4 it suffices to show that $H^C(x)$ is irreducible and $H^C(x)$ is the zero matrix. Now since $x, y \in R$ for some rectangle $R \in \mathcal{R}'$, it follows from the definition of \mathcal{R}' that $\mathbf{1}(H^C(x)) = \mathbf{1}(H^C(y))$. Since, for a given matrix A , irreducibility and being the zero matrix are properties that can be identified using the matrix $\mathbf{1}(A)$, it follows that K is a recurrent communication class of $M^C(x)$. \square

Fix two TTC paths γ and γ' . Our goal is to show that they clear the same sets of cycles, $\Sigma(\gamma) = \Sigma(\gamma')$, or equivalently that $\Sigma(\gamma) \cup \Sigma(\gamma') = \Sigma(\gamma) \cap \Sigma(\gamma')$. We will do this by showing that for every cycle $\sigma \in \Sigma(\gamma) \cup \Sigma(\gamma')$, if all cycles in $\Sigma(\gamma) \cup \Sigma(\gamma')$ that block σ are in $\Sigma(\gamma) \cap \Sigma(\gamma')$, then $\sigma \in \Sigma(\gamma) \cap \Sigma(\gamma')$.

We first show that this is true in a special case, which can be understood intuitively as the case when the cycle σ appears during the run of $TTC(\gamma)$ and also appears during the run of $TTC(\gamma')$. In terms of the continuum model, if $\sigma = (K, \underline{x}, \bar{x})$, then having σ appear in $TTC(\gamma)$ corresponds to γ passing through \underline{x} and K being a recurrent communication class of the Markov chain at \underline{x} , and having σ appear in $TTC(\gamma')$ corresponds to γ' passing through some point \underline{x}' with $\underline{x}_K = \underline{x}'_K$ and K being a recurrent communication class of the Markov chain at \underline{x}' . We make these ideas formal in the following lemma.

Lemma 7. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy, and let γ and γ' be two TTC paths for this economy. Let $K \subseteq \mathcal{C}$ and time \underline{t} be such that at time \underline{t} , γ has available schools C , γ' has available schools C' , the paths are at the same point when projected onto the coordinates K , $\gamma(\underline{t})_K = \gamma'(\underline{t})_K$, and K is a recurrent communication class of $M^C(\gamma(\underline{t}))$ and of $M^{C'}(\gamma'(\underline{t}))$. Suppose that for all schools $c \in K$ and cycles $\sigma' \triangleright \sigma$ involving school c , if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. Suppose also that cycle $\sigma = (K, \underline{x}, \bar{x})$ is cleared in $TTC(\gamma)$, where $\gamma(\underline{t}) = \underline{x}$, but at most measure 0 of σ has been cleared by time t in $TTC(\gamma')$. Then σ is also cleared in $TTC(\gamma')$.*

Proof. We define the ‘interior’ of the cycle σ by $X = \{x : \underline{x}_c \leq x_c \leq \bar{x}_c \forall c \in K, x_{c'} \geq \underline{x}_{c'} \forall c' \notin K\}$. We first show that in either run of TTC, if a point on the TTC path is in the interior of the cycle (i.e. $x \in X$) then K is a recurrent communication class of the Markov chain for the set of available schools at that point. Precisely, if

$\gamma(u) \in X$ and the set of available schools at time u in $TTC(\gamma)$ is D , then K is a recurrent communication class of $M^D(\gamma(u))$, and similarly if $\gamma'(u) \in X$ and the set of available schools at time u in $TTC(\gamma)$ is D' , then K is a recurrent communication class of $M^{D'}(\gamma'(u))$. The former claim follows from the fact that σ is cleared in $TTC(\gamma)$, $\sigma \in \Sigma(\gamma)$. It remains to show that the latter is true.

In order to show that K is a recurrent communication class with a given set of available schools D' at a point x , by Lemma 4 it suffices to show that $H^{D'}(x)_{K,K}$ is irreducible and $H^{D'}(x)_{K,D'\setminus K}$ is the zero matrix. We will use the fact that K is a recurrent communication class of $M^{C'}(\gamma'(u))$. Since $\gamma'(t)$ and $\gamma'(u)$ are both in the same rectangle R for some $R \in \mathcal{R}'$, it holds that $\mathbf{1}(H^{C'}(\gamma'(t))) = \mathbf{1}(H^{C'}(\gamma'(u)))$.

We first examine the difference between C' and D' , and the resulting differences between $H^{C'}(\gamma'(t))$ and $H^{D'}(\gamma'(u))$. As $TTC(\gamma')$ progresses from $\gamma'(t)$ to $\gamma'(u)$, some schools clear, changing the set of available schools. If $C' = D'$, it follows that $\mathbf{1}(H^{C'}(\gamma'(t))) = \mathbf{1}(H^{C'}(\gamma'(u))) = \mathbf{1}(H^{D'}(\gamma'(u)))$. If $C' \neq D'$ then $C' \supsetneq D'$. The matrices $H^{C'}(\gamma'(u))_{K,C'\setminus K}$ and $H^{D'}(\gamma'(u))_{K,D'\setminus K}$ are given by the measures of people pointed to by schools in K whose top choice school out of the available set C' (respectively D') is not in K . Since K is a recurrent communication class of $M^{C'}(\gamma'(u))$, it follows that $\mathbf{1}(H^{C'}(\gamma'(u))_{K,C'\setminus K}) = 0$, so for all students their top choice out of C' is in K . This means that their top choice out of D' is also in K , and so $\mathbf{1}(H^{D'}(\gamma'(u))_{K,D'\setminus K}) = 0$. The matrices $H^{C'}(\gamma'(u))_{K,K}$ and $H^{D'}(\gamma'(u))_{K,K}$ are given by the measures of people pointed to by schools in K whose top choice school out of the available set C' (respectively D') is in K . Since $\mathbf{1}(H^{C'}(\gamma'(u))_{K,C'\setminus K}) = 0$, it follows that all students' top choice out of C' is in K , so all students' top choice schools are the same irrespective of whether C' or D' is the set of available schools. Hence $H^{C'}(\gamma'(u))_{K,K} = H^{D'}(\gamma'(u))_{K,K}$ and both matrices are irreducible. Hence, whether $C' = D'$ or $C' \supsetneq D'$, it holds that $H^{D'}(\gamma'(u))_{K,K}$ is irreducible and $H^{D'}(\gamma'(u))_{K,D'\setminus K}$ is the zero matrix, and so K is a recurrent communication class of $M^{D'}(\gamma'(u))$.

We now invoke Theorem 9 to show that in each of the two paths, all the students in the cycle σ clear with each other. In other words, there exists a time t such that $\gamma(t) = \bar{x}_c \forall c \in K$, and similarly there exists a time t' such that $\gamma'(t')_c = \bar{x}'_c \forall c \in K$.

The argument is as follows. While the path γ is in the 'interior' of the cycle, that is $\gamma(t) \in X$, it follows from Theorem 9 that the projection of the gradient of γ to K is a rescaling of some vector $d^K(\gamma(t))$, where $d^K(\cdot)$ depends on $H(\cdot)$ but not on γ . Similarly, while $\gamma'(t') \in X$, it holds that the projection of the gradient of γ'

to K is a rescaling of the vector $d^K(\gamma'(t'))$, for the same function $d^K(\cdot)$. Hence if we take the section of γ in the ‘interior’ and project it to the coordinates in K , and similarly take the section of γ' in the ‘interior’ and project it to the coordinates in K , then the two projections are identical. In other words, if $\pi_K(x)$ is the projection of a vector x to the coordinates indexed by schools in K , then $\pi_K(\gamma(\gamma^{-1}(\underline{x}, \bar{x}))) = \pi_K(\gamma'(\gamma'^{-1}(\underline{x}, \bar{x})))$.

Let $\tilde{\gamma} := \pi_K(\gamma(\gamma^{-1}(\underline{x}, \bar{x})))$ and $\tilde{\gamma}' = \pi_K(\gamma'(\gamma'^{-1}(\underline{x}, \bar{x})))$ denote these projections respectively. Now K is a recurrent communication class at time t during $TTC(\gamma)$ for any $t \in \gamma^{-1}(\underline{x}, \bar{x}]$, and similarly at any time t' during $TTC(\gamma')$ for any $t' \in (\gamma')^{-1}(\underline{x}, \bar{x}]$. Suppose $\pi_K(\gamma(t)) = \pi_K(\gamma'(t'))$. Then this implies that for all schools $c \in K$, the same measure of students are assigned to K from time \underline{t} to t under $TTC(\gamma)$, as from time \underline{t} to t' under $TTC(\gamma')$.

Recall that we have assumed that for all schools $c \in K$ and cycles $\sigma' \triangleright \sigma$ involving school c , if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. This implies that for all $c \in K$, the measure of students assigned to c from time 0 to \underline{t} under $TTC(\gamma)$ is the same as the measure of students assigned to c from time 0 to \underline{t} under $TTC(\gamma')$.

Since $TTC(\gamma)$ clears σ the moment it exits the interior, this implies that $TTC(\gamma')$ also clears σ the moment it exits the interior. \square

We are now ready to prove that the TTC allocation is unique. As the proof takes several steps, we separate it into sections for readability.

Proof of uniqueness. Let γ and γ' be two TTC paths. Denote the sets of cycles associated with $TTC(\gamma)$ and $TTC(\gamma')$ respectively by $\Sigma = \Sigma(\gamma)$ and $\Sigma' = \Sigma(\gamma')$. Since the set of cycles of a TTC mechanism define the allocation up to a set of students of η -measure 0, it suffices to show that $\Sigma = \Sigma'$.

Let $\sigma = (K, \underline{x}, \bar{x})$ be a cycle in $\Sigma \cup \Sigma'$ such that the following assumption holds:

Assumption 2. *For all $\tilde{\sigma} \triangleright \sigma$ it holds that either $\tilde{\sigma}$ is in both Σ and Σ' or $\tilde{\sigma}$ is in neither.*

We show that if σ is in either Σ and Σ' , it is in both Σ and Σ' . Since Σ and Σ' are finite sets, this will be sufficient to show that $\Sigma = \Sigma'$. Without loss of generality we may assume that $\sigma \in \Sigma$.

We give here an intuitive overview of the proof. Let $\Sigma_{\triangleright\sigma} = \{\tilde{\sigma} \in \Sigma : \tilde{\sigma} \triangleright \sigma\}$ denote the set of cycles that are comparable to σ and cleared before σ in $TTC(\gamma)$. Assumption (2) about σ implies that $\Sigma_{\triangleright\sigma} \subseteq \Sigma'$. We will show that this implies that no students in σ start clearing under $TTC(\gamma')$ until all the students in σ have the same top available school in $TTC(\gamma')$ as when they clear in $TTC(\gamma)$, or in other words, that if some students in σ start clearing under $TTC(\gamma')$ at time t , then the cycle σ appears at time t . We will then show that once some of the students in σ start clearing under $TTC(\gamma')$ then all of them start clearing. It then follows from Lemma 7 that σ clears under both $TTC(\gamma)$ and $TTC(\gamma')$.

Let ℓ denote the round of $TTC(\gamma)$ in which σ is cleared. We define the times in $TTC(\gamma)$ and $TTC(\gamma')$ when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared, by

$$\begin{aligned}\bar{t}_{\triangleright\sigma} &= \min \left\{ t : \gamma(t) \leq (\tilde{x}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \tilde{x}, (\tilde{x}) \right) \in \Sigma_{\triangleright\sigma} \text{ and } H(\gamma(t)) \neq \mathbf{0} \right\}, \\ \bar{t}'_{\triangleright\sigma} &= \min \left\{ t : \gamma'(t) \leq (\tilde{x}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \tilde{x}, (\tilde{x}) \right) \in \Sigma_{\triangleright\sigma} \text{ and } H(\gamma'(t)) \neq \mathbf{0} \right\}.\end{aligned}$$

We define also the times in $TTC(\gamma)$ when σ starts to be cleared and finishes clearing,

$$\underline{t}_{\sigma} = \max \{ t : \gamma(t) \geq \bar{x} \}, \quad \bar{t}_{\sigma} = \min \{ t : \gamma(t) \leq \underline{x} \}$$

and the times in $TTC(\gamma')$ when students in σ start to be cleared and finish clearing,

$$\underline{t}'_{\sigma} = \max \{ t : \gamma'(t) \geq \bar{x} \}, \quad \bar{t}'_{\sigma} = \min \{ t : \gamma'(t) \leq \underline{t} \}.$$

Let C denote the set of available schools in $TTC(\gamma)$ at time \underline{t}_{σ} , and let C' denote the set of available schools in $TTC(\gamma')$ at time \underline{t}'_{σ} . We remark that part of the issue, carried over from the discrete setting, is that these times when $\Sigma_{\triangleright\sigma}$ stop clearing and σ starts clearing might not match up. In particular, other incomparable cycles could clear during, before or after these times. In the continuum model, there may also be sections on the TTC curve at which no school is pointing to a positive density of students. However, all the issues in the continuum case can be addressed using the intuition from the discrete case. We show that under $TTC(\gamma')$ all the students in σ eventually point to their top choice school in σ , and only a zero η -measure set of students in σ clear before this occurs.

We begin by showing that the cycle σ is appears in $TTC(\gamma)$ from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when σ starts to be cleared, and similarly for

$TTC(\gamma')$. We split the analysis of the cycle appearances into sections, first showing that this is true for schools under $TTC(\gamma)$ and $TTC(\gamma')$ respectively, and then for students.

