

## ADVANCES IN COMPUTATIONAL LYAPUNOV ANALYSIS USING SUM-OF-SQUARES PROGRAMMING

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**ABSTRACT.** The stability of an equilibrium point of a nonlinear dynamical system is typically determined using Lyapunov theory. This requires the construction of an energy-like function, termed a Lyapunov function, which satisfies certain positivity conditions. Unlike linear dynamical systems, there is no algorithmic method for constructing Lyapunov functions for general nonlinear systems. However, if the systems of interest evolve according to polynomial vector fields and the Lyapunov functions are constrained to be sum-of-squares polynomials then stability verification can be cast as a semidefinite (convex) optimization programme. In this paper we describe recent advances in sum-of-squares programming that facilitate advanced stability analysis and control design.

**1. Introduction.** One of the most fundamental questions in systems and control theory is that of determining the stability of an equilibrium point of a dynamical system. Consider the linear time invariant (LTI) system  $\dot{x}(t) = Ax(t)$  where  $x(t) \in \mathbb{R}^n$  is the state vector and  $A \in \mathbb{R}^{n \times n}$ . The equilibrium point  $x^* = 0$  is asymptotically stable if and only if  $A$  is Hurwitz, i.e. all eigenvalues of  $A$  lie in the open left half plane. In 1892 the Russian mathematician A. M. Lyapunov showed that for a nonlinear system  $\dot{x}(t) = f(x(t))$  with  $f(0) = 0$  and  $f$  locally Lipschitz, if there exists a radially unbounded positive definite function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $V(0) = 0$  and  $\dot{V} \triangleq \langle \frac{\partial V}{\partial x}, f(x) \rangle < 0$  for all  $x \neq 0 \in \mathbb{R}^n$  then the 0 equilibrium is globally asymptotically stable. For LTI systems this is translates to the following:  $A$  is Hurwitz if and only if for a given square, positive definite matrix  $Q$  there exists a positive definite matrix  $P \in \mathbb{R}^{n \times n}$  that satisfies the *Lyapunov equation*  $A^T P + PA + Q = 0$ . Equivalently, one may search for a positive definite matrix  $P$  such that the inequality  $A^T P + PA \prec 0$  is satisfied. The inequality follows by considering the quadratic Lyapunov function candidate  $V(x) = x^T P x$ . Clearly if the previous inequality holds then the Lie derivative  $x^T (A^T P + PA)x$  is negative definite. The search for the matrix  $P$  subject to  $A^T P + PA \prec 0$  is a convex optimisation problem [7], specifically a Semidefinite Program (SDP) [6], which can be solved efficiently.

For general nonlinear systems, there is no convex optimization method that can be used to search for a Lyapunov function (should one exist) certifying stability of

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an equilibrium point. In fact there is no general algorithm that can construct a Lyapunov function even when a converse theorem guarantees that such a certificate exists. However, if we restrict our attention to dynamical systems whose vector fields evolve according to polynomial functions and only consider Lyapunov function candidates that are representable as the sum of squared polynomial functions, then tractable convex optimization techniques do exist. Although not pursued here, other restrictions on the system such as exponential stability and  $f \in \mathcal{C}^2$  allow for convex optimization based methods, see for example [11]. The link between convexity and the sum-of-squares decomposition was established in the thesis of Parrilo [26]. Broadly speaking, sum-of-squares programming can be seen as a nonlinear (polynomial) expansion of the Linear Matrix Inequality (LMI) framework.

Fifteen years after sum-of-squares techniques were first introduced, the field is still rapidly developing and finding new application areas. In this paper we describe recent advances in the area with application to advanced computational stability analysis, particularly with a view to control applications. For a broader introduction to sum-of-squares theory the reader is referred to [3] for an algebraic flavour and [9] for a robust control perspective.

**1.1. Notation.**  $\mathbb{R}^n$  denotes the standard Euclidean  $n$ -dimensional space,  $\mathbb{R}^{n \times m}$  the set of real matrices of dimension  $n \times m$  and  $\mathcal{S}^n$  the space of symmetric  $n \times n$  matrices. The  $j^{\text{th}}$  row and column of a matrix  $A \in \mathbb{R}^{n \times n}$  are denoted by  $A_{j,:}$  and  $A_{:,j}$  respectively. Given  $x \in \mathbb{R}^n$  we denote the ring of multivariable polynomials with real coefficients by  $\mathbb{R}[x]$  and the subset of sum-of-squares polynomials in the variable  $x$  by  $\Sigma[x]$ . Sometimes it may be necessary to indicate the maximum degree of a polynomial or sum-of-squares polynomial in which case we use the subscript notation  $\mathbb{R}_d[x]$  or  $\Sigma_d[x]$  where  $d$  is a positive integer. All other notation will be introduced as and when needed.

**2. The sum-of-squares decomposition.** We first begin by defining the primary objects of interest.

**Definition 2.1 (Monomial).** Given an  $n$ -tuple of non-negative integers  $(\alpha_1, \dots, \alpha_n)$ , a monomial in  $x_1, \dots, x_n$  takes the form  $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ . The degree of a monomial is given by  $\partial(\alpha) = \sum_{i=1}^n \alpha_i$ .

**Definition 2.2 (Real Polynomial).** A polynomial  $f$  in  $x_1, \dots, x_n$  with coefficients in  $\mathbb{R}$  is a finite linear combination of monomials written in the form

$$f(x) = \sum_i a_{\alpha_i} x^{\alpha_i}, \quad a_{\alpha_i} \in \mathbb{R}.$$

The degree of  $f$ , denoted  $\partial(f)$ , is the maximum  $\alpha_i$  corresponding to a non-zero  $a_{\alpha_i}$ . The set of all such polynomials is denoted  $\mathbb{R}[x]$ .

**Definition 2.3 (Sum-of-Squares Polynomial).** A polynomial  $P \in \mathbb{R}[x]$  is a sum-of-squares (SOS) if it can be decomposed into  $N$  polynomials and expressed as

$$P(x) = \sum_{i=1}^N p_i^2(x)$$

where  $p_1, \dots, p_N \in \mathbb{R}[x]$ .

From Definition 2.3 it follows that both the degree,  $\partial(P)$ , and the minimum degree monomial must both be even in order for the polynomial  $P$  to admit a

sum-of-squares decomposition. It is clear from Definition 2.3 that sum-of-squares polynomials are nonnegative on  $\mathbb{R}^n$ , i.e.  $P(x) \geq 0$  for all  $x \in \mathbb{R}^n$ . The following theorem by Parrilo [27] is the key to providing computational methods that allow us to construct sum-of-squares decompositions.

**Theorem 2.4.** *The existence of a sum-of-squares decomposition of a polynomial in  $n$  variables and of degree  $2d$  can be decided by solving a semidefinite programming feasibility problem. Assuming no structure or sparsity, the dimensions of the linear matrix inequality constraint are  $\binom{n+d}{d} \times \binom{n+d}{d}$ .*

We can now relate Definition 2.3 with Theorem 2.4: If  $P(x)$  is a sum-of-squares polynomial of degree  $2d$  and  $Z(x)$  is a column vector containing all monomials up to degree  $d$  then there exists a positive semidefinite matrix  $Q$  such that

$$P(x) = Z(x)^T Q Z(x). \tag{1}$$

Furthermore, when the coefficients of  $P$  are not known, they can be searched for using (1) subject to the constraint  $Q \succeq 0$ .

The significance of this result is that semidefinite programming feasibility problems are convex. For completeness we describe the general structure of a semidefinite programme, but as writing out the SDPs explicitly can become cumbersome, several software packages such as SOSTOOLS [23], Gloptipoly [15] and YALMIP [21] have been developed, that automate the conversion between sum-of-squares programs and SDPs. Throughout this work SOSTOOLS will be used to formulate and solve all sum-of-squares programmes.

