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Robust nonlinear stability and performance analysis of an F/A-18 aircraft model using sum of squares programming

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SUMMARY

In this paper, we develop algorithms for the computational analysis of stability and robust performance properties of nonlinear models governed by polynomial differential equations in quasi-Linear Parameter Varying form subject to parametric uncertainty. The methods presented use the sum of squares decomposition and ideas from real algebraic geometry to represent polynomial non-negativity over closed sets to compute various system properties such as $\mathcal{L}_2$ gain, regions of attraction, reachable sets and nonlinear Hankel norm approximations. The methods we present are then illustrated on a nonlinear model of an F/A-18 aircraft short-period dynamics subject to uncertainty in the aerodynamic coefficients. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

One of the fundamental problems in systems and control theory is that of verifying whether or not an equilibrium point of a nonlinear dynamical system is stable in the face of uncertainties. For finite dimensional nominal linear systems, this question can be answered both analytically and algorithmically. However, linear models can only approximate a nonlinear system locally and computational nonlinear stability analysis is a nontrivial task. If the systems under study are also uncertain, stability assessment becomes harder and even running exhaustive simulations becomes a challenge. Furthermore, all performance metrics that we may wish to compute assume the existence of stability certificates for these systems.

A major problem encountered when dealing with uncertain nonlinear system models (from a computational viewpoint) is that the uncertainty can shift the equilibrium point of interest. The simplest example of this occurs when dealing with additive uncertainty. Computational verification thus becomes difficult as the stability of the equilibrium point must be assessed for all allowable disturbances. In this paper, we consider nonlinear uncertain systems modeled by quasi-Linear Parameter Varying (qLPV) dynamics [1]. The qLPV framework allows us to maintain the nonlinear character of the dynamic equations but guarantees that the equilibrium is fixed for all allowable parameters. This approach strikes a balance between model accuracy and algorithm complexity, and for these reasons, it is frequently used in the aerospace industry, see for example [2, 3].

In this paper, we consider the problem of computationally verifying stability and performance measures of an aircraft feedback control system. This is an essential part of the engineering design cycle and is a requirement for clearance of flight control laws. The verification stage can be

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computationally demanding as it relies on a combination of extensive high-fidelity simulations of the nonlinear system and robust analysis of linearized models corresponding to varying aerodynamic conditions. An alternative method would be to consider robustness of nonlinear models directly; however, such an approach is in general intractable even for low dimensional models. This problem is further compounded by the fact that realistic aircraft models often contain nonlinearities and operate in an uncertain environment subject to disturbances. Furthermore, such models are usually parameterized by aerodynamic coefficients that vary over the intended flight envelope.

Traditional methods for analyzing robust stability and performance assume that the system model has been linearized and that the uncertainty and model can be represented by a linear fractional representation [4]. Well-studied methods from robust control can then be used to construct robust flight control laws and analyze performance. Many such techniques are described in the collective work [5]. Representative methods that one would typically call *traditional* methods include μ-analysis for structured uncertainty [6], $\mathcal{H}_\infty$ loop shaping [7] and gain scheduling [8].

More recently, thanks to developments in the seemingly unrelated fields of convex optimization and real algebraic geometry, algorithmic methods based on polynomial non-negativity permit the analysis of nonlinear systems directly [9–11]. On the basis of these methods, the region of attraction of an aircraft model is investigated in [12], and adaptive control laws for aircraft models with delays are investigated in [13].

The main result of this work is a collection of sum of squares (SOS) optimization problems that can be used to verify the robust stability and performance of an uncertain nonlinear model of an F/A-18 in qLPV form. In particular, we consider the closed loop stability of the F/A-18 model subject to uncertainty in its short-period dynamics under the feedback control of a nonlinear dynamic inversion controller [14]. This work can be considered as an extension to the work in [15], which considers an qLPV model subject to polytopic uncertainty. Algorithmic methods for obtaining Lyapunov functions and estimating regions of attraction, reachable sets and nonlinear Hankel norm bounds for the closed loop nonlinear uncertain system are presented that use the SOS decomposition and dissipation inequalities [16]. Preliminary results obtained in this paper appeared in [17].

The paper is set out as follows: In Section 2, the SOS decomposition is described. Next, in Section 3, SOS techniques are used to obtain various stability and performance certificates. The nonlinear aircraft model and NDLI controller derived and the SOS programmes formulated in Section 4 are applied before the paper is concluded in Section 5.

2. PRELIMINARIES

$\mathcal{R}^n$ denotes the n-dimensional Euclidean space; for a vector $x \in \mathcal{R}^n$, the standard Euclidean 2-norm is denoted by $|x|$. For time-varying signals, $\|\cdot\|$ denotes the $\mathcal{L}_2$ norm.

Let $\mathcal{R}_m[x] \triangleq \mathcal{R}[x_1, \ldots, x_n]$ denote the commutative ring of real valued multivariable polynomials in $x_1, \ldots, x_n$ of maximum degree m. When it is clear from context, we will simply use $\mathcal{R}[x]$. Furthermore, let $\Sigma[x]$ denote the set of polynomials that are SOS of fixed degree. A polynomial, $p(x)$, is said to be an SOS if it can be written as

$$p(x) = \sum_{i=1}^{M} h_i^2(x).$$

Clearly, $p(x) \in \Sigma[x] \Rightarrow p(x) \geqslant 0$ for all $x \neq 0$ (the converse is not necessarily true). Equivalently [9], $p(x) \in \Sigma[x]$ if and only if there exists a vector of monomials $Z(x)$ and a positive semidefinite matrix Q such that

$$p(x) = Z^T(x) Q Z(x).$$

This formulation allows SOS decompositions to be computed using convex optimization techniques, specifically semidefinite programming [18]. Throughout this work, we will formulate and solve all SOS optimization problems using the freely available MATLAB toolbox SOSTOOLS [19] in conjunction with the SDP solver SeDuMi [20]. The methods presented in this paper rely on using the

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SOS decomposition to construct Lyapunov functions for nonlinear systems; detailed descriptions of how this can be carried out are presented in [11, 21].

