Efficient Rare Event Simulation for Heavy-tailed Multiserver Queues

Blanchet, J., Glynn, P. and Liu, J. C.


Abstract

This paper develops an efficient importance sampling algorithm for computing steady-state tail probabilities for customer waiting times in a $G/G/2$ queue in which the service time requirements are heavy-tailed random variables. Under appropriate regularity conditions, we prove that our estimator has bounded relative variance whenever the rate at which work arrives to the system is less than 2 (but not equal to one). The argument hinges on the construction of an appropriate Lyapunov function that can be used to bound the second moment of our estimator. The construction followed here uses fluid heuiristics to guess the form of the Lyapunov function, after which tuning constants that define the algorithmic implementation are then chosen in order to force the function to satisfy the required Lyapunov inequality. In addition, the Lyapunov function can be used to obtain upper bounds for large deviation probabilities; thereby not only allowing to study the performance of the algorithm but also provide large deviations estimates. Our approach takes advantage of the regenerative representation for the steady-state distribution. We therefore also discuss the use of importance sampling, and related efficiency issues, in conjunction with regenerative steady-state simulation.

1 Introduction

In this paper, we consider a positive recurrent two-server $G/G/2$ queue, with first-in first-out (FIFO) queue discipline, in which the service time requirements are heavy-tailed. Our interest here is in the development of an efficient rare-event simulation algorithm, based on the use of importance sampling, for computing tail probabilities associated with the distribution of customer waiting times. This model presents several major challenges, relative to the existing literature. First, unlike the single-server $G/G/1$ queue, there is no alternative probabilistic representation of the steady-state waiting time in terms of a simple functional of random walk (like the all-time maximum). This implies that one must build a rare-event simulation algorithm that
applies directly to the dynamics of the specified $G/G/2$ queueing model, so that the approach followed here must necessarily be more general than the existing algorithms for $G/G/1$ that all leverage off the special random walk structure that is available there. Secondly, when the rate $\rho$ at which work enters a $G/G/2$ queue is less than one, the most likely path leading to a large customer waiting time involves a busy period in which two customers, both with large service requirements, simultaneously block both servers. Development of a good importance sampler, and verification of its efficiency, is more difficult when such compound events are the typical way in which the rare event of interest occurs.

The approach that we follow for applying importance sampling to the steady-state of the $G/G/2$ queue is to exploit the fact that the steady-state of such a process can be represented as the ratio of two expectations defined in terms of cycles involving times at which the corresponding Markov chain visits some fixed set $K$. Unlike conventional rare-event simulation, this regenerative-like representation of the steady-state leads to importance sampling estimators that are biased. As a consequence, Section 2 of this paper proposes a generalization of rare-event computational efficiency, within the context of steady-state simulation for general state space Harris recurrent Markov chains, that covers biased estimators. Sections 2 and 3 further discuss the use of state-dependent importance sampling, in conjunction with the regenerative-like steady-state representation, in this general setting, and describe an approach (based on Lyapunov functions) for verifying algorithmic efficiency.

As indicated above, our main contribution in this paper is the introduction of the first algorithm to be proposed for rare-event simulation for a heavy-tailed multi-server queue, in particular the $G/G/2$ queue. Sections 4 and 5 develop an efficient algorithm for the probability of observing one customer that experiences delay and later, in Section 6, we show rigorously that this algorithm is also efficient for the steady-state waiting time. More precisely, Section 4 proposes an algorithm for the heavily loaded case in which the expected rate $\rho$ at which work arrives is greater than one, and uses Lyapunov-based ideas to verify its efficiency. The heavily loaded case is, of course, the easier case, because the presence of just one large customer service time requirement within a busy cycle leads to a path in which customers experience long waiting times. We structure our discussion in this section so that the general principles underlying both our choice of the “tuning constants” associated with our algorithm and our development of the Lyapunov bound on the corresponding estimator’s variance are visible. Section 5 turns to the more challenging setting in which the queue is lightly loaded, so that two large customer service requirements within a busy cycle are needed to induce long customer waiting times. Using similar ideas to those introduced in Section 4, estimator efficiency is again verified under reasonably broad hypotheses on the problem data for this heavy-tailed model. Of course, because of the presence of the compound rare event, the associated calculations are more involved. In Section 7, we study our algorithms empirically and report their performance.

This paper builds on the significant and expanding literature on rare-event sim-
ulation for heavy-tailed queues. Several different proposals have been made for simulation of steady-state tail probabilities in the context of the heavy-tailed $M/G/1$ queue, in which the associated algorithms have an opportunity to take advantage of the Pollaczek-Khintchine formula; see, in particular, Asmussen and Binswanger (1997), Asmussen, Binswanger and Hojgaard (2000), Juneja and Shahabuddin (2002), Asmussen and Kroese (2006) and Dupuis, Leder and Wang (2006). An algorithm for rare-event simulation in the $G/G/1$ queue setting can be found in Blanchet and Glynn (2007). It should be noted, however, that the latter paper actually proceeds by converting the tail estimation problem for $G/G/1$ into a level crossing problem for random walk (based on the probabilistic equivalence of the $G/G/1$ waiting with the all-time maximum of a one-dimensional random walk). It should be further noted that we use here a mixture sampling idea (due to Dupuis, Leder and Wang (2006) and proposed as a means of computing tail probabilities for heavy-tailed sums) in order to avoid certain variate generation difficulties that arise in the context of the recently proposed $G/G/1$ algorithm. Related literature also includes Foss and Korshunov (2006), in which the first tail asymptotics for the $G/G/2$ steady-state waiting time distribution were obtained, as well as the algorithm proposed by Sadowsky (1990) to compute tail probabilities for $G/G/s$ queues in which the service time requirements are light-tailed. In contrast to the heavy-tailed $G/G/2$ setting, the rare-event path for a light-tailed $G/G/s$ queue is very similar to the one followed by a single-server queue with a server that serves at rate $s$ (rather than unit rate). As a consequence, the analysis for the light-tailed $G/G/s$ queue can take advantage, to a greater degree, of related analyses for the single-server queue.

2 Computing Steady-State Rare Event Probabilities

In the context of rare-event simulation of the $G/G/1$ queue, virtually all existing algorithms take advantage of the fact that the steady-state distribution of the waiting time sequence of the $G/G/1$ queue is identical to that of the all-time maximum of an associated negative drift random walk process. Consequently, computing a tail probability for the waiting time sequence of the $G/G/1$ queue is equivalent to computing a level crossing probability for a random walk. This turns out to be a highly convenient fact in developing good importance sampling algorithms in the $G/G/1$ setting.

Unfortunately, no such equivalence to a random walk maximum exists for the $G/G/s$ queue with $s \geq 2$. Hence, one must develop a rare-event importance sampling that applies directly to the $G/G/s$ itself (rather than a stochastically equivalent model). As noted in the Introduction, our focus here is on the waiting time sequence of the $G/G/2$ queue. This sequence, as we shall see in Section 4, can be represented via the so-called Kiefer-Wolfowitz process as a Markov chain taking values in $\mathbb{R}^2_+$ (see
Amussen (2003), p. 341.) We therefore need to discuss the use of the importance sampling in computing steady-state rare-event probabilities for Markov chains.

Let $W = (W_n : n \geq 0)$ be an $S$-valued Harris recurrent Markov chain with stationary distribution $\pi$. For $x \in S$, let $P_x(\cdot)$ and $E_x(\cdot)$ be the probability distribution and expectation operator corresponding to $W$, conditional on $W_0 = x$. Suppose that our goal is to compute $z = \pi(B)$, where $z$ is small. In the setting of Markov chain Monte Carlo (MCMC) algorithms on $S = \mathbb{R}^d$ (see, for example, Liu (2001)), one can take advantage of the fact that $\pi$’s density is known up to a normalizing constant, so that $\pi(dx) = c\nu(x)dx$ for $x \in \mathbb{R}^d$, where the density (with respect to Lebesgue measure) $\nu(\cdot)$ is known and the constant $c$ is unknown. One approach to applying importance sampling here is to modify the MCMC algorithm’s transition function so that its unique stationary distribution has known density $\phi$ (with no unknown normalizing constant), and to estimate $\pi(B)$ via the ratio

$$
\frac{n^{-1} \sum_{i=0}^{n-1} I(W_i \in B) \nu(W_i) / \phi(W_i)}{n^{-1} \sum_{i=0}^{n-1} I(W_i \in B) \nu(W_i) / \phi(W_i)}.
$$

(Note that the troublesome normalization constant no longer appears.) The idea now is to choose an easily simulated transition function for $W$ for which the density $\phi(\cdot)$ is roughly in proportion to $\nu(x)I(x \in B)$. Of course, if one can generate variates directly from the density $\phi(\cdot)$, one possible choice for the MCMC sampler is to draw iid samples from $\phi(\cdot)$ (thereby avoiding the complications associated with dependent sampling schemes).

In typical non-MCMC setting, little is generally known a priori about $\pi$. In particular, in performance evaluation contexts, $\pi$ is not known up to a normalization constant. In this context, an application of importance sampling to computing $z$ must be implemented by altering the dynamics of the sample paths that are generated (so that $B$ is hit more frequently). Note that for $K$ with $\pi(K) > 0$, $\pi(B)$ can be represented as

$$
\pi(B) = \frac{E_{\pi_K} \left( \sum_{j=0}^{T_{K}^{-1}} I(W_j \in B) \right)}{E_{\pi_K}(T_K)},
$$

where $\pi_K(\cdot) = \pi(\cdot \cap K) / \pi(K)$, $E_{\pi_K}(\cdot) = \int_K \pi_K(dx) E_x(\cdot)$, $P_{\pi_K}(\cdot) = \int_K \pi_K(dx) P_x(\cdot)$, and $T_K = \inf\{n \geq 1 : W_n \in K\}$ is the first return time to $K$; see Orey (1959). When $W$ returns to a “regeneration state”, say $x_0 \in S$ infinitely often, a particularly convenient choice of $K$ is $K \in \{x_0\}$, in which case (2) is the familiar ratio formula for the steady-state distribution of a regenerative process; see, for example, Asmussen (2003).

Since $B$ is rare, the probability $P_{\pi_K}(T < T_K)$ is small, where $T = \inf\{n \geq 0 : W_n \in B\}$ is the first hitting time of $B$. Note that the event $\{T < T_K\}$ is an exit probability for which a substantial importance sampling literature exists; see, for example, Glynn and Iglehart (1989) and Blanchet and Glynn (2007). This suggests that we compute $\pi(B)$ by developing a good importance sampler for the event $\{T <
This approach, in conjunction with the use of (2) in settings where \(K\) is not a “regeneration set”, has been explored previously; see, for example, Goyal et al (1992).

It is clearly of some interest to explore the degree to which a good importance sampler for \(f \ll T < K\) can be expected to efficiently compute \(B\). To explore this question, we need to develop a measure of efficiency for rare-event simulation. To this end, let \((P_b : b \in \Gamma)\) be a family of problem instances. It is a rare-event family if \(z_b \triangleq P_b > 0\) for each \(b \in \Gamma\) and \(\inf_{b \in \Gamma} z_b = 0\). All problem instances in the remainder of this paper are assumed to be rare-event families. We wish to develop a notion of efficiency that permits us to describe the estimators that are biased (as in (1)). For each \(b \in \Gamma\), let \(Z_b (c)\) be an estimator for computing \(z_b\) that can be computed in \(c\) units of computer time (as measured, for example, by the number of floating point operations or the number of random variates generated).

**Definition 1** Fix \(r > 0\). We say that the family of estimators \((Z_b (c) : b \in \Gamma, c > 0)\) is \(r\)-efficient for \(\Gamma\) if

\[
\sup_{b \in \Gamma} z_b^{-r} E |Z_b (c) - z_b|^r \rightarrow 0
\]

as \(c \nearrow \infty\). The family is said to be efficient if it is 2-efficient. The family is said to have an efficiency gap of (no larger than) \(s\) if

\[
\sup_{b \in \Gamma} z_b^{-2+s} E |Z_b (c) - z_b|^2 < \infty.
\]

By Markov’s inequality, it follows that if the family \((Z_b (c) : b \in \Gamma, c > 0)\) is \(r\)-efficient for \(r > 0\), then for \(\varepsilon > 0\),

\[
\sup_b P (|Z_b (c) - z_b| \geq \varepsilon z_b) \rightarrow 0
\]

as \(c \nearrow \infty\), so that the relative error goes to zero uniformly as \(c \nearrow \infty\). In other words, the amount of computer time required to evaluate \(z_b\) to a relative precision \(\varepsilon\) with a given probability \(\delta > 0\) is uniformly bounded across all problem instances \(b \in \Gamma\).