A standard semidefinite programme (SDP) takes the form

$$\begin{aligned} p^* = \inf_{X \in \mathcal{S}^n} & \quad \langle C, X \rangle \\ \text{s.t.} & \quad \langle A_i, X \rangle = b_i \\ & \quad X \succeq 0 \end{aligned} \tag{2}$$

where  $X \in \mathcal{S}^n$  is the decision variable,  $\langle \cdot, \cdot \rangle$  defines the inner product  $\langle A, B \rangle := \text{trace}(A^T B)$ ,  $A_i \in \mathcal{S}^n$  for all  $i = 1, \dots, m$ ,  $C \in \mathcal{S}^n$ ,  $b \in \mathbb{R}^m$  and the partial ordering  $X \succeq 0$  is interpreted as the matrix  $X$  is positive semidefinite. Optimization problem (2) is referred to as the *primal* problem and  $p^*$  is the optimal value for this problem. The dual problem is

$$\begin{aligned} d^* = \sup_{y \in \mathbb{R}^m} & \quad \langle b, y \rangle \\ \text{s.t.} & \quad \sum_{i=1}^m y_i A_i \preceq C \end{aligned} \tag{3}$$

where  $y \in \mathbb{R}^m$  is the dual decision variable and  $\langle \cdot, \cdot \rangle$  denotes the standard Euclidean inner product. From a geometrical stand point the feasible region of (2) is the intersection of an affine subspace and the positive semidefinite cone. As the positive semidefinite cone is *self-dual* we can appeal to duality theory to derive (3). Semidefinite programmes of the form given above have a *weak duality* property, that is, feasible primal solutions give upper bounds on the dual solution and feasible solutions to dual problem give lower bounds on the primal problem. For a thorough treatment of SDPs and their numerical interpretations the reader is referred to [41, 37]. As with other convex optimization programming solvers there are a wide variety of free SDP solvers and parsers available, for example SeDuMi [34], SDPT3 [38] and CVX [12].

**2.1. Real algebraic geometry.** One of the main advantages of the sum-of-squares decomposition is that it can be used as a computational relaxation to polynomial non-negativity. It will be shown in Section 3 why this is desirable for the case of nonlinear stability verification. Frequently however, we will not necessarily require a function to be positive on the whole state space, but rather on a bounded domain. A standard example is that of trying to find a Lyapunov function for a nonlinear system that contains multiple equilibria. In this case it is clear that the Lyapunov inequalities will only hold over a given region around the equilibrium of interest. Similar arguments appear many times in robust control theory.

A result from real algebraic geometry [4] called the Positivstellensatz, which we will now introduce, provides a feasibility test for determining the non-negativity of polynomials over semialgebraic sets. The Positivstellensatz establishes a relationship between geometric objects (*affine varieties*) and the validity of an algebraic relationship (*polynomial ideal*) [10].

We first define what a semialgebraic set is, which we will use to define the domain over which we wish to determine the non-negativity of a given polynomial:

**Definition 2.5** (Semialgebraic Set). The basic closed semialgebraic set is a subset of  $\mathbb{R}^n$  defined by the existence of a finite number of polynomial inequalities:

$$\{x \in \mathbb{R}^n \mid f_i(x) \geq 0, f_i \in \mathbb{R}[x], \forall i = 1, \dots, j\}.$$

**Definition 2.6** (Ideal). Given the multivariate polynomials  $g_1, \dots, g_m \in \mathbb{R}[x]$ , the *ideal* generated by  $g_i$  for  $i = 1, \dots, m$  is the set

$$\mathcal{I}(g_1, \dots, g_m) = \left\{ \sum_{i=1}^m t_i g_i \mid t_1, \dots, t_m \in \mathbb{R}[x] \right\}.$$

**Definition 2.7** (Multiplicative Monoid). Given polynomials  $h_1, \dots, h_u \in \mathbb{R}[x]$  the *multiplicative monoid*, denoted  $\mathcal{M}(h_1, \dots, h_u)$ , generated by  $h_i$  for  $i = 1, \dots, u$  is the set of all finite products of  $h_i$  including 1.

**Definition 2.8** (Cone). Given the multivariate polynomials  $f_1, \dots, f_s \in \mathbb{R}[x]$  the *algebraic cone* generated by  $f_i$  for  $i = 1, \dots, s$  is the set

$$\mathcal{C}(f_1, \dots, f_s) = \left\{ s_0 + \sum_{i=1}^r s_i F_i \mid F_i \in \mathcal{M}(f_1, \dots, f_m), s_i \in \Sigma[x] \right\}$$

where  $r$  denotes the number of polynomials in  $\mathcal{M}$ .

Using the preceding definitions we can now state the Positivstellensatz theorem:

**Theorem 2.9** (Positivstellensatz). Let  $\{f_1, \dots, f_s\}$ ,  $\{g_1, \dots, g_m\}$  and  $\{h_1, \dots, h_u\}$  be finite multivariate polynomials in  $\mathbb{R}[x]$ . Denote by  $\mathcal{C}$  the algebraic cone generated by  $\{f_1, \dots, f_s\}$ ,  $\mathcal{I}$  the ideal generated by  $\{g_1, \dots, g_m\}$  and  $\mathcal{M}$  the multiplicative monoid generated by  $\{h_1, \dots, h_u\}$ . Then the following statements are equivalent:

- The set

$$\left\{ x \in \mathbb{R}^n \mid \begin{array}{l} f_1(x) \geq 0, \dots, f_s(x) \geq 0 \\ g_1(x) = 0, \dots, g_m(x) = 0 \\ h_1(x) \neq 0, \dots, h_u(x) \neq 0 \end{array} \right\}$$

is empty.

- There exist  $f \in \mathcal{C}$ ,  $g \in \mathcal{I}$  and  $h \in \mathcal{M}$  such that

$$f + g + h^2 = 0.$$

For a more detailed treatment and the proof of the Positivstellensatz (and algebraic geometry) see [4]. The Positivstellensatz provides an equivalence relationship between sets defined by polynomial inequalities and algebraic objects. Thus through semidefinite programming we can verify set emptiness. Note that there is no mention of the bound of the degree on any of the polynomials or sum-of-squares polynomials in Theorem 2.9. In order to use the SDP machinery we must place bounds on the maximum degree of all functions.

The following example taken from [28] illustrates the ideas presented above. Consider the set of polynomial relations:

$$f(x, y) := x - y^2 + 3 \geq 0, \quad g(x, y) := y + x^2 + 2 = 0 \tag{4}$$

We wish to prove that there is no  $(x, y) \in \mathbb{R}^2$  such that (4) is satisfied. Using Theorem 2.9, system (4) has no solution if and only if there exist sum-of-squares polynomials  $s_1, s_2 \in \Sigma[x, y]$  and a polynomial  $t \in \mathbb{R}[x, y]$  that satisfy  $s_1 + s_2 f + t g = -1$ . If the left hand side of this expression is evaluated for any feasible solution of (4) then the result would be positive, which is a contradiction. Two polynomials that certify no solution exists to (4) are  $t = -6$ ,  $s_2 = 2$  and  $s_1 = 1/3 + 2(y + 3/2)^2 + 6(x - 1/6)^2$ . These can be found using SOSTOOLS.

Many of the sum-of-squares based theorems in the following sections can be formulated as an application of the Positivstellensatz, but we will not explicitly provide these formulations.

**3. Computational Lyapunov stability analysis.** In this section we will introduce the class of systems that we will focus on in the rest of the paper. We will present several problems related to this system and then introduce a number of theorems which address these problems. Once these theorems are formulated, a sum-of-squares relaxation that solves the relevant problems will be presented. In general, complete proofs of theorems will not be provided unless it is of immediate benefit to the reader or will be helpful to the reader should they want to generalise the result to a broader class of systems or problems.

The primary system of interest is described by a set of coupled differential equations of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathcal{D} \tag{5}$$

where  $f : \mathcal{D} \rightarrow \mathbb{R}^n$  is a vector of polynomial functions,  $x(t) \in \mathbb{R}^n$  is the state vector and  $\mathcal{D} \subseteq \mathbb{R}^n$  is a neighbourhood of the origin. Without loss of generality it is assumed that the equilibrium point of interest (which we denote by  $x^*$ ) is at the origin, i.e.  $f(x^*) = 0$ ,  $x^* = 0$  and therefore  $x^* \in \mathcal{D}$ . Furthermore we take for granted the fact that  $f$  is locally Lipschitz in  $\mathcal{D}$  and thus solutions to (5) exist and are unique (locally).

**Definition 3.1.** The equilibrium point  $x^*$  of (5) is said to be

- *stable*, if for all  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon) > 0$  such that  $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon$ , for all  $t \geq 0$
- *asymptotically stable* if it stable as defined above, and  $\delta$  may be chosen such that  $\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = x^*$
- *unstable* if it is not stable.

**Definition 3.2.** The equilibrium point  $x^*$  of (5) is said to be *locally exponentially stable* if there exist positive constants  $c, K$  and  $\gamma$  such that for all  $x(t_0) \in \mathcal{D}$  with  $\|x(t_0)\| < c$  we have

$$\|x(t)\| \leq K \|x(t_0)\| e^{-\gamma(t-t_0)},$$

and *globally exponentially stable* if the conditions above hold for all  $x(t_0) \in \mathbb{R}^n$ .