3. RESULTS

In this paper, we will consider qLPV system models, which can be thought of as an extension to the class of models covered by differential inclusions [22]. A qLPV model is given by

$$\dot{x} = A(x;\theta)x + B(x;\theta)u \quad \triangleq \quad f(x,u;\theta) \tag{1a}$$

$$y = C(x;\theta) \quad \triangleq \quad h(x;\theta) \tag{1b}$$

where $x \in \mathcal{R}^n$ is the state vector, $y \in \mathcal{R}^q$ is the output, $u \in \mathcal{R}^k$ is the input and $\theta \in \Theta \subset \mathcal{R}^p$ is the vector of parameters where Θ is a semialgebraic set given by

$$\Theta = \{\theta \in \mathcal{R}^p | g_i(\theta) \leqslant 0, \quad i = 1, \dots, j\} \tag{2}$$

where the functions g_i are polynomial in θ for $i = 1, \dots, j$. The matrices A, B, C are polynomial in their arguments θ and x. We also define a domain of the state space that we are interested in analyzing the system properties over, which contains the origin, by

$$\mathcal{D} = \{x \in \mathcal{R}^n | g_i(x) \leqslant 0, \quad i = j+1, \dots, k\}, \tag{3}$$

with g_i for $i = j+1, \dots, k$ being polynomial functions of x. One of the advantages of the qLPV model representation (1) is that the equilibrium point $x^* = 0$ by intent does not change as a function of the uncertainty in the system.

3.1. Robust stability

The first system property we are interested in is that of *robust stability* of the unforced dynamical system

$$\dot{x} = f(x, 0; \theta) \tag{4}$$

corresponding to system (1a) with $u = 0$. It is assumed that without loss of generality, the equilibrium point of interest is $x^* = 0$. Throughout this paper, it is also assumed that the vector field f is sufficiently smooth so as to ensure local existence and uniqueness of solutions for all $x_0 \in \mathcal{D}$. Such an assumption is justified as we are dealing with polynomial vector fields.

The analysis questions posed in this work will be formulated using the Lyapunov framework [23]. Frequently, we will search for a function $V(x)$, which is positive definite, that is, $V(x) > 0, \forall x \neq 0$. However, SOS programming only guarantees that $V(x) \geqslant 0$. The requirement that $V(x)$ be positive definite rather than positive semidefinite can be ensured by imposing the requirement that $V(x) - \varphi(x) \geqslant 0$, $\forall x$ where $\varphi(x)$ is positive definite. Typical choices of $\varphi(x)$ include

$$\varphi(x) = \epsilon |x|^2, \quad \epsilon > 0 \tag{5a}$$

$$\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^{m} \epsilon_{ij} > \gamma \quad \forall i = 1, \dots, n, \quad \gamma > 0, \quad \epsilon_{ij} \geqslant 0 \quad \forall i, j. \tag{5b}$$

It then follows that $V(x) - \varphi(x) \geqslant 0 \Rightarrow V(x) \geqslant \varphi(x) > 0$.

We begin by presenting a generalization of the Lyapunov stability theorem:

Proposition 1 ([21]*)*
Let $x^* = 0$ be an equilibrium point of (4). By assumption, we have $x^* \in \mathcal{D}$ where $\mathcal{D}$ is defined as in (3). Let there exist a continuously differentiable polynomial function $V(x)$ and positive definite functions $\varphi_i(x)$, $i = 1, 2$ and non-negative polynomials $q_i(x, \theta)$, $r_l(x, \theta)$, for $i = 1, \dots, j$ and $l = j+1, \dots, k$ such that

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1. $V(x) - \varphi_1(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad V(0) = 0,$
2. $-\frac{\partial V}{\partial x} f(x, 0; \theta) + \sum_{i=1}^{j} q_i(x, \theta) g_i(\theta) + \sum_{l=j+1}^{k} r_l(x, \theta) g_l(x) - \varphi_2(x) \geqslant 0$,

then x^* is robustly asymptotically stable.

Proof
The first condition imposes that $V(x)$ is positive definite; that is, $V(x) > 0\ \forall x$, apart from $x = 0$, where $V(0) = 0$. Denote by $\dot{V}(x)$, the derivative of $V(x)$ with respect to time, that is, $\dot{V}(x) = \frac{\partial V}{\partial x} f(x, 0; \theta)$. The second condition can be written as

$$-\dot{V}(x) > -\sum_{i=1}^{j} q_i(x, \theta) g_i(\theta) - \sum_{l=j+1}^{k} r_l(x, \theta) g_l(x), \tag{6}$$

which when $\theta \in \Theta$ and $x \in \mathcal{D}$, the non-negativity of q_i and r_i makes the right-hand side of (6) positive. Thus, $\dot{V} < 0$ when $\theta \in \Theta$ and $x \in \mathcal{D}$. Therefore, $V(x)$ is a Lyapunov function for (4) and the equilibrium point is robustly asymptotically stable. □

This proposition can be thought of as a generalized nonlinear version of the S-procedure [24] using Positivstellensatz, a result from real algebraic geometry [25]. It is by relaxing the polynomial inequality constraints in Proposition 1 to the existence of an SOS decomposition that it is possible to construct Lyapunov functions algorithmically using convex optimization.

Programme 1
Select even integers $a, m_i, n_i > 0$ and search for a polynomial $V(x) \in \mathcal{R}_a[x]$, where $V(0) = 0$, SOS polynomials $q_i \in \Sigma_{n_i}[x, \theta]$ for $i = 1, \ldots, j$ and $r_i \in \Sigma_{m_i}[x, \theta]$ for $i = j+1, \ldots, k$, and positive definite functions $\varphi_i(x)$, $i = 1, 2$ such that

$$V(x) - \varphi_1(x) \in \Sigma[x]$$

$$-\frac{\partial V}{\partial x} f(x, 0; \theta) + \sum_{i=1}^{j} q_i(x, \theta) g_i(\theta) + \sum_{l=j+1}^{k} r_l(x, \theta) g_l(x) - \varphi_2(x) \in \Sigma[x, \theta]$$

where the functions $\varphi_i(x)$ are of the form (5a) or (5b), then x^* is a robustly stable equilibrium point and $V(x)$ is a Lyapunov function for (4).

It is assumed that the degree bounds (a, m_i, n_i) have been chosen *a priori*. If Programme 1 returns an infeasible solution, less conservative solutions may be found by increasing the degree of the polynomials, that is, by increasing a, m_i and n_i. This assumption is true for all SOS programmes although we shall only state it here once. The trade-off is that as the degree bounds are increased, the computational cost does so too. Work toward reducing the computational burden of high-order SOS programmes based on system decomposition is presented in [26].

In general, it is not possible to analytically obtain bounds on the degree of any of the polynomial multipliers or indeed the Lyapunov function for uncertain systems. However, progress toward this goal is being made. It has been shown that if a system is nominally exponentially stable on a bounded region of the state space, then a polynomial Lyapunov function must exist [27]. Furthermore, a bound on the degree of an SOS Lyapunov function has also been identified [28]. Work along similar lines has shown that when the Lyapunov function degree is fixed, SOS techniques can fail to find a Lyapunov function when one is known to exist; however, if the degree is allowed to increase and the vector field is homogenous, then existence of a polynomial Lyapunov function implies existence of an SOS Lyapunov function [29].