We first show that in $TTC(\gamma)$, from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when σ starts to be cleared, the students pointed to by schools in K remain constant (up to a set of η -measure 0).

Claim 1. In $TTC(\gamma)$, let $\tilde{\Theta}$ denote the set of students cleared in time $[\bar{t}_{\triangleright\sigma}, \underline{t}_\sigma)$ who are preferred by some school in $c \in K$ to the students in σ , that is, θ satisfying $r_c^\theta > \bar{x}_c$. Then $\eta(\tilde{\Theta}) = 0$.

The idea is that if this set has positive measure, there must be a cycle $\tilde{\sigma}$ containing a positive η -measure of such students. Then $\tilde{\sigma}$ is comparable to σ and, by assumption (2), must be cleared before σ , contradicting that $\tilde{\sigma}$ is cleared after time $\bar{t}_{\triangleright\sigma}$. We present this argument formally below.

Suppose $\eta(\tilde{\Theta}) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma)$, there exists some cycle $\tilde{\sigma} = (\tilde{K}, \tilde{x}, (\tilde{x})) \in \Sigma(\gamma)$ containing a positive η -measure of students in $\tilde{\Theta}$. Then $\tilde{\sigma}$ is clearing at some time in $[\bar{t}_{\triangleright\sigma}, \underline{t}_\sigma)$, so $\tilde{\sigma} \neq \sigma$ by the definition of \underline{t}_σ . Moreover, since $\tilde{\sigma}$ contains a positive η -measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in [\bar{t}_{\triangleright\sigma}, \underline{t}_\sigma)$ and a school $c \in K$ for which $\tilde{x}_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq (\tilde{x})_c$. Hence

$$\tilde{x}_c \leq \gamma(\underline{t}_\sigma)_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \tilde{x}_c,$$

so $\tilde{\sigma} \triangleright \sigma$ and must be cleared before σ . But

$$(\tilde{x})_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \gamma(\bar{t}_{\triangleright\sigma})_c,$$

so $\tilde{\sigma}$ is not cleared before $\bar{t}_{\triangleright\sigma}$, contradicting the definition of $\bar{t}_{\triangleright\sigma}$.

We also show that in $TTC(\gamma')$, from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when some students in σ start to be cleared, the students pointed to by schools in K remain constant (up to a set of η -measure 0).

Claim 2. In $TTC(\gamma')$, let $\tilde{\Theta}$ denote the set of students cleared in time $[\bar{t}'_{\triangleright\sigma}, \underline{t}'_\sigma)$ who are preferred by some school in $c \in K$ to the students in σ , that is, θ satisfying $r_c^\theta > \bar{x}_c$. Then $\eta(\tilde{\Theta}) = 0$.

The idea is that if this set has positive measure, then there must be a cycle $\tilde{\sigma}$ containing a positive η -measure of such students. Any such cycle is smaller than σ

and in $\Sigma' \setminus \Sigma$, contradicting assumption (2) on σ . We present this argument formally below.

Suppose $\eta(\tilde{\Theta}) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma')$, there exists some cycle $\tilde{\sigma} = (\tilde{K}, \tilde{\underline{x}}, (\tilde{\bar{x}})) \in \Sigma(\gamma')$ containing a positive η -measure of students in $\tilde{\Theta}$. As in the proof of (1), $\tilde{\sigma}$ is clearing at some time in $[\tilde{t}'_{\triangleright\sigma}, \underline{t}'_{\sigma})$, so $\tilde{\sigma} \neq \sigma$ by the definition of \underline{t}'_{σ} . Moreover, since $\tilde{\sigma}$ contains a positive η -measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in [\tilde{t}'_{\triangleright\sigma}, \underline{t}'_{\sigma})$ for which $\tilde{\underline{x}}_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq (\tilde{\bar{x}})_c$. Hence

$$\tilde{\underline{x}}_c \leq \gamma'(\underline{t}'_{\sigma})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \tilde{\bar{x}}_c,$$

so $\tilde{\sigma} \triangleright \sigma$ and must be cleared before σ . Moreover,

$$(\tilde{\underline{x}})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \gamma(\tilde{t}'_{\triangleright\sigma})_c,$$

so it follows from the definition of $\tilde{t}'_{\triangleright\sigma}$ that $\tilde{\sigma} \notin \Sigma_{\triangleright\sigma}$.

Since we have assumed that $\tilde{\sigma} \in \Sigma'$, it follows that $\tilde{\sigma} \in \Sigma' \setminus \Sigma$, contradicting assumption (2) on σ .

We now show that in $TTC(\gamma)$ (and $TTC(\gamma')$), from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when some students in σ start to be cleared, the schools pointed to by students in σ remain constant. We do this by showing that any schools that students in σ prefer to their favorite school in K become unavailable before all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared.

Claim 3. Let $\sigma = (K, \underline{x}, \bar{x}) \in \Sigma$ satisfy Assumption 2. Suppose there is a school c that some student in σ prefers to all the schools in K . Then school c is unavailable in $TTC(\gamma)$ at any time $t \geq \bar{t}_{\triangleright\sigma}$, and unavailable in $TTC(\gamma')$ at any time $t \geq \tilde{t}'_{\triangleright\sigma}$.

The idea is that school c is unavailable after all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared, which is $\bar{t}_{\triangleright\sigma}$ in $TTC(\gamma)$ and $\tilde{t}'_{\triangleright\sigma}$ in $TTC(\gamma')$.

We know that c is unavailable at time \underline{t}_{σ} in $TTC(\gamma)$. Suppose that school c is available in $TTC(\gamma')$ after all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared. Then there exists some cycle $\tilde{\sigma}$ clearing in time $\tilde{t} \in (\bar{t}_{\triangleright\sigma}, \underline{t}_{\sigma})$ in $TTC(\gamma)$ involving school c . But this means that $\tilde{\sigma} \triangleright \sigma$ so $\tilde{\sigma} \in \Sigma_{\triangleright\sigma}$. Hence the measure of students in $\Sigma_{\triangleright\sigma}$ assigned to school c is q_c . The result follows from the definitions of $\bar{t}_{\triangleright\sigma}$ and $\tilde{t}'_{\triangleright\sigma}$ as the times when all cycles in $\Sigma_{\triangleright\sigma}$ are cleared.

Claims (1), (2) and (3) show that the set of students pointed to by schools in

K is constant up until the moment when students in the cycle σ start clearing. We now show that σ appears at the moment when students in the cycle σ start clearing under both $TTC(\gamma)$ and $TTC(\gamma')$. We formalize this in the continuum model by considering the coordinates of the path γ at the time \underline{t}_σ when the cycle σ starts clearing, and showing that, for all coordinates indexed by schools in K , this is equal to \bar{x} .

Claim 4. $\pi_K(\gamma(\underline{t}_\sigma)) = \pi_K(\bar{x})$.

The minimality of \bar{x} implies that $\gamma(\underline{t}_\sigma)_c \geq \bar{x}_c$ for all $c \in K$. Suppose that there exists some $c \in K$ such that $\gamma(\underline{t}_\sigma)_c > \bar{x}_c$. Since σ starts clearing at time \underline{t}_σ , for all $\varepsilon > 0$ school c must point to a non-zero measure in σ over the time period $[\underline{t}_\sigma, \underline{t}_\sigma + \varepsilon]$. The set of students that school c points to in this time is a subset of those with score r_c^θ satisfying $\gamma(\underline{t}_\sigma)_c \geq r_c^\theta \geq \gamma(\underline{t}_\sigma + \varepsilon)_c$, where continuity of $\gamma(\cdot)$ and the assumption that $\gamma(\underline{t}_\sigma)_c > \bar{x}_c$ implies that $\gamma(\underline{t}_\sigma + \varepsilon)_c > \bar{x}_c$ for sufficiently $\varepsilon > 0$. But the set of θ cleared in σ with $r_c^\theta > \bar{x}_c$ has η -measure 0, which is a contradiction.

We now show that the coordinates of the path γ' at time \underline{t}'_σ indexed by schools in K are equal to the corresponding coordinates of \bar{x} . In the discrete case, this part of the proof follows almost immediately from the fact that every student in σ is pointing to the same school under σ and $TTC(\gamma)$, and similarly every school in σ is pointing to the same student under σ and $TTC(\gamma)$ as shown in (4), and the fact that all cycles that come before σ under \triangleright are cleared in both $TTC(\gamma)$ and $TTC(\gamma')$, and so the students and schools in σ are pointing to the same schools and students under both $TTC(\gamma)$ and $TTC(\gamma')$. In the continuum, we will have to work a little harder to show that this is true, but the idea of the proof is the same.

Claim 5. $\pi_K(\gamma'(\underline{t}'_\sigma)) = \pi_K(\bar{x})$.

The minimality of \bar{x} implies that $\gamma'(\underline{t}'_\sigma)_c \geq \bar{x}_c = \gamma(\underline{t}_\sigma)_c$ for all $c \in K$. Since we cannot assume that σ is the cycle that is being cleared at time \underline{t}'_σ in $TTC(\gamma')$, the argument is more complicated than that for the previous claim and relies on the fact that K is a recurrent communication class in $TTC(\gamma)$, and that all cycles comparable to σ are already cleared in $TTC(\gamma')$. As already stated, the underlying concept is very simple in the discrete model, but is complicated in the continuum by the definition of the TTC path in terms of specific points, as opposed to measures of students, and the need to account for sets of students of η -measure 0. The idea will be to link the existence of positive measures of students pointed to by schools,

as measured by the entries of the matrix H , to the coordinates of $\gamma'(\underline{t}'_\sigma)$ and $\gamma(\underline{t}_\sigma)$.

Let $K_=$ be the set of coordinates in K at which equality holds, $\gamma'(\underline{t}'_\sigma)_c = \gamma(\underline{t}_\sigma)_c$, and let $K_>$ be the set of coordinates in K where strict inequality holds, $\gamma'(\underline{t}'_\sigma)_c > \gamma(\underline{t}_\sigma)_c$. If $K_>$ is empty then the claim holds and $\pi_K(\gamma'(\underline{t}'_\sigma)) = \pi_K(\bar{x})$. The rest of this proof will be dedicated to showing that $K_>$ is empty. The idea is the following. Under $TTC(\gamma')$, at time \underline{t}'_σ , every school in $K_=$ points to a non-zero measure of students who point to other schools in K , and some school in $K_=$ is involved in a cycle clearing at time \underline{t}'_σ . Moreover, since the two TTC paths have the same c -coordinates for all $c \in K_=$, if, at time \underline{t}_σ in $TTC(\gamma)$ a school $c \in K_=$ points to a non-zero measure of students whose top choice is in $K_>$, then the same is true at time \underline{t}'_σ in $TTC(\gamma')$. (This is the part of the argument that looks at the entries of the H matrices.) However, at time \underline{t}'_σ in $TTC(\gamma')$ every school in $K_>$ points to a zero measure of students, which contradicts the trade balance equations for cycles clearing at time \underline{t}'_σ .

Recall that C is the set of available schools in $TTC(\gamma)$ at time \underline{t}_σ , and C' is the set of available schools in $TTC(\gamma')$ at time \underline{t}'_σ . We prove formally the above results about $K_=$ and $K_>$. Note that if $c \in K_=$ and $c' \in K_>$, by assumption, $\gamma'(\underline{t}'_\sigma)_c = \gamma(\underline{t}_\sigma)_c = \bar{x}_c$ and $\gamma'(\underline{t}'_\sigma)_{c'} > \gamma(\underline{t}_\sigma)_{c'} = \bar{x}_{c'}$. Note also that since some students in σ are being cleared in $TTC(\gamma')$ at time \underline{t}'_σ , there exists a coordinate $c \in K$ for which equality holds, so $K_=$ is nonempty.

Claim 6. Suppose that $c' \in K_>$. Then there exists $\varepsilon > 0$ such that in $TTC(\gamma')$, the set of students pointed to by school c' in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ has η -measure 0.

The proof of (6) is as follows. Since $c' \in K_>$ it holds that $\gamma'(\underline{t}'_\sigma)_{c'} > \bar{x}_{c'}$, and since γ' is continuous, for sufficiently small ε it holds that $\gamma'(\underline{t}'_\sigma + \varepsilon)_{c'} > \bar{x}_{c'}$. Hence the set of students that school c' points to in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ is a subset of those with score $r_{c'}^\theta$ satisfying $\gamma'(\underline{t}'_\sigma)_{c'} \geq r_{c'}^\theta \geq \gamma'(\underline{t}'_\sigma + \varepsilon)_{c'} \bar{x}_{c'}$. Suppose $\tilde{\sigma}$ is a cycle clearing some of these students. Since $\tilde{\sigma}$ is cleared after $\Sigma_{\triangleright\sigma}$ and before σ , assumption (2) on σ implies that $\tilde{\sigma}$ is incomparable to σ . If $\tilde{\sigma}$ contains a positive measure of students pointed to by school c' , then $\tilde{\sigma}$ blocks σ via the blocking school c' . Hence all cycles clear at most measure 0 of the students that school c' points to in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$, and since there are a finite number of cycles this set has measure 0.

Suppose that $c \in K_=$. Then for all sufficiently small ε , the set of students pointed to by advancing the cutoff for school c by ε in $TTC(\gamma)$ contains all but a 0 η -measure

set of students pointed to by advancing the cutoff for school c by ε in $TTC(\gamma')$. That is,

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma))$$

has η -measure 0.