Throughout this work all analysis questions pertaining to stability will be formulated in the Lyapunov framework and a computational procedure that uses sum-of-squares will be described for verification purposes.

**3.1. Lyapunov function construction.** In this section we introduce the central theoretical tool that will form the core of all results described in this review. Two notions of Lyapunov stability will be presented. Using the sum-of-squares decomposition, computational methods for stability verification based upon these stability definitions will be presented. In essence the majority of the control problems considered in this paper follow the same two steps: i) Recasting the *primal* problem as a Lyapunov-type problem and then, ii) constructing a sum-of-squares relaxation to the problem.

**Theorem 3.3.** *Let  $x^* = 0 \in \mathcal{D} \subset \mathbb{R}^n$  be an equilibrium point of (5). If there exists a function  $V : \mathcal{D} \rightarrow \mathbb{R}$  that is continuously differentiable such that*

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \quad \forall x \in \mathcal{D} \setminus \{0\} \\ \dot{V}(x) &\leq 0 \quad \forall x \in \mathcal{D} \end{aligned}$$

*then  $x^*$  is a stable. Moreover,  $x^*$  is asymptotically stable if*

$$\dot{V}(x) < 0 \quad \forall x \in \mathcal{D} \setminus \{0\}.$$

When such a function exists,  $x^*$  is said to be (asymptotically) stable in the sense of Lyapunov.

**Definition 3.4.** If there exists a function  $V$  that satisfies the asymptotic stability requirements of Theorem 3.3 with  $\mathcal{D} = \mathbb{R}^n$  and  $\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$  then  $x^*$  is said to be globally asymptotically stable.

A stronger notion of stability, at least in terms of convergence properties, is that of *exponential stability*.

**Theorem 3.5.** *Let  $x^* = 0$  be a an equilibrium point for (5) and assume that  $\mathcal{D} \subset \mathbb{R}^n$  is a given domain which includes  $x^*$ . Assume there exists a continuously differentiable function  $V : \mathcal{D} \rightarrow \mathbb{R}$  and positive constants  $k_1, k_2, k_3$  such that*

$$\begin{aligned} k_1 \|x\|^p \leq V(x) &\leq k_2 \|x\|^p \\ \dot{V}(x) &\leq -k_3 \|x\|^p \end{aligned}$$

*for all  $t \geq 0$  and  $x \in \mathcal{D}$  where  $p$  is a positive integer. Then,  $x^*$  is exponentially stable. Furthermore, if the assumptions hold when  $\mathcal{D} = \mathbb{R}^n$  then  $x^*$  is globally exponentially stable.*

The following theorem illustrates how sum-of-squares programming can be used to construct a polynomial stability certificate, in this case a Lyapunov function, for an equilibrium point of (5). In accordance with Theorem 3.3 the domain of interest,  $\mathcal{D}$  must be defined. In particular  $\mathcal{D}$  must be representable as a semi-algebraic set. In this work we will focus on the simplified case of a single polynomial inequality. The simplest example is a ball centred at the origin with radius  $\sqrt{r}$ ,  $r > 0$  which we denote by  $B_r \triangleq \{x \in \mathbb{R}^n \mid \|x\|_2^2 \leq r\}$ . The polynomial inequality expression associated with  $B_r$  is  $\hat{\beta}(x) \triangleq x^T x - r \leq 0$ . For the remainder of this paper we

will assume that the domain of interest  $\mathcal{D}$  is represented by all points that satisfy  $\beta(x) \leq 0$  where  $\beta \in \mathbb{R}[x]$ .

**Theorem 3.6.** *If there exists a polynomial function  $V$ , sum-of-squares polynomials  $r_1, r_2$  and positive definite polynomial functions  $\varphi_1, \varphi_2$  all of bounded degree, such that*

$$\begin{aligned} V(x) + r_1(x)\beta(x) - \varphi_1(x) &\in \Sigma[x] \\ -\dot{V}(x) + r_2(x)\beta(x) - \varphi_2(x) &\in \Sigma[x] \end{aligned}$$

*then the equilibrium point  $x^*$  of (5) is asymptotically stable.*

*Proof.* The first condition ensures that  $V(x)$  is positive on  $\mathcal{D}$ . To see this, consider the following. Since  $\beta(x)$  is negative for  $x \in \mathcal{D}$  and  $r_1(x)$  is non-negative by construction, the term  $r_1(x)\beta(x)$  is non-positive for  $x \in \mathcal{D}$  and since by definition  $\varphi_1(x) > 0$  we have that  $V(x) - \varphi_1(x) \in \Sigma[x] \Rightarrow V(x) - \varphi_1(x) \geq 0 \Rightarrow V(x) \geq \varphi_1(x) > 0$ . Therefore the first condition satisfies the first equality and inequality of Theorem 3.3. Identical arguments on the second condition enforce the derivative inequality in Theorem 3.3. This concludes the proof.  $\square$

Note that if the degree of the polynomial and sum-of-squares variables in Theorem 3.6 are not bounded then the optimisation problem is convex but infinite dimensional and thus not computationally tractable.

A few questions immediately come to mind: Do stable polynomial systems always admit a sum-of-squares Lyapunov function? How conservative are we being by limiting ourselves to positive polynomials that admit a sum-of-squares decomposition? Can we determine *a priori* the degree of the Lyapunov function required? The answer to the first question is simply, no. In [1] a globally stable polynomial system is presented that cannot admit any polynomial Lyapunov function. Of course there are several subtleties to this result: The search is limited to global stability analysis and local stability is not considered. Secondly, the authors are referring to asymptotic stability. As will be shown in Theorem 3.8, locally exponentially stable systems do admit polynomial Lyapunov functions and it is possible to obtain a degree bound *a priori* [29, 30]. We first present a sum-of-squares programme for verifying exponential stability.

**Theorem 3.7.** *Let  $x^* = 0$  be an equilibrium point of (5) which is contained in  $\mathcal{D} \subset \mathbb{R}^n$ . Suppose there exists a polynomial  $V$ , sum-of-squares polynomial  $r$  and scalars  $\kappa_1, \kappa_2 > 0$  such that*

$$\begin{aligned} V(x) - \kappa_1\|x\|^2 &\in \Sigma[x] \\ -\dot{V}(x) - \kappa_2\|x\|^2 + r(x)\beta(x) &\in \Sigma[x] \end{aligned}$$

*then  $x^*$  is a locally exponentially stable equilibrium point.*

In Theorem 3.7 we have used the fact that it is known that the Lyapunov function  $V$  can be assumed to be a sum-of-squares polynomial [30], thus the sum-of-squares multiplier used in the first condition of Theorem 3.6 is not required here. The following theorem [30] shows that all systems of the form (5) that are exponentially stable on a bounded domain admit a sum-of-squares polynomial Lyapunov function.

**Theorem 3.8.** *For system (5), suppose that  $f$  is polynomial,  $\partial(f) = q$  and that  $x^* = 0$  is exponentially stable on  $B_r$  for some  $r > 0$  with  $\|x(t)\| \leq K\|x(0)\|e^{-\gamma t}$ .*

Then there exist  $\kappa_1, \kappa_2, \kappa_3 > 0$  and a sum-of-squares polynomial  $V$  such that for all  $x \in B_r$  the following inequalities

$$\begin{aligned}\kappa_1 \|x\|^2 &\leq V(x) \leq \kappa_2 \|x\|^2 \\ \dot{V}(x) &\leq -\kappa_3 \|x\|^2\end{aligned}$$

are satisfied. Furthermore,  $\partial(V) < 2q^{Nk-1}$  where  $k(L, \gamma, K)$  is any integer such that  $c(k) < K$  where

$$c(k) \triangleq \sum_{i=0}^{N-1} (e^{TL} + K(TL)^k)^i K^2 (TL)^k$$

and satisfies

$$\begin{aligned}c(k)^2 + \frac{\log 2K^2}{2\gamma} K \frac{(TL)^k}{T} (1 + c(k))(K + c(k)) &< \frac{1}{2}, \\ c(k)^2 &< \frac{\gamma}{KL \log 2K^2} (1 - (2K^2)^{-\frac{L}{\gamma}})\end{aligned}$$

where  $L$  denotes the Lipschitz bound of  $f$  on  $B_{4Kr}$  and  $N(L, \gamma, K)$  is any integer such that  $NT > (\log 2K^2 / 2\gamma)$  and  $T < \frac{1}{2L}$ .