In addition to stability, it is often desirable to determine the Region of Attraction (RoA), denoted R_A, of an equilibrium point of the unforced system (4). Let $\phi(t; x_0, \theta)$ be the solution to (4) with initial state x_0 at $t = 0$ and $\theta \in \Theta$. Note that this implies existence and uniqueness of ϕ. The region of attraction corresponds to the volume of state space such that

$$R_A = \{x_0 \in \mathcal{D} | \forall \theta \in \Theta, \phi(t; x_0, \theta) \to 0 \text{ as } t \to \infty\}.$$

Determining the set R_A exactly is a nontrivial task, and no analytic solutions exist for general nonlinear systems. Most region of attraction algorithms attempt to (under)approximate R_A by simple, more computationally amenable set descriptions such as ellipses and polytopes using the linear differential inclusion framework [22]. More recently, SOS methods have been applied to RoA analysis [10,30], although these typically lead to bilinear matrix inequalities, which are non-convex [31]. The most common method for approximating R_A is to determine the largest level curve of a Lyapunov function that is completely contained in $\mathcal{D}$. Define the set enclosed by the level curve $V(x) = c$ by

$$\Omega_c = \{x \in \mathcal{R}^n | V(x) \leqslant c\}. \tag{7}$$

When $\Omega_c \subset \mathcal{D}$ and $V(x)$ is a Lyapunov function of (4), it follows that Ω_c is a compact positively invariant subset of $\mathcal{D}$ and is thus an estimate of the RoA. We now propose an algorithm based on the SOS decomposition and the Positivstellensatz that computes the largest level set of V, that is, the maximum c, such that $\Omega_c \subset \mathcal{D}$.

Programme 2
Given a Lyapunov function $V \in \mathcal{R}_a[x]$ for (4) and a non-negative integer κ, search for SOS polynomials $d_i \in \Sigma_{b_i}[x]$ for $i = j+1, \ldots, k$ that solve

$$\begin{aligned} \max_{d_i \in \Sigma_{b_i}[x], \gamma > 0} \quad & \gamma \\ \text{s.t.} \quad & |x|^{2\kappa}(V(x) - \gamma) - d_l(x) g_l(x) \in \Sigma[x], \quad l = j+1, \ldots, k. \end{aligned}$$

It then follows that $\Omega_\gamma \subset \mathcal{D}$ and the maximum level set of V contained in $\mathcal{D}$ is given by $\{x \in \mathcal{R}^n | V(x) = \gamma\}$.

Less conservative solutions can be obtained by increasing κ and the degree of the SOS polynomial multipliers b_i. A full derivation of this SOS programme is provided in the Appendix.

3.2. *Input-to-state properties*

In this section, we turn our attention to analyzing input-to-state properties of the qLPV system (1a). We are interested in determining the set of reachable states in $\mathcal{R}^n$ when the input energy is bounded by $\epsilon > 0$, that is, $\int_0^T u^T u dt \leqslant \epsilon$. The set of reachable states is denoted by R_ϵ and formally defined as

$$R_\epsilon \triangleq \left\{ x(T) \,\middle|\, (x,u) \text{ satisfy (1a)}, x(0) = 0, \int_0^T u^T u dt \leqslant \epsilon,\ T \geqslant 0 \right\}.$$

Following the same methodology as applied in Programme 2, an estimate of the reachable set is the ϵ-level set of an appropriately chosen Lyapunov function; $R_\epsilon \subset \{x \in \mathcal{R}^n | V(x) \leqslant \epsilon\} \subset \mathcal{D}$.

Proposition 2 ([17]*)*
Given an qLPV system of the form (1a) and a positive definite function $\varphi(x)$, if there exists a continuously differentiable function $V(x)$ such that

1. $V(x) - \varphi(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad V(0) = 0,$
2. $\frac{\partial V}{\partial x} f(x,u;\theta) \leqslant u^T u, \forall (x,\theta) \in \mathcal{D} \times \Theta,$
3. The set $\{x \in \mathcal{R}^n | V(x) \leqslant \epsilon\} \subset \mathcal{D},$

then $R_\epsilon \subset \{x \in \mathcal{R}^n | V(x) \leqslant \epsilon\}$ when the input signal satisfies $\|u\|_2^2 \leqslant \epsilon$.

Proof
For the case when $u = 0$, it is clear that the first two conditions ensure that $V(x)$ is a Lyapunov function for the unforced system $\dot{x} = f(x, 0; \theta)$. For the more interesting case where $u \neq 0$, we use a storage function argument as follows: Integrating both sides of Condition 2 from 0 to T gives

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$$V(x(T)) - V(x(0)) \leqslant \int_0^T u^T u dt.$$

By construction, we have that $V(x(0)) = V(0) = 0$ and noting that $\int_0^T u^T u dt \leqslant \int_0^\infty u^T u dt = \|u\|_2^2$, we obtain

$$V(x(T)) \leqslant \|u\|_2^2 \leqslant \epsilon$$

for every $T \geqslant 0$ and $\|u\|_2^2 \leqslant \epsilon$. Finally, Condition 3 ensures that $\{x \in \mathcal{R}^n | V(x) \leqslant \epsilon\} \subset \mathcal{D}$ and therefore contains the subset R_ϵ. □

The results of Proposition 2 are implemented in the following SOS programme:

Programme 3
Given a non-negative integer κ, even integers $a, m_i, n_i, p_i > 0$, search for a positive definite function $\varphi(x)$, a polynomial function $V \in \mathcal{R}_a[x]$, with $V(0) = 0$, SOS polynomials $q_i \in \Sigma_{n_i}[x, \theta]$ for $i = 1, \ldots, j$, $r_i \in \Sigma_{m_i}[x, \theta]$ for $i = j+1, \ldots, k$, $s_i \in \mathcal{R}_{p_i}[x]$ for $i = j+1, \ldots, k$ and an $\epsilon > 0$ such that:

$$\begin{aligned}
\max_{\substack{\epsilon > 0, r_i \in \Sigma_{m_i}[x,\theta] \\ q_i \in \Sigma_{n_i}[x,\theta], s_i \in \Sigma_{p_i}[x]}} \quad & \epsilon \\
\text{s.t.} \quad & V(x) - \varphi(x) \in \Sigma[x] \\
& -\frac{\partial V}{\partial x} f(x, u; \theta) + u^T u + \sum_{i=1}^{j} q_i(x, \theta) g_i(\theta) \\
& + \sum_{l=j+1}^{k} r_l(x, \theta) g_l(x) \in \Sigma[x, \theta] \\
& |x|^{2\kappa}(V(x) - \epsilon) - s_i(x) g_i(x) \in \Sigma[x], \quad i = j+1, \ldots, k.
\end{aligned}$$

The function φ should be constructed as described by (5a) or (5b). The reachable set R_ϵ is contained within the set $\{x \in \mathcal{R}^n | V(x) \leqslant \epsilon\}$.