**Remark 1.** Note that (in contrast to much of the existing literature) these definitions involve no limiting operations that depend on a (possibly artificial) parameterization of the family \(\Gamma\), nor do the definitions assume any functional form (e.g. exponential or power law decay) of the probabilities as a function of the parameterization that is used.

**Remark 2.** We generalize the notion of efficiency to \(r\)-efficiency for several reasons. This generalization is convenient theoretically (see, for example, the statement of Proposition 2 below). It further can be used to establish that the computer time required to compute a rare event probability to a given relative precision is uniform across a family of problem instances, even in cases in which the estimator has infinite variance (by utilizing \(r\)-efficiency with \(r < 2\)). Finally, such a generalization can be useful in proving that the sample variance of an estimator is well-behaved across \(\Gamma\).
Recall that a family of non-negative rv’s (random variables) \( (\psi_\beta : \beta \in \Theta) \) (or, equivalently, probability distributions on \( \mathbb{R}_+ \)) has \textit{bounded relative variance} if there exists \( m > 0 \) such that
\[
\text{Var}(\psi_\beta) \leq m \left( E\psi_\beta \right)^2
\]
for all \( \beta \in \Theta \); see Juneja and Shahabuddin (2006) for more information on standard definitions related to performance analysis of rare-event simulation algorithms. Efficiency is, of course, equivalent to assuming bounded relative variance when, for each \( b \in \Gamma \), \( z_b(c) \) is a sample mean of \( n(c) \) iid copies of \( Y_b(c) \), where \( n(c) \not\to \infty \) as \( c \not\to \infty \) and \( Y_b(c) \) is unbiased for \( z_b \).

A slight weakening of \( r \)-efficiency is provided next.

**Definition 2** Fix \( r > 0 \). We say that the family of estimators \( (Z_b(c) : b \in \Gamma, c > 0) \) is \( r \)-logarithmically efficient for \( \Gamma \) if, for each \( \varepsilon > 0 \),
\[
\sup_{b \in \Gamma} z_b^{r+\varepsilon} E|Z_b(c) - z_b|^r \to 0
\]
as \( c \not\to \infty \). The family is said to be logarithmically efficient if it is 2-logarithmically efficient.

We note that \( r \)-logarithmic efficiency implies that
\[
\lim_{n \to \infty} \frac{\log E|Z_{bn}(c)|^r}{\log z_{bn}} \geq r
\]
for any subsequence \( (P_{bn}(B_{bn}) : n \geq 1, b_n \in \Gamma) \) of problem instances for which \( Z_{bn} \to 0 \) as \( n \not\to \infty \).

Returning now to the efficient computation of \( \pi(B) \), consider the identity (2) in the case in which \( K = \{x_0\} \), where \( x_0 \) is chosen as a regeneration state of \( W \). Assuming that we can find a good importance sampler for the event \( \{T < T_K\} \), the idea is to use importance sampling to compute the numerator of (2), and crude Monte Carlo to calculate its denominator. We denote the likelihood ratio associated with simulating \( W \) up to \( T \land T_K \) by the rv \( L \). We then generate \( \lfloor nu \rfloor \) independent iid copies of \( T_K \), starting from \( x_0 \), under \( W \)'s nominal dynamics to estimate the denominator. Let \( \hat{R}_n \) and \( \overline{D}_{\lfloor un \rfloor} \) be the sample means for
the numerator and denominator respectively, and observe that since \( D_{\lfloor un \rfloor} \geq 1 \) a.s.,

\[
E \left( \frac{\overline{R}_n}{D_{\lfloor un \rfloor}} - z \right)^2 \leq E \left( \overline{R}_n - zD_{\lfloor un \rfloor} \right)^2 \\
= Var \left( \overline{R}_n - zD_{\lfloor un \rfloor} \right) \\
= Var \left( \overline{R}_n \right) + z^2 Var \left( D_{\lfloor un \rfloor} \right). \tag{3}
\]

Consider now a decreasing sequence of sets \((B_b : b \in \Gamma)\) and the associated family of problem instances \((\pi(B_b) : b \in \Gamma)\) corresponding to a given Markov chain \(W\). Given a computer budget \(c\) and problem instance \(b\), the sample size \(n = n_b(c)\) is increasing (no faster than linearly) in \(c\). (Note that in general, the computer time per numerator replication may depend strongly in \(b\).) Our notions of efficiency and logarithmic efficiency require bounding \(E(Z_b(c) - z_b)^2\) in terms of \(Z_b^2\). In view of (3) and the fact that \(Var(T_K)\) is independent of \(b\), the key issue is the variance of \(\sum_{j=0}^{T_K-1} I(W_j \in B_b)L_b\) under \(\overline{P}_b\), where the notation \(L_b\), \(\overline{P}_b\) and \(T_b\) reflects the dependence of \(L\), \(\overline{P}\) and \(T\) on \(b\). Put

\[
N_b = \sum_{j=0}^{T_K-1} I(W_j \in B_b)
\]

and let \(N_b(x)\) be a rv with the distribution \(P_x(N_b \in \cdot)\). In addition, let \(\iota_{b,j}(x) = E_xN_b^j\).

**Proposition 1** Suppose that

i.) \((N_b(x) : x \in B_b, b \in \Gamma)\) has bounded relative variance;

ii.) \((\iota_{b,1}(W_{T_b})L_bI(T_b < T_K) : b \in \Gamma)\) has bounded relative variance.

Then, \((N_bL_b : b \in \Gamma)\) has bounded relative variance.

**Proof.**

\[
\tilde{E} \left( N_b^2L_b^2 \right) \\
= \tilde{E} \left( \tilde{E} \left( N_b^2 \mid W_0, ..., W_{T_b} \right) L_b^2I(T_b < T_K) \right) \\
= \tilde{E} \left( \iota_{b,2}(W_{T_b})L_b^2I(T_b < T_K) \right) \\
\leq k_1 \tilde{E} \left( \iota_{b,1}(W_{T_b})L_b^2I(T_b < T_K) \right) \\
\leq k_1k_2 \tilde{E} \left( \iota_{b,1}(W_{T_b})L_bI(T_b < T_K) \right)^2 \\
= k_1k_2z_b^2,
\]

where the two inequalities follow from i.) and ii.) respectively. \(\blacksquare\)
Note that the first condition above asserts that the “clump size” \( N_b \) of the number of visits to \( B_b \) over an \( x \)-cycle has bounded relative variance over \( B_b \). This condition is broadly valid, and is typically satisfied in both light-tailed and heavy-tailed models. As for the second condition, this is closely related to the efficient estimation of \( P(T_b < T_K) \). The next proposition makes this connection clearer.

**Proposition 2** Suppose that

i.) \((\ell_{b,1}(W_{T_b})|T_b < T_K : b \in \Gamma)\) is 4-efficient;

ii.) \((I(T_b < T_K) L_{T_b} : b \in \Gamma)\) is 3-efficient.

Then, \((\ell_{b,1}(W_{T_b}) L_b I(T_b < T_K) : b \in \Gamma)\) has bounded relative variance.

**Proof.** Note that

\[
\begin{align*}
\tilde{E}(\ell_{b,1}(W_{T_b})^2 L_b^2 I(T_b < T_K)) \\
\leq \tilde{E}(\ell_{b,1}(W_{T_b})^4 L_b I(T_b < T_K))^{1/2} \tilde{E}(L_b^2 I(T_b < T_K))^{1/2} \\
\leq k_3 \tilde{E}(\ell_{b,1}(W_{T_b})^4 I(T_b < T_K))^{1/2} \tilde{E}(L_b I(T_b < T_K))^{3/2} \\
= k_3 \tilde{E}(\ell_{b,1}(W_{T_b})^4|T_b < T_K)^{1/2} P(T_b < T_K)^2 \\
\leq k_3 k_4 \tilde{E}(\ell_{b,1}(W_{T_b})|T_b < T_K)^2 P(T_b < T_K)^2 = k_3 k_4 z_b^2,
\end{align*}
\]

where the second and third inequalities above follow from ii.) and i.) respectively. \( \square \)

Note that condition i.) of the previous proposition is again a uniformity hypothesis controlling the clump size that can be expected to hold across many light-tailed and heavy-tailed models. Condition ii.) (when viewed in combination with the condition i.’s of the previous two propositions) is the only hypothesis needed for bounded relative efficiency of our numerator estimators that concerns the importance sampler, and is an assertion that the importance sampler is efficient for computing \( P(T_b < T_K) \) (actually efficient in the somewhat stronger sense of 3-efficiency). We conclude that estimation of \( \pi(B_b) \) essentially comes down to developing an efficient importance sampler for \( P(T_b < T_K) \).

### 3 State-dependent Importance Sampling

To compute the rare-event probability \( u_b^*(x) = P_x(T_b < T_K) \) we note that \((u_b^*(x) : x \in S)\) solves the equation

\[
u_b^*(x) = E_x u_b^*(W_1) \equiv \int_S P(x, dy) u_b^*(y) \tag{4}\]

8
subject to the boundary condition \( u^*_b(x) = 1 \) for \( x \in B_b \) and \( u^*_b(x) = 0 \) for \( x \in K \). The conditional distribution of \( W \) given the event \( \{T_b < T_K\} \), is that \( W \)'s conditional dynamics form a Markov chain with modified transition kernel

\[
R_b(x, dy) = P(x, dy) \frac{u^*_b(y)}{u^*_b(x)}
\]

for \( x, y \notin K \). Given that \( u^*_b \) is unknown, one possible means to developing a rare-event simulation algorithm is to substitute an approximation \( v_b \) for \( u^*_b \). Since \( v_b \) is not an exact solution to (4), the normalization constant

\[
w^*_b(x) = \int_S P(x, dy) v_b(y)
\]

does not equal \( v_b(x) \), and the approximating transition kernel \( \tilde{R}_b \) takes the form

\[
\tilde{R}_b(x, dy) = P(x, dy) \frac{v_b(y)}{w^*_b(x)}
\]

This approach to developing an importance sampler for computing \( P_{x_0}(T_b < T_K) \) was recently implemented in the \( G/G/1 \) case by Blanchet and Glynn (2007). This implementation was based on the known heavy-tailed asymptotics for the steady-state distribution of the waiting time sequence \( W \). A similar idea would likely work in the \( G/G/2 \) setting, in which case the approximation \( v_b \) would be based on asymptotics due to Foss and Korshunov (2006). However, one difficulty with this idea that is likely to be troublesome in the \( G/G/2 \) context (or other higher dimensional problems) is the need to develop an efficient algorithm for simulating transitions from the kernel \( \tilde{R}_b \). The difficulty is that the kernel \( \tilde{R}_b \) is not a priori constructed in such a way that the path generation problem has an immediate solution.

An alternative is to take advantage of known problem structure relating, in particular, to the probabilistic mechanism that generates a visit to \( B_b \) prior to returning to the regeneration state \( K = \{x_0\} \). For example, in light-tailed queues, one knows that such paths occur when the associated random walk is exponentially twisted according to a twisting parameter \( \theta \) (that, in principle, can be chosen in a state-dependent way). On the other hand, for heavy-tailed queues, one expects that the associated random walk proceeds according to increments that are chosen as a mixture of a big jump (say greater than \( y \)) and a small jump (smaller than \( y \)). The modified increment distribution is characterized by \( \beta = (p, y) \), where \( p \) is the mixture probability. As in the light-tailed setting, the choice of \( p \) may be state-dependent. This induces a one-step transition kernel \( \tilde{K} = \left( \tilde{K}_\beta(x) : x, z \in S \right) \), where \( \tilde{K}_\beta(x, dz) \) can be represented, for each choice of \( \beta = (p, y) \), in the form \( \tilde{K}_\beta(x, dz) = k^{-1}_b(x, y) P(x, dz) \). Note that transitions of \( K \) can be simulated by exponential twisting in the light-tailed setting and via mixture sampling in the heavy-tailed context, so that constraining \( \tilde{K} \) to be of this form forces the Markov chain to be easily simulatable under the importance distribution.
This key variate generating insight is due to Dupuis and Wang (2004?). They further recognized that the state-dependent choice of \((\beta(x) : x \in S)\) that minimizes the \(r\)-th moment for \((r > 1)\) of \(I(T_b < T_K) L_b\) under the importance sampler is the optimal control associated with the Hamilton-Jacobi-Bellman equation

\[
V_b(x) = \min_{\beta} E_x[k_b(\beta, x, W_1)^r-1 V_b(W_1)]
\]

subject to \(V_b(x) = 1\) on \(B_b\). (They specifically point out this connection in minimizing the variance for light-tailed uniformly geometrically ergodic Markov chains and in the context of heavy-tailed regularly varying sums, see Dupuis, Leder and Wang (2006)).