The proof of this theorem is constructive and is based upon the Picard iteration. The highlights of the theorem are that exponentially stable systems have a sum-of-squares polynomial Lyapunov function, the degree of which is bounded, and furthermore the degree bound is a function of the continuity properties of  $f$  and the convergence rate of (5) in the sense of Definition 3.2 (with  $t_0 = 0$ ).

This section is concluded with an illustrative example taken from [2]. Consider the nonlinear system described the following set of nonlinear ODEs:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2.\end{aligned}\tag{6}$$

This system has two saddle points at  $(\pm\sqrt{3}, 0)$  as well as a stable focus at the origin. Here we are interested in trajectories of (6) with initial conditions in the circular disc centred on the origin with radius 1:  $\mathcal{D} = \{x \in \mathbb{R}^2 \mid x^T x - 1 \leq 0\}$ , i.e.  $B_1$ . Using SOSTOOLS [23], Theorem 3.6 was used to construct a Lyapunov function verifying local asymptotic stability. The phase portrait for the system and the level sets of the Lyapunov function are shown in Figure 1.

In addition to stability, it is often desirable to determine the Region of Attraction (RoA) denoted  $R_A$ , of  $x^*$  belonging to (5). Let  $\phi(t; x_0)$  be the solution to (5) with initial state  $x_0$  at  $t = 0$ . The region of attraction corresponds to the volume of state space  $R_A \subseteq \mathbb{R}^n$  such that

$$R_A = \{x_0 \in \mathbb{R}^n \mid \phi(t; x_0) \text{ exists and is unique, } \phi(t; x_0) \rightarrow x^* \text{ as } t \rightarrow \infty\}.$$

Determining the set  $R_A$  exactly is a nontrivial task and no analytic solutions exist for general nonlinear systems. Most region of attraction algorithms attempt to (under)approximate  $R_A$  by simple, more computationally amenable set descriptions such as ellipses and polytopes using the linear differential inclusion framework [5]. More recently, sum-of-squares methods have been applied to RoA analysis [8, 39] although these typically lead to bilinear matrix inequalities which are non-convex [36]. The most common method for approximating  $R_A$  is to determine the largest level curve of a Lyapunov function that is completely contained in some pre-specified

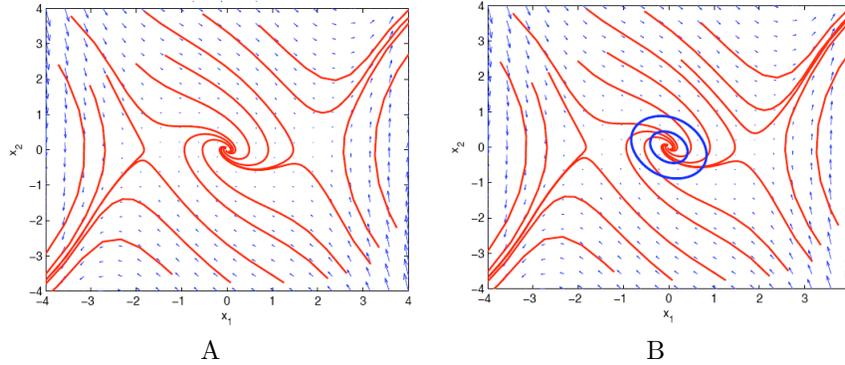


FIGURE 1. A: Trajectories of system (6) initiated from various initial conditions (red lines) can be seen to either converge to the stable equilibrium point at the origin or rapidly diverge if their initial state is close to the two saddle points at  $(\pm\sqrt{3}, 0)$ . B: Level curves of the Lyapunov function (blue circles).

subset of  $\mathbb{R}^n$ . Denote such a subset by  $\mathcal{X}$  and assume that it can be represented by a semialgebraic set of the form

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \quad i = 1, \dots, k\}.$$

For expositional purposes we will assume that  $\mathcal{X} = \mathcal{D}$ , that is, we will assume that we are searching for an approximation of  $R_A$  on the same set that was used for stability analysis. For completeness we will provide a general algorithm at the end of the example that attempts the more complicated task of estimating  $R_A$  on the whole of  $\mathbb{R}^n$ . The reader is referred to the references previously mentioned for various implementations of the algorithm. A new method based upon set invariance which provides an estimate of  $R_A$  that is not given by a Lyapunov function level-set is presented in [40].

Define the point set enclosed by the level curve  $V(x) = c$  where  $c > 0$  by

$$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}. \tag{7}$$

When  $\Omega_c \subset \mathcal{D}$  and  $V(x)$  is a Lyapunov function of (5) it follows that  $\Omega_c$  is compact and positively invariant and hence a region of attraction for the zero equilibrium point,  $x^*$ , of (5). We now propose a sum-of squares programme based on the Positivstellensatz that computes the largest level-set of  $V$ , i.e. the maximum  $c$ , such that  $\Omega_c \subset \mathcal{D}$ . We denote this set by  $\Omega_{c^*}$ .

Given a Lyapunov function  $V \in \mathbb{R}_a[x]$  and non-negative integers  $\kappa, b_1, \dots, b_k$ , search for sum-of-squares polynomials  $d_i \in \Sigma_{b_i}[x]$  for  $i = 1, \dots, k$  that solve:

$$\begin{aligned} & \max_{d_i \in \Sigma_{b_i}[x], c > 0} && c && (8) \\ & \text{s.t.} && \|x\|^{2\kappa}(V(x) - c) - d_i(x)g_i(x) \in \Sigma[x], && i = 1, \dots, k, \end{aligned}$$

and let  $c^*$  denote the optimal value of the objective function. The maximum level set of  $V$  contained within  $\mathcal{D}$  is thus  $V(x) = c^*$ .

Less conservative (although computationally more demanding) solutions can be obtained by increasing  $\kappa$  and the degree of the sum-of-squares polynomial multipliers  $b_i$ .

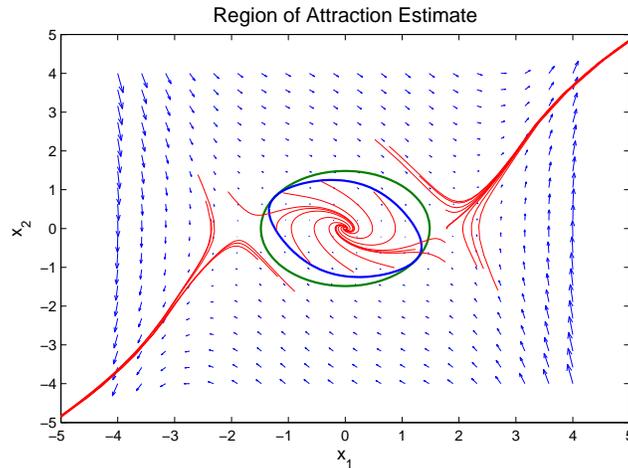


FIGURE 2. An estimate of the RoA of (6) obtained by solving optimization (8). The green (outer) curve indicates the domain  $\mathcal{D}$  we are interested in, the blue (inner) curve is the largest level set of the Lyapunov function contained in  $\mathcal{D}$ .

Let us return to the example system (6). Consider now the domain  $\mathcal{D} = \{x \in \mathbb{R}^2 \mid x^T x - 2.2 \leq 0\}$  shown by the green level curve in Figure 2. Following the steps described previously we obtain the quartic Lyapunov function:

$$\begin{aligned} V(x) = & 3.421x_1^2 + 1.7217x_1x_2 + 2.8584x_2^2 + 0.45219x_1^4 \\ & + 1.318x_2x_1^3 + 1.5945x_2^2x_1^2 + 0.20294x_1x_2^3 + 0.86584x_2^4. \end{aligned}$$

Setting  $\kappa = 1$  and  $\partial(d_1) = 2$ , solving (8) we find that the largest level curve of  $V$  contained in  $\mathcal{D}$ , and thus an approximation of the RoA on  $\mathcal{D}$ , is  $\{x \in \mathbb{R}^2 \mid V(x) \leq 6.308\}$  which is shown in Figure 2. Improved estimates of the true region of attraction may be obtained by iterating between adjusting the domain  $\mathcal{D}$  and then computing the largest Lyapunov level curve inside that region. The pseudo-code for such an algorithm is:

1. Define a domain  $\mathcal{X}$  in the state space described by a semialgebraic set.
2. Construct a Lyapunov function  $V(x)$  on  $\mathcal{X}$  as described earlier.
3. Compute  $\Omega_{c^*}$  using (8). Given a scalar  $\Delta > 0$ , set  $\mathcal{X}' = \Omega_{c^* + \Delta}$  and go to Step 2 replacing  $\mathcal{X}$  with  $\mathcal{X}'$ .