3.3. *Output energy*

The dual problem to determining the set of reachable states from a finite energy input is that of determining the maximum output energy achievable from a given initial condition over the uncertainty set Θ. Given an initial condition x_0, the maximum output energy, denoted by ϵ^*, of an qLPV system of the form (1a–1b), is given by

$$\epsilon^* = \max\left\{ \int_0^\infty y^T y dt \,\middle|\, \dot{x} = f(x, 0; \theta), \quad y = h(x; \theta), \quad x(0) = x_0 \right\}, \tag{8}$$

where the maximum is taken over the set Θ.

Proposition 3
Consider the unforced ($u = 0$) qLPV system (1a–1b) and positive definite function $\varphi(x)$. If there exists a continuously differentiable function $V(x)$ such that

1. $V(x) - \varphi(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad V(0) = 0,$
2. $\frac{\partial V}{\partial x} f(x, 0; \theta) \leqslant -y^T y$, $\forall (x, \theta) \in \mathcal{D} \times \Theta$ and (x, y) satisfying (1) with $u = 0$,

then $V(x_0)$ is an upper bound for ϵ^* over the uncertainty set Θ for the initial condition $x(0) = x_0 \in \mathcal{D}$.

Proof
Integrating both sides of Condition 2 from 0 to $T \geqslant 0$ gives

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$$V(x(T)) - V(x(0)) \leqslant -\int_0^T y^T y dt$$

which is true for all $T \geqslant 0$. Because $V(x)$ is a Lyapunov function, we have that $V(x(T)) \geqslant 0$; thus, it follows that $V(x(0)) \geqslant \int_0^T y^T y dt$. Note that as $T \to \infty$ so $x(T) \to 0$ and $V(x(T)) \to 0$ and so the upper bound tightens. □

In the following equations, we formulate an SOS programme for computing an upper bound on ϵ^* using the previous proposition.

Programme 4
Given an initial condition $x_0 \in \mathcal{D}$ and positive definite function $\varphi(x)$, find a polynomial function $V \in \mathcal{R}_a[x]$ where $V(0) = 0$ and SOS polynomials $q_i \in \Sigma_{n_i}[x,\theta]$ for $i = 1,\ldots,j$ and $r_i \in \Sigma_{m_i}[x,\theta]$ for $i = j+1,\ldots,k$ that solve

$$\begin{aligned}
\min_{V\in\mathcal{R}_a[x],q_i\in\Sigma_{n_i}[x,\theta],r_l\in\Sigma_{m_i}[x,\theta]} \quad & V(x_0) \\
\text{s.t.} \quad & V(x) - \varphi(x) \in \Sigma[x] \\
& -\frac{\partial V}{\partial x} f(x,0;\theta) - y^T y + \sum_{i=1}^{j} q_i(x,\theta) g_i(\theta) \\
& + \sum_{l=j+1}^{k} r_l(x,\theta) g_l(x) \in \Sigma[x,\theta].
\end{aligned}$$

Then $V(x(0)) \geqslant \epsilon^*$.

3.4. $\mathcal{L}_2 \to \mathcal{L}_2$ gain

The input-to-output gain of a system is the maximum ratio of the output signal energy to the input signal energy where energy is defined as the $\mathcal{L}_2$ norm of a signal; hence, it also denoted as the $\mathcal{L}_2 \to \mathcal{L}_2$ gain. The $\mathcal{L}_2 \to \mathcal{L}_2$ norm of an qLPV system of the form (1) is given by

$$\sup_{\|u\|_2\neq 0,\theta\in\Theta} \frac{\|y\|_2}{\|u\|_2}, \quad x(0) = 0. \tag{9}$$

The following proposition uses a Storage function argument to compute bounds on the $\mathcal{L}_2 \to \mathcal{L}_2$ gain of a system.

Proposition 4
Given an qLPV system of the form (1) and a positive definite function $\varphi(x)$ such that

1. $V(x) - \varphi(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad V(0) = 0,$
2. $-\frac{\partial V}{\partial x} f(x,u;\theta) - y^T y + \gamma^2 u^T u \geqslant 0 \; \forall (x,\theta) \in \mathcal{D} \times \Theta$

then the $\mathcal{L}_2 \to \mathcal{L}_2$ gain of (1) is less than γ.

Proof
Integrating Condition 2 from 0 to $T \geqslant 0$ and rearranging terms, we obtain

$$\begin{aligned}
-(V(x(T)) - V(x(0))) &\geqslant \int_0^T \left(y^T y - \gamma^2 u^T u\right) dt \\
&= \int_0^T y^T y dt - \gamma^2 \int_0^T u^T u dt.
\end{aligned}$$

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By definition $x(0) = 0$ and from Condition 1, we have $V(x(0)) = 0$ and $V(x(T)) \geqslant 0$. Rearranging the previous inequality gives

$$\frac{\int_0^T y^T y dt}{\int_0^T u^T u dt} + \frac{V(x(T))}{\int_0^T u^T u dt} \leqslant \gamma^2$$

for all $T \geqslant 0$. As $T \to \infty$ stability of the equilibrium point means $V(x(T)) \to 0$ and thus

$$\frac{\|y\|_2}{\|u\|_2} \leqslant \gamma.$$

□

An SOS programme to compute an upper bound of the input–output gain based on the previous proposition is now given.

Programme 5
Given the qLPV system (1) search for a continuously differentiable polynomial function $V \in \mathcal{R}_a[x]$, a positive definite function $\varphi(x)$ given by (5a) or (5b), and SOS polynomials $q_i \in \Sigma_{n_i}[x, \theta]$ for $i = 1, \ldots, j$ and $r_i \in \Sigma_{m_i}[x, \theta]$ for $i = j+1, \ldots, k$ that solve

$$\begin{aligned}
\min_{\substack{\gamma > 0, V \in \mathcal{R}_m[x], \\ q_i \in \Sigma_{n_i}[x,\theta], r_i \in \Sigma_{m_i}[x,\theta]}} \quad & \gamma \\
\text{s.t.} \quad & V(x) - \varphi(x) \in \Sigma[x] \\
& -\frac{\partial V}{\partial x} f(x,u) - y^T y + \gamma u^T u + \sum_{i=1}^{j} q_i(x,\theta) g_i(\theta) \\
& + \sum_{l=j+1}^{k} r_l(x,\theta) g_l(x) \in \Sigma[x,\theta]
\end{aligned}$$

then the $\mathcal{L}_2 \to \mathcal{L}_2$ gain of (1) is upper bounded by $\sqrt{\gamma}$.