Because (5) is generally not solvable, an alternative is to seek a “control” \(\beta = (\beta(x) : x \in S)\) that is \(r\)-efficient but not necessarily optimal (in the sense of (5)). To verify \(r\)-efficiency, one must bound \(E_{x_0} I(T_b < T_K) L_b^r\) (over \(b\)). Given a state-dependent selection \((\beta_b(x) : x \in S)\) of \(\beta\), this requires bounding the \(r\)-th moment quantity.

\[
s_b^{(r)}(x_0) \triangleq \tilde{E}_{x_0} \left( I(T_b < T_K) \prod_{j=1}^{T_b} k_b(\beta_b(W_{j-1}), W_{j-1}, W_j)^r \right)
\]

\[
= E_{x_0} \left( I(T_b < T_K) \prod_{j=1}^{T_b} k_b(\beta_b(W_{j-1}), W_{j-1}, W_j)^{r-1} \right)
\]

More generally, given that finally we are interested in the efficiency of the waiting time sequence, we are also concerned with bounding

\[
s_{b,\chi}^{(r)}(x_0) \triangleq E_{x_0} \left( I(T_b < T_K) \prod_{j=1}^{T_b} k_b(\beta_b(W_{j-1}), W_{j-1}, W_j)^{r-1} \chi(W_{T_b}) \right),
\]

for a \(\chi : S \rightarrow [0, \infty)\).

**Proposition 3** Suppose that there exists a function \(h_b : S \rightarrow [0, \infty)\) satisfying

i.) \(E_x k_b(\beta_b(x), x, W_1)^{r-1} h_b(W_1) I(W_1 \in K^c) \leq h_b(x)\)

for \(x \in B_b^c\);  
ii) \(h_b(x) \geq \varepsilon \chi(x)\) for \(x \in B_b\).

Then, \(s_{b,\chi}^{(r)}(x) \leq \varepsilon^{-1} h_b(x)\) for \(x \in S\).

**Proof.** Let \(M = (M_n : n \geq 1)\) be defined via

\[
M_n = \prod_{j=1}^{T_b \wedge n} k_b(\beta_b(W_{j-1}), W_{j-1}, W_j)^{r-1} h_b(W_{T_b \wedge n}) I(T_K > T_b \wedge n).
\]
Note that $M$ is a non-negative supermartingale that is adapted to the filtration generated by the chain $W$. Since $P_x(T_K < \infty) = 1$ for $x \in S$ we have that

$$M_n \to \prod_{j=1}^{T_b} k_b(\beta_b(W_{j-1}), W_{j-1}, W_j)^{-1} h_b(W_{T_b}) I(T_K > T_b)$$

as $n \nearrow \infty$. Fatou’s lemma then yields the stated bound. ■

The function $h_b$ is called a Lyapunov function (for bounding the $r$-th moment).

**Remark 3** To bound $E(I(T_b < T_K) L_b^r)$ for any initial distribution that is supported on $K$ (e.g. the distribution of $W$ that initiates a regenerative cycle), an obvious sufficient condition for finiteness is to require that $\sup \{h_b(x) : x \in K\} < \infty$.

The next result shows how the previous Lyapunov bounds immediately yield upper bounds on rare-event probabilities for heavy-tailed models. Such upper bounds are often the most challenging part of any asymptotic calculation.

**Corollary 1** The Lyapunov function satisfying Proposition 3 yields an upper bound on $P_x(T_b < T_K)$, namely $P_x(T_b < T_K) \leq (h_b(x) / \varepsilon)^{1/r}$.

**Proof.**

$$P_x(T_b < T_K)^r = \left( \bar{E}_x I(T_b < T_K) L_b \right)^r \leq \bar{E}_x I(T_b < T_K) L_b^r = s_b^{(r)}(x) \leq h_b(x) / \varepsilon.$$ ■

Recall that the zero-variance importance sampler for the event $\{T_b < T_K\}$ involves sampling from the conditional distribution of $W$ given $\{T_b < T_K\}$; see for example, Glynn and Iglehart (1989). In the presence of sufficient model structure, one often has intuition about the conditional dynamics of $W$ given $\{T_b < T_K\}$. This potentially can be used to reduce the search for an $r$-efficient control $(\beta(x) : x \in S)$ to that of a parameterized family of controls $(\beta(\gamma, x) : x \in S)$ for some finite-dimensional set of parameters $\gamma$. This reduction leads to a finite-dimensional search (rather than an infinite-dimensional one).

We illustrate this idea via the single-server $G/G/1$ queue. Let $(W_n : n \geq 0)$ be the waiting time sequence (exclusive of service) for this queue under a first-in first-out (FIFO) queue discipline. Thus, $W$ satisfies

$$W_{n+1} = (W_n + X_{n+1})^+$$
for an iid sequence \((X_n : n \geq 1)\). Suppose that \(B_b = [b, \infty)\) and \(K = \{0\}\). We assume that the chain is positive recurrent with \(EX_1 < 0\). We further assume that \(X_1\) has a heavy right tail, so that

\[
P(X_1 > x) = L(t) t^{-\alpha}
\]

for \(t > 0\) and \(\alpha > 0\), where \(L(\cdot)\) is slowly varying (i.e. \(L(tm)/L(t) \to 1\) as \(t \to \infty\) for each \(m > 0\)). In this heavy-tailed setting, we expect an exceedence of \(b\) to occur through a large jump of order \(b\). This suggests that the importance sampler should try to induce a large jump. One intuitively reasonable mechanism for inducing such large jump is, conditional on \(\{n < T_b \land T_K\}\) and \(W_n = w\), to generate \(W_{n+1}\) from a mixture of the “large jump increment distribution” \(P(X_1 \in \cdot | X_1 > a (b - x))\) and a “moderate increment distribution” \(P(X_1 \in \cdot | X_1 \leq a (b - x))\) for some \(a \in (0,1)\) (this particular mixture form was introduced in Dupuis, Leder and Wang (2006)). We choose \(a \in (0,1)\) to reflect the fact that there are paths of significant probability leading to \(\{T_b < T_K\}\) that involve large jumps that take \(W\) to a position just below \(b\).

For the mixture probability \(p_b(x)\) associated with the large increments, note that conditional on \(\{W_n = x, n < T_b \land T_K\}\), we expect one large jump to occur before the queue empties at time \(T_K\). When \(x\) is close to \(K\), \(p(\cdot)\) should be bounded away from zero (for, otherwise, the regenerative cycle will end, with high probability, without an exceedence of \(b\)). On the other hand, when \(x\) is large, the chain tends to drift down to \(K\) at linear rate so that the system has roughly order \(x\) remaining time units within which to generate a large jump. Hence, in order that the likelihood of a large jump over the remaining regenerative cycle be bounded away from zero, we should choose \(p(x)\) to be roughly of order \(1/x\). These two observations suggest that a suitable functional form for the function \(p(\cdot)\) to be used in the importance sampler is to choose \(p(x) = \theta/(d + x)\) for appropriately chosen constants \(\theta\) and \(d\). We therefore restrict our search for controls \((\beta(x) : x \geq 0)\) for the \(G/G/1\) queue to a parametrically defined family of controls \((\beta(\gamma, x))\) where the parameter \(\gamma\) is given by \(\gamma = (a, \theta, d)\).

The goal is now to find a choice of \((\alpha, \theta, d)\) for which the Lyapunov bound of Proposition 3 is of the order required for \(r\)-efficiency. Note that the zero-variance change-of-measure for \(\{T_b < T_K\}\) is Markovian and (obviously) \(r\)-efficient, so that \(s_{\gamma}^{(r)}(x)\) is then given by \(P_x(T_b < T_K)^r\). Since we are developing our (hopefully) \(r\)-efficient change-of-measure so as to mimic the zero-variance Markovian conditional distribution, this suggests that \(\mathbb{E}_x \mathbb{I} (T_b < T_K) L_b\) should behave (roughly) like \(P_x(T_b < T_K)^r\). In the presence of good intuition (or known asymptotics) for the model, this recommends the choice of Lyapunov function \(h_b(x) = v_b(x)^r\), where \(v_b(x)\) is our approximation to \(P_x(T_b < T_K)^r\). Note that our chosen approximation will often be poor when \(x\) is close to \(B_b\). Because Proposition 3 demands that the appropriate inequality be satisfied everywhere on \(B_b^c\), it is useful to introduce some additional parameters into \(v_b(x)^r\) so as to satisfy the Lyapunov inequality. The development of a practically implementable and theoretically justifiable \(r\)-efficient importance sampler then comes...
down to choosing $\beta$ and the parameters of the Lyapunov function in such a way that the Lyapunov inequality is satisfied (and so that $v_b(x)$ is of the order of magnitude of $P_x(T_b < T_K)$).

Returning to the $G/G/1$ example, the system tends to drift down to $K$ linearly. At each such step along the path to $K$ there is an approximate probability $P(X_1 > b - y)$ of entering $B_b$ on that step (given current position $y$). This suggests approximating $P_x(T_b < T_K)$ via a function of the form

$$c_0 \int_0^{x+d_0} P(X_1 > b - x + s) \, ds;$$

the two constants $c_0$ and $d_0$ are the parameters introduced so as to provide additional degrees of freedom in satisfying the Lyapunov bound. For this example, appropriate constants $a$, $d$, $\theta$, $c_0$ and $d_0$ can be found so as to meet all the above requirements. This provides an alternative rare-event simulation algorithm for the $G/G/1$ queue to that proposed in Blanchet and Glynn (2007). We omit the details, in view of the fully developed $G/G/2$ discussion that will be presented in the remaining sections of this paper.

4 $G/G/2$ Queue $\rho > 1/2$

Let $(V_n : n \geq 1)$ be an iid sequence of non-negative rv’s representing service requirements of an arrival stream. We assume that arrivals join the system according to a renewal process governed by a sequence of iid inter-arrival times $(\tau_n : n \geq 1)$ which are independent of the $V_n$’s. We shall suppose in addition that the $\tau_n$’s have unbounded support.

Define the traffic intensity, $\rho$, via $\rho = EV_n/(2E\tau_n)$. In this section we assume that $\rho > 1/2$. We introduce the random variables $(X_n : n \geq 1)$ defined via $X_n = V_n - \tau_{n+1}$ and we write $\overline{F}(x) = P(X > x)$, $F(x) = 1 - \overline{F}(x)$ for the tail probability and distribution functions of $X$ evaluated at $x$ respectively. We shall assume that $X_1 = V_1 - \tau_2$ has a regularly varying density, $f(\cdot)$, in particular

$$f(t) = t^{-(\alpha+1)} L(t),$$

for $\alpha > 1$ and $L(kt)/L(t) \longrightarrow 1$ as $t \not\to \infty$ for each $k > 0$. In addition, we assume that $f(\cdot)$ is eventually decreasing and that $Var(X) < \infty$. Finally, we define the integrated tail function $(G(x) : x \in \mathbb{R})$ via $G(x) = \int_x^\infty \overline{F}(s) \, ds$.