To prove this, we rely mostly on the fact that $\gamma(\underline{t}_\sigma) = \gamma'(\underline{t}'_\sigma) = \bar{x}_c$, so that the set $A = (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma))$ is a subset of the slice $x : \bar{x}_c - \varepsilon < x_c \leq \bar{x}_c$, and if $x \in A$ then $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$ for all $c' \in \mathcal{C}$. The intuition is that the set of students in A that in the cycle σ has η -measure 0, since $A \subseteq (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma))$, and the set of students in A that are not in the cycle σ has η -measure 0, since $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$.

Specifically, suppose $\varepsilon < \bar{x}_c - \underline{x}_c$. Then at most η -measure 0 of the students in

$$(\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma))$$

are not cleared by cycle σ . Hence at most η -measure 0 of the subset

$$(\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)) \setminus (\gamma'(\underline{t}'_\sigma) + \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma))$$

is not in the cycle σ .

We now consider the measure of the subset that is in cycle σ . Since $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$ for all $c' \in \mathcal{C}$, it follows that all students in A are cleared under $TTC(\gamma')$ by time \underline{t}'_σ , and by the definition of \underline{t}'_σ it follows that the set of students in A that are also in cycle σ has η -measure 0. Hence the η -measure of A is 0.

Claim 7. Suppose that $c \in K_=$, $b \in \mathcal{C}$ and $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$. Then for sufficiently small ε , the set of students in σ whose scores are in the set

$$(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma))$$

has $\eta^{b|C}$ -measure of $\Omega(\varepsilon)$.

The idea behind the proof is to consider the set of student in σ whose scores are in the set $(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma))$, show that this set of students has $\eta^{b|C}$ -measure of $\Omega(\varepsilon)$, and that it differs from the measure of the set that we want by a set of η -measure 0.

Since $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$, it follows that

$$\tilde{H}_c^b(x) \doteq \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^\theta \in (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)] \text{ and } Ch_\theta(C^{(\ell)}) = b\}) > 0$$

and hence

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$$

has $\eta^{b,C}$ -measure $\Omega(\varepsilon)$ for sufficiently small ε .

Moreover, at most η -measure 0 of the students in $(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$ are not in the cycle σ . Finally, $(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)] \supseteq (\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus A$, where

$$A = ((\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) + \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]).$$

Hence the $\eta^{b,C}$ -measure of students in σ who are in $(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)]$ is at least

$$\begin{aligned} & \Omega(\varepsilon) - \eta^{b,C}(A) \\ &= \Omega(\varepsilon) \text{ (by (B.3))} \end{aligned}$$

Claim 8. Suppose that $c \in K_-$, $b \in K$ and $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$. Then $H^{C'}(\gamma'(\underline{t}'_\sigma))_{cb} > 0$.

The proof is as follows. Since every $H^C(\gamma'(\underline{t}'_\sigma))_{cb}$ is a positive multiple of $\tilde{H}_c^{b|C}(\gamma'(\underline{t}'_\sigma))$, it suffices to show that

$$\tilde{H}_c^{b|C'}(\gamma'(\underline{t}'_\sigma)) > 0.$$

Since $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$, it follows from (7) that there is an $\eta^{b|C}$ -measure $\Omega(\varepsilon)$ of students in σ with ranks in

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$$

Suppose for the sake of contradiction that

$$\tilde{H}_c^{b|C'}(\gamma'(\underline{t}'_\sigma)) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^\theta \in (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)] \text{ and } Ch_\theta(C') = b\}) = 0.$$

Then for sufficiently small ε it holds that

$$\eta^{b|C'}((\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)]) = o(\varepsilon).$$

Hence there is an η -measure $\Omega(\varepsilon)$ of students in σ with ranks in $(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)]$ whose top choice school in C is b , but whose top choice school in C' is not b . Let one such student in σ be of type $\theta \in \Theta^{b|C} \setminus \Theta^{b|C'}$. Let school b' be such that the student chooses b' out of C' , that is, $\theta \in \Theta^{b'|C'}$. Since $b' \in C'$ it is available in $TTC(\gamma')$ at time \underline{t}'_σ , and since we have shown that no student in σ prefers a school in $C' \setminus K$ to all the schools in K it holds that $b' \in K$. Moreover, since $\theta \in \Theta^{b|C}$ by construction, it holds that θ prefers school b to all other schools in K , so $b = b'$. Finally, we have assumed that $\theta \notin \Theta^{b|C'}$, so $b \neq b'$. This gives the required contradiction.

We are now ready to prove (5). Recall that $K = K_{=} \cup K_{>}$, where $K_{=}$ is nonempty and it suffices to prove that $K_{>}$ is empty. Suppose for the sake of contradiction that $K_{>}$ is nonempty.

Consider the schools K' involved in a cycle at time \underline{t}'_σ in $TTC(\gamma')$. It follows from the definition of \underline{t}'_σ that $K' \cap K_{=}$ is nonempty. Moreover, if $c \in K' \cap K_{=}$ and $H^{C'}(\gamma'(\underline{t}'_\sigma))_{cc'} > 0$ then $c' \in K'$.

Let $c \in K' \cap K_{=}$. Since $K = K_{=} \cup K_{>}$ is a recurrent communication class of $H^{C'}(\gamma(\underline{t}_\sigma))$, it holds that there exists a chain $c = c_0 - c_1 - c_2 - \dots - c_n$ such that $H^{C'}(\gamma(\underline{t}_\sigma))_{c_i c_{i+1}} > 0$ for all $i < n$, $c_i \in K_{=}$ for all $i < n$, and $c_n \in K_{>}$. Since K' is a recurrent communication class, it follows that $c_i \in K'$ for all $i \leq n$. Hence c_n is involved in a cycle at time \underline{t}'_σ . But since $c_n \in K_{>}$, there exists $\varepsilon > 0$ such that in $TTC(\gamma')$, the set of students pointed to by school c_n in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ has η -measure 0, which is a contradiction. Hence we have shown that $\gamma'(\underline{t}'_\sigma)_K = \gamma(\underline{t}_\sigma)_K$.

We have shown in (5) that for our chosen $\sigma = (K, \underline{x}, \bar{x})$, it holds that $\gamma(\underline{t}_\sigma)_K = \gamma'(\underline{t}'_\sigma)_K = \underline{x}_K$. Invoking Lemma 7 shows that σ is cleared under both $TTC(\gamma)$ and $TTC(\gamma')$. Hence $\Sigma = \Sigma'$, as required. \square

B.4 Proof of Proposition 3

In this section, we show that given a discrete economy, the outcome of TTC in the continuum embedding gives the same assignment as TTC on the discrete model, $\mu_d = \hat{\mu}_d$.

The intuition behind this result is that TTC is essentially performing the same assignments in both models, with discrete TTC assigning students to schools in discrete steps, and continuum TTC assigning students to schools continuously, in

fractional amounts. Moreover, the structure of the embedding implies that continuum TTC assigns at most one student to a given school at any point in time, and does not begin assigning a new student until to a given school until one student has been completely assigned. Hence, by restricting continuum TTC to the discrete time steps when individual students are fully assigned, we obtain the same outcome as discrete TTC.

Proof. The formal proof is as follows. We construct a discrete cycle selection rule ψ and TTC path γ such that TTC on the discrete economy E with cycle selection rule ψ gives the same allocation as $TTC(\gamma)$. Since the assignment of discrete TTC is unique (Shapley and Scarf, 1974), and the assignment in the continuum model is unique (Proposition 1), this proves the theorem.

The discrete cycle selection rule ψ is defined as follows. At each step of discrete TTC, take all available cycles in the graph obtained by having students point to their favorite school, and schools to their favorite student.

The TTC path γ is defined as follows. We first define $\mathbf{d}(x)$. At each point x , let C be the set of available schools, let $K(x)$ be the set of all students in recurrent communication classes of $H(x)$, and let $d_c(x) = \frac{1}{|K(x)|}$ if $c \in K$ and 0 otherwise. Let X be the set of points x such that x_c is a multiple of $\frac{1}{N}$ for all $c \notin K(x)$. Note that each student s 's cube I^s has equal density, and the number of cubes intersecting any axis-parallel hyperplane is at most one, so for all $x \in X$ the entries of the matrix $H(x)$ are all 0's and $\frac{1}{N}$'s. We remark that this means that $N \times H(x)$ is the adjacency matrix of the pointing graph (where school b points to school c if some student pointed to by b wants c). Hence $K(x)$ and the schools desired by the students in $K(x)$ form a maximal union of cycles in the pointing graph when the set of available schools is C and the set of available students is those whose cubes contain part of $\{\theta : r^\theta \geq x\}$. It follows that $\mathbf{d}(x) = \mathbf{d}(x) \cdot H(x)$ for all $x \in X$.

Now consider the TTC path γ satisfying $\gamma'(t) = \mathbf{d}(\gamma(t))$. We show that $\gamma(t) \in X$ for all t . The path starts at $\gamma(0) = 0$. Moreover, at any time t , if $\gamma(t) \in X$ then the derivative of the TTC path is equal to $\mathbf{d}(\gamma(t))$, which points along the diagonal in the projection onto the coordinates K , and is 0 along all other coordinates. Hence $\gamma(t) \in X$ for all t .

Let t_1, t_2, \dots be the discrete set of times when a student s is first fully assigned, that is $\{t_i\} = \cup_s \{\inf \{t \mid \exists c \in C \text{ s.t. } \gamma_c(t) \leq r_c^\theta \forall \theta \in I^s\}\}$.

We note that for every two students s, s' and school c it holds that the projections

I_c^s and $I_c^{s'}$ of I^s and $I^{s'}$ onto the c th coordinates are non-overlapping, i.e. either for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta < r_c^{\theta'}$, or for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta > r_c^{\theta'}$. Since all the capacities are multiples of $\frac{1}{N}$, it holds that for all t_i , every student is either fully assigned or fully unassigned, i.e. $\forall s \in \mathcal{S} \exists c \in C$ s.t. $\gamma_c(t) \leq r_c^\theta \forall \theta \in I^s$ or $\gamma_c(t) \geq r_c^\theta \forall \theta \in I^s, c \in C$. Moreover, since all the capacities are multiples of $\frac{1}{N}$, it follows that schools fill at a subset of the set of times $\{t_i\}$.

In other words, we have shown that for every i , if S is the set students who are allocated a seat at time t_i , then $S \cup \mu(S)$ are the agents in the maximal union of cycles in the pointing graph at t_{i-1} . Hence γ finishes clearing the cubes corresponding to the same set of cycles at t_i as ψ does in step i . It also follows that every student s who is fully assigned is fully assigned to exactly one school, that is if $\mu(\theta) \neq \emptyset$ for some $\theta \in I^s$ then $\exists c$ s.t. $\forall \theta \in I^s \mu(\theta) = c$. Hence $\hat{\mu}_d = \mu_d$. \square

B.5 Proof of Theorem 6

To prove Theorem 6, we will want some way of comparing two TTC paths γ and $\tilde{\gamma}$ obtained under two continuum economies differing only in their measures η and $\tilde{\eta}$. Intuitively, we want to pick points on the paths such that there exists a school c where the number of seats offered by school c is less under γ than $\tilde{\gamma}$, but the number of students who are offered some seat and want school c is more under γ than $\tilde{\gamma}$. Since these are difficult to compare under different measures, we instead focus on the ranks of students who are offered seats by school c , and the ranks of students who are offered some seat and want school c . The two conditions that we want then correspond exactly with there being a school c and an interval τ in which, if we re-parametrize so that $\gamma_c(t) = \tilde{\gamma}_c(t)$ for all $t \in \tau$, then $\gamma_b(t) \leq \tilde{\gamma}_b(t)$ for all b and for all $t \in \tau$. We formally define this notion below.

Definition 9. Let γ and $\tilde{\gamma}$ be increasing continuous functions from $[0, 1]$ to $[0, 1]^C$ with $\gamma(0) = \tilde{\gamma}(0)$. Then $\gamma(t)$ *dominates* $\tilde{\gamma}(t)$ *via school* c if

$$\begin{aligned} \gamma_c(t) &= \tilde{\gamma}_c(t), \text{ and} \\ \gamma_b(t) &\leq \tilde{\gamma}_b(t) \text{ for all } b \in C. \end{aligned}$$

We remark that we (somewhat unintuitively) require $\gamma(\cdot) \leq \tilde{\gamma}(\cdot)$, since more students are offered seats under γ than γ' and higher ranks give more restrictive sets.

We also say that γ *dominates* $\tilde{\gamma}$ *via school* c *at time* t . If γ and γ' are TTC paths, we can interpret this as school c being more demanded under γ , since with the same rank at c , in γ students are competitive with more ranks at other schools b . In other words, high ranks at school c are a more valuable commodity under γ than under γ' .

We now show that any two non-increasing continuous paths γ, γ' starting and ending at the same point can be re-parametrized so that for all t there exists a school $c(t)$ such that γ dominates γ' via school $c(t)$ at time t . We first show that, if $\gamma(0) \leq \tilde{\gamma}(0)$, then there exists a re-parametrization of γ such that γ dominates $\tilde{\gamma}$ on some interval starting at 0.