3.5. *Hankel Norm estimation*

Hankel norm operators are frequently used in linear control analysis to synthesize lower-order plant models prior to designing a controller. Related work has extended the concept of control and observability gramians to nonlinear systems [32, 33]. The Hankel norm of an qLPV system is defined as

$$\sup_{u \in \mathcal{L}_2[0,T], \theta \in \Theta} \frac{\sqrt{\int_T^\infty y^T y dt}}{\sqrt{\int_0^T u^T u dt}}.$$

The quantity that we wish to obtain an estimate of is

$$\phi = \max \left\{ \int_T^\infty y^T y dt \;\middle|\; \begin{array}{l} \int_0^T u^T u dt \leqslant \epsilon, \quad x(0) = 0 \\ u(t) = 0 \text{ for } t > T \geqslant 0 \\ (x, u, y) \text{ satisfy (1)} \end{array} \right\}.$$

A Hankel norm estimate is then $\sqrt{\phi}$.

Proposition 5
Given the qLPV system (1) and positive definite functions $\varphi_i(x)$, $i = 1, 2$, if there exist continuously differentiable functions $\{V, W\} \in \mathcal{R}[x]$ such that:

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1. $V(x) - \varphi_1(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad V(0) = 0,$
2. $W(x) - \varphi_2(x) \geqslant 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}, \quad W(0) = 0,$
3. $-\frac{\partial V}{\partial x} f(x, u; \theta) + u^T u \geqslant 0, \quad \forall (x, \theta) \in \mathcal{D} \times \Theta,$
4. $-\frac{\partial W}{\partial x} f(x, 0; \theta) - y^T y \geqslant 0, \quad \forall (x, \theta) \in \mathcal{D} \times \Theta,$

then $\phi \leqslant \sup\{W(x(T)) | V(x(T)) \leqslant \epsilon\}$ and $\sqrt{\phi}$ is an upper bound for the robust nonlinear Hankel norm of (1).

Proof

The proof combines results from Propositions 2 and 3. From the proof of Proposition 2, select an $\epsilon > 0$, then we have $V(x(T)) \leqslant \epsilon$, and from the proof of Proposition 3, we obtain $W(x(T)) \geqslant \int_T^\infty y^T y dt$. Therefore, an estimate of the Hankel norm is

$$\begin{aligned} \phi &\leqslant \sup \left\{ \int_T^\infty y^T y dt \middle| V(x(T)) \leqslant \epsilon \right\} \\ &\leqslant \sup \{ W(x(T)) | \, V(x(T)) \leqslant \epsilon \} . \end{aligned}$$

□

Programme 6

Compute $V(x)$ using SOS programme 3 for some $\epsilon > 0$.
Compute $W(x)$ using SOS programme 4.
Define $h(x) \triangleq V(x) - \epsilon \leqslant 0$. Given a positive integer κ, search for an SOS polynomial $r \in \Sigma[x]$ that solves

$$\begin{aligned} \max_{\delta>0, r \in \Sigma[x]} \quad & \delta \\ \text{s.t.} \quad & |x|^{2\kappa}(W(x) - \delta) - r(x)h(x) \in \Sigma[x, \theta]. \end{aligned}$$

Then $\sqrt{\delta}$ is an upper bound for ϕ.

In the next section, the ideas presented in this section will be illustrated on a nonlinear model of an F/A-18 aircraft.

4. F/A-18 EXAMPLE

4.1. Short-period dynamics

The qLPV model we analyze in this paper approximates the F/A-18 short-period dynamics of a nonlinear, aeroservoelastic aircraft model derived in [34]. The model consists of two states q and $\Delta\alpha$ that represent pitch rate and angle of attack, respectively. The dynamics are given by

$$\begin{bmatrix} \dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{Iyy} M_\alpha(\alpha)_{unc} \\ 1 + \frac{\cos(\alpha)}{mV_o} Z_q(\alpha)_{unc} & \frac{\cos(\alpha)}{mV_o} Z_\alpha(\alpha)_{unc} \end{bmatrix} \begin{bmatrix} q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} \frac{1}{Iyy} M_\epsilon(\alpha)_{unc} \\ \frac{\cos(\alpha)}{mV_o} Z_\epsilon(\alpha)_{unc} \end{bmatrix} \Delta elev \tag{10}$$

where $elev$ is the elevator deflection and $\Delta elev = elev - elev_o$ represents large deviations from the steady-state value $elev_o$. Similarly, $\Delta\alpha = \alpha - \alpha_o$. The terms I_{yy} and m represent the moment of inertia and aircraft mass, respectively. The model is valid around an operating point characterized by the velocity V_o and altitude h_o.

The functions $M_\alpha(\alpha)_{unc}, M_\epsilon(\alpha)_{unc}, Z_q(\alpha)_{unc}, Z_\alpha(\alpha)_{unc}$ and $Z_\epsilon(\alpha)_{unc}$ represent the weighted additive uncertainty in the aerodynamic coefficients given by

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$$
\begin{aligned}
M_\alpha(\alpha)_{unc} &= M_\alpha(\alpha) + \widehat{M}_\alpha w_{M_\alpha}\delta_{M_\alpha} \\
M_\epsilon(\alpha)_{unc} &= M_\epsilon(\alpha) + \widehat{M}_\epsilon w_{M_\epsilon}\delta_{M_\epsilon} \\
Z_q(\alpha)_{unc} &= Z_q(\alpha) + \widehat{Z}_q w_{Z_q}\delta_{Z_q} \\
Z_\alpha(\alpha)_{unc} &= Z_\alpha(\alpha) + \widehat{Z}_\alpha w_{Z_\alpha}\delta_{Z_\alpha} \\
Z_\epsilon(\alpha)_{unc} &= Z_\epsilon(\alpha) + \widehat{Z}_\epsilon w_{Z_\epsilon}\delta_{Z_\epsilon}
\end{aligned}
$$

where $M_\alpha(\alpha), \ldots, Z_\epsilon(\alpha)$ are polynomial functions of α obtained from a least-squares optimization on the raw data from the aerodynamic lookup table [34]. The functions δ are time-varying and satisfy $|\delta_i(t)| \leqslant 1$, the constants $w_{M_\alpha}, \ldots, w_{Z_\epsilon}$ satisfy $w_i > 0$, and finally, the constant terms $\widehat{M}_\alpha, \ldots, \widehat{Z}_\epsilon$ represent the average moment or force over the range α (pitch rate).