The dynamics of the $G/G/2$ queue are described at the time of the $n$-th arrival via the Kiefer-Wolfowitz process $\left((W_n^{(1)}, W_n^{(2)}) : n \geq 0\right)$, defined via

$$W_{n+1}^{(1)} = \min\left((W_n^{(1)} + X_{n+1})^+, (W_n^{(2)} - \tau_{n+1})^+\right),$$

$$W_{n+1}^{(2)} = \max\left((W_n^{(1)} + X_{n+1})^+, (W_n^{(2)} - \tau_{n+1})^+\right).$$
where \(X_{n+1} = V_n - \tau_{n+1}\). The component \(W_{n+1}^{(1)}\) describes the waiting time in queue (excluding service) of the \((n+1)\)-th customer that arrives into the system. The second component is basically \(W_{n+1}^{(1)}\) plus the remaining time of the customer that is being served right at the time when the \((n+1)\)-th customer is about to be served. Note that one can recover a single-server queue by formally placing an infinite amount of initial workload into one of the queues, in such case we obtain \(W_{n+1}^{(2)} = \infty\) and we recover Lindlay’s equation for the \(G/G/1\) queue, namely \(W_{n+1}^{(1)} = \left(W_{n+1}^{(1)} + X_{n+1}\right)^+\).

In order to evaluate one step transition expectations given values \(0 \leq w_1 \leq w_2\), it is useful to introduce the notation

\[
W_1' = \min\left((w_1 + X)^+, (w_2 - \tau)^+\right),
\]

\[
W_2' = \max\left((w_1 + X)^+, (w_2 - \tau)^+\right),
\]

where \((X, \tau)\) has the same distribution as \((V_1, \tau_1, \tau_2)\).

Set \(B_b = [b, \infty) \times [b, \infty)\) and \(K = \{(0,0)\}\). We are interested in developing an efficient rare-event simulation algorithm for the estimation of the probability

\[
P_{(0,0)}(T_b < T_K)
\]
as \(b \to \infty\).

The first step in our program is to select an appropriate family of changes-of-measure that captures the large deviations behavior of the system. In particular, it is intuitively plausible to expect that in this case the rare-event will involve a large service time that will block one of the servers for long time. When a server is blocked by a long service time, the second server operates much as an unstable \(G/G/1\) queue (this is because \(\rho > 1/2\) which implies \(EX > 0\)). Hence, it seems appropriate to consider a family of importance samplers that with some small probability induces, at each arrival, a service time that is long enough to trigger a build up in the second queue which eventually induces a waiting time larger than \(b\).

More precisely, if \(h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}\) is any bounded (measurable) function it seems appropriate to consider a family of transition Kernels \(Q_{a,p}(\cdot)\) such that (for \(w_2 < b\))

\[
E_{(w_1, w_2)}^{Q_{a,p}} h\left(W_1^{(1)}, W_2^{(2)}\right) = p\left(w_1, w_2\right) \frac{E\left(h\left(W_1', W_2'\right); X > a\left(b - w_1 + b - w_2\right)\right)}{F\left(a\left(b - w_1 + b - w_2\right)\right)}
\]

\[
+ \left(1 - p\left(w_1, w_2\right)\right) \frac{E\left(h\left(W_1', W_2'\right); X \leq a\left(b - w_1 + b - w_2\right)\right)}{F\left(a\left(b - w_1 + b - w_2\right)\right)},
\]

where \(a \in (0,1)\) and \(p\left(w_1, w_2\right) \in (0,1)\) will be chosen depending on the state of the Kiefer-Wolfowitz process. We expect the probability \(p\left(w_1, w_2\right)\) to behave qualitatively as the probability that the rare-event be caused by the next arrival. Note that if the next service time is roughly \((b - w_1) + (b - w_2)\), then the largest component of the
Kiefer-Wolfowitz vector will be above level $b$ by an amount of order $(b - w_2)$, while the smallest component will be roughly at level $w_2$ and can be expected to have enough time to reach level $b$ under the original/nominal dynamics of the system. The parameter $a \in (0, 1)$ plays a fundamental role in controlling the contribution to the likelihood ratio of those sample paths for which the rare-event occurs without a single large jump. Those sample paths have been noted in the literature on rare-event simulation for heavy-tailed systems to play an important role in the variance of importance sampling estimators (see Asmussen, Binswanger and Højgaard (2000)); therefore, introducing $a \in (0, 1)$ is natural in our parametric family of importance samplers estimators considered.

Our goal is to find a positive function $g(\cdot)$ such that if we define

$$J_1 = \frac{E(g(W'_1, W'_2); X > a (2b - w_1 - w_2)) \overline{F}(a (2b - w_1 - w_2))}{p(w_1, w_2)},$$

$$J_2 = \frac{E(g(W'_1, W'_2); X \leq a (2b - w_1 - w_2)) F(a (2b - w_1 - w_2))}{1 - p(w_1, w_2)},$$

then $g(\cdot)$ must satisfy

$$J_1 + J_2 \leq g$$

on the set $w_1 < b$ and $g(w_1, w_2) \geq \varepsilon$ for $w_1 \geq b$. The rest of this section is devoted to the construction of $g$ and the characterization of $p$. Our program proceeds in the following steps.

Step 1) Pick a parametric family of functions $g(\cdot)$ that are plausible candidates for a good Lyapunov function. Select the parameters of $g(\cdot)$ based on the following steps.

Step 2) Analyze $J_2$ in a large region of the space. In particular, assuming that $w_2 \geq \kappa_2$ for some constant $\kappa_2 > 0$ specified later, $w_1 \leq b$ and $2b - w_1 - w_2 \geq \kappa$.

Step 3) Analyze $J_1$ on the region indicated in Step 2)

Step 4) Combine the estimates obtained in the previous two steps to show that the Lyapunov bound is satisfied on the region $w_2 \geq \kappa_2$ and $w_1 \leq b$.

Step 5) Extend the validity of the Lyapunov bound on the region $w_2 \leq \kappa_2$.

Let us proceed with Step 1) of our program. If our family of importance samplers is well designed, we expect $g(\cdot)$ to behave much as $P_{(w_1, w_2)} (T_b < T_K)^2$. We can use a heuristic analysis based on sample path arguments for heavy-tailed processes to guess qualitatively the form of $P_{(w_1, w_2)} (T_b < T_K)$. In particular, it is plausible to expect this probability to be upper bounded by a function of the form

$$\int_0^{w_2 + w_1 + d} \overline{F}(b - w_1 + b - w_2 + s) \, ds = G(2b - w_1 - w_2) - G(2b + d)$$

when $b$ is large. The fact that $w_1$ and $w_2$ are exchangeable in the previous expression will useful to develop the desired Lyapunov bound. Nevertheless, such feature is likely
to be problematic when $w_1$ is relatively small compared to $w_2$ because such bound will tend to be very loose. For instance, it turns out that this problem will manifest itself right at the boundary, namely, when $w_1$ is close to zero and $w_2$ is large. Therefore, we consider an additional correction term of the form

$$h_\xi (w_1, w_2) = \int_0^{w_1} \xi (s) \bar{F} (2b - w_2 - s) \, ds,$$

where $\xi (\cdot)$ is defined for some $\kappa_0 > 1$ and $\zeta \in (0, 1)$ as

$$\xi (s) = I (s \leq \kappa_0) + (\zeta (\kappa_0 - s) + 1) I (\kappa_0 \leq s \leq \kappa_0 + 1/\zeta).$$

We propose searching for a Lyapunov function $g$ of the form

$$g (w_1, w_2) = 1 \wedge cg_1 (w_1, w_2),$$  \hspace{1cm} \text{(7)}

where

$$g_1 (w_1, w_2) = h_1 (w_1, w_2)^2$$

and

$$h_1 (w_1, w_2) = h_0 (w_1, w_2) - h_\xi (w_1, w_2),$$

with

$$h_0 (w_1, w_2) = G (2b - w_1 - w_2) - G (2b + d).$$

The constants $c$, $d$, $\zeta$ and $\kappa_0$ will be chosen appropriately to make $g$ satisfy (6).

It is important to note that

$$h_0 (w_1, w_2) - h_\xi (w_1, w_2) > 0$$ \hspace{1cm} \text{(8)}

for all $w_1 \leq w_2$. To see this, note that

$$0 \leq h_\xi (w_1, w_2) \leq \int_0^{w_1 \wedge (\kappa_0 + 1/\zeta)} \bar{F} (2b - w_2 - s) \, ds$$

$$= G (2b - w_2 - w_1 \wedge (\kappa_0 + 1/\zeta)) - G (2b - w_2).$$ \hspace{1cm} \text{(9)}

This implies that

$$h_0 (w_1, w_2) - h_\xi (w_1, w_2)$$

$$\geq G (2b - w_2 - w_1) - G (2b - w_2 - w_1 \wedge (\kappa_0 + 1/\zeta))$$

$$+ G (2b - w_2) - G (2b + d),$$

which clearly implies (8) because $G (\cdot)$ is decreasing. Similarly, by taking partial derivatives of $h_1$ it follows that $g$ is a non-decreasing function in both of its arguments.

Throughout the rest of the paper, we shall denote

$$\widetilde{W}_1 \triangleq w_1 + \eta_1, \quad \widetilde{W}_2 \triangleq w_2 + \eta_2,$$
where
\[
\begin{align*}
\tilde{W}_1 & \in (w_1 \wedge W'_1, w_1 \vee W'_1), \\
\tilde{W}_2 & \in (w_2 \wedge W'_2, w_2 \vee W'_2)
\end{align*}
\]  
(10)
as generic points in any Taylor expansion that we perform. For instance, using Taylor’s theorem with remainder, we have that if \( U_1 \) is a uniform random variable independent of \( V \) and \( \tau \), then (note that we just need to assume that \( g(\cdot, w_2) \) is absolutely continuous)
\[
E g(W'_1, W'_2) = E g(w_1, W'_2) + E (\partial_{w_2} g(w_1 + U_1 \Delta_1, W'_2) \Delta_1),
\]
where \( \Delta_1 \) and \( \Delta_2 \) are defined via
\[
\Delta_1 \triangleq W'_1 - w_1, \quad \Delta_2 \triangleq W'_2 - w_2.
\]
(11)
Applying the same procedure to \( E g(w_1, W'_2) \) we obtain
\[
E g(W'_1, W'_2) = g(w_1, w_2) + E (\partial_{w_2} g(w_1, w_2 + U_1 \Delta_2) \Delta_2)
\]
\[
+ E (\partial_{w_1} g(w_1 + U_1 \Delta_1, W'_2) \Delta_1).
\]
We will allow ourselves abusing the notation by writing the previous expression as
\[
E g(W'_1, W'_2) = g(w_1, w_2) + E (\partial_{w_2} g(\tilde{W}_1, \tilde{W}_2) \Delta_2)
\]
\[
+ E (\partial_{w_1} g(\tilde{W}_1, \tilde{W}_2) \Delta_1).
\]
The quantities \( \tilde{W}_1 \) and \( \tilde{W}_2 \) have been used to denote generic points that lie in the intervals specified by (10); this is the only property that we shall use in our future estimates.