The above steps should be iterated until the optimization problems become infeasible. Note that by employing Theorem 2.9 it is possible to ensure that  $\mathcal{X}'$  contains  $\mathcal{X}$  c.f. [40].

In the following subsection we shall consider the problem of constructing a state feedback controller. One of the goals of feedback control is to enlarge the region of attraction of a desired equilibrium point.

**3.2. State feedback control design.** The standard control synthesis problem can be stated as follows: given a (possibly unstable) dynamical system of the form  $\dot{x}(t) = f(x(t), u(t))$  find a control law of the form  $u(x)$  such that the closed loop is asymptotically stable. Typically the controlled system should seek to minimise a given convex cost function. For LTI systems the solution to such problems are

well understood and there are well known results that give necessary and sufficient conditions for the existence of the controller and optimality conditions. The non-linear counterpart is however still an open problem. We will describe one approach based on the work in [32] to solve the state feedback case, extensions to more complicated synthesis problems including output feedback and estimator design have been investigated [42, 43, 35], for results related to the general nonlinear control setup see [17, 22]. We first consider a relaxed version of the problem stated above by assuming that the dynamical system can be written in input-affine form:

$$\dot{x} = f(x) + g(x)u \tag{9}$$

where  $f$  is a vector of polynomial functions such that  $f(0) = 0$  and  $g$  is a matrix of polynomial functions and the polynomial control law to be constructed is  $u = h(x)$ . Initially we will just consider the stabilisability problem and ignore optimal synthesis. Under these assumptions, state-feedback control amounts to finding a polynomial Lyapunov function such that

$$V(x) - \phi(x) \in \Sigma[x], \tag{10a}$$

$$-\frac{\partial V}{\partial x}(f(x) + g(x)h(x)) \in \Sigma[x], \tag{10b}$$

where  $\phi(x)$  is a positive definite polynomial function. Clearly constructing a sum-of-squares decomposition for the derivative condition (10b) is challenging as the problem is not jointly convex in  $V(x)$  and  $h(x)$ . Convex methods that avoid this problem include density functions [33], and moment relaxations [20]. However, the method presented here follows the Lyapunov approach but utilises a recasting of system (9) into a form that is reminiscent of the LTI set up. Let  $Z(x)$  be an appropriately chosen vector of monomials of dimension  $N \times 1$  and of degree at least 1, then (9) can be written as

$$\dot{x} = A(x)Z(x) + B(x)u \tag{11}$$

where  $A(x)$  and  $B(x)$  are polynomial matrices of appropriate dimension. By definition  $Z(x) = 0$  if and only if  $x = 0$ . Let us now define the matrix  $M(x) \in \Sigma[x]^{N \times n}$  such that

$$M_{ij}(x) \triangleq \frac{\partial Z_i}{\partial x_j}(x).$$

Define the index set  $\mathcal{I}_B = \{b_1, b_2, \dots, b_k\}$  to denote the row indices of  $B(x)$  that contain only zeros. Finally let us define the vector  $\tilde{x} = [x_{b_1}, \dots, x_{b_k}]^T$ . The state-feedback problem is now to find a function  $u(x) = h(x) := F(x)Z(x)$  such that the equilibrium point  $x^* = 0$  of the closed loop system is stable. After describing a solution to the state feedback problem a method for including a cost function will be described.

**Theorem 3.9.** *Given the dynamical system (11), if there exists a symmetric polynomial matrix  $P(\tilde{x}) \in \mathbb{R}[x]^{N \times N}$ , a polynomial matrix  $H(x) \in \mathbb{R}[x]^{n \times N}$  a positive scalar  $\epsilon$  and a sum-of-squares polynomial  $s \in \Sigma[x]$  such that*

$$z^T(P(\tilde{x}) - \epsilon I)z \in \Sigma[z, \tilde{x}] \tag{12a}$$

$$\begin{aligned} & z^T(P(\tilde{x})A^T(x)M^T(x) + M(x)A(x)P(\tilde{x}) \dots \\ & + H^T(x)B^T(x)M^T(x) + M(x)B(x)H(x) \dots \in \Sigma[z, x] \tag{12b} \\ & - \sum_{i \in \mathcal{I}_B} \frac{\partial P}{\partial x_j}(\tilde{x})A_{i,:}Z(x) + s(x)I)z \end{aligned}$$

with  $z \in \mathbb{R}^N$ . Then the state feedback problem admits a solution, furthermore, one such stabilizing controller is

$$u(x) = H(x)P^{-1}(\tilde{x})Z(x).$$

*Proof.* An outline of the proof is provided for completeness. Assume the polynomial matrices  $P$  and  $H$  defined in Theorem 3.9 exist and that conditions (12a)–(12b) are satisfied. Consider the Lyapunov function  $V(x) = Z^T(x)P^{-1}(\tilde{x})Z(x)$ . The proof proceeds by showing that  $V$  is indeed a Lyapunov function for the closed loop system

$$\dot{x} = [A(x) + B(x)H(x)P^{-1}(\tilde{x})] Z(x). \tag{13}$$

Positive definiteness of  $V$  is guaranteed by the fact that (12a) ensures  $P$  and thus  $P^{-1}$  is positive definite. What is left to do now is show that the derivative condition is non-positive. First, differentiate  $V$  along the trajectories of (13), then using the identity

$$\frac{\partial P}{\partial x_i}(x) = -P(x)\frac{\partial P^{-1}}{\partial x_i}(x)P(x)$$

it is easily shown that  $V$  is a valid Lyapunov function and that condition (12b) enforces non-positivity.  $\square$

When condition (12b) in Theorem 3.9 holds, if  $s(x) > 0$  for all  $x \neq 0$  then the origin is asymptotically stable. Furthermore if  $P(\tilde{x})$  is a constant the origin of the closed loop system is globally stable.

The results of this section are now extended to the optimal control setting. We focus on *nonlinear*  $\mathcal{H}_\infty$  state-feedback control<sup>1</sup>. Here we consider dynamical systems which can be written in the form

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(x) & B_1(x) & B_2(x) \\ C_1(x) & 0 & 0 \\ C_2(x) & 0 & I \end{bmatrix} \begin{bmatrix} Z(x) \\ w \\ u \end{bmatrix}, \tag{14}$$

where the monomial vector  $Z(x)$  takes the same form as in the state-feedback problem and  $z_1 \in \mathbb{R}^{m_1}, z_2 \in \mathbb{R}^{m_2}$ . The index vector  $\mathcal{I}_B$  is extended to denote the zero rows of the matrix  $[B_1(x) B_2(x)]$ . The control objective in this case is to minimise the induced  $\mathcal{L}_2$ -gain from  $w$  to  $z = [z_1^T, z_2^T]^T$  by choosing a control law  $u(x) = F(x)Z(x)$ . Below we present a relaxation to the problem, instead of finding a controller that finds the minimum achievable  $\mathcal{L}_2$ -gain we specify an acceptable gain,  $\gamma > 0$  and then determine if a controller that achieves this bound exists. This is commonly referred to as *sub-optimal* control.

**Theorem 3.10.** *Given a nonlinear system of the form (14). If there exist a symmetric polynomial matrix  $P \in \mathbb{R}[\tilde{x}]^{N \times N}$ , a positive definite sum-of-squares polynomial  $s \in \Sigma[x]$  and a constant  $\epsilon > 0$  such that*

$$\begin{aligned} & \nu_1^T(P(\tilde{x}) - \epsilon I)\nu_1 \in \Sigma[\nu_1, \tilde{x}] \\ - \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}^T & \begin{bmatrix} \Psi & PC_1^T & MB_1 \\ C_1P & -(\gamma - s)I & 0 \\ B_1^T M^T & 0 & -(\gamma - s)I \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in \Sigma[\nu, x] \end{aligned}$$

<sup>1</sup>The term nonlinear  $\mathcal{H}_\infty$  is a somewhat confusing term for an inherently linear object, however it is frequently used in practice and so we adopt it here. It refers to the largest achievable gain of system (9) from an input signal belonging to the function space  $\mathcal{L}_2[0, \infty]$  to an output signal in  $\mathcal{L}_2[0, \infty]$ .

where  $\Psi = M\hat{A}P + P\hat{A}M^T - MBB^T M^T - \sum_{i \in \mathcal{I}_B} \frac{\partial P}{\partial x_i}(A_i Z) + sI$  and  $\hat{A} = A - B_2 C_2$ , with  $\nu_1 \in \mathbb{R}^N, \nu_2 \in \mathbb{R}^{m_1+m_2}$ . (Note that we have dropped the dependence on variables in the second constraint in order to reduce the notational burden.) Then the state feedback law

$$u(x) = - [\gamma B_2^T(x) M^T(x) P^{-1}(\tilde{x}) + C_2(x)] Z(x)$$

renders  $x^*$  of the closed loop system asymptotically stable and the  $\mathcal{L}_2$ -gain from  $w$  to  $z$  is less than  $\gamma$ . If, additionally,  $P(\tilde{x})$  is a constant matrix then the  $\mathcal{L}_2$ -gain bound and asymptotic stability hold globally.