This choice of uncertainty model is best illustrated by way of example. In Figure 1, we vary the weight w_{M_ϵ} and plot the uncertainty bands created in $M_\epsilon(\alpha)$. This corresponds to uncertainty in the moment due to the elevator deflection over the range $\alpha \in [-7, 14]$.

4.2. *Nonlinear Dynamic Inversion control law*

Nonlinear Dynamic Inversion (NDI) control laws are a popular choice for control design of aerospace vehicles as the controllers derived are valid for the entire flight envelope, thus avoiding the need for gain scheduling methods. Given the nonlinear model of the F/A-18 short-period dynamics (10), the NDI control law we consider is given by

$$\Delta_{elev} = \left(\frac{M_\epsilon(\alpha)}{I_{yy}}\right)^{-1}\left(v - \frac{M_\alpha(\alpha)}{I_{yy}}\Delta\alpha\right) \tag{11}$$

which is designed to achieve pitch rate control based on the *nominal* model. The command pitch acceleration, v, is the external signal, which is produced via a simple proportional control law, $v = \omega_q(q_{com} - q)$, where the scalar gain ω_q is selected such that the input–output response of the nominal model is

$$\frac{q(s)}{q_{com}(s)} = \frac{5}{s+5}. \tag{12}$$

It is by combining the uncertain short-period dynamics model (10), the NDI control law (11) and the proportional controller (12) that the closed loop system in qLPV form becomes

$$\begin{bmatrix} \dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \left(A_{inv}(\alpha,\delta) - w_q[B_{inv}\ 0_{2\times 1}]\right)\begin{bmatrix} q \\ \Delta\alpha \end{bmatrix} + w_q B_{inv}(\alpha,\delta) q_{com}, \tag{13}$$

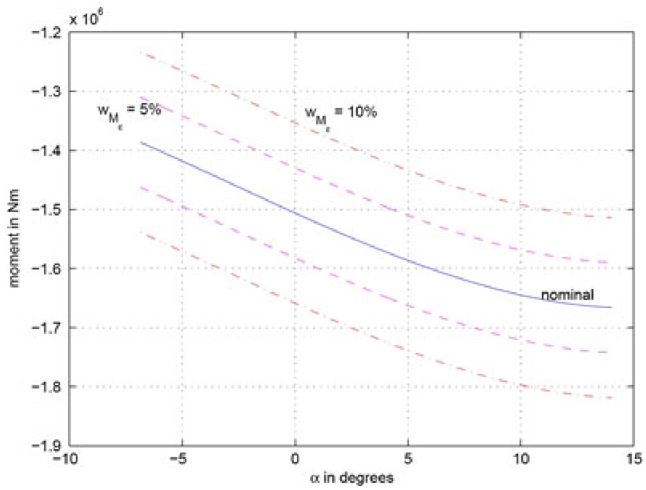

Figure 1. Uncertainty in the aerodynamic coefficient $M_\epsilon(\alpha)$ for choices of w_{M_ϵ} corresponding to a 5% and 10% deviation from nominal operating conditions.

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where $\delta \in \mathcal{R}^5$ is the vector of uncertain parameters $\delta = \left[\delta_{m_\alpha}, \delta_{m_\epsilon}, \delta_{z_q}, \delta_{z_\alpha}, \delta_{z_\epsilon}\right]^T$ and

$$A_{inv}(\alpha, \delta) = \begin{bmatrix} 0 & A_{inv}^{12} \\ A_{inv}^{21} & A_{inv}^{22} \end{bmatrix}, \tag{14}$$

with

$$A_{inv}^{12} = \frac{\widehat{M}_\alpha}{I_{yy}} w_{M_\alpha} - \frac{M_\alpha(\alpha)}{M_\epsilon(\alpha)} \frac{\widehat{M}_\epsilon}{I_{yy}} w_{M_\epsilon} \delta_{M_\epsilon},$$

$$A_{inv}^{21} = 1 + cs(\alpha)\left(\frac{Z_q(\alpha)}{mV_o} + \frac{\widehat{Z}_q}{mV_o} w_{Z_q} \delta_{Z_q}\right),$$

$$A_{inv}^{22} = cs(\alpha)\left(\frac{Z_\alpha(\alpha)}{mV_o} + \frac{\widehat{Z}_\alpha}{mV_o} w_{Z_\alpha} \delta_{Z_\alpha}\right) - cs(\alpha)\frac{M_\alpha(\alpha)}{M_\epsilon(\alpha)}\left(\frac{Z_\epsilon(\alpha)}{mV_o} + \frac{\widehat{Z}_\epsilon}{mV_o} w_{Z_\epsilon} \delta_{Z_\epsilon}\right),$$

and

$$B_{inv}(\alpha, \delta) = \begin{bmatrix} 1 + \frac{\widehat{M}_\epsilon}{M_\epsilon(\alpha)} w_{M_\epsilon} \delta_{M_\epsilon} \\ cs(\alpha)\frac{I_{yy}}{M_\epsilon(\alpha)}\left(\frac{Z_\epsilon(\alpha)}{mV_o} + \frac{\widehat{Z}_\epsilon}{mV_o} w_{Z_\epsilon} \delta Z_\epsilon\right) \end{bmatrix}, \tag{15}$$

where $cs(\alpha) = \left(1 - \frac{1}{2}(\Delta\alpha + a_0)^2\right)$ is an approximation of $\cos(\alpha)$.

4.3. Robust stability

We begin by analyzing the robust stability of the qLPV model of the closed loop F/A-18 short-period dynamics under the influence of an NLDI controller described by (13) with $q_{com} = 0$. Throughout this section, it is assumed that $V_o = 150$ m/s at sea-level, for $w_{M_\alpha} = \cdots = w_{Z_\epsilon} = w$. In Figure 2(A), the stability of the nominal system model is analyzed, and the region of attraction estimate is plotted using a quadratic and quartic Lyapunov function. The RoA estimate is computed using SOS programme 1 to compute the Lyapunov function and SOS programme 2 to enlarge the RoA. It is clear that the higher-order Lyapunov function produces a larger RoA estimate, that is, a less conservative estimate of the RoA.

Introducing uncertainty into the model at levels $w_i = 0.2$ and $w_i = 0.4$ for $i = 1, \ldots, 5$, the RoA estimates decrease as the uncertainty increases. This is shown in Figure 2(B) using a fourth-order Lyapunov function. For comparison, the nominal estimate obtained previously is also shown.