Lemma 8. *Suppose $\gamma, \tilde{\gamma}$ are a pair of non-increasing functions $[0, 1] \rightarrow [0, 1]^c$ such that $\gamma(0) \leq \tilde{\gamma}(0)$. Then there exist coordinates c, b , a time \bar{t} and an increasing function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $\gamma_b(g(t)) = \tilde{\gamma}_b(t)$, and for all $t \in [0, \bar{t}]$ it holds that*

$$\gamma_c(g(t)) = \tilde{\gamma}_c(t) \text{ and } \gamma(g(t)) \leq \tilde{\gamma}(t).$$

That is, if we renormalize the time parameter t of $\gamma(t)$ so that γ and $\tilde{\gamma}$ agree along the c th coordinate, then γ dominates $\tilde{\gamma}$ via school c at all times $t \in [0, \bar{t}]$, and also dominates via school b at time \bar{t} .

Proof. The idea is that if we take the smallest function g such that $\gamma_c(g(t)) = \tilde{\gamma}_c(t)$ for some coordinate c and all t sufficiently small, then $\gamma(g(t)) \leq \tilde{\gamma}(t)$ for all t sufficiently small. The lemma then follows from continuity. We make this precise.

Fix a coordinate c . Let $g^{(c)}$ be the renormalization of γ so that γ and $\tilde{\gamma}$ agree along the c th coordinate, i.e. $\gamma_c(g^{(c)}(t)) = \tilde{\gamma}_c(t)$ for all $t \in [0, T]$.

For all t , we define the set $\kappa_{>}^{(c)}(t)$ of schools, or coordinates, along which the γ curve renormalized along coordinate c has larger value at time t than $\tilde{\gamma}$ has at time t , that is,

$$\kappa_{>}^{(c)}(t) = \{b \mid \gamma_b(g^{(c)}(t)) > \tilde{\gamma}_b(t)\},$$

and similarly define the sets $\kappa_{<}^{(i)}(t)$ and $\kappa_{=}^{(i)}(t)$ where the renormalized γ curve is smaller than $\tilde{\gamma}$ and equal to $\tilde{\gamma}$ respectively,

$$\begin{aligned} \kappa_{<}^{(c)}(t) &= \{b \mid \gamma_b(g^{(c)}(t)) < \tilde{\gamma}_b(t)\}, \\ \kappa_{=}^{(c)}(t) &= \{b \mid \gamma_b(g^{(c)}(t)) = \tilde{\gamma}_b(t)\}. \end{aligned}$$

Since γ and $\tilde{\gamma}$ are continuous, there exists some $\bar{t}^{(c)} > 0$ such that the functions $\kappa_{>}^{(c)}(\cdot)$, $\kappa_{<}^{(c)}(\cdot)$ and $\kappa_{\leq}^{(c)}(\cdot)$ are constant over the interval $(0, \bar{t}^{(c)})$. Let $C_{>}^{(c)} = \kappa_{>}^{(c)}(t)$ for all $t \in (0, \bar{t}^{(c)})$ and similar define $C_{<}^{(c)}$ and $C_{\leq}^{(c)}$.

If $\gamma(g^{(c)}(t)) = \tilde{\gamma}(t)$ in the interval $(0, \bar{t}^{(c)})$, then by continuity we may take $\bar{t} = \bar{t}^{(c)}$ and b to be any other coordinate.

Hence we may assume that for all c , $\gamma(g^{(c)}(t)) \neq \tilde{\gamma}(t)$ in the interval $(0, \bar{t}^{(c)})$, so at least one of $C_{>}^{(c)}$ and $C_{<}^{(c)}$ is nonempty for all c . Let $\bar{t} = \min_{c \in \mathcal{C}} \bar{t}^{(c)}$.

Suppose that schools c and b satisfy that $b \in C_{>}^{(c)}$. We claim that $g^{(b)}(t) > g^{(c)}(t)$ for all $t \in (0, \bar{t})$. This is because γ is increasing and

$$\begin{aligned} \gamma_b(g^{(b)}(t)) &= \tilde{\gamma}_b(t) \text{ (for all } t \text{ by the definition of } g^{(b)}) \\ &> \gamma_b(g^{(c)}(t)) \text{ (since } b \in C_{>}^{(c)}, \text{ and hence } b \in \kappa_{>}^{(c)}(t) \text{ for } t \in (0, \bar{t}^{(c)}). \end{aligned}$$

Suppose that $C_{>}^{(c)} \neq \emptyset$ for all $c \in \mathcal{C}$. Then for all c , there exists b such that $g^{(b)}(t) > g^{(c)}(t)$ for all $t \in (0, \bar{t})$, which is impossible since there are a finite number of elements $c \in \mathcal{C}$ and hence a finite number of values $g^{(c)}(t)$. Hence $C_{>}^{(c)} = \emptyset$ for some coordinate c .

Let c be a coordinate for which $C_{>}^{(c)} = \emptyset$. Then $C_{<}^{(c)}$ is nonempty, and $\kappa_{<}^{(c)}(\bar{t}^{(c)}) \neq C_{<}^{(c)}$. Hence by continuity there exists $b \in C_{<}^{(c)}$, such that $b \in \kappa_{\leq}^{(c)}(\bar{t}^{(c)})$, and the coordinates c, b , time $\bar{t}^{(c)}$ and function $g^{(c)}$ satisfy the required conditions. \square

We are now ready to show that there exists a re-parametrization of γ such that γ always dominates $\tilde{\gamma}$ via some school.

Lemma 9. *Suppose $\gamma, \tilde{\gamma}$ are a pair of non-increasing functions $[0, 1] \rightarrow [0, 1]^{\mathcal{C}}$ such that $\gamma(0) = \tilde{\gamma}(0) = 1$ and $\gamma(1) = \tilde{\gamma}(1) = 0$. Then there exists an increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $t \in [0, 1]$, there exists a school $c(t)$ such that $\gamma(g(t))$ dominates $\tilde{\gamma}(t)$ via school $c(t)$.*

Proof. Fix a coordinate c . Let $g^{(c)}$ be the renormalization γ so that γ and $\tilde{\gamma}$ agree along the c th coordinate, i.e. $\gamma_c(g^{(c)}(t)) = \tilde{\gamma}_c(t)$ for all $t \in [0, T]$. Let $A^{(c)}$ be the set of times t such that $\gamma(g^{(c)}(t))$ dominates $\tilde{\gamma}(t)$. By continuity, $A^{(c)}$ is closed. Consider the set $B^{(c)}$ which we define to be the closure of the interior of $A^{(c)}$. Notice that, since $A^{(c)}$ is closed, it contains $B^{(c)}$. Moreover, since the interior of $A^{(c)}$ is open, it is

a countable union of open intervals, and hence $B^{(c)}$ is a countable union of disjoint closed intervals. We show that $\cup_{c \in \mathcal{C}} B^{(c)} = [0, 1]$, which shows that $\cup_{c \in \mathcal{C}} A^{(c)} = [0, 1]$.

Suppose that $\cup_{c \in \mathcal{C}} B^{(c)} \neq [0, 1]$. Then there exists some school c and points $\underline{t} < \bar{t}$ such that $\gamma(g^{(c)}(\underline{t}))$ dominates $\tilde{\gamma}(\underline{t})$ via school c , and for all b there is no interval τ in $[\underline{t}, \bar{t}]$ such that $\gamma(g^{(b)}(t))$ dominates $\tilde{\gamma}(t)$ via school b for all $t \in \tau$. But this contradicts Lemma 8.

We now construct a function g that satisfies the required properties as follows. Write $[0, 1] = \cup_n \{\tau_n\}$ as a countable union of closed intervals such that any pair of intervals intersects at most at their endpoints, and each interval τ_n is a subset of $B^{(c)}$ for some c . For each τ_n fix some $c = c(n)$ so that $\tau \subseteq B^{(c)}$.

We now define g . If $t \in \tau_n \subseteq B^{(i)}$, let $g(t) = g^{(c)}(t)$. Then by definition $\gamma(g(t))$ dominates $\tilde{\gamma}(t)$ via school $c(t) = c(n)$. Now g is defined on all of $[0, 1]$ since $\cup_{c \in \mathcal{C}} B^{(c)} = [0, 1]$. Moreover g is well-defined since if t is in two different intervals τ_n, τ_m , then domination via $c(n)$ implies that $\gamma(g^{(c(m))}(t)) \geq \tilde{\gamma}(t) = \gamma(g^{(c(n))}(t))$ and domination via $c(m)$ implies that $\gamma(g^{(c(n))}(t)) \geq \tilde{\gamma}(t) = \gamma(g^{(c(n))}(t))$, and so $\gamma(g^{(c(m))}(t)) = \gamma(g^{(c(n))}(t))$ and we can pick one value for g that satisfies all required properties. This completes the proof. \square

Consider two continuum economies $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ and $\tilde{\mathcal{E}} = (\mathcal{C}, \Theta, \tilde{\eta}, q)$, where the measures η and $\tilde{\eta}$ satisfy the assumptions given in Section (3), namely the given normalization, an excess of students, and piecewise Lipschitz continuity (Assumption 1). Suppose also that the measure η and $\tilde{\eta}$ have total variation distance ε and have full support. Let γ be a TTC path for economy \mathcal{E} with run-out sequence $\{(\mathcal{C}^{(\ell)}, t^{(\ell)})\}_\ell$, and let $\tilde{\gamma}$ be a TTC path for economy $\tilde{\mathcal{E}}$ with run-out sequence $\{(\tilde{\mathcal{C}}^{(\ell)}, \tilde{t}^{(\ell)})\}_\ell$. Consider any school c and any points x, \tilde{x} such that x is cleared in the first round of $TTC(\gamma)$, i.e. $\gamma^{-1}(t) \in [0, t^{(1)}]$, \tilde{x} is cleared in the first round of $TTC(\tilde{\gamma})$, i.e. $\tilde{\gamma}^{-1}(t) \in [0, \tilde{t}^{(1)}]$ and $x_c = \tilde{x}_c$. We show that the set of students allocated to school c when running $TTC(\gamma)$ up to x differs from the set of students allocated to school c when running $TTC(\tilde{\gamma})$ up to \tilde{x} by a set of measure $O(\varepsilon|\mathcal{C}|)$.

Proposition 11. *Suppose that $\gamma, \tilde{\gamma}$ are TTC paths in one round of the continuum economies \mathcal{E} and $\tilde{\mathcal{E}}$ respectively, where the set of available schools is the same in these rounds of $TTC(\gamma)$ and $TTC(\tilde{\gamma})$. Suppose also that γ and $\tilde{\gamma}$ end at x and \tilde{x} respectively, where $x_b = \tilde{x}_b$ for some $b \in \mathcal{C}$ and $x_c \leq \tilde{x}_c$ for all $c \in \mathcal{C}$. Then for all $c \in \mathcal{C}$, the set of students with ranks better than x under \mathcal{E} and ranks better than \tilde{x}*

under $\tilde{\mathcal{E}}$ who are assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has measure $O(\varepsilon |C|)$.⁴²

Proof. By Lemma 9, we may assume without loss of generality that γ and $\tilde{\gamma}$ are parametrized such that $x = \gamma(1)$, $\tilde{x} = \tilde{\gamma}(1)$ and for all times $t \leq 1$ there exists a school $c(t)$ such that $\gamma(t)$ dominates $\tilde{\gamma}(t)$ via school $c(t)$.

Let $\tau_c = \{t \leq 1 : c(t) = c\}$ be the times when γ dominates $\tilde{\gamma}$ via school c . We remark that, by our construction in Lemma 9, we may assume that τ_c is the countable union of disjoint closed intervals, and that if $c \neq c'$ then τ_c and $\tau_{c'}$ have disjoint interiors.

Let $\tau = [\underline{t}, \bar{t}]$ be an interval. Recall that $\mathcal{T}_c(\gamma; \tau) = \mathcal{T}_c(\gamma; \bar{t}) \setminus \mathcal{T}_c(\gamma; \underline{t})$ is the set of students who were offered a seat by school c at some time $t \in \tau$.

If $\tau = \cup_n \tau_n$ is a union of disjoint closed intervals, we define $\mathcal{T}_c(\gamma; \tau) = \mathcal{T}_c(\gamma; \tau_n)$ to be the set of students who were offered a seat by school c at some time $t \in \tau$, and $\mathcal{T}^{clC}(\tau, \gamma) = \cup_n \mathcal{T}^{clC}(\tau_n, \gamma)$ to be the set of students who were assigned to a school c at some time $t \in \tau$, given a set of available schools C . Since γ is a TTC path for \mathcal{E} and $\tilde{\gamma}$ is a TTC path for $\tilde{\mathcal{E}}$, the following trade balance equations hold,

$$\eta(\mathcal{T}_c(\gamma; \tau_c)) = \eta(\mathcal{T}^{clC}(\gamma; \tau_c)) \text{ for all } c \in C. \quad (6)$$

$$\tilde{\eta}(\mathcal{T}_c(\tilde{\gamma}; \tau_c)) = \tilde{\eta}(\mathcal{T}^{clC}(\tilde{\gamma}; \tau_c)) \text{ for all } c \in C. \quad (7)$$

Since γ dominates $\tilde{\gamma}$ via school c at all times $t \in \tau_c$, we have that

$$\mathcal{T}_c(\gamma; \tau_c) \subseteq \mathcal{T}_c(\tilde{\gamma}; \tau_c). \quad (8)$$

Moreover, by the choice of parametrization, $\cup_c \tau_c = [0, 1]$ and so, since $x \leq \tilde{x}$,

$$\cup_c \mathcal{T}(\gamma; \tau_c) \supseteq \cup_c \mathcal{T}(\tilde{\gamma}; \tau_c). \quad (9)$$

⁴²This is according to both measures η and $\tilde{\eta}$.