*Proof.* The proof follows from the classic LPV synthesis algorithms c.f.[16] and the obvious differences are covered by Theorem 3.9. □

**4. Non-polynomial system analysis.** In this section we will describe how sum-of-squares techniques can be used to verify the stability of dynamical systems that do not evolve according to polynomial vector fields. This overview will be informal and is intended primarily for pedagogical reasons. Two simple examples will motivate the problem and then a general algorithm will be presented. The method we describe involves taking a non-polynomial system and *recasting* it as a constrained polynomial system. Stability of the new system then implies stability of the original system. The caveat is that the state dimension of the new polynomial system is likely to be larger than that of the original system and include new state constraints[25].

Consider the dynamical system

$$\dot{z}(t) = f(z(t)) \tag{15}$$

where  $f(z^* = 0) = 0$  and  $z(t) \in \mathbb{R}^n$ . Unlike in (5)  $f$  is not necessarily a vector of polynomial functions. For brevity we drop the dependence on time,  $t$ , from the notation. Let us now define the augmented state vectors  $\tilde{x}_1, \tilde{x}_2$ :

$$\begin{aligned} \tilde{x}_1 &\triangleq [x_1, \dots, x_n]^T = z, \\ \tilde{x}_2 &\triangleq [x_{n+1}, \dots, x_{n+m}]^T, \end{aligned}$$

which after applying a recasting algorithm form the system

$$\dot{\tilde{x}}_1 = f_1(\tilde{x}_1, \tilde{x}_2) \tag{16a}$$

$$\dot{\tilde{x}}_2 = f_2(\tilde{x}_1, \tilde{x}_2) \tag{16b}$$

where  $f_1, f_2$  are polynomial vector functions. In order for the system (16) to accurately capture the behaviour of (15) the recasting process must introduce some constraints on (16) (or more concretely on the new state variables  $\tilde{x}_2$ ). Essentially this restricts the state evolution of (16) to a manifold. We write these constraints in vector form

$$\tilde{x}_2 = F(\tilde{x}_1) \tag{17}$$

with the equality being applied in an element-wise manner. Clearly for the recasting process to be of any use  $F$  must be polynomial in  $\tilde{x}_1$ . Such a recasting process may indirectly induce some non-polynomial constraints of the form

$$G_1(\tilde{x}_1, \tilde{x}_2) = 0 \tag{18a}$$

$$G_2(\tilde{x}_1, \tilde{x}_2) \leq 0 \tag{18b}$$

which are automatically satisfied when (17) holds.

In order to verify stability of an equilibrium point of the non-polynomial system (15) using sum-of-squares programming we instead construct a Lyapunov function for the constrained, recast system (16). A theorem that proves this condition is sufficient and sum-of-squares programme that implements the results will be provided after two illustrative examples are presented.

The first example is an *exact* recasting. Exactness refers to the fact that the transformed system has the same state dimension as the original system. Take the non-polynomial scalar system  $\dot{x}(t) = ce^{-\alpha x(t)}$ . Choose  $p(t) = ce^{-\alpha x(t)}$  then  $\dot{p}(t) = -\alpha p^2(t)$ . Next we present a more involved example taken from [25]. Let  $\Gamma$  be the static saturation function:

$$\Gamma(w) = \frac{w}{\sqrt{1+w^2}}.$$

Now consider the system

$$\dot{x}_1 = x_2, \tag{19a}$$

$$\dot{x}_2 = -\Gamma(x_1 + x_2), \tag{19b}$$

which has one equilibrium point,  $x^*$  at the origin. To recast this as a polynomial system, introduce the following variables:

$$\begin{aligned} u_1 &= \sqrt{1 + (x_1 + x_2)^2}, & u_3 &= \sqrt{1 + x_1^2}, \\ u_2 &= \frac{1}{u_1}, & u_4 &= \frac{1}{u_3}. \end{aligned}$$

Note that these variables are the induced equality constraints described by (18a). The state equations from (19) can now be written as:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -(x_1 + x_2)u_2, \\ \dot{u}_1 &= (x_1 + x_2)(x_2 - x_1u_2 - x_2u_2)u_2, \\ \dot{u}_2 &= -(x_1 + x_2)(x_2 - x_1u_2 - x_2u_2)u_2^3, \\ \dot{u}_3 &= x_1x_2u_4, \\ \dot{u}_4 &= -x_1x_2u_4^3, \end{aligned} \tag{20}$$

with the constraints

$$\begin{aligned} u_i &\geq 0 \quad i \in \{1, 2, 3, 4\}, \\ u_1^2 &= 1 + (x_1 + x_2)^2, \\ u_1u_2 &= 1, \\ u_3^2 &= 1 + x_1^2, \\ u_3u_4 &= 1. \end{aligned} \tag{21}$$

It should be clear that the polynomial constraints (21) above encode the variables  $u_1, \dots, u_4$  defined earlier. The Lyapunov theorems presented thus far do not take into account constrained systems of the form (20)–(21). First, a generalised constrained polynomial system is defined and then a sum-of-squares programme for stability verification is presented.

A constrained polynomial dynamical system takes the following form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), \theta(t)) \\ a_i(x, \theta) &\leq 0, \quad \text{for } i = 1, \dots, N_1, \\ b_j(x, \theta) &= 0, \quad \text{for } j = 1, \dots, N_2, \\ \int_0^T c_k(x, \theta) dt &\leq 0, \quad \text{for } k = 1, \dots, N_3, \end{aligned} \tag{22}$$

where  $x(t) \in \mathbb{R}^n$  is the usual state vector and  $\theta(t) \in \mathbb{R}^m$  represents any needed auxiliary variables. It is assumed that the functions  $a_i, b_j, c_k$  are all polynomials. The domain of interest is  $\mathcal{D} \subset \mathbb{R}^{m+n}$  which we assume has a semialgebraic representation:

$$\mathcal{D} \triangleq \{ (x, \theta) \in \mathbb{R}^{n+m} \mid a_i(x, \theta) \leq 0, b_j(x, \theta) = 0, \forall i, j \}.$$

We further assume that  $f(x, \theta) = 0$  for  $x = x^* = 0$  and  $\theta \in \mathcal{D}_\theta^0$  where  $\mathcal{D}_\theta^0 \triangleq \{ \theta \in \mathbb{R}^m \mid (0, \theta) \in \mathcal{D} \}$ . The following Lyapunov Theorem from [25] provides a sum-of-squares programme for verifying the stability of the equilibrium point  $x^*$  of (22).

**Theorem 4.1.** *Given system (22), where  $f$  is allowed to be a rational function with no singularities in  $\mathcal{D}$ . Suppose there exist polynomials  $V(x)$ ,  $w(x, \theta)$ ,  $p_i(x, \theta)$ ,  $q_j(x, \theta)$  and scalars  $r_k > 0$  such that  $V(x^*) = 0$ ,  $V(x) > 0$  for all  $x \neq x^*$  around  $x^*$  and that  $w(x, \theta) > 0$  and  $p_i(x, \theta) \geq 0$  on  $\mathcal{D}$ . Then if*

$$-\frac{\partial V}{\partial x} f(x, \theta) + \sum_{i=1}^{N_1} p_i(x, \theta) a_i(x, \theta) + \sum_{j=1}^{N_2} q_j(x, \theta) b_j(x, \theta) + \sum_{k=1}^{N_3} r_k c_k(x, \theta) \geq 0, \tag{23}$$

or if there are no integral constraints

$$-w(x, \theta) \frac{\partial V}{\partial x} f(x, \theta) + \sum_{i=1}^{N_1} p_i(x, \theta) a_i(x, \theta) + \sum_{j=1}^{N_2} q_j(x, \theta) b_j(x, \theta) \geq 0, \tag{24}$$

then the equilibrium point  $x^* = 0$  is stable.