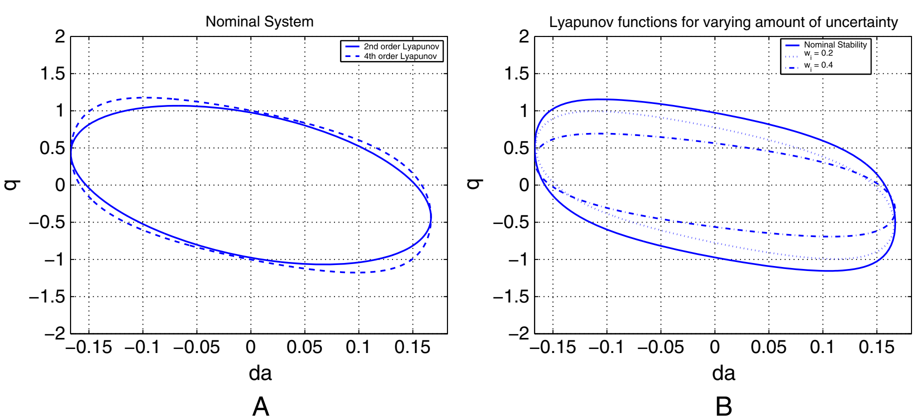

Figure 2. (A): Region of attraction estimates for the nominal system obtained using quadratic (solid line) and fourth-order (dashed) Lyapunov functions. (B): RoA estimates for uncertainty levels $w_i = 0.2$ (dotted line) and $w_i = 0.4$ (dash-dot line) using a fourth-order Lyapunov function.

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4.4. *Input-state-output performance*

We now turn our attention to analyzing the input–output properties of the F/A-18 aircraft. To begin with, let us determine the set of reachable states (R_ϵ) in the state space given a bounded $\mathcal{L}_2$-norm input q_{com}. The region of the state space we consider for this analysis is

$$\mathcal{D} = \{\Delta\alpha | -0.166 \text{ rad} \leqslant \Delta\alpha \leqslant 0.166 \text{ rad}\}.$$

It is by using SOS programme 3 that the R_ϵ can be estimated. In Figure 3, the maximum deviation of the angle of attack, $\Delta\alpha$, for increasing levels of uncertainty is plotted as a function of input energy. Consider for example the uncertainty level $w = 0.1$: using Figure 3, it can be seen that when $\|q_{com}\| < 0.15$ then $\Delta\alpha$ will not deviate by more than 6 degrees from its equilibrium value. If the level of uncertainty is increased to $w = 0.4$, less energy in the input ($\|q_{com}\| < 0.1$) can be tolerated to keep $\Delta\alpha$ within the same range.

It is by using Programme 5 that an upper bound on the $\mathcal{L}_2 \rightarrow \mathcal{L}_2$ gain from input, q_{com}, to output (which we select as pitch rate q) can be computed. The results for different levels of uncertainty are shown in Figure 4(A). To interpret these plots, they must be considered in conjunction with the previous plot. Let us continue with the running example: for all pitch rate reference signals that

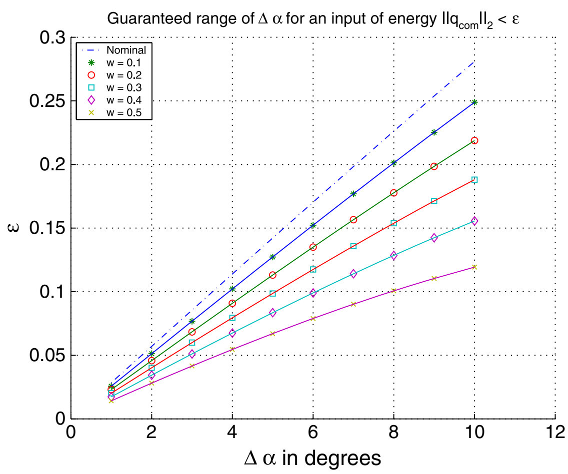

Figure 3. Guaranteed range of $\Delta\alpha$ for an input energy $\|q_{com}\| \leqslant \epsilon$ for varying magnitudes of parametric uncertainty.

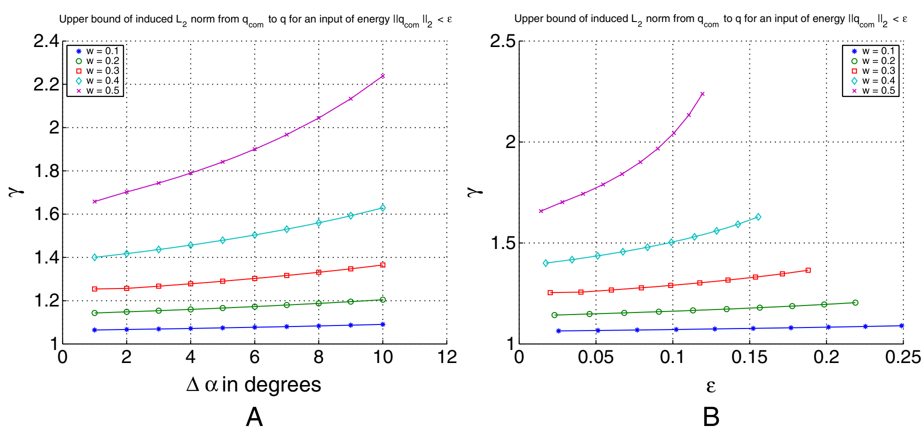

Figure 4. (A):Robust performance plotted as a function of the deviation in the angle of attack, $\Delta\alpha$, for various uncertainty levels. The deviation in angle of attack is shown on the x-axis and an upper bound of the $\mathcal{L}_2 \rightarrow \mathcal{L}_2$-gain on the y-axis. (B): Robust performance plotted as a function of the magnitude of the input signal, $\|q_{com}\| < \epsilon$, for varying levels of uncertainty.

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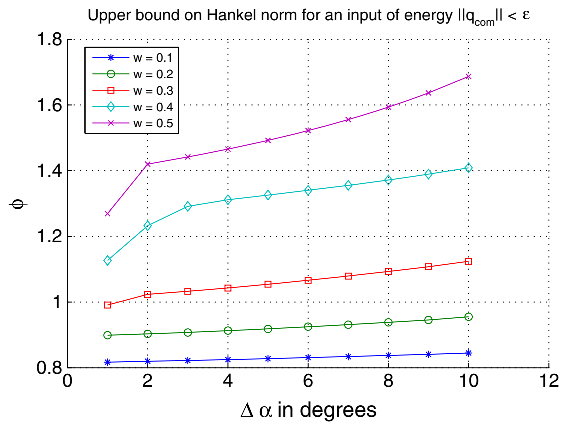

Figure 5. Hankel norm estimates for varying levels of uncertainty plotted as a function of $\Delta\alpha$.

satisfy $\|q_{com}\| \leqslant 0.15$, it is guaranteed that the angle of attack will not deviate by any more than 6 degrees from its equilibrium value and the $\mathcal{L}_2$-norm of the output will satisfy $\|q\| < 1.07\|q_{com}\|$ for any amount of uncertainty $w \leqslant 0.1$.