Hence

$$\begin{aligned}
& \eta(\mathcal{T}^{c|C}(\gamma; \tau_c) \setminus \mathcal{T}^{c|C}(\tilde{\gamma}; \tau_c)) \\
&= \eta(\mathcal{T}^{c|C}(\gamma; \tau_c)) - \eta(\mathcal{T}^{c|C}(\tilde{\gamma}; \tau_c)) \quad (\text{by (9)}) \\
&\leq \eta(\mathcal{T}^{c|C}(\gamma; \tau_c)) - \tilde{\eta}(\mathcal{T}^{c|C}(\tilde{\gamma}; \tau_c)) + \varepsilon \quad (\text{since } \eta, \tilde{\eta} \text{ have total variation } \varepsilon) \\
&= \eta(\mathcal{T}_c(\gamma; \tau_c)) - \tilde{\eta}(\mathcal{T}_c(\tilde{\gamma}; \tau_c)) + \varepsilon \quad (\text{by (6) and (7)}) \\
&\leq \eta(\mathcal{T}_c(\gamma; \tau_c)) - \eta(\mathcal{T}_c(\tilde{\gamma}; \tau_c)) + 2\varepsilon \quad (\text{since } \eta, \tilde{\eta} \text{ have total variation } \varepsilon) \\
&\leq 2\varepsilon \quad (\text{by (8)}),
\end{aligned}$$

that is,

$$\eta(\mathcal{T}^{c|C}(\gamma; \tau_c) \setminus \mathcal{T}^{c|C}(\tilde{\gamma}; \tau_c)) \leq 2\varepsilon. \quad (10)$$

Also, for all schools $b \neq c$, since η has full support, it holds that

$$\eta(\mathcal{T}^{c|C}(\gamma; \tau_b) \setminus \mathcal{T}^{c|C}(\tilde{\gamma}; \tau_b)) \leq \frac{M}{m} \eta(\mathcal{T}^{b,C}(\gamma; \tau_b) \setminus \mathcal{T}^{b,C}(\tilde{\gamma}; \tau_b)). \quad (11)$$

Hence

$$\begin{aligned}
& \eta(\mathcal{T}^{c|C}(\gamma; 1) \setminus \mathcal{T}^{c|C}(\tilde{\gamma}; 1)) \\
&= \eta(\mathcal{T}^{c|C}(\gamma; 1)) - \eta(\mathcal{T}^{c|C}(\tilde{\gamma}; 1)) \quad (\text{by (9)}) \\
&= \eta(\mathcal{T}^{c|C}(\gamma; \cup_b \tau_b)) - \eta(\mathcal{T}^{c|C}(\tilde{\gamma}; \cup_b \tau_b)) \\
&= \sum_{b \in C} (\eta(\mathcal{T}^{c|C}(\gamma; \tau_b)) - \eta(\mathcal{T}^{c|C}(\tilde{\gamma}; \tau_b))) \\
&\leq \sum_{b \in C} \eta(\mathcal{T}^{c|C}(\gamma; \tau_b) \setminus \mathcal{T}^{c|C}(\tilde{\gamma}; \tau_b)) \\
&\leq \sum_{b \in C} \frac{M}{m} \eta(\mathcal{T}^{b,C}(\gamma; \tau_b) \setminus \mathcal{T}^{b,C}(\tilde{\gamma}; \tau_b)) \quad (\text{by (11)}) \\
&\leq 2|C| \varepsilon \frac{M}{m} \quad (\text{by (10)}).
\end{aligned}$$

That is, given a school c , the set of students assigned to school c with score $r^\theta \not\leq x$ under γ and not assigned to school c with score $r^\theta \not\leq \tilde{x}$ under $\tilde{\gamma}$ has η -measure $O(\varepsilon |C|)$. A similar argument shows that the set of students assigned under $\tilde{\gamma}$ but not γ has $\tilde{\eta}$ -measure $O(\varepsilon |C|)$. \square

We are now ready to prove Theorem 6.

Proof of Theorem 6. Assume without loss of generality that the schools are c_1, c_2, \dots , where school c_ℓ reaches capacity in round ℓ of $TTC(\gamma)$. We show by induction on ℓ that for all schools c , the set of students assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ by the end of round ℓ has η -measure $O(\varepsilon\ell|\mathcal{C}|)$. This will prove the theorem.

The base case $\ell = 1$ follows directly from Proposition 11.

We now show the inductive step, proving for $\ell + 1$ assuming true for $1, 2, \dots, \ell$. For each $1 \leq i \leq \ell$ let $p^{(i)} = \gamma(t^{(i)})$ be the cutoffs obtained from $TTC(\gamma)$ in round i , and let \tilde{t}_i be the largest time t such that $\gamma(t^{(i)})$ dominates $\tilde{\gamma}(i)$ via some school available in round i of $TTC(\gamma)$. Since the measure of students assigned to c_i under $TTC(\gamma)$ and not to $TTC(\tilde{\gamma})$ by this point is $O(\varepsilon\ell|\mathcal{C}|)$ for all i , and the measure $\tilde{\eta}$ is bounded away from 0, if the two measures are sufficiently close, that is, for sufficiently small ε (dependent on the capacities) it holds that the schools reach capacity in the order c_1, c_2, \dots in $TTC(\tilde{\gamma})$ at times $\tilde{t}^{(i)}$. We may invoke Proposition 11 to show that the difference in allocations to school c in time $[t^{(\ell)}, t^{(\ell+1)}]$ under $TTC(\gamma)$ and in time $[\tilde{t}^{(\ell)}, \tilde{t}^{(\ell+1)}]$ under $TTC(\gamma')$ is of order $O(\varepsilon|\mathcal{C}|)$, and invoke the inductive hypothesis to show that the difference in allocations to school c in times $[0, t^{(\ell)}]$ under $TTC(\gamma)$ and in times $[0, \tilde{t}^{(\ell)}]$ under $TTC(\tilde{\gamma})$ is of order $O(\varepsilon\ell|\mathcal{C}|)$. This completes the induction. \square

B.6 Derivation of the Instantaneous Trade Balance Equations (3)

In this section, we show that the instantaneous trade balance equations (3) hold. The idea is the following. The trade balance equations must hold over any time interval within a round, and for small enough intervals $[t, t + \varepsilon]$ we can approximate the set of students assigned over the interval by a simple set, and likewise we can approximate the set of students offered by a simple set. Through these simple sets we turn the trade balance equations into linear equations that depend only on η via the values $\tilde{H}_b^c(x)$. We formalize this below.

For $b, c \in \mathcal{C}$, $x \in [0, 1]^{\mathcal{C}}$, $\alpha \in \mathbb{R}$ we define the set ⁴³

$$T_b^c(x, \alpha) \doteq \{\theta \in \Theta \mid r^\theta \in [x - \alpha e^b, x] \text{ and } Ch_\theta(\mathcal{C}^{(\ell)}) = c\}.$$

⁴³We use the notation $[\underline{x}, \bar{x}] = \{z \in \mathbb{R}^n \mid \underline{x}_i \leq z_i < \bar{x}_i \forall i\}$ for $\underline{x}, \bar{x} \in \mathbb{R}^n$, and $e^c \in \mathbb{R}^{\mathcal{C}}$ is a vector whose c -th coordinate is equal to 1 and all other coordinates are 0.

We may think of $T_b^c(x, \alpha)$ as the set of students in the next α students on school b 's priority list who are unassigned when $\gamma(t) = x$, and want school c . We remark that the sets used in the definition of the $\tilde{H}_b^c(x)$ are precisely the sets $T_b^c(x, \varepsilon)$.

We can use the sets $T_b^c(x, \alpha)$ to approximate expressions involving $\mathcal{T}_c(\gamma; t)$ and $\mathcal{T}^c(\gamma; t)$. Specifically, consider the run of the TTC algorithm in a round ℓ from $\gamma(t) = x$ to $\gamma(t + \tau) = x - \delta$. During the interval $[t, t + \tau]$ the students $\mathcal{T}_c(\gamma; t + \tau) \setminus \mathcal{T}_c(\gamma; t)$ were offered a seat at school c , and the students $\mathcal{T}^c(\gamma; t + \tau) \setminus \mathcal{T}^c(\gamma; t)$ were assigned to school c . We relate these sets to the sets $T_b^d(x, \alpha)$ in the following lemma.

Lemma 10. *Consider the interval $[t, t + \tau]$, and let $\gamma(t) = x$ and $\delta(\tau) = \gamma(t) - \gamma(t + \tau)$. During the interval $[t, t + \tau]$, the set of students who were assigned to school c is*

$$\mathcal{T}^c(\gamma; t + \tau) \setminus \mathcal{T}^c(\gamma; t) = \bigcup_b T_b^c(x, \delta(\tau)_b) = \left\{ \theta \in \Theta \mid r^\theta \in [\gamma_b(t + \tau), \gamma_b(t)) \text{ and } Ch_\theta(\mathcal{C}^{(\ell)}) = c \right\}$$

and the set of students who were offered a seat at school c is

$$\mathcal{T}_c(\gamma; t + \tau) \setminus \mathcal{T}_c(\gamma; t) = \bigcup_d T_c^d(x, \delta(\tau)_c) \cup \Delta$$

for some small set $\Delta \subset \Theta$. Further, it holds that $\lim_{\tau \rightarrow 0} \frac{1}{\tau} \cdot \eta(\Delta) = 0$, and for any $c \neq c', d \neq d' \in \mathcal{C}$ we have $\lim_{\tau \rightarrow 0} \frac{1}{\tau} \cdot \eta(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'})) = 0$ and $T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'}) = \phi$.

Proof. The first two equations are easily verified, and the fact that the last intersection is empty is also easy to verify. To show the bound on the measure of Δ , we observe that it is contained in the set $\bigcup_{c'} \bigcup_d (T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'}))$, so it suffices to show that $\lim_{\tau \rightarrow 0} \frac{1}{\tau} \cdot \eta(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'})) = 0$. This follows from the fact that the density defining η is upper bounded by M , so $\eta(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'})) \leq M |\gamma_c(t + \tau) - \gamma_c(t)| |\gamma_{c'}(t + \tau) - \gamma_{c'}(t)|$. Since for all schools c the function γ_c is continuous and has bounded derivative, it is also Lipschitz continuous, so

$$\frac{1}{\tau} \eta(\Delta) \leq \frac{1}{\tau} \eta(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^{d'}(x, \delta(\tau)_{c'})) \leq M L_c L_{c'} \tau$$

for some Lipschitz constants L_c and $L_{c'}$ and the lemma follows. \square

We are now ready to write the trade balance equations in terms of the entries of the matrix $H(x)$. In the interval $[t, t + \tau]$, the trade balance equations are given

by $\eta(\mathcal{T}^c(\gamma; t + \tau) \setminus \mathcal{T}^c(\gamma; t)) = \eta(\mathcal{T}_c(\gamma; t + \tau) \setminus \mathcal{T}_c(\gamma; t))$. Let us take the difference, divide by τ and take the limit as $\tau \rightarrow 0$. Then on the left hand side we obtain

$$\begin{aligned}
& \lim_{\tau \rightarrow 0} \frac{1}{\tau} \eta(\mathcal{T}^c(\gamma; t + \tau) \setminus \mathcal{T}^c(\gamma; t)) \\
&= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \eta \left(\bigcup_b T_b^c(x, \delta(\tau)_b) \right) \quad (\text{by Lemma 10}) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_b \frac{1}{\tau} \eta(T_b^c(x, \delta(\tau)_b)) + O\left(\frac{1}{\tau} (\|\gamma(t) - \gamma(t + \tau)\|_\infty)^2\right) \right] \quad (\text{since } \nu \text{ is bounded above}) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_b \frac{1}{\tau} \eta(T_b^c(x, \delta(\tau)_b)) \right] \quad (\text{since } \gamma \text{ is Lipschitz continuous}) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_b \frac{\delta(\tau)_b}{\tau} \cdot \frac{1}{\delta(\tau)_b} \eta \left(\left\{ \theta \in \Theta \mid r^\theta \in [x - \delta(\tau)_b e^b, x] \text{ and } Ch_\theta(\mathcal{C}^\ell) = c \right\} \right) \right] \\
&= \sum_b \frac{\partial \gamma(t)_b}{\partial t} \tilde{H}_b^c(x)
\end{aligned}$$

On the right hand side we obtain

$$\begin{aligned}
& \lim_{\tau \rightarrow 0} \frac{1}{\tau} \eta(\mathcal{T}_c(\gamma; t + \tau) \setminus \mathcal{T}_c(\gamma; t)) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_d \frac{1}{\tau} \eta(T_c^d(x, \delta(\tau)_c)) + O\left(\frac{1}{\tau} (\|\gamma(t) - \gamma(t + \tau)\|_\infty)^2\right) \right] \quad (\text{by Lemma 10}) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_d \frac{1}{\tau} \eta(T_c^d(x, \delta(\tau)_c)) \right] \quad (\text{since } \gamma \text{ is Lipschitz continuous}) \\
&= \lim_{\tau \rightarrow 0} \left[\sum_d \frac{\delta(\tau)_c}{\tau} \cdot \frac{1}{\delta(\tau)_c} \eta \left(\left\{ \theta \in \Theta \mid r^\theta \in [x - \delta(\tau)_c e^c, x] \text{ and } Ch_\theta(\mathcal{C}^\ell) = d \right\} \right) \right] \\
&= \frac{\partial \gamma(t)_c}{\partial t} \sum_d \tilde{H}_c^d(x).
\end{aligned}$$

Hence taking the limit in the trade balance equations gives us the following instantaneous trade balance equations at time t ,

$$\sum_b \frac{\partial \gamma(t)_b}{\partial t} \tilde{H}_b^c(x) = \frac{\partial \gamma(t)_c}{\partial t} \sum_d \tilde{H}_c^d(x) \text{ for all } c \in C,$$

or equivalently, if we let $d = \frac{\partial \gamma(t)}{\partial t}$, then

$$\sum_b d_b \cdot \tilde{H}_b^c(x) = \sum_b d_c \cdot \tilde{H}_c^b(x),$$

as required.