In order to explain our strategy, we estimate \( J_2 \) using Taylor’s theorem. We obtain
\[
E (g(W'_1, W'_2); X \leq a (2b - w_1 - w_2))
\]
\[
= P (X \leq a (2b - w_1 - w_2))
\]
\[
+ E (\partial_{w_1} g(w_1 + \eta_1, w_2 + \eta_2) \Delta_1; X \leq a (2b - w_1 - w_2))
\]
\[
+ E (\partial_{w_2} g(w_1 + \eta_1, w_2 + \eta_2) \Delta_2; X \leq a (2b - w_1 - w_2)).
\]
Our strategy takes advantage of the fact that \( g(\cdot) \) is a non-decreasing in both \( w_1 \) and \( w_2 \). On the other hand, we have that the Kiefer-Wolfowitz vector decreases on average on a large region of the space \( w_1 \leq w_2 \) (although not at the boundary, when both \( w_1 \) and \( w_2 \) are small, it will be important to keep this in mind later). We then expect a negative contribution appearing in the previous Taylor expansion which we
can use to our advantage in order to design (22). In particular, if $g$ is well chosen, we aim to have an expression similar to (assuming $b - w_1$ large enough and that $w_1$ is bounded away from zero)

$$1 \approx \frac{J_1}{g} + \frac{J_2}{g}$$

$$\approx \frac{F(a (b - w_1))^2 P_{w_1,w_2} (T_b < T_K | X > b - w_1)^2}{p(w_1, w_2) g}
\left(1 + \frac{\partial_{w_1} g}{g} E \Delta_1 + \frac{\partial_{w_2} g}{g} \Delta_2 \right) \left(1 - p(w_1, w_2)\right). \tag{12}$$

Note that we have

$$\partial_{w_1} g \approx \partial_{w_2} g.$$

Since we expect to have $E (\Delta_1 + \Delta_2) < 0$ on when $w_1$ is bounded away from zero, then we plan to control the behavior of the term corresponding to $J_1$ by selecting

$$p(w_1, w_2) \approx \frac{\partial_{w_2} g}{g} = \partial_{w_2} \log g.$$

A similar story happens when $w_1$ is close to zero and $w_2$ is sufficiently large. In such case, $E \Delta_2 < 0$, so we must select $\xi(\cdot)$ in order to control the contribution of $\partial_{w_1} g$. Finally, when $w_2$ is close to zero, we will use the fact that $p$ must be bounded away from zero because, otherwise, the regenerative cycle will end with high probability. Note that if $g$ is picked close enough to the square of $P_{w_1,w_2} (T_b < T_K)^2$ the choice of $p(\cdot)$ indicated above has a natural interpretation as a hazard rate at which the rare event is expected to occur.

In order to continue with Step 2) of our program we need a couple of results. The first of these results establishes the negative drift of the Kiefer-Wolfowitz vector.

**Proposition 4** Define $\Delta_1$ and $\Delta_2$ as in (24). Then, given $0 < \varepsilon_0 < 2E \tau - EV$ we can compute $\kappa_1, \kappa > 0$ such that if $w_1 \geq \kappa_1$ and $2b - w_1 - w_2 \geq \kappa$ then

$$E_{(w_1,w_2)} (\Delta_1 + \Delta_2; X \leq a (2b - w_1 - w_2)) \leq -\varepsilon_0 < 0.$$ 

In addition, there exists $\kappa_2 > 0$ such that if $w_1 \leq \kappa_1$, $\kappa_2 \leq w_2$ and $2b - w_1 - w_2 \geq \kappa$, then

$$E_{(w_1,w_2)} (\Delta_2; X \leq a (2b - w_1 - w_2)) \leq -\varepsilon'_0$$

for $\varepsilon'_0 \in (0, E \tau)$.

Now we are interested in studying the derivatives of $h_0$ and $h_A$, this will allow us to perform a Taylor expansion for $J_2$. We have that

$$\partial_{w_1} h_0 (w_1, w_2) = F (2b - w_1 - w_2) = \partial_{w_2} h_0 (w_1, w_2).$$
In addition,
\[ \partial_{w_2} h_\xi (w_1, w_2) = \int_0^{w_1} \xi (w_1) f (2b - w_2 - s) \, ds, \]
\[ \partial_{w_1} h_\xi (w_1, w_2) = \xi (w_1) \bar{F} (2b - w_2 - w_1). \]

Let us now proceed with our analysis of \( J_2 \). In particular, assume that \( 2b - w_1 - w_2 \geq \kappa \) and \( \kappa \) large enough. Then, using Taylor’s theorem with remainder we obtain
\[
E (g (W'_1, W'_2); X \leq a (2b - w_1 - w_2)) \\
= g (w_1, w_2) P (X \leq a (2b - w_1 - w_2)) \\
+ c E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_1} h_0 (\tilde{W}_1, \tilde{W}_2) \Delta_1; X \leq a (2b - w_1 - w_2)) \\
+ c E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_2} h_0 (\tilde{W}_1, \tilde{W}_2) \Delta_2; X \leq a (2b - w_1 - w_2)) \\
- c E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_1} h_\xi (\tilde{W}_1, \tilde{W}_2) \Delta_1; X \leq a (2b - w_1 - w_2)) \\
- c E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_2} h_\xi (\tilde{W}_1, \tilde{W}_2) \Delta_2; X \leq a (2b - w_1 - w_2)). \quad (13)
\]

The following lemmas (whose proof is given in the appendix at the end of this section) summarizes useful estimates about the random variables involved in expression (13).

**Lemma 1** Assume that \( EX^2 < \infty \). Then, for each \( \varepsilon > 0 \) we can select \( \kappa > 0 \) and \( d > 0 \) such that if \( 2b - w_1 - w_2 \geq \kappa \), then
\[
E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_1} h_0 (\tilde{W}_1, \tilde{W}_2) \Delta_1; X \leq a (2b - w_1 - w_2)) \\
\leq h_1 (w_1, w_1) \partial_{w_1} h_0 (w_1, w_2) (E (\Delta_1) + \varepsilon/4)
\]
and
\[
E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_2} h_0 (\tilde{W}_1, \tilde{W}_2) \Delta_2; X \leq a (2b - w_1 - w_2)) \\
= h_1 (w_1, w_1) \partial_{w_2} h_0 (w_1, w_2) (E (\Delta_2) + \varepsilon/4).
\]

**Lemma 2** Suppose that \( EX^2 < \infty \). Then, for each \( \varepsilon > 0 \) there exists \( \kappa > 0 \) such that if \( 2b - w_1 - w_2 \geq \kappa \), then
\[
- E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_2} h_\xi (\tilde{W}_1, \tilde{W}_2) \Delta_2; X \leq a (2b - w_1 - w_2)) \\
\leq h_1 (w_1, w_1) \partial_{w_1} h_\xi (w_1, w_2) \varepsilon/4.
\]

In addition, given \( \varepsilon > 0 \) we can compute \( \kappa > 0 \) and \( d > 0 \) such that if \( 2b - w_1 - w_2 \geq \kappa \), then
\[
E (h_1 (\tilde{W}_1, \tilde{W}_2) \partial_{w_1} h_\xi (\tilde{W}_1, \tilde{W}_2) \Delta_1; X \leq a (2b - w_1 - w_2)) \\
\geq h_1 (w_1, w_2) \partial_{w_1} h_\xi (w_1, w_2) \cdot (E (\xi (\tilde{W}_1) \Delta_1) - \varepsilon/4).
\]
The previous lemmas together with expression (13) immediately imply the following proposition.

**Proposition 5** For each \( \varepsilon > 0 \) if is possible to compute \( d > 0 \) and \( \kappa > 0 \) so that if \( 2b - w_1 - w_2 \geq \kappa \) then

\[
E \left( g(W'_1, W'_2); X \leq a(2b - w_1 - w_2) \right) \\
\leq g(w_1, w_2) P(X \leq a(2b - w_1 - w_2)) \\
+ \varepsilon h_1(w_1, w_2) \partial_{w_1} h_0(w_1, w_2) c \\
- h_1(w_1, w_2) \partial_{w_1} h_0(w_1, w_2) c E \left( \xi \left( \tilde{W}_1 \right) \Delta_1; X \leq a(2b - w_1 - w_2) \right) \\
- h_1(w_1, w_2) \partial_{w_1} h_0(w_1, w_2) c E \left( \Delta_1 + \Delta_2; X \leq a(2b - w_1 - w_2) \right).
\]

Proposition 5 suggests that the Lyapunov bound may be satisfied if one chooses \( \xi (\cdot) \) judiciously. It is not hard to see (by virtue of Proposition 4) that selecting \( \xi = 0 \) allows us to satisfy the Lyapunov bound on a region of the form \( w_1 \geq \kappa_1, w_2 \geq \kappa_2 \) and \( 2b - w_1 - w_2 \geq \kappa \) (after choosing \( \varepsilon > 0 \) small enough). However, the problem with selecting \( \xi = 0 \) is that the sign \( E(\Delta_1 + \Delta_2) \) cannot be controlled outside of this region (in particular when \( w_1 \) is close to zero, regardless of the value of \( w_2 \)). Our job is to make sure that \( \xi (\cdot) \) can be found such that for all \( u \in [0, 1] \)

\[
E(\Delta_2) + E(\Delta_1(1 - \xi(w_1 + u\Delta_1))) \leq -2\varepsilon, \tag{14}
\]

uniformly over \( w_1 \leq b \) for some \( \varepsilon > 0 \). Now, it turns out that it suffices to satisfy (14) uniformly on the region \( w_1 \leq b \) and \( w_2 \geq \kappa_2 \) for some \( \kappa_2 \in (0, \infty) \). The reason is that when \( w_2 \) is small it is possible to pick \( p(w_1, w_2) \) bounded away from zero, because in that region the regenerative cycle can terminate with relatively high probability without any visit to \( b \); we will provide the details of the case \( w_2 \leq \kappa_2 \) later. The following lemma, which is proved at the end of the section, indicates that (14) is indeed satisfied.

**Lemma 3** If \( \kappa_2 \) is large enough, we can choose \( \kappa_0 \) and \( \zeta \) such that there exists some \( \varepsilon > 0 \) for which (14) is satisfied when \( w_2 > \kappa_2 \).

Lemma 3 in combination with Proposition 5 complete Step 2) of our outlined program. We now proceed to Step 3), namely, the estimation of \( J_1 \). In particular, we have the following Proposition.

**Proposition 6** Assuming that \( \kappa_0 \) and \( \zeta \) are given. Then, it is possible to select \( d > 0 \) and \( \kappa > 0 \) so that if \( 2b - w_1 - w_2 \geq \kappa \), then

\[
\frac{J_1}{g} \leq \frac{2F(a(2b - w_1 - w_2))^2}{p(w_1, w_2)c[G(2b - w_1 - w_2) - G(2b + d)]}.
\]
Propositions 5 and 6 combined with a convenient selection of \( p(w_1, w_2) \) is basically all what is required to complete Step 4. More precisely, let
\[
p(w_1, w_2) = \frac{\theta F(a \left(2b - w_1 - w_2\right))}{G(2b - w_1 - w_2) - G(2b + d)},
\]
for \( \theta > 0 \). We will indicate how to pick \( \theta > 0 \) so that that the Lyapunov bound (6) is guaranteed to hold on \( w_2 \geq \kappa_2, w_1 \leq b \) and \( 2b - w_1 - w_2 \geq \kappa \). In order to do this, first let us pick \( m_F \) such that, for all \( t \geq 0 \),
\[
\bar{F}(at) \leq m_F F(t).
\]
Also, note that
\[
h_1(w_1, w_2) \leq G(2b - w_1 - w_0) - G(2b + d)
\]
and (16). The result is stated assuming \( \text{Propositions 5 and 6} \) combined with a convenient selection of \( m_F \) is selected according to Lemma 3 and (14), is basically all what is required to complete Step 4. More precisely, let
\[
p(w_1, w_2) \geq \frac{m_F}{\theta m_F}.
\]
We now are ready to state the next result which summarizes Step 4) of our program.

**Proposition 7** Let \( \theta \leq \varepsilon/(3m_F) \) and \( c \geq 2/\theta^2 \). Assume that \( p(w_1, w_2) \leq 1/2 \), then
\[
J_1 + J_2 \leq g
\]
as long as \( w_1 \leq b, 2b - w_1 - w_2 \geq \kappa_2 \) and \( w_2 \geq \kappa_4 \).

**Remark 4** The constant \( \varepsilon \) is selected according to Lemma 3 and (14), \( \kappa_2 > 0 \) is chosen according to Lemma 3. The constant \( m_F \) is to be selected as indicated in 6 and (16). The result is stated assuming \( p(w_1, w_2) \leq 1/2 \) just for convenience in the proof, if the ratio (15) is larger than \( 1/2 \) just select \( p(w_1, w_2) = 1/2 \).

**Remark 5** The previous computations are done assuming that \( 2b - w_1 - w_2 \geq \kappa \), which in particular implies that \( g < 1 \) if \( \kappa \) is large enough. Note that all the estimates involving the derivatives are independent of \( c \). So, the parameter \( c \) only appears to control the size of \( \theta \) and the analysis above remains valid as long as \( c \geq 2/\theta^2 \).