The previous results can be used to show how the $\mathcal{L}_2$-norm bound on the reference input affects the full $\mathcal{L}_2 \rightarrow \mathcal{L}_2$ system gain. This is shown in Figure 4(B) where it is clear that the performance degrades, that is, the gain from input-to-output increases, as the uncertainty in the model is increased. Furthermore, as the uncertainty level increases, the *rate* at which the gain increases (as a function of the input signal magnitude, ϵ) also increases. These results are plotted for the same uncertainty levels as in Figure 4(A).

The F/A-18's performance with respect to the nonlinear Hankel norm estimation is shown in Figure 5. As with the $\mathcal{L}_2 \rightarrow \mathcal{L}_2$ performance shown in Figures 4(A–B), it can be seen that the performance decreases as the uncertainty in the model increases. Here, we show how the performance varies with respect to a change in the angle of attack.

5. CONCLUSION

In this paper, it has been shown how robust and nonlinear analysis of qLPV systems can be performed using the SOS decomposition of mutivariable polynomials. Algorithmic methods based on Lyapunov techniques and convex optimization are explicitly derived and presented for a variety of input–output system properties such as $\mathcal{L}_2$-gain, reachable sets and nonlinear Hankel norm estimations. The results are illustrated on a model of the F/A-18 short-period dynamics under a nonlinear dynamic inversion feedback control law.

APPENDIX

Here, we present a version of the Positivstellensatz theorem from real algebraic geometry. This result is then used to derive the RoA SOS programme 2. The following definitions are required before stating the main theorem:

Definition 1 (Ideal)
Given the multivariate polynomials $\{g_i, \ldots, g_m\} \in \mathcal{R}[x]$, the *Ideal* generated by g_i for $i = 1, \ldots, m$ is the set

$$\mathcal{I}(g_i, \ldots, g_m) = \left\{ \sum_{i=1}^{m} t_i g_i \,\middle|\, t_1, \ldots, t_m \in \mathcal{R}[x] \right\}.$$

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Definition 2 (Multiplicative monoid)
Given polynomials $\{h_i, \ldots, h_m\} \in \mathcal{R}[x]$, the *Multiplicative Monoid*, denoted $\mathcal{M}(h_1, \ldots, h_m)$, generated by h_i for $i = 1, \ldots, m$ is the set of all finite products of h_i including 1.

Definition 3 (Cone)
Given the multivariate polynomials $\{k_1, \ldots, k_m\} \in \mathcal{R}[x]$, the Algebraic Cone generated by k_i for $i = 1, \ldots, m$ is the set

$$\mathcal{C}(k_1, \ldots, k_m) = \left\{ s_0 + \sum_{i=1}^{r} s_i F_i \,\middle|\, F_i \in \mathcal{M}(k_1, \ldots, k_m), s_i \in \Sigma[x] \right\}$$

where r denotes the number polynomials in $\mathcal{M}$ and $\Sigma[x]$ denotes the cone of SOS polynomials in x.

Theorem 1 (Positivstellensatz)
Let $(f_i)_{\{i=1,\ldots,l\}}$, $(g_j)_{\{j=1,\ldots,m\}}$ and $(h_k)_{\{k=1,\ldots,p\}}$ be finite families of polynomials in $\mathcal{R}[x]$. Denote by $\mathcal{C}$ the algebraic cone generated by $(f_i)_{\{i=1,\ldots,l\}}$, $\mathcal{I}$ the ideal generated by $(g_j)_{\{j=1,\ldots,m\}}$ and $\mathcal{M}$ the multiplicative monoid generated by $(h_k)_{\{k=1,\ldots,p\}}$. Then the following statements are equivalent:

- The set
$$\left\{ x \in \mathcal{R}^n \,\middle|\, \begin{array}{ll} f_i(x) \geqslant 0, & i = 1, \ldots, l \\ g_j(x) = 0, & j = 1, \ldots, m \\ h_k(x) \neq 0, & k = 1, \ldots, p \end{array} \right\}$$
is empty.
- There exist $f \in \mathcal{C}$, $g \in \mathcal{I}$ and $h \in \mathcal{M}$ such that $f + g + h^2 = 0$.

We now show how the previous result can be used to find the largest level set of a Lyapunov function contained in a domain.

Assume that a polynomial Lyapunov function $V(x)$ has been obtained using SOS programme 1 and the domain of interest $\mathcal{D}$ is defined by a single polynomial $p_1(x) \leqslant 0$. We wish to find the largest $\gamma > 0$ such that the set $\{x \in \mathcal{R}^n | V(x) < \gamma, p_1(x) \geqslant 0\} = \emptyset$. Equivalently, this set can be written as

$$\{x \in \mathcal{R}^n | -(V(x) - \gamma) \geqslant 0, p_1(x) \geqslant 0, V(x) - \gamma \neq 0\} = \emptyset. \quad (16)$$

It is by applying the second equivalence relation in Theorem 1 that the empty set (16) is equivalent to the existence of SOS polynomials $r_1, r_2, r_3, r_4 \in \Sigma[x]$ and positive integer k, such that

$$r_1 - r_2(V - \gamma) + r_3 p_1 - r_4(V - \gamma) p_1 + (V - \gamma)^{2k} = 0. \quad (17)$$

Setting $r_1 = r_3 = 0$ and choosing $k = 1$ gives

$$-r_2(V - \gamma) - r_4(V - \gamma) p_1 + (V - \gamma)^2 = 0,$$

dividing through by $(V - \gamma)$ and rearranging, we obtain

$$(V - \gamma) - r_4 p_1 = r_2. \quad (18)$$

This is now a convex SOS programme. The largest level set of V contained in $\mathcal{D}$ is obtained by maximizing γ over (18) which is the condition presented in SOS programme 2. It is by multiplying the $(V - \gamma)$ term by $|x|^{2\kappa}$ (where κ is a positive integer) that a nested hierarchy of tests is obtained. Uncertainty descriptions are included in exactly the same manner.

For more complicated domain descriptions that consist of multiple polynomial inequalities, that is, when $\mathcal{D} = \{p_1(x) \leqslant 0, \ldots, p_m(x) \leqslant 0\}$, the level set $V(x) - \gamma$ must be contained in each $p_i(x) \leqslant 0$; thus, the SOS programme must determine the largest γ that (18) holds *for all* p_i's.

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