Let us now write these equations in terms of the matrix $H(x)$. Recall that $v_c = \sum_b \tilde{H}_c^b(x)$ is the measure of marginal students that will get an offer from school c . We rewrite the instantaneous trade balance equations as follows.

$$\begin{aligned} \sum_b d_b \cdot \tilde{H}_b^c(x) &= d_c \sum_b \tilde{H}_c^b(x) \\ \sum_b d_b \left(\frac{1}{\hat{v}} \tilde{H}_b^c(x) \right) &= d_c \frac{v_c}{\hat{v}} \\ \sum_b d_b \left(\frac{1}{\hat{v}} \tilde{H}_b^c(x) + \mathbf{1}_{b=c} \left(1 - \frac{v_c}{\hat{v}} \right) \right) &= d_c \left(\frac{v_c}{\hat{v}} + \left(1 - \frac{v_c}{\hat{v}} \right) \right) \\ \sum_b d_b H_b^c(x) &= d_c \end{aligned}$$

and since this holds for all b , we obtain the matrix equation

$$dH(x) = d.$$

C Proofs for Applications Section

C.1 Optimal Investment in School Quality

In this section, we prove the results stated in Section 5.1. We will assume that the total measure of students is 1, and speak of student measures and student proportions interchangeably.

Proofs for Section 5.1

Proof of Proposition 4. Let $\gamma, p, \{t^{(1)}, t^{(2)}\}$ be the TTC path, cutoffs and runout times with quality δ , and let $\hat{\gamma}, \hat{p}, \{\hat{t}^{(1)}, \hat{t}^{(2)}\}$ be the TTC path, cutoffs and runout times with quality $\hat{\delta}$. When we change δ_ℓ to $\hat{\delta}_\ell$, this increases the relative popularity of school ℓ .

Consider first when $\ell = 1$. As there are only two schools, $|d_1(x)|$ decreases and $|d_2(x)|$ increases for all x . It follows that if $\gamma_1(t) = \hat{\gamma}_1(\hat{t})$ then $\gamma_2(t) \geq \hat{\gamma}_2(\hat{t})$, and if $\gamma_2(t) = \hat{\gamma}_2(\hat{t})$ then $\gamma_1(t) \leq \hat{\gamma}_1(\hat{t})$. Suppose that $p_2^1 \leq \hat{p}_2^1$. Then there exists $t \leq t^{(1)}$

such that $p_2^1 = \gamma_2(t^{(1)}) \leq \gamma_2(t) = \hat{\gamma}_2(\hat{t}^{(1)})$, and so

$$p_1^1 = \gamma_1(t^{(1)}) \leq \gamma_1(t) \leq \hat{\gamma}_1(\hat{t}^{(1)}) = \hat{p}_1^1$$

as required. Hence it suffices to show that $p_2^1 \leq \hat{p}_2^1$.

Suppose for the sake of contradiction that $p_2^1 > \hat{p}_2^1$. Then there exists $t < \hat{t}^{(1)}$ such that $p_2^1 = \gamma_2(t^{(1)}) = \hat{\gamma}_2(t) > \hat{\gamma}_2(\hat{t}^{(1)})$, and so

$$\mathcal{T}_2(\gamma; t^{(1)}) \subseteq \mathcal{T}_2(\hat{\gamma}; t) \subset \mathcal{T}_2(\hat{\gamma}; \hat{t}^{(1)}).$$

Similarly

$$\mathcal{T}_1(\gamma; t^{(1)}) \supset \mathcal{T}_1(\hat{\gamma}; \hat{t}^{(1)}).$$

It follows that

$$\eta\left(\left\{\theta \in \mathcal{T}_2(\gamma; t^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 1\right\}\right) < \hat{\eta}\left(\left\{\theta \in \mathcal{T}_2(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 1\right\}\right),$$

since the set increased and more students want school 1, and similarly

$$\eta\left(\left\{\theta \in \mathcal{T}_1(\gamma; t^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 2\right\}\right) > \hat{\eta}\left(\left\{\theta \in \mathcal{T}_1(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 2\right\}\right),$$

However, the trade balance equations give that

$$\eta\left(\left\{\theta \in \mathcal{T}_2(\gamma; t^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 1\right\}\right) = \eta\left(\left\{\theta \in \mathcal{T}_1(\gamma; t^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 2\right\}\right) \text{ and}$$

$$\hat{\eta}\left(\left\{\theta \in \mathcal{T}_2(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 1\right\}\right) = \hat{\eta}\left(\left\{\theta \in \mathcal{T}_1(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\prec_{\theta}}\{1, 2\} = 2\right\}\right),$$

which gives the required contradiction.

The fact that p_2^2 decreases follows from the fact that p_1^1 increases, since the total number of assigned students is the same. □

Proof of Proposition 5.

TTC Cutoffs We calculate the TTC cutoffs under the logit model for different student choice probabilities by using the TTC paths and trade balance equations. In round 1, the marginals $\tilde{H}_b^c(x)$ for $b, c \in \mathcal{C}$ at each point $x \in [0, 1]$ are given by

$\tilde{H}_b^c(x) = e^{\delta_c} \prod_{c' \neq b} x_{c'}$. Hence $v_b = \sum_c \tilde{H}_b^c(x) = (\sum_b e^{\delta_c}) \prod_{c' \neq b} x_{c'} = \prod_{c' \neq b} x_{c'}$, so $v = \frac{\prod_c x_c}{\min_c x_c}$ and the matrix $H(x)$ is given by

$$H_{b,c}(x) = e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} + \mathbf{1}_{b=c} \left(1 - \frac{\min_{c'} x_{c'}}{x_b} \right) = \begin{cases} 1 - (1 - e^{\delta_c}) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise,} \end{cases}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x) H(x) = d(x)$. To solve for this, we observe that this equation is the same as $d(x) (H(x) - I) = \mathbf{0}$, where I is the n -dimensional identity matrix, and $[H(x) - I]$ has (b, c) th entry

$$[H(x) - I]_{b,c} = \begin{cases} - (1 - e^{\delta_b}) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise.} \end{cases}$$

Since this has rank $n - 1$, the nullspace is easily obtained by replacing the last column of $H(x) - I$ with ones, inverting the matrix and left multiplying it to the vector $e^{|\mathcal{C}|}$ (the vector with all zero entries, other than a 1 in the $|\mathcal{C}|$ th entry). This yields the valid direction $d(x)$ with c th component

$$d_c(x) = - \frac{e^{\delta_c} x_c}{\sum_b e^{\delta_b} x_b}.$$

We now find a valid TTC path γ using the instantaneous trade balance equations 4. Since the ratios of the components of the gradient $\frac{d_b(x)}{d_c(x)}$ only depend on x_b, x_c and the $\delta_{c'}$, we solve for x_c in terms of x_1 , using the fact that the path starts at $(1, 1)$. This gives the path γ defined by $\gamma_c(\gamma_1^{-1}(x_1)) = x_1^{e^{\delta_c} - \delta_1}$ for all c .

Recall that the schools are indexed so that school c_1 is the most demanded school, that is, $\frac{e^{\delta_1}}{q_1} = \max_c \frac{e^{\delta_c}}{q_c}$. Since we are only interested in the changes in the cutoffs $\gamma(t^{(1)})$ and not in the specific time, let us assume without loss of generality that $\gamma_1(t) = 1 - t$. Then school c_1 fills at time $t^{(1)} = 1 - (1 - \frac{q_1}{e^{\delta_1}} (\sum_{c'} e^{\delta_{c'}}))^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}} = 1 - (1 - \rho_1 (\sum_{c'} e^{\delta_{c'}}))^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}}$. Hence the round 1 cutoffs are

$$p_b^1 = (1 - t^{(1)})^{e^{\delta_b} - \delta_1} = \left(1 - \rho_1 \left(\sum_{c'} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}} = \left(1 - \rho_1 \left(\sum_{c'} e^{\delta_{c'}} \right) \right)^{\pi^{b|c}}. \quad (12)$$

It can be shown by projecting onto the remaining coordinates and using induction

that the round i cutoffs are given by

$$p_b^c = \begin{cases} \left(\prod_{c' < c} \frac{1}{p_{c'}} \right)^{\pi^{b|c}} \left(\prod_{c'} p_{c'}^{c-1} - \rho_c \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\pi^{b|c}} & \text{if } b \geq c \\ p_b^b & \text{if } b \leq c. \end{cases}$$

TTC Cutoffs - Comparative Statics We perform some comparative statics calculations for the TTC cutoffs under the logit model. For $b \neq \ell$ it holds that the TTC cutoff p_b^1 for using priority at school b to receive a seat at school 1 is decreasing in δ_ℓ ,

$$\begin{aligned} \frac{\partial p_b^1}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}} \right] \\ &= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}} \ln \left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right) \right] \\ &= p_b^1 \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}} \right) \ln \left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right) + \left(\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right)}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} \right] \\ &= -p_b^1 \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^1)^2} \right) \left[-\ln \left(\frac{1}{1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1} \right) + \frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} - 1 \right] \end{aligned}$$

is negative, since $0 < \frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} < 1$, where for brevity we define $\Delta^c = \sum_{b \geq c} e^{\delta_b}$.

We can decompose this change as

$$\frac{\partial p_b^1}{\partial \delta_\ell} = -p_b^1 \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^1)^2} \right) \left[\ln \left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right) \right] - p_b^1 \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^1)^2} \right) \left[\frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} - 1 \right] < 0,$$

where the first term is the increase in p_b^1 due to the fact that relatively fewer students are pointed to and cleared by school b for every marginal change in rank, and the second term is the decrease in p_b^1 due to the fact that school 1 is relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}$ in order for school 1 to reach capacity.

For $b = \ell$ the TTC cutoff p_ℓ^1 is again decreasing in δ_ℓ ,

$$\begin{aligned}
\frac{\partial p_\ell^1}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}}} \right] \\
&= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{\delta_\ell}{\sum_{c'} e^{\delta_{c'}}}} \ln \left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right) \right] \\
&= p_\ell^1 \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}} \right) \ln \left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right) + \left(\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - \frac{q_1}{e^{\delta_1}} \left(\sum_{c'} e^{\delta_{c'}} \right) \right)}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} \right] \\
&= -p_b^1 \left(\frac{e^{\delta_\ell} (\Delta^1 - e^{\delta_\ell})}{(\Delta^1)^2} \right) \ln \left(\frac{1}{1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1} \right) - p_b^1 \left(\frac{e^{2\delta_\ell + \delta_b}}{(\Delta^1)^2} \right) \left(\frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}} \right) \Delta^1 \right)} - 1 \right)
\end{aligned}$$

is negative since both terms are negative.

Similarly, for $c < \ell$ and $b \neq \ell$ the TTC cutoff p_ℓ^c is decreasing in δ_ℓ ,

$$\begin{aligned}
\frac{\partial p_b^c}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_b}}{\sum_{c' \geq c} e^{\delta_{c'}}}} \right] \\
&= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{\delta_b}{\sum_{c' \geq c} e^{\delta_{c'}}}} \ln \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right) \right] \\
&= p_b^c \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_b}}{\sum_{c' \geq c} e^{\delta_{c'}}} \right) \ln \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right) \right] \\
&\quad + p_b^c \left[\left(\frac{e^{\delta_b}}{\sum_{c' \geq c} e^{\delta_{c'}}} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)}{\left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} \right] \\
&= -p_b^c \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^c)^2} \right) \left[-\ln \left(\frac{1}{1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c} \right) + \frac{1}{\left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} - 1 \right] \\
&\quad - p_b^c \left[\frac{e^{\delta_b} \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \frac{\partial P^c}{\partial \delta_\ell}}{\left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} \right]
\end{aligned}$$

is negative, where $P^c = \prod_{c' < c} \frac{1}{p_{c'}}$, since $0 < 1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c < 1$ and

$$\frac{\partial P^c}{\partial \delta_\ell} = P^c \left(\sum_{c' < c} \frac{-\frac{\partial p_{c'}}{\partial \delta_\ell}}{p_{c'}} \right) > 0 \text{ so both terms are negative.}$$

We can decompose this change as follows. Let $P^c = \prod_{c' < c} \frac{1}{p_{c'}}$. For $c < \ell$ and $b \geq c$, $b \neq \ell$ it holds that

$$\begin{aligned} \frac{\partial p_b^c}{\partial \delta_\ell} &= -p_b^c \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^c)^2} \right) \left[\ln \left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right) \right] \\ &\quad - p_b^c \left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^c)^2} \right) \left[\frac{1}{\left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} - 1 \right] - p_b^c \left[\frac{e^{\delta_b} \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \frac{\partial}{\partial \delta_\ell} P^c}{\left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} \right] \end{aligned}$$

which is negative. The first term is the increase in p_b^c due to the fact that relatively fewer students are pointed to and cleared by school j for every marginal change in rank, and the second and third terms are the decrease in p_b^c due to the fact that schools 1 through c are relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots, \mathcal{C}^{(c)}$ in order for schools 1 through c to reach capacity.