*Proof.* This is a simple extension of Theorem 3.3. Integrating either (23) or (24) from 0 to  $T$  it is easily verified that  $V(x(0)) - V(x(T)) \geq 0$  on  $\mathcal{D}$  when the constraints are satisfied, thus ensuring stability.  $\square$

The conditions in Theorem 4.1 can easily be implemented as sum-of-squares programmes in exactly the same manner as those of Theorems 3.6, 3.7 The final theorem in this section provides sufficient conditions for verifying the stability of (15) by constructing a Lyapunov function for (16). As we've done elsewhere functional dependance on variables has been suppressed where possible to improve readability.

**Theorem 4.2.** *Let  $\mathcal{D}_1 \subset \mathbb{R}^n$  and  $\mathcal{D}_2 \subset \mathbb{R}^m$  be open sets with  $z^* = 0 \in \mathcal{D}_1$  and  $F(\mathcal{D}_1) \subset \mathcal{D}_2$  and define  $\tilde{x}_2^0 = F(0)$ . If there exists a function  $\tilde{V} : \mathcal{D}_1 \times \mathcal{D}_2 \rightarrow \mathbb{R}$  and column vectors of functions  $\lambda_1(\tilde{x}_1, \tilde{x}_2)$ ,  $\lambda_2(\tilde{x}_1, \tilde{x}_2)$ ,  $s_1(\tilde{x}_1, \tilde{x}_2)$  and  $s_2(\tilde{x}_1, \tilde{x}_2)$  such that*

$$\begin{aligned} \tilde{V}(0, \tilde{x}_2^0) &= 0 \\ \tilde{V} + \lambda_1^T G_1 + s_1^T G_2 - \phi(\tilde{x}_1, \tilde{x}_2) &\geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2 \\ -\frac{\partial \tilde{V}}{\partial \tilde{x}_1} f_1 - \frac{\partial \tilde{V}}{\partial \tilde{x}_2} f_2 + \lambda_2^T G_1 + s_2^T G_2 &\geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2 \\ s_1(\tilde{x}_1, \tilde{x}_2) &\geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^{n+m} \\ s_2(\tilde{x}_1, \tilde{x}_2) &\geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^{n+m} \end{aligned}$$

with  $\phi(\tilde{x}_1, \tilde{x}_2)$  any scalar function such that  $\phi(\tilde{x}_1, F(\tilde{x}_1)) > 0$  for all  $(\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2$ . Then  $z^*$  is a stable equilibrium point of (15).

Replacing positivity constraints in Theorem 4.2 with sum-of-squares relaxations and assuming that  $\mathcal{D}_1$  and  $\mathcal{D}_2$  can be represented as semialgebraic sets allows for the conditions above to be algorithmically verified.

**4.1. Absolute stability.** An alternative method for analysing non-polynomial systems using the sum-of-squares framework was developed in [14] using the notion of *functional sum-of-squares decompositions* and *absolute stability* arguments [18, Ch. 7]. Recall the definition of a sum-of-squares polynomial (Definition 2.3), consider the functional  $p(x, \zeta(x))$  where  $x \in \mathbb{R}^n$  and  $\zeta : \mathcal{D}_x \rightarrow \mathbb{R}^p$ . We would like to determine if  $p(x, \zeta(x)) \geq 0$  on  $\mathcal{D}_x \subset \mathbb{R}^n$ . Extending the idea of a sum-of-squares decomposition of a polynomial to the functional case, a relaxation for positivity is to search for a decomposition of the form

$$p(x, \zeta(x)) = \sum_{i=1}^N g_i(x, \zeta(x))^2. \tag{25}$$

There is no implication here that the functions  $g_i$  should be polynomial. This idea can be taken further and incorporated into a semidefinite programme to computationally verify positivity of  $p$ . To do so we treat  $\zeta$  as an independent variable, then if  $p$  is polynomial in  $(x, \zeta)$  with  $\partial(p) = 2d$  and a decomposition of the form  $p(x, \zeta) = \sum_{i=1}^N g_i^2(x, \zeta)$  exists where the functions  $g_i$  are polynomial in  $x$  and  $\zeta$ , then computing a functional-decomposition can be cast in a similar manner to the polynomial case. Let

$$p(x, \zeta) = Z(x, \zeta)^T Q Z(x, \zeta) \tag{26}$$

where  $Z$  is a vector of monomials in  $x$  and  $\zeta$  of degree less than or equal to  $d$  and the decision variable  $Q$  is a symmetric matrix of coefficients to be determined. If there exists a positive semidefinite  $Q$  then  $p$  is said to possess a functional sum-of-squares decomposition and satisfies  $p(x, \zeta) \geq 0$  for all  $x, \zeta$ . The following example illustrates this idea. The problem is to determine iff the expression

$$p(x) = 2x^4 + 2x^3 \sin x - x^2 \sin^2 x + 5 \sin^4 x \tag{27}$$

is non-negative. Non-negativity is confirmed if a functional sum-of-squares decomposition of  $p$  can be constructed: Set  $\zeta(x) = \sin x$  then (27) can be written as  $p(x, \zeta) = 2x^4 + 2x^3\zeta - x^2\zeta^2 + 5\zeta^4$ , now define  $Z(x, \zeta) = [x^2, \zeta^2, x\zeta]^T$  then (26) tells us that (27) is a functional sum-of-squares polynomial if there exists a  $\lambda$  such that

$$Q = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is positive semidefinite. Clearly multiple values of  $\lambda$  satisfy this requirement, in fact the solution space is an affine subspace. Taking  $\lambda = -3$  and performing a Cholesky decomposition and a variable substitution gives

$$p(x) = \frac{1}{2}(2x^2 - 3 \sin^2 x + x \sin x)^2 + \frac{1}{2}(\sin^2 x + 3x \sin x)^2.$$

The notion of functional sum-of-squares decompositions will be used to address the absolute stability problem. Let us consider the system

$$\begin{aligned} \dot{x} &= f(x, u) \quad x(0) = x_0 \in \mathcal{D}_x \\ y &= h(x) \\ u &= \zeta(t, y) \end{aligned} \tag{28}$$

where  $x \in \mathcal{D}_x \subset \mathbb{R}^n$ ,  $u \in \mathcal{D}_u \subset \mathbb{R}^p$ ,  $y \in \mathcal{D}_y \subset \mathbb{R}^r$  and  $f : \mathcal{D}_x \times \mathcal{D}_u \rightarrow \mathbb{R}^n$  is a polynomial function. We assume that the system (28) is well-posed and that  $\mathcal{D}_x$  is an open set containing the equilibrium point  $x^* = 0$ , furthermore (28) is assumed to be locally Lipschitz in  $x$  on  $\mathcal{D}_x$  and continuous in  $t$  and  $x \in \mathcal{D}_x$  for all  $t \geq 0$ . The function  $\zeta : \mathbb{R} \times \mathcal{D}_y \rightarrow \mathcal{D}_u$  is assumed to be nonlinear and  $f(x^*, \zeta(t, h(x^*))) = 0$ . The vector of functions

$$\zeta(t, y) = \begin{bmatrix} \zeta_1(t, y) \\ \vdots \\ \zeta_s(t, y) \end{bmatrix}, \quad \zeta_i(t, y) : \mathbb{R} \times \mathcal{D}_y \rightarrow \mathbb{R}^{p_i},$$

where  $\sum_{k=1}^s p_k = p$  are placed in a polynomial generalised sector:

$$r_k(y, \zeta) \triangleq [\zeta(t, y) - \alpha_k(y)]^T [\zeta(t, y) - \beta_k(y)] \leq 0, \quad k = 1, \dots, s \tag{29}$$

with  $\alpha_k, \beta_k \in \mathbb{R}^{p_i}[y]$  for all  $i$ . Finally let  $r = [r_1^T, \dots, r_s^T]^T$  and define the set  $H_r$  to contain all the functions  $\phi(t, y)$  such that  $r(y, \phi(t, y)) \leq 0$  for all  $y \in \mathcal{D}_y$  and all  $t \geq 0$ .

**Definition 4.3.** System (28) is said to be *locally absolutely stable* if  $\zeta_k$  satisfies the polynomial generalised sector constraint (29) for all  $x \in \mathcal{D}_x$  and  $x^*$  is asymptotically stable for all  $\zeta \in H_r$ . If the previous conditions all hold and  $\mathcal{D}_x = \mathbb{R}^n$  then we say the system is *absolutely stable*.