Finally, we need to make sure that the Lyapunov bound is satisfied on \( w_2 \leq \kappa \) and \( 2b - w_1 - w_2 > \kappa \) as well. In order to do this, we shall do a slight modification to the family of changes-of-measure that we utilize. In particular, if \( w_2 \leq \kappa \) we propose sampling from a transition kernel \( Q'_{a,p}(\cdot) \) such that if \( h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is any bounded (measurable) function,
\[
E^{Q'_{a,p}}_{(w_1,w_2)} h \left(W_1^{(1)}, W_1^{(2)} \right)
= p(w_1, w_2) \frac{E \left(h \left(W_1', W_2' \right) ; X > a \left(2b - w_1 - w_2\right)\right)}{\bar{F}(a \left(2b - w_1 - w_2\right))}
+ (1 - p(w_1, w_2)) \frac{E \left(h \left(W_1', W_2' \right) ; X \leq a \left(2b - w_1 - w_2\right), W_2' > 0\right)}{P \left(X \leq a \left(2b - w_1 - w_2\right), W_2' > 0\right)}.
\]
Note that it is feasible to use $Q'_{a,b} (\cdot)$ as a valid transition kernel because (given $W_0^{(1)} = w_1$ and $W_0^{(2)} = w_2$) the event $\{T_b < T_\kappa\} \subseteq \{W_1^{(2)} > 0\}$. It is not difficult to see if we select $p(w_1, w_2)$ according to (15), with $\theta$ is sufficiently small, the Lyapunov bound will be satisfied also on $w_2 \leq \kappa_2$. However, we now are interested in computing an upper bound on $\theta$.

A useful property to note is that since $\tau$ has unbounded support we have that

$$P_{w_1, w_2}(W'_2 = 0) \geq \inf_{w_2 \leq \kappa_2} P_{w_1, w_2}(X + w_1 \leq 0, w_2 - \tau \leq 0) \geq P(X \leq -\kappa_1, \tau \geq \kappa_1) > 0.$$ 

Equivalently, we have if we define

$$\bar{\pi} = 1 - P(X \leq -\kappa_1, \tau \geq \kappa_1),$$

then

$$\sup_{w_2 \leq \kappa_2} P_{w_1, w_2}(W'_2 > 0) \leq \bar{\pi} < 1.$$ 

We need to show that if $w_1, w_2 \leq \kappa_2$ and $2b - w_1 - w_2 \geq \kappa$ then

$$1 \geq \frac{E(g(W'_1, W'_2); X > a(2b - w_1 - w_2))}{p(w_1, w_2) g(w_1, w_2) \overline{F}(a(2b - w_1 - w_2))^{-1}} \frac{E(g(W'_1, W'_2); X \leq a(2b - w_1 - w_2), W'_2 > 0)}{(1 - p(w_1, w_2)) g(w_1, w_2) P(X \leq a(2b - w_1 - w_2), W'_2 > 0)^{-1}}$$

A completely analogous asymptotic analysis as the one described during Step 2) can be done in this case. First, we note that

$$E_{(w_1, w_2)}(\Delta_1 + \Delta_2) \leq E X^+.$$ 

If we select $p(w_1, w_2)$ as in (15) then we obtain

$$\frac{E(g(W'_1, W'_2); X > a(2b - w_1 - w_2))}{p(w_1, w_2) g(w_1, w_2) \overline{F}(a(2b - w_1 - w_2))^{-1}} + \frac{1 - p(w_1, w_2)}{\bar{\pi} \overline{F}(2b - w_1 - w_2) E X^+}$$

$$\leq \frac{E(g(W'_1, W'_2); X \leq a(2b - w_1 - w_2))}{c\theta[G(2b - w_1 - w_2) - G(2b + d)]} + \frac{\bar{\pi}}{1 - p(w_1, w_2)} \frac{1}{E X^+}$$

Note that as

$$\frac{\overline{F}(a(2b - w_1 - w_2))}{c\theta[G(2b - w_1 - w_2) - G(2b + d)]} \sim a^{-\alpha} \frac{1}{c\theta(w_1 + w_2 + d)}$$

22
\( b \neq \infty \) uniformly over \( w_2 \leq \kappa_2. \) Therefore, we obtain that given \( \delta \in (0,1), \) if 
\[ 2b - w_2 - w_1 \geq \kappa \text{ and } d \geq 4 \left( 1 + \delta \right) \] then (18) is bounded by
\[
\left( 1 + \delta \right) \alpha^{-\alpha} \left( 1 + \frac{2 + \delta}{d} E X^+ \right) \left( 1 + \frac{\alpha^{-\alpha} \left( 1 + \delta \right)}{d} \right) \tag{19}
\]

It follows that we can fix the value of \( \theta \) and \( c \) (in particular, we can select \( \theta \) as in Proposition 7). Then, select \( \delta \in (0,1) \) and \( d \) large enough so that the (19) is less than one. This guarantees that the Lyapunov bound is satisfied on the region \( 2b - w_1 - w_2 \geq \kappa \) and \( w_1 \leq b. \)

We are almost done with the construction of the Lyapunov bound. An important remaining issue is that the Lyapunov bound must be satisfied on \( w_1 \leq b \) and we currently have the bound satisfied on \( w_1 \leq b \) and \( 2b - w_1 - w_2 \geq \kappa. \) So, for instance, if the algorithm ends up in a situation where, say \( w_2 \geq 2b - \kappa \) and \( w_1 = 0, \) then we would not be able to justify the use of importance sampling in on region. This is important because it is not difficult to construct examples for which \( P_{0,2b}(T_b < T_K) \) is still a rare event. So, if we stop our importance sampler when we hit a region of the form \( w_2 \geq 2b - \kappa, \) \( w_1 = 0 \) and apply naive Monte Carlo from that time onwards, we may end up with an algorithm that has unbounded relative error. This difficulty, however, is very easy to resolve in our context. As we indicated in Remark 5, the estimates obtained impose a lower bound on \( c \) and none of the estimates derived before depend on \( c \) as long as the lower bound is satisfied. Note that the estimates are derived under the assumption that \( g(w_1, w_2) < 1. \) On the other hand, if \( g(w_1, w_2) = 1, \) the Lyapunov bound is immediately satisfied if one does not apply any importance sampling at all on the region \( g(w_1, w_2). \) Therefore, in order to extend the domain in which the Lyapunov bound is satisfied, we just increase the value of \( c \) so that, given our estimated parameter \( \kappa > 0, \) we have that \( g(w_1, w_2) = 1 \) if \( 2b - w_1 - w_2 \leq \kappa. \) Note that this fix basically does not change the implementation procedure suggested by this analysis.

Let us collect all of our estimates in the form of an algorithm. The specification of parameters such as the \( \kappa_j \)'s the \( \varepsilon \)'s, \( \zeta \) and \( \delta \) are not difficult to do either numerically (or analytically) depending on specific problems. We shall discuss a detailed example that involves the more complicated case in which \( \rho < 1/2 \) (which is discussed in the next section).

**Algorithm 1**

The objective of this algorithm is to estimate \( P_{0,0}(T_{b^*} < K). \) Set \( b^* \geq 0 \) and fix \( a \in (0,1) \) (say \( a = 1/2). \) Initialize \( w_1 = 0 = w_2, \) \( REACH = 0 \) and \( L = 1. \) Pick \( \theta \) as indicated in Proposition 7 and the parameters \( c, \kappa_0, d_0, \theta, \kappa \) and \( \kappa_2. \)

**STEP 1**

While \( REACH = 0 \)

If \( g(w_1, w_2) = 1 \) then sample \( (V, \tau) \) according to the nominal distribution and let \( X = V - \tau. \)
Elseif set
\[ p = \frac{\theta \bar{F}(a(2b' - w_1 - w_2))}{G(2b' - w_1 - w_2) - G(2b' + d)} \land 1/2. \]

If \( w_2 \leq \kappa_2 \), sample \((V, \tau)\) as follows. With probability \( p \) generate \((V, \tau)\) with law \( \mathcal{L}(\cdot | V - \tau \geq a(2b - w_1 - w_2))\), with probability \( 1 - p \) sample \((V, \tau)\) with law \( \mathcal{L}(\cdot | W'_2 > 0, X \leq a(2b - w_1 - w_2))\), where \( W'_2 = (w_1 + X)^+ \lor (w_2 - \tau)^+ \) and \( X = V - \tau \). Then, update

\[
L \leftarrow L \cdot \left[ p(w_1, w_2)^{-1} \bar{F}(a(2b - w_1 - w_2)) I(X \geq a(2b - w_1 - w_2)) + (1 - p(w_1, w_2))^{-1} P(W'_2 > 0, X \leq a(2b - w_1 - w_2)) \right].
\]

Else, sample \((V, \tau)\) according to the following mixture. With probability \( p \) generate \((V, \tau)\) with law \( \mathcal{L}(\cdot | V - \tau \geq a(2b - w_1 - w_2))\), with probability \( 1 - p \) sample \((V, \tau)\) with law \( \mathcal{L}(\cdot | V - \tau < a(2b - w_1 - w_2))\). Let \( X = V - \tau \) and update

\[
L \leftarrow L \cdot \left[ p(w_1, w_2)^{-1} \bar{F}(a(2b - w_1 - w_2)) I(X \geq a(2b - w_1 - w_2)) + (1 - p(w_1, w_2))^{-1} F(2b - w_1 - w_2) \right].
\]

Endif

Update

\[
\begin{align*}
    w_1 &\leftarrow \min((w_1 + X)^+ , (w_2 - \tau)^+) , \\
    w_2 &\leftarrow \max((w_1 + X)^+ , (w_2 - \tau)^+) .
\end{align*}
\]

Endif

If \( w_1 \notin (0, b] \) then \( \text{REACH} \leftarrow 1 \)

Endif

Loop

**STEP 2** Set \( L \leftarrow L \cdot I(w_1 > b) \) and RETURN \( L \).

**Theorem 1** If \( s_b^{(2)}(0, 0) = \tilde{E}_{(0, 0)}(L^2) \), where \( \tilde{E}(\cdot) \) is the probability measure induced by the importance sampling scheme indicated in **Algorithm 1** and \( L \) is the final output indicated in **STEP 2**. Then,

\[
\sup_{b > 0} \frac{\tilde{E}_{(0, 0)}(L^2)}{P_{(0, 0)}(T_b < T_K)^2} < \infty.
\]

**Proof.** Our previous analysis combined with Proposition 3 yields

\[
s_b^{(2)}(0, 0) \leq g(0, 0)
\]
(note that \(c\) was selected so that \(g\left(W_{T_{b}}^{(1)}, W_{T_{b}}^{(2)}\right) = 1\)). On the other hand, it follows (isolating the event of a large enough service time that triggers a large enough delay) that

\[
\lim_{b \to \infty} \frac{P_{(0,0)}(T_{b} < T_{K})}{P(X > b)} > 0.
\]

Consequently, using Corollary 1, we obtain that there exists \(\delta > 0\) such that

\[
\delta F(b) \leq P_{(0,0)}(T_{b} < T_{K}) \leq g(0,0)^{1/2}.
\]

Since

\[
g(0,0)^{1/2} \sim c^{1/2}[G(2b) - G(2b + d)] \sim c^{1/2}F(2b)d
\]

as \(b \to \infty\). These observations imply (by virtue of regular variation) the statement of the Theorem. 