For $c < \ell$ and $b = \ell$ the TTC cutoff p_ℓ^c is also decreasing in δ_ℓ ,

$$\begin{aligned} \frac{\partial p_\ell^c}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}^c} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_\ell}}{\sum_{c' \geq c} e^{\delta_{c'}}}} \right] \\ &= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{e^{\delta_\ell}}{\sum_{c' \geq c} e^{\delta_{c'}}}} \ln \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}^c} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right) \right] \\ &= p_\ell^c \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_\ell}}{\sum_{c' \geq c} e^{\delta_{c'}}} \right) \ln \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}^c} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right) \right] \\ &\quad + p_\ell^c \left[\left(\frac{e^{\delta_\ell}}{\sum_{c' \geq c} e^{\delta_{c'}}} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}^c} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)}{\left(1 - \left(\prod_{c' < c} \frac{1}{p_{c'}^c} \right) \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} \right] \\ &= -p_\ell^c \left[\left(\frac{e^{\delta_\ell} (\Delta^c - e^{\delta_\ell})}{(\Delta^c)^2} \right) \ln \left(\frac{1}{1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c} \right) \right] \\ &\quad - p_\ell^c \left(\frac{e^{\delta_\ell}}{\Delta^c} \right) \left[\frac{\left(P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) e^{\delta_\ell} \right) + \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \frac{\partial P^c}{\partial \delta_\ell}}{\left(1 - P^c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \right)} \right] \end{aligned}$$

which is negative, since $\frac{\partial P^c}{\partial \delta_\ell} = P^c \left(\sum_{c' < c} -\frac{\partial p_{c'}^c}{\partial \delta_\ell} \cdot \frac{1}{p_{c'}^c} \right) > 0$ so both terms are negative.

When $c = \ell$, the effects of changing δ_ℓ on the cutoffs required to obtain a seat at school ℓ are a little more involved. For $c = \ell$ and $b \neq \ell$,

$$\begin{aligned}
\frac{\partial p_b^\ell}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \left(\prod_{c' < \ell} \frac{1}{p_{c'}^\ell} \right) \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_b}}{\sum_{c' \geq \ell} e^{\delta_{c'}}}} \right] \\
&= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{e^{\delta_b}}{\sum_{c' \geq \ell} e^{\delta_{c'}}}} \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) (\sum_{c' \geq \ell} e^{\delta_{c'}}) \right) \right] \\
&= p_b^\ell \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) + \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_b^\ell \left[\left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^\ell)^2} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_b^\ell \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \left[\left(\frac{e^{\delta_\ell}}{\Delta^\ell} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right]
\end{aligned}$$

where $P_\ell = \prod_{c' < \ell} \frac{1}{p_{c'}^\ell}$, the first term is positive, and the second term has the same sign as its numerator $\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)$. Similarly for $c = \ell$ and $b = \ell$,

$$\begin{aligned}
\frac{\partial p_\ell^\ell}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_\ell}}{\sum_{c' \geq \ell} e^{\delta_{c'}}}} \right] \\
&= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{e^{\delta_\ell}}{\sum_{c' \geq \ell} e^{\delta_{c'}}}} \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) (\sum_{c' \geq \ell} e^{\delta_{c'}}) \right) \right] \\
&= p_\ell^\ell \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_\ell}}{\Delta^\ell} \right) \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) + \left(\frac{e^{\delta_\ell}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_\ell^\ell \left[- \left(\frac{e^{\delta_\ell} (\Delta^\ell - e^{\delta_\ell})}{(\Delta^\ell)^2} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) \right] \\
&\quad + p_\ell^\ell \left[\left(\frac{e^{\delta_\ell}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_\ell^\ell \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \left[- \left(\frac{\Delta^\ell - e^{\delta_\ell}}{\Delta^\ell} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right]
\end{aligned}$$

where $P_\ell = \prod_{c' < \ell} \frac{1}{p_{c'}^\ell}$, the first term is negative, and the second term has the

same sign as its numerator $\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)$. Since $\frac{\partial}{\partial \delta_\ell} \left(\prod_{b \geq \ell} p_b^\ell \right) > 0$, it follows that $\frac{\partial p_b^\ell}{\partial \delta_\ell} > 0$ for all $b \neq \ell$, and there are regimes in which $\frac{\partial p_\ell^\ell}{\partial \delta_\ell}$ is positive, and regimes where it is negative. \square

Proofs for Section 5.1

Proof of Proposition 6.

Welfare Expressions We derive the welfare expressions corresponding to these cutoffs. Let $\mathcal{C}^{(c)} = \{c, c+1, \dots, n\}$. Since the schools are ordered so that $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \dots \leq \frac{q_n}{e^{\delta_n}}$, it follows that the schools also fill in the order $1, 2, \dots, n$.

Suppose that the total mass of students is 1. Then the mass of students with budget set $\mathcal{C}^{(1)}$ is given by $N^1 = q_1 \left(\frac{\sum_b e^{\delta_b}}{e^{\delta_1}} \right)$, and the mass of students with budget set $\mathcal{C}^{(2)}$ is given by $N^2 = \left(q_2 - \frac{e^{\delta_2}}{\sum_b e^{\delta_b}} N^1 \right) \left(\frac{\sum_{b \geq 2} e^{\delta_b}}{e^{\delta_2}} \right) = \left(\frac{q_2}{e^{\delta_2}} - \frac{q_1}{e^{\delta_1}} \right) \left(\sum_{b \geq 2} e^{\delta_b} \right)$. An inductive argument shows that the proportion of students with budget set $\mathcal{C}^{(c)}$ is

$$N^c = \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left(\sum_{b \geq c} e^{\delta_b} \right).$$

which depends only on δ_b for $b \geq c-1$.

Moreover, each such student with budget set $\mathcal{C}^{(c)}$, conditional on their budget set, has expected utility Small and Rosen (1981)

$$U^c = \mathbb{E} \left[\max_{c' \in \mathcal{C}^{(c)}} \{ \delta_b + \varepsilon_{\theta c'} \} \right] = \ln \left[\sum_{b \geq c} e^{\delta_b} \right],$$

which depends only on δ_b for $b \geq c$.

Hence the expected social welfare from fixed qualities δ_c is given by

$$\begin{aligned} U_{TTC} &= \sum_c N^c \cdot U^c \\ &= \sum_c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \ln \Delta^c, \end{aligned}$$

where $\Delta^c = \sum_{b \geq c} e^{\delta_b}$.

Welfare - Comparative Statics Taking derivatives, we obtain that

$$\begin{aligned}\frac{dU_{TTC}}{d\delta_\ell} &= \sum_c \left(\frac{dN^c}{d\delta_\ell} \cdot U^c + N^c \cdot \frac{dU^c}{d\delta_\ell} \right) \\ &= \sum_{c \leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c + \sum_{c \leq \ell} N^c \cdot \frac{dU^c}{d\delta_\ell},\end{aligned}$$

where

$$\sum_{c \leq \ell} N^c \cdot \frac{dU^c}{d\delta_\ell} = e^{\delta_\ell} \sum_{c \leq \ell} \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) = q_\ell.$$

It follows that

$$\frac{dU_{TTC}}{d\delta_\ell} = q_\ell + \sum_{c \leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c.$$

□

Proof of Proposition 7. We solve for the social welfare maximising budget allocation. For a fixed runout ordering (i.e. $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \dots \leq \frac{q_n}{e^{\delta_n}}$), the central school board's investment problem is given by the program

$$\begin{aligned}\max_{\kappa_1, \kappa_2, \dots, \kappa_n} \quad & \sum_i \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_i} \right) \left(\sum_{j \geq i} \kappa_j \right) \ln \left(\sum_{j \geq i} \kappa_j \right) \\ \text{s.t.} \quad & \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_{i-1}} \quad \forall i \\ & \sum_i \kappa_i = K \\ & q_0 = 0.\end{aligned} \tag{13}$$

We can reformulate this as the following program,

$$\begin{aligned}\max_{\kappa_2, \dots, \kappa_n} \quad & \left(\frac{q_1}{K - \sum_i \kappa_i} \right) K \ln K + \left(\frac{q_2}{\kappa_2} - \frac{q_1}{K - \sum_i \kappa_i} \right) U_2 \ln U_2 + \sum_{i \geq 3} \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) U_i \ln U_i \\ \text{s.t.} \quad & \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_i} \quad \forall i \geq 3 \\ & \frac{q_1}{K - \sum_i \kappa_i} \leq \frac{q_2}{\kappa_2},\end{aligned} \tag{14}$$

where $U_i = \sum_{j \geq i} \kappa_j$.

The reformulated problem (14) has objective function

$$U(\boldsymbol{\kappa}) = \left(\frac{q_1}{K - \sum_i \kappa_i} \right) K \ln K + \left(\frac{q_2}{\kappa_2} - \frac{q_1}{K - \sum_i \kappa_i} \right) \Delta^2 \ln \Delta^2 + \sum_{i \geq 3} \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) \Delta^i \ln \Delta^i,$$

where $\Delta^i = \sum_{j \geq i} \kappa_j$. Taking the derivatives with respect to the budget allocations κ_k gives

$$\begin{aligned} \frac{\partial U}{\partial \kappa_k} &= \left(\frac{q_1}{(K - \sum_i \kappa_i)^2} \right) (K \ln K - \Delta^2 \ln \Delta^2) + \left(\frac{q_2}{\kappa_2} - \frac{q_1}{K - \sum_i \kappa_i} \right) (1 + \ln \Delta^2) \\ &\quad + \left(-\frac{q_k}{(\kappa_k)^2} \right) (\Delta^k \ln \Delta^k - \Delta^{k+1} \ln \Delta^{k+1}) + \sum_{3 \leq i \leq k} \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) (1 + \ln \Delta^i) \\ &= \left(\frac{q_1}{(K - \sum_i \kappa_i)^2} \right) \left(K \ln \frac{K}{\Delta^2} - (K - \Delta^2) \right) + \sum_{2 \leq i < k} \frac{q_i}{\kappa_i} \ln \frac{\Delta^i}{\Delta^{i+1}} + \frac{q_k}{(\kappa_k)^2} \left(\kappa_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}} \right), \end{aligned}$$

where

$$K \ln \frac{K}{\Delta^2} - (K - \Delta^2) = K \left(\ln \frac{K}{\Delta^2} - 1 + \frac{\Delta^2}{K} \right) = K \left(\ln x - 1 + \frac{1}{x} \right) \geq 0,$$

$$\ln \frac{\Delta^i}{\Delta^{i+1}} = \ln \left(1 + \frac{\kappa_i}{\kappa_{i+1} + \dots + \kappa_n} \right) \geq 0, \text{ and}$$

$$\kappa_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}} = \Delta^{k+1} \left(\frac{\kappa_k}{\Delta^{k+1}} - \ln \left(1 + \frac{\kappa_k}{\Delta^{k+1}} \right) \right) = \Delta^{k+1} (x - \ln x) \geq 0$$

and so $\frac{\partial U}{\partial \kappa_k} \geq 0$ for all k .

Moreover, if $\frac{q_{i-1}}{\kappa_{i-1}} = \frac{q_i}{\kappa_i}$, then defining a new problem with $n-1$ schools, capacities

$$\tilde{q}_j = \begin{cases} q_j & \text{if } j < i-1 \\ q_{i-1} + q_i & \text{if } j = i-1 \\ q_{j+1} & \text{if } j > i-1 \end{cases}$$

and assigning a budget K by

$$\tilde{\kappa}_j = \begin{cases} \kappa_j & \text{if } j < i-1 \\ \kappa_{i-1} + \kappa_i & \text{if } j = i-1 \\ \kappa_{j+1} & \text{if } j > i-1 \end{cases}$$

leads to a problem with the same objective function, since

$$\begin{aligned}
& \left(\frac{q_{i-1}}{\kappa_{i-1}} - \frac{q_{i-2}}{\kappa_{i-2}} \right) \Delta^{i-1} \ln \Delta^{i-1} + \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) \Delta^i \ln \Delta^i + \left(\frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_i}{\kappa_i} \right) \Delta^{i+1} \ln \Delta^{i+1} \\
&= \left(\frac{q_{i-1}}{\kappa_{i-1}} - \frac{q_{i-2}}{\kappa_{i-2}} \right) \Delta^{i-1} \ln \Delta^{i-1} + \left(\frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_i}{\kappa_i} \right) \Delta^{i+1} \ln \Delta^{i+1} \\
&= \left(\frac{q_{i-1} + q_i}{\kappa_{i-1} + \kappa_i} - \frac{q_{i-2}}{\kappa_{i-2}} \right) \Delta^{i-1} \ln \Delta^{i-1} + \left(\frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_{i-1} + q_i}{\kappa_{i-1} + \kappa_i} \right) \Delta^{i+1} \ln \Delta^{i+1}.
\end{aligned}$$

Hence if there exists i for which $\frac{q_i}{\kappa_i} \neq \frac{q_{i-1}}{\kappa_{i-1}}$, we may take i to be minimal such that this occurs, decrease each of $\kappa_1, \dots, \kappa_{i-1}$ proportionally so that $\kappa_1 + \dots + \kappa_{i-1}$ decreases by ε and increase κ_i by ε and increase resulting value of the objective. It follows that the objective is maximized when $\frac{q_1}{\kappa_1} = \frac{q_2}{\kappa_2} = \dots = \frac{q_n}{\kappa_n}$, i.e. when the money assigned to each school is proportional to the number of seats at the school. \square

C.2 Design of TTC Priorities

We demonstrate how to calculate the TTC cutoffs for the two economies in Figure 10 by using the TTC paths and trade balance equations.