An interpretation of the sector inequalities is that they are used to encode parametric uncertainty and functions for which we have polynomial upper and lower bounds. The simplest example is the case of a single parameter  $\delta \in \mathbb{R}$  such that  $a \leq \delta \leq b$  where  $a$  and  $b$  are known. The sector inequality (29) takes the form

$$[\zeta_k - a]^T [\zeta_k - b] \leq 0$$

with  $\zeta_k = \delta$ . In [14] the absolute stability problem is addressed where both  $f$  and the sector inequalities may be rational functions. For simplicity we only focus on the polynomial case.

**Theorem 4.4.** Assume that  $\mathcal{D}_x$  is represented by a finite set of polynomial inequalities  $\mathcal{D}_x \triangleq \{x \in \mathbb{R}^n \mid a_i(x) \leq 0\}$  where  $a_i \in \mathbb{R}[x]$  for  $i = 1, \dots, b$ . Let  $V(x)$  be a polynomial function such that  $V(0) = 0$  and define the vectors  $s_1 \in \Sigma[x, \zeta]^{p \times 1}$ ,  $s_2 \in \Sigma^b[x, \zeta]$  and  $\sigma_1(x), \sigma_2(x)$  are positive definite polynomial functions. For  $r(y, \zeta) \leq 0$  defined by (29), if

$$\begin{aligned} V(x) - \sigma_1(x) &\in \Sigma[x] \\ -\frac{\partial V}{\partial x} f(x, \zeta) - \sigma_2(x) + s_1(x, \zeta)^T r(h(x), \zeta) + s_2(x, \zeta)^T a(x) &\in \Sigma[x, \zeta] \end{aligned}$$

where  $a = [a_1, \dots, a_b]^T$ . Then the equilibrium point  $x^* = 0$  of (28) is locally absolutely stable, additionally if  $\mathcal{D}_x = \mathbb{R}^n$  then it is absolutely stable.

If  $f$  is a rational function then a simple modification to the definition of the sector inequalities (29) and a minor change to Theorem 4.4 allows for stability to be verified using the sum-of-squares decomposition. Note that system (19) is a specific example of a non-polynomial system that can be modelled and analysed using this methodology.

**4.2. Systems with delays.** The final class of systems we consider in this paper, is that of systems with time-delays: in this case, the state is a function rather than a point in  $\mathbb{R}^n$ , so that the time evolution depends on the history of the state. Over the past few years, a lot of progress has been made in the stability analysis of linear time-delay systems using Lyapunov-Krasovskii (L-K) and frequency domain methods [13, 19]. In the case of linear time-delay systems, converse Lyapunov functional theorems exist, which provide information on the structure of the Lyapunov function candidates that should be considered when constructing Lyapunov functionals. These searches can be cast as LMIs and solved efficiently, however, the analysis of nonlinear time delay systems is much more difficult.

In this review paper we discuss recent advances that allow the construction of L-K functionals for time-delay systems using sum-of-squares. More details on the methodology can be found in [24, 31].

We build on the notation used in this paper and borrow more from [13]. For  $b > a$  denote  $C([a, b], \mathbb{R}^n)$  the Banach space of continuous functions mapping the interval  $[a, b]$  into  $\mathbb{R}^n$  with the topology of uniform convergence. For  $\phi \in C([a, b], \mathbb{R}^n)$  the norm of  $\phi$  is defined as  $\|\phi\| = \sup_{a \leq \theta \leq b} |\phi(\theta)|$ , where  $|\cdot|$  denotes a norm on  $\mathbb{R}^n$ . Finally, we let  $C^\gamma = \{\phi \in C : \|\phi\| < \gamma\}$ .

We consider autonomous Retarded Functional Differential Equations (RFDEs) of the form

$$\dot{x}(t) = f(x_t). \quad (30)$$

where  $f : \Omega \rightarrow \mathbb{R}^n$ ,  $\Omega \subset C$ ,  $\dot{\cdot}$  represents the right-hand derivative<sup>2</sup> and  $x_t \in \Omega$ ,  $x_t(\theta) = x(t + \theta)$ ,  $\theta \in [-r, 0]$ . Definitions of stability of the steady-state  $x^*$  of this system satisfying  $f(x^*) = 0$  can be found in [13].

Let  $\Omega \subset C^\gamma$ , define  $V : \Omega \rightarrow \mathbb{R}$  a continuous function and let  $\dot{V}$  denote the *Upper Right Dini Derivative*, defined as

$$\dot{V}(\phi) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(x_h(\phi)) - V(\phi)].$$

We then have the following theorem [19]:

**Theorem 4.5.** (*Lyapunov-Krasovskii*) Let  $\Omega \subset C^\gamma$ . Suppose  $V : \Omega \rightarrow \mathbb{R}$  is continuous and there exist nonnegative functions  $a(s)$  and  $b(s)$  mapping  $\mathbb{R}^+$  to  $\mathbb{R}^+$  such that  $a(s) \rightarrow \infty$  as  $|s| \rightarrow \infty$ , and  $a(0) = b(0) = 0$  such that

$$a(|\phi(0)|) \leq V(\phi), \quad \dot{V}(\phi) \leq -b(|\phi(0)|) \quad \forall \phi \in \Omega. \quad (31)$$

Then the solution  $x = 0$  of (30) is uniformly stable. If, in addition,  $b(s)$  is positive definite, then the solution  $x = 0$  of (30) is uniformly asymptotically stable.

Consider now the following functional:

$$V(x_t) = V_0(x(t)) + \int_{-r}^0 V_1(\theta, x(t), x(t + \theta)) d\theta + \int_{-r}^0 \int_{t+\theta}^t V_2(x(\zeta)) d\zeta d\theta$$

<sup>2</sup>The right hand derivative of a function  $g(s)$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$  is defined as  $\frac{dg}{ds} = \lim_{h \rightarrow 0^+} \frac{g(s+h) - g(s)}{h}$ .

for the system of the form (30). This functional is composed of three terms; the first resembles the term that is usually used for ODEs; the second term is a functional, to take care of the effect of delay, while the last term is added for reasons that will become obvious in the sequel. Sufficient conditions for the (global) stability of the zero equilibrium can now be formulated as follows:

**Theorem 4.6.** *Let  $0$  be an equilibrium for the system given by (30). Let there exist polynomials  $V_0$ ,  $V_1$  and  $V_2$  and a positive definite polynomial  $\varphi(x(t))$  such that:*

1.  $V_0(x(t)) - \varphi(x(t)) \geq 0$ ,
2.  $V_1(\theta, x(t), x(t + \theta)) \geq 0$  for  $\theta \in [-r, 0]$ ,
3.  $V_2(x(\zeta)) \geq 0$ ,
4.  $r \frac{\partial V_1}{\partial x(t)} f + \frac{dV_0}{dx(t)} f - r \frac{\partial V_1}{\partial \theta} + rV_2(x(t)) - rV_2(x(t + \theta)) + V_1(0, x(t), x(t)) - V_1(-r, x(t), x(t - r)) \leq 0$  for  $\theta \in [-r, 0]$ .

Then the equilibrium  $0$  of the system given by (30) is globally uniformly stable.

A proof can be found in [24].

To impose the conditions  $\theta \in [-r, 0]$ , we use a process similar to the region of attraction section. The polynomial  $V_1(\theta, x(t), x(t + \theta))$  needs to be non-negative only when  $h(\theta) = \theta(\theta + r) \leq 0$ . We therefore rewrite condition (2) in Theorem 4.6, as follows:

$$V_1(\theta, x(t), x(t + \theta)) + p(\theta, x(t), x(t + \theta))h(\theta) \in \Sigma[x, \theta].$$

Condition (4) can be verified in a similar manner, resulting into a sum-of-squares programme. We can also consider different Lyapunov structures, depending on the system at hand.

When dealing with nonlinear systems with more than one equilibria or with state-space/parameter constraints, we will only be able to construct a L-K functional within a restricted region of the state-space. In this case, we will still need to specify  $\Omega = \{x_t \in C : \|x_t\| \leq \gamma\}$ , and adjoin the relevant conditions on  $x(t)$ ,  $x(t - r)$  and  $x(t + \theta) \forall \theta \in [-r, 0]$  to the relevant kernels of the Lyapunov functionals in much the same way that the conditions  $\theta \in [-r, 0]$  were adjoined in Conditions (2) and (4) of Theorem 4.6.

**5. Conclusions.** In this paper we have described how the sum-of-squares decomposition of multivariate polynomials can be used to computationally verify stability and performance properties of nonlinear dynamical systems. In particular we have illustrated how to construct Lyapunov functions that verify local and exponential stability of polynomial systems, synthesise state-feedback controllers and address stability issues related to non-polynomial systems and infinite dimensional systems.

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