4.1 Technical Proofs \(\rho > 1/2\)

Proof of Proposition 4. The proposition follows immediately after noting that

\[
\Delta_1 + \Delta_2 = (w_1 + X)^+ - w_1 + (w_2 - \tau)^+ - w_2
\]

\[
= \max(X, -w_1) + \max(-\tau, -w_2).
\]

And that

\[
\Delta_2 \leq -\tau I(w_2 - w_1 \geq V) + X \cdot I(w_2 - w_1 < V).
\]

Proof of Lemma 1. Note that

\[
\Delta_1 + \Delta_2 = \max(-w_1, X) + \max(-w_2, -\tau).
\]

Therefore,

\[
\Delta_1 + \Delta_2 \leq X^+.
\]

We have that

\[
\frac{h_1(W_1,\tilde{W}_2)}{h_1(w_1, w_2)} I(X \leq a(2b - w_1 - w_2))
\]

\[
\leq \frac{G(2b - w_1 - w_2 - \eta_1 - \eta_2) - G(2b + d)}{G(2b - w_1 - w_2) - G(2b + d)}
\]

\[
\cdot I(X \leq a(2b - w_1 - w_2)).
\]

(20)
Also,
\[
\frac{\partial w_i h_0 \left( \tilde{W}_1, \tilde{W}_2 \right)}{\partial w_i h_0 \left( w_1, w_2 \right)} I \left( X \leq a \left( 2b - w_1 - w_2 \right) \right) \quad \frac{\partial h_0 \left( w_1, w_2 \right)}{\partial h_0 \left( w_1, w_2 \right)} I \left( X \leq a \left( 2b - w_1 - w_2 \right) \right) = \frac{F \left( 2b - w_1 - w_2 - \eta_1 - \eta_2 \right)}{F \left( 2b - w_1 - w_2 \right)} I \left( X \leq a \left( 2b - w_1 - w_2 \right) \right).
\] (21)

Now, since both \( h_1 \left( w_1, w_2 \right) \) and \( \partial h_0 \left( w_1, w_2 \right) \) are non decreasing in both coordinates we obtain that if \( X \leq 0 \) then \( \Delta_1 + \Delta_2 \leq 0 \), so the ratios (20) and (21) are bounded by 1. Let us consider

\[
0 \leq X \leq a \left( 2b - w_1 - w_2 \right),
\]

which implies \( 0 \leq \eta_1 + \eta_2 \leq a \left( 2b - w_1 - w_2 \right) \). Pick \( m'_0 > 0 \) such that

\[
1 \leq \frac{F' \left( 2b - w_1 - w_2 - \eta_1 - \eta_2 \right)}{F \left( 2b - w_1 - w_2 \right)} \leq 1 + m'_0 \frac{f \left( 2b - w_1 - w_2 \right) X^+}{F \left( 2b - w_1 - w_2 \right)}.
\]

Note that such \( m'_0 > 0 \) can be selected to satisfy

\[
\sup_{0 \leq s \leq t \leq a, t \geq 0} \frac{f \left( t - s \right)}{f \left( t \right)} \leq m'_0.
\]

Similarly, we can pick \( m'_1 > 0 \) such that

\[
1 \leq \frac{G \left( 2b - w_1 - w_2 - \eta_1 - \eta_2 \right) - G \left( 2b + d \right)}{G \left( 2b - w_1 - w_2 \right) - G \left( 2b + d \right)} \leq 1 + m'_1 \frac{f \left( 2b - w_1 - w_2 \right) X^+}{G \left( 2b - w_1 - w_2 \right) - G \left( 2b + d \right)}.
\]

In this case, it suffices to select \( m'_1 > 0 \) such that

\[
\sup_{0 \leq s \leq t \leq a, t \geq 0} \frac{F \left( t - s \right)}{F \left( t \right)} \leq m'_1.
\]

We obtain

\[
E \left( \frac{h_1 \left( \tilde{W}_1, \tilde{W}_2 \right)}{h_1 \left( w_1, w_2 \right)} \frac{\partial w_i h_0 \left( \tilde{W}_1, \tilde{W}_2 \right)}{\partial w_i h_0 \left( w_1, w_2 \right)} \Delta_1; X \leq a \left( 2b - w_1 - w_2 \right) \right)
\]

\[
= \quad E \left( \frac{h_1 \left( \tilde{W}_1, \tilde{W}_2 \right)}{h_1 \left( w_1, w_2 \right)} \frac{\partial w_i h_0 \left( \tilde{W}_1, \tilde{W}_2 \right)}{\partial w_i h_0 \left( w_1, w_2 \right)} \Delta_1; X \leq 0 \right)
\]

\[
+ E \left( \frac{h_1 \left( \tilde{W}_1, \tilde{W}_2 \right)}{h_1 \left( w_1, w_2 \right)} \frac{\partial w_i h_0 \left( \tilde{W}_1, \tilde{W}_2 \right)}{\partial w_i h_0 \left( w_1, w_2 \right)} \Delta_1; X \in \left( 0, a \left( 2b - w_1 - w_2 \right) \right) \right).
\]
On the set $X \leq 0$ we have that $\Delta_1 \leq 0$. We also indicated above that on that set the ratios corresponding to $h_1$ and $\partial_{w_1} h_0$ are bounded by 1. In addition, these two ratios are always positive and converge to 1 as $2b - w_1 - w_2 \not\to \infty$. Therefore, for each $\varepsilon > 0$ there exists $\kappa > 0$ such that if $2b - w_1 - w_2 \geq \kappa$ then

$$
E \left( \frac{h_1 \left( \tilde{W}_1, \tilde{W}_2 \right)}{h_1 (w_1, w_2)} \frac{\partial_{w_1} h_0 \left( \tilde{W}_1, \tilde{W}_2 \right)}{\partial_{w_1} h_0 (w_1, w_2)} \Delta_1; X \leq 0 \right)
\leq E (\Delta_1; X \leq 0) + \varepsilon / 8.
$$

The analysis above shows that expectation involving $X \in (0,a(2b - w_1 - w_2))$ is upper bounded by (using that $\Delta_1 \leq X^+$)

$$
E (\Delta_1; X \in (0,a(2b - w_1 - w_2)))
+ m_1' \frac{f (2b - w_1 - w_2) EX^2}{F (2b - w_1 - w_2)}
+ m_0' \frac{G (2b - w_1 - w_2) - G (2b + d)}{F (2b - w_1 - w_2)}
+ m_1' \frac{f (2b - w_1 - w_2) E (X^3; X \in (0,a(2b - w_1 - w_2)))}{(G (2b - w_1 - w_2) - G (2b + d))}.
$$

It follows that by choosing $d$ large enough the previous quantity can be made less or equal to $E (\Delta_1; X \geq 0) + \varepsilon / 8$ as long as $2b - w_1 - w_2 \geq \kappa$. Note that it is not required to assume $E |X|^3 < \infty$ (just $EX^2 < \infty$) in the last line of the previous display because we can divide by $2b - w_1 - w_2$ in order to control the behavior of $X$ on the set $\{X \in (0,a(2b - w_1 - w_2))\}$. Combining the previous estimates, we obtain the conclusion of the lemma.

**Proof of Lemma 2.** The proof of this lemma proceeds following the same steps as in the previous results. First put $m_0 = \kappa_0 + 1/\zeta$ and note that assuming that $f$ is eventually decreasing there exists a constant $m_+ > 0$ such that

$$
\partial_{w_2} h_\xi \left( \tilde{W}_1, \tilde{W}_2 \right)
\leq \int_0^{\tilde{W}_1 \wedge m_0} f \left( 2b - \tilde{W}_2 - s \right) ds
\leq \int_0^{m_0} f \left( 2b - \tilde{W}_2 - s \right) ds
\leq m_+ m_0 f \left( 2b - \tilde{W}_2 - m_0 \right).
$$

On the other hand, on $X \leq a(2b - w_1 - w_2)$ we have

$$
2b - \tilde{W}_2
\geq 2b - \max (w_2, a(2b - w_1 - w_2) + w_1)
\geq 2b - \max (w_2, a(2b - w_2) + w_2)
= (2b - w_2) - a(2b - w_2)^+.
$$
We also have that
\[ \partial_{w_1} h_0 (w_1, w_2) = F (2b - w_2 - w_1) \geq F (2b - w_2). \]
Therefore, on \( X \leq a (2b - w_1 - w_2) \) again using that \( f (\cdot) \) is eventually decreasing, we have
\[
0 \leq \frac{\partial_{w_2} h_\xi (\widetilde{W}_1, \widetilde{W}_2)}{\partial_{w_1} h_0 (w_1, w_2)} \leq \frac{m_0 m_+ f ((2b - w_2) - a (2b - w_2)^+ - m_0)}{F (2b - w_2)}.
\]
Similar estimates as those given in the proof of Lemma 1 combined with the previous analysis imply that for each \( \varepsilon > 0 \) there exists \( \kappa > 0 \) such that if \( 2b - w_1 - w_2 \geq \kappa \), then
\[
E \left( h_1 (\widetilde{W}_1, \widetilde{W}_2) \partial_{w_2} h_\xi (\widetilde{W}_1, \widetilde{W}_2) \Delta_1 I (X \leq a (2b - w_1 - w_2)) \right) \\
\leq h_1 (w_1, w_2) \partial_{w_1} h_0 (w_1, w_2) \varepsilon / 4.
\]
Now, we consider the ratio
\[
\frac{h_1 (\widetilde{W}_1, \widetilde{W}_2) \partial_{w_1} h_\xi (\widetilde{W}_1, \widetilde{W}_2) - \Delta_1 I (X \leq a (2b - w_1 - w_2))}{h_1 (w_1, w_2) \partial_{w_1} h_0 (w_1, w_2)}
\]
Note that
\[
\frac{\partial_{w_1} h_\xi (\widetilde{W}_1, \widetilde{W}_2)}{\partial_{w_1} h_0 (w_1, w_2)} = \xi (\widetilde{W}_1) \frac{F (2b - \widetilde{W}_1 - \widetilde{W}_2)}{F (2b - w_1 - w_2)}
\]
The analysis of this ratio follows along the same lines as the argument given for (21) in the proof of Lemma 1. The contribution of the ratio involving \( h_1 \) is handled exactly in the same way as we did for (20). The conclusion of the lemma thus follows as a consequence of these estimates.

**Proof of Lemma 3.** We first note that
\[
\Delta_1 \xi (w_1 + u_\Delta_1) \geq \Delta_1 (\xi (w_1) - \zeta \Delta_1).
\]
Therefore,
\[
E (\Delta_1 \xi (w_1 + u_\Delta_1)) \geq E (\Delta_1 (\xi (w_1) - \zeta \Delta_1)) \\
= \xi (w_1) E (\Delta_1) - \zeta E (\Delta_1^2).
\]
Also note that regardless \( (w_1, w_2) \),
\[
E (\Delta_1^2) \leq E \tau^2 + EV^2
\]
We choose $\zeta$ small enough such that $E (\Delta_1 \xi (w_1 + u \Delta_1)) > \xi (w_1) E (\Delta_1) - \bar{\varepsilon}$. Therefore,

$$E (\Delta_1 + \Delta_2) - E (\Delta_1 \xi (w_1 + u \Delta_1)) < E (\Delta_2) + E (\Delta_1) (1 - \xi (w_1)) + \bar{\varepsilon}.$$  

First we choose $\kappa_0 = \kappa_1$. Recall that $E (\Delta_1 + \Delta_2) < -2\varepsilon_0$ whenever $w_1 > \kappa_1$. Then choose $m$ and $\kappa_2$ together such that $E (\Delta_2) < E (\varepsilon) - \varepsilon_0$ whenever $w_2 > \kappa_2$ and $w_1 < \kappa_2 - m$ (the feasibility of this step can be justified similarly as in Proposition 4).

When $w_1 \leq \kappa_0 = \kappa_1$, $\xi (w_1) = 1$, therefore

$$E (\Delta_1 + \Delta_2) - E (\Delta_1 \xi (w_1 + u \Delta_1)) < E (\Delta_2) + \bar{\varepsilon}.$$ 

Choose $\bar{\varepsilon}$ and $\varepsilon$ small enough such that

$$E (\Delta_2) + \bar{\varepsilon} < -2\varepsilon.$$ 

When $\kappa_2 - m > w_1 > \kappa_1$, if $E (\Delta_1) \geq 0$,

$$E (\Delta_1 + \Delta_2) - E (\Delta_1 \xi (w_1 + u \Delta_1)) < E (\Delta_2) + E (\Delta_1) (1 - \xi (w_1)) + \bar{\varepsilon} \leq E (\Delta_1 + \Delta_2) + \bar{\varepsilon} < -2\varepsilon$$

If $E (\Delta_1) < 0$

$$E (\Delta_1 + \Delta_2) - E (\Delta_1 \xi (w_1 + u \Delta_1)) < E (\Delta_2) + E (\Delta_1) (1 - \xi (w_1)) + \bar{\varepsilon} \leq E (\Delta_2) + \bar{\varepsilon} < -2\varepsilon.$$ 

When $w_1 > \kappa_2 - m > \kappa_0 + 1/\zeta$ (we can enlarge $\kappa_2$ if necessary), then $\xi (w_1) = 0$.

$$E (\Delta_1 + \Delta_2) - E (\Delta_1 \xi (w_1 + u \Delta_1)) \leq E (\Delta_1 + \Delta_2) + \bar{\varepsilon} < -2\varepsilon.$$ 

Before giving the Proof of Proposition 6, we need the following auxiliary result.