Consider the economy \mathcal{E}_0 , where the top priority students have ranks uniformly distributed in $[m, 1]^2$. If $x = (x_1, x_1)$ is on the diagonal, then $\tilde{H}_i^j(x) = \frac{x_1}{2}$ for all $i, j \in \{1, 2\}$. Hence

$$H_{i,j}(x) = \frac{1}{x_1} \left(\frac{x_1}{2} \right) + \mathbf{1}_{i=j} \left(1 - \frac{x_1}{x_1} \right) = \frac{1}{2} \quad \forall i, j \in \{1, 2\}$$

and so there is a unique valid direction $d(\vec{x}) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Moreover, $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2} \right)$ satisfies $\frac{d\gamma(t)}{dt} = \mathbf{d}(\gamma(t))$ for all t and hence Theorem (5) implies that $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2} \right)$ is the unique TTC path, and the cutoff points $p_b^c = \sqrt{1 - 2q}$ give the unique TTC allocation.

Consider now the economy \mathcal{E}_1 , where top priority students have ranks uniformly distributed in the $\tilde{r} \times \tilde{r}$ square $(1 - \tilde{r}, 1] \times (m, m + \tilde{r}]$ for some small \tilde{r} .

If x is in $(1 - \tilde{r}, 1] \times [m + \tilde{r}, 1]$ then $\tilde{H}_1^j(x) = \frac{1}{2} \left(m + (1 - m) \frac{1-m}{\tilde{r}} \right)$ for all j and $\tilde{H}_2^j(x) = \frac{m}{2}$ for all j . Hence $v_1 = m \left(1 + (1 - m) \frac{1-m}{\tilde{r}m} \right)$, $v_2 = m$, $v = v_1$. So

$$H(x) = \frac{1}{v_1} \begin{bmatrix} \frac{v_1}{2} & \frac{v_1}{2} \\ \frac{v_2}{2} & v_1 - \frac{v_2}{2} \end{bmatrix}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x)H(x) = d(x)$. Solving for this,

$$\begin{aligned} d(x)(H(x) - I) &= \mathbf{0} \\ d(x) \frac{1}{v_1} \begin{bmatrix} -\frac{v_1}{2} & \frac{v_1}{2} \\ \frac{v_2}{2} & -\frac{v_2}{2} \end{bmatrix} &= \mathbf{0} \\ d(x) &= \frac{1}{v_1 + v_2} \begin{bmatrix} -v_2 \\ -v_1 \end{bmatrix} = \frac{1}{2 + \frac{r^2}{\tilde{r}(1-r)}} \begin{bmatrix} -1 \\ -1 - \frac{r^2}{\tilde{r}(1-r)} \end{bmatrix}. \end{aligned}$$

If x is in $(m, 1 - \tilde{r}] \times (m, 1]$ then $\tilde{H}_i^j(x) = \frac{m}{2}$ for all i, j . Hence $v_1 = v_2 = v = m$ and

$$H_{i,j}(x) = 1 + \frac{\tilde{r}}{1-m} + \mathbf{1}_{i=j} \left(1 - \frac{m}{m}\right) = 1 + \frac{\tilde{r}}{1-m} \quad \forall i, j,$$

and so there is a unique valid direction $d(x) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

Finally, if $x = (x_1, x_2)$ is in $[0, 1] \setminus (m, 1]^2$ then $\tilde{H}_1^j(x) = \frac{1}{2}x_2$ and $\tilde{H}_2^j = \frac{1}{2}x_1$ for all j . Hence $v_1 = x_2, v_2 = x_1$ and $v = \max\{x_1, x_2\}$. So

$$H(x) = \frac{1}{v} \begin{bmatrix} \frac{x_2}{2} + (v - x_2) & \frac{x_2}{2} \\ \frac{x_1}{2} & \frac{x_1}{2} + (v - x_1) \end{bmatrix} = \frac{1}{v} \begin{bmatrix} v - \frac{x_2}{2} & \frac{x_2}{2} \\ \frac{x_1}{2} & v - \frac{x_1}{2} \end{bmatrix}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x)H(x) = d(x)$. Solving for this,

$$\begin{aligned} d(x)(H(x) - I) &= \mathbf{0} \\ d(x) \frac{1}{v} \begin{bmatrix} -\frac{x_2}{2} & \frac{x_2}{2} \\ \frac{x_1}{2} & -\frac{x_1}{2} \end{bmatrix} &= \mathbf{0} \\ d(x) &= \frac{1}{x_1 + x_2} \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}. \end{aligned}$$

Hence the TTC path $\gamma(t)$ has gradient $\frac{1}{2+\frac{(1-m)^2}{\tilde{r}m}} \begin{bmatrix} -1 \\ -1 - \frac{(1-m)^2}{\tilde{r}m} \end{bmatrix}$ from the point $(1, 1)$ to the point $\left(1 - \tilde{r}, 1 - \tilde{r} - \frac{r^2}{1-r}\right)$, gradient $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ from the point $\left(1 - \tilde{r}, 1 - \tilde{r} - \frac{r^2}{1-r}\right)$ to the point $\left(m + \frac{r^2}{1-r}, m\right)$ and gradient $\frac{1}{2+\frac{(1-m)^2}{m^2}} \begin{bmatrix} -1 - \frac{(1-m)^2}{m^2} \\ -1 \end{bmatrix}$ from the point $\left(m + \frac{(1-m)^2}{m}, m\right)$ to the point $\left(\sqrt{\frac{1-2q}{1-2m+2m^2}}, \sqrt{(1-2q)(1-2m+2m^2)}\right) = (\bar{p}, \underline{p})$.

Finally, we show that if economy \mathcal{E}_2 is given by perturbing the relative ranks of students in $\{\theta \mid r_c^\theta \geq m \forall c\}$, then the TTC cutoffs for \mathcal{E}_2 are given by $p_1^1 = p_1^2 = x$, $p_2^1 = p_2^2 = y$ where $x \leq \bar{p} = \sqrt{\frac{1-2q}{1-2m+2m^2}}$ and $y \geq \underline{p} = \sqrt{(1-2q)(1-2m+2m^2)}$. (By symmetry, it follows that $\underline{p} \leq x, y \leq \bar{p}$.) Let γ_1 and γ_2 be the TTC paths for \mathcal{E}_1 and \mathcal{E}_2 respectively. Consider the point (x_{bound}, m) on γ_2 . The TTC path γ_2 for \mathcal{E}_2 has gradient $\frac{1}{x_{bound}+m} \begin{bmatrix} -x_{bound} \\ -m \end{bmatrix}$ from (x_{bound}, m) to (x, y) .

Consider the aggregate trade balance equations for students assigned before the TTC path reaches (x_{bound}, m) . They stipulate that the measure of students in $[0, m] \times [m, 1]$ who prefer school 1 is at most the measure of students who are either perturbed, or in $[x_{bound}, 1] \times [0, m]$, and who prefer school 2. This means that

$$\begin{aligned} \frac{1}{2}m(1-m) &\leq \frac{1}{2} \left((1-m)^2 + m(1-x_{bound}) \right), \\ mx_{bound} &\leq 1 - 2m + 2m^2 \\ x_{bound} &\leq m + \frac{(1-m)^2}{m}. \end{aligned}$$

Hence γ_2 lies above γ_1 ⁴⁴ and so $x \leq \bar{p}$ and $y \geq \frac{1-2q}{\bar{p}} = \underline{p}$.

C.3 Comparing Top Trading Cycles and Deferred Acceptance

In this section, we derive the expressions for the TTC and DA cutoffs given in Section 5.3.

Consider the TTC cutoffs for the neighborhood priority setting. We prove by induction on ℓ that $p_j^\ell = 1 - \frac{q_\ell}{2q}$ for all ℓ, j such that $j \geq \ell$.

⁴⁴That is, for each x_1 , if (x_1, y_1) lies on γ_1 and (x_1, y_2) lies on γ_2 , then $y_2 \geq y_1$.

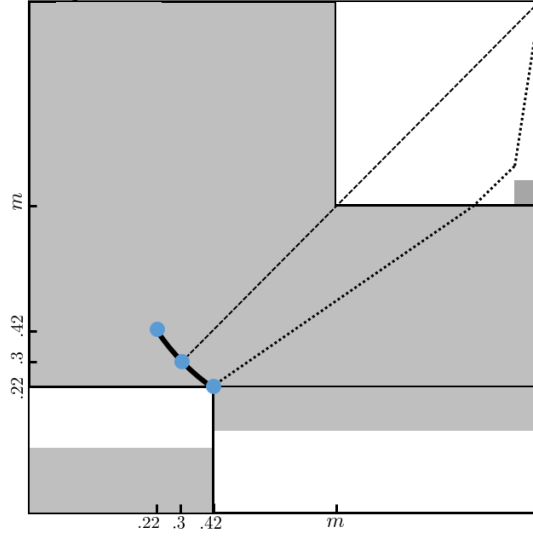


Figure 13: Economy \mathcal{E}_1 from Example 8. The black borders partition the space of students into four regions. The density of students is zero on white areas, and constant on each of the shaded areas within a bordered region. In each of the four regions, the total measure of students within is equal to the total area (white and shaded) within the borders of the region.

Base case: $\ell = 1$.

For each school i , there are measure q of students whose first choice school is i , αq of whom have priority at i and $\frac{(1-\alpha)q}{n-1}$ of whom have priority at school j , for all $j \neq i$.

The TTC path is given by the diagonal, $\gamma(t) = \left(1 - \frac{t}{\sqrt{n}}, 1 - \frac{t}{\sqrt{n}}, \dots, 1 - \frac{t}{\sqrt{n}}\right)$. At the point $\gamma(t) = (x, x, \dots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2(1-x)$ of students from each neighborhood have been assigned. Since the same proportion of students have each school as their top choice, this means that the quantity of students assigned to each school i is $2(1-x)q$. Hence the cutoffs are given by considering school 1, which has the smallest capacity, and setting the quantity assigned to school 1 equal to its capacity q_1 . It follows that $p_j^1 = x^*$ for all j , where $2(1-x^*)q = q_1$, which yields

$$p_j^1 = 1 - \frac{q_1}{2q} \text{ for all } j.$$

Inductive step.

Suppose we know that the cutoffs $\{p_j^i\}_{i,j:i \leq \ell}$ satisfy $p_j^i = 1 - \frac{q_i}{2q}$. We show by induction that the $(\ell + 1)$ th set of cutoffs $\{p_j^{\ell+1}\}_{j>\ell}$ are given by $p_j^{\ell+1} = 1 - \frac{q_{\ell+1}}{2q}$.

The TTC path is given by the diagonal when restricted to the last $n - \ell$ coordi-

nates, $\gamma(t^{(\ell)} + t) = \left(p_1^1, p_2^2, \dots, p_\ell^\ell, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, \dots, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}\right)$.

Consider a neighborhood i . If $i > \ell$, at the point $\gamma(t) = (p_1^1, p_2^2, \dots, p_\ell^\ell, x, x, \dots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2(p_\ell^\ell - x)$ of (all previously assigned and unassigned) students from neighborhood i have been assigned in round $\ell + 1$. If $i \leq \ell$, no students from neighborhood i have been assigned in round $\ell + 1$.

Consider the set of students S who live in one of the neighborhoods $\ell + 1, \ell + 2, \dots, n$. The same proportion of these students have each remaining school as their top choice out of the remaining schools. This means that for any $i > \ell$, the quantity of students assigned to school i in round $\ell + 1$ by time t is a $\frac{1}{n-\ell}$ fraction of the total number of students assigned in round $\ell + 1$ by time t , and is given by $(n - \ell) q \frac{1}{n-\ell} = 2(p_\ell^\ell - x)q$. Hence the cutoffs are given by considering school $\ell + 1$, which has the smallest residual, and setting the quantity assigned to school $\ell + 1$ equal to its residual capacity $q_{\ell+1} - q_\ell$. It follows that $p_j^{\ell+1} = x^*$ for all $j > \ell$ where $2(p_\ell^\ell - x^*)q = q_{\ell+1} - q_\ell$, which yields

$$p_j^{\ell+1} = p_\ell^\ell - \frac{q_{\ell+1} - q_\ell}{2q} = 1 - \frac{q_\ell}{2q} - \frac{q_{\ell+1} - q_\ell}{2q} = 1 - \frac{q_{\ell+1}}{2q} \text{ for all } j > \ell.$$

This completes the proof that the TTC cutoffs are given by $p_j^i = p_i^j = 1 - \frac{q_i}{2q}$ for all $i \leq j$.

Now consider the DA cutoffs. We show that the cutoffs $p_i = 1 - \frac{q_i}{2q}$ satisfy the supply-demand equations.

We first remark that the cutoff at school i is higher than all the ranks of students without priority at school i , $p_i \geq \frac{1}{2}$. Since every student has priority at exactly one school, this means that every student is either above the cutoff for exactly one school and is assigned to that school, or is below all the cutoffs and remains unassigned. Hence there are $q2(1 - p_i) = q_i$ students assigned to school i for all i , and the supply-demand equations are satisfied.