**Lemma 4** For each $\varepsilon' > 0$ there exists $b_0 > 0$, $\kappa > 0$ and $d > 0$ such that

$$\frac{h_\xi (w_1, w_2)}{h_0 (w_1, w_2)} \leq \varepsilon'$$

uniformly over $w_2 \leq w_1$, $b > b_0$ and $2b - w_2 \geq \kappa$.

**Proof.** Write $m_0 = \kappa_0 + 1/\zeta$. Then, as indicated in (9),

$$0 \leq h_\xi (w_1, w_2) \leq G (2b - w_2 - m_0) - G (2b - w_2).$$
Note that
\[ G(2b - w_2 - m_0) - G(2b - w_2) \leq F(2b - w_2 - m_0) m_0. \]

We have that
\[ \frac{h_\xi(w_1, w_2)}{h_0(w_1, w_2)} \leq \frac{F(2b - w_2 - m_0) m_0}{G(2b - w_2) - G(2b + d)}. \]

Now, fix \( \delta \in (0, 1) \) and note that if \( w_2 \leq \delta b \), then
\[ \frac{h_\xi(w_1, w_2)}{h_0(w_1, w_2)} \leq \frac{F(2b(1 - \delta) - m_0)}{G(2b) - G(2b + d)} m_0. \]

It is also the case that
\[ G(2b) - G(2b + d) \geq F(2b + d) d. \]

Therefore,
\[ \frac{h_\xi(w_1, w_2)}{h_0(w_1, w_2)} \leq \frac{F(2b(1 - \delta) - m_0)}{F(2b + d) d}. \]

It is then clear that \( d \) can be chosen so that there exists \( b_0 > 0 \) for which
\[ \frac{h_\xi(w_1, w_2)}{h_0(w_1, w_2)} \leq \varepsilon'. \]

if \( b \geq b_0 \). On the other hand, if \( w_2 > 2\delta b \), then for every \( \delta_1 \in (0, 1) \) there exists \( b_0 > 0 \) such that for all \( b \geq b_0 \),
\[ G(2b - w_2) \geq G(2b(1 - \delta)) \geq \delta_1^{-1} G(2b) \geq \delta_1^{-1} G(2b + d). \]

Therefore,
\[ G(2b - w_2) - G(2b + d) \geq (1 - \delta_1) G(2b - w_2). \]

Thus, we conclude that for every \( \varepsilon' > 0 \) there exists \( \kappa > 0 \) such that if \( 2b - w_2 \geq \kappa \), then
\[ \frac{h_\xi(w_1, w_2)}{h_0(w_1, w_2)} \leq \frac{F(2b - w_2 - m_0) m_0}{(1 - \delta_1) G(2b - w_2)} \leq \varepsilon'. \]

\[ \right] \]

\[ \textbf{Proof of Proposition 6.} \]

\[ \frac{J_1}{g} = \frac{E[g(W_1^1, W_2^1) | X > a(2b - w_1 - w_2)]}{p(w_1, w_2) g(w_1, w_2)} \frac{F(a(2b - w_1 - w_2))^2}{F(a(2b - w_1 - w_2))^2} \leq \frac{F(a(2b - w_1 - w_2))^2}{p(w_1, w_2) g(w_1, w_2)}. \]
Now, Lemma 4 indicates that by picking \(d > 0\) and \(\kappa\) large enough so that \(2b - w_2 - w_1 \geq \kappa\) we can guarantee
\[
G(2b - w_1 - w_2) - G(2b + d) - \int_0^{w_1} \xi(s) F(2b - w_2 - s) \, ds \\
\geq (G(2b - w_1 - w_2) - G(2b + d))/2.
\]
And we conclude that
\[
\frac{J_1}{g} \leq \frac{2F(a(2b - w_1 - w_2))^2}{p(w_1, w_2)c[G(2b - w_1 - w_0) - G(2b + d)]},
\]
which yields the proof of the result. \(\square\)

**Proof of Proposition 7.** Combining Proposition 5, inequality (14) and Proposition 6 we obtain
\[
\frac{J_1}{g} + \frac{J_2}{g} \leq \frac{m_0F(a(2b - w_1 - w_2))}{\theta c[G(2b - w_1 - w_0) - G(2b + d)]} + \frac{P(X \leq a(2b - w_1 - w_2))}{1 - p(w_1, w_2)} - \frac{\varepsilon}{1 - p(w_1, w_2)} \frac{\partial \psi_0}{h_0}(w_1, w_2) \frac{\partial h}{h_1}(w_1, w_2).
\]
Assuming that \(p(w_1, w_2) \leq 1/2\) and using (17) we obtain
\[
\frac{J_1}{g} + \frac{J_2}{g} \leq 1 + p(w_1, w_2) \left( \frac{2}{\theta^2 c} + 2 - \frac{\varepsilon}{\theta m_1} \right) .
\]
Selecting \(\theta \leq \varepsilon/(3m_1)\) and \(c \geq 2/\theta^2\) we obtain that
\[
\frac{J_1}{g} + \frac{J_2}{g} \leq 1,
\]
which is equivalent to the statement of the proposition. \(\square\)

5 **G/G/2 Queue \(\rho < 1/2\)**

We continue using the same notation and assumptions imposed in the previous section except that here we are concerned with the case \(\rho = EV_n/(2E\tau_n) < 1/2\).

As in the previous section, the idea is to select an appropriate change-of-measure from a parametric family of importance sampling distributions that captures the large deviations behavior of the system. In particular, it is intuitively plausible to expect that in this case event \(\{T_b < T_K\}\) will be cause by two rare events. First, it will
involve a large service time that will block one of the servers for $O(b)$ units of time – we shall refer to this epoch of the system’s large deviations evolution as phase 1. Then, during the time at which such server is blocked, another large customer will block the second queue for $O(b)$ units of time causing customers to experience long delays – this corresponds to phase 2. The intuition just explained suggests considering a family of changes-of-measure that induces large service times with given positive probability at each time step.

More precisely, if $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is any bounded (measurable) function, we are interested in considering a family of transition Kernels $Q_{a,p}(\cdot)$ such that (for $w_2 < b$)

$$E_{(w_1,w_2)}^{Q_{a,p}} h \left( W_1^{(1)}, W_2^{(2)} \right)$$

$$= p(w_1,w_2) \frac{E (h(W_1',W_2') ; X > a (b - w_1))}{F(a (b - w_1))}$$

$$+ (1 - p(w_1,w_2)) \frac{E (h(W_1',W_2') ; X \leq a (b - w_1))}{F(a (b - w_1))},$$

where $a \in (0,1)$ and $p(w_1,w_2) \in (0,1)$ will be chosen depending on the state of the Kiefer-Wolfowitz process. The intuition of this change-of-measure is analogous to that explained in the previous section.

Our goal is to find a positive function $g(\cdot)$ such that if we define

$$J_1 = \frac{E (g(W_1',W_2') ; X > a (b - w_1))}{p(w_1,w_2)},$$

$$J_2 = \frac{E (g(W_1',W_2') ; X \leq a (b - w_1))}{(1 - p(w_1,w_2))},$$

then $g(\cdot)$ must satisfy

$$J_1 + J_2 \leq g(w_1,w_2)$$

(22)

on the set $w_1 < b$ and $g(w_1,w_2) \geq \varepsilon$ for $w_1 \geq b$.

As before, if our family of importance samplers is well designed, we expect $g(\cdot)$ to behave much as $P_{(w_1,w_2)}(T_b < T_K)^2$. We can use a heuristic analysis based on sample path arguments for heavy-tailed processes to guess qualitatively the form of $P_{(w_1,w_2)}(T_b < T_K)$. We shall appeal to the intuition that indicates the way in which the rare-event occurs. To this end, it is useful to define the stopping time $T_b^* = \inf \{ n \geq 1 : W_n^{(2)} > b \}$. As we indicated previously, the most likely way in which the event $\{ T_b < T_K \}$ should occur is by a series of two rare events (corresponding to the two phases described before). The first phase corresponds to the event $\{ T_b^* < T_K \}$, then we need to trigger the event $\{ T_b < T_K' \}$. Now, observe that

$$P_{w_1,w_2}(T_b < T_K)$$

$$= P_{w_1,w_2}(T_b^* < T_K) P_{w_1,w_2}(T_b < T_K | T_b^* < T_K).$$

32
Using a fluid analysis, the event \( \{ T'_b < T_K \} \) occurs with probability roughly
\[
\int_0^{w_1 + d_0} F(b - w_1 + s) \, ds + (w_2 - w_1) F(b).
\]

Now, given that the event \( \{ T'_b < T_K \} \) occurs, because the tails of \( X \) are regularly varying, the overshoot over level \( b \) for the component \( W^{(2)} \) is \( O((b - w_1)) \). Therefore, a plausible guess for the form of the corresponding conditional probability \( P_{w_1,w_2}(T_b < T_K | T'_b < T_K) \) could be
\[
\int_0^{c(b-w_1)} F(b - w_2 + s) \, ds.
\]

(23)

Since the previous function is not increasing in \( w_1 \) (which, as we saw in the previous section is convenient because the process has negative drift), instead of trying a Lyapunov function of the form indicated in the previous display, we try something of the form
\[
\int_0^{(b-w_2)+d} F(b - w_2 + s) \, ds,
\]
which has a functional form that is at most of the order given by (23).

Using these functions, we define
\[
h_0 (w_1, w_2) = \int_0^{w_2 + d_0} F(b - w_1 + s) \, ds
\]
\[
= G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)
\]
\[
h_1 (w_2) = \int_0^{(b-w_2)+d_1} F(b - w_2 + s) \, ds
\]
\[
= G(b - w_2) - G(2(b - w_2) + d_1),
\]
then
\[
g_0 (w_1, w_2) = c_0 h_0 (w_1, w_2)^2 \wedge 1
\]
\[
g_1 (w_2) = c_1 h_1 (w_2)^2 \wedge 1.
\]

Finally, set
\[
g (w_1, w_2) = g_0 (w_1, w_2) g_1 (w_2).
\]

As before, the constants \( c_0, c_1, d_0 \) and \( d_1 \) are to be determined later.

In order to explain our strategy, note that \( J_2 \) can be estimated using Taylor’s theorem as
\[
E(g(W'_{1,1}, W_{2,2}) \wedge X \leq a(b - w_1)) = F(a(b - w_1)) g(w_1, w_2)
\]
\[
+ E(\partial_{w_1} g(w_1 + \eta_1, w_2 + \eta_2) \Delta_1; X \leq a(b - w_1))
\]
\[
+ E(\partial_{w_2} g(w_1 + \eta_1, w_2 + \eta_2) \Delta_2; X \leq a(b - w_1)),
\]

33
as in the previous section, we write
\[ \Delta_1 = W'_1 - w_1, \quad \Delta_2 = W'_2 - w_2, \]
and
\[
\begin{align*}
\tilde{W}_1 &\triangleq w_1 + \eta_1 \in (w_1 \wedge W'_1, w_1 \vee W'_1) \\
\tilde{W}_2 &\triangleq w_2 + \eta_2 \in (w_2 \wedge W'_2, w_2 \vee W'_2) .
\end{align*}
\]

As in the case \( \rho > 1/2 \), we expect a negative contribution appearing in the Taylor expansion due to the fact that \( g \) is non-decreasing (in both arguments) and the Kiefer-Wolfowitz process decreases on average. As in the previous section, our strategy is to consider an expression that roughly takes the form (assuming \( b - w_1 \) large enough)

\[
1 \approx \frac{J_1}{g} + \frac{J_2}{g} \approx \frac{F(a(b - w_1))^2 P_{w_1,w_2}(T_b < T_K | X > b - w_1)^2}{p(w_1, w_2) g} \left( 1 + E(\Delta_1) \frac{\partial w_1 g}{g} + E(\Delta_2) \frac{\partial w_2 g}{g} \right) / (1 - p(w_1, w_2)) .
\]

(25)

Since we have \( E\Delta_1 < 0 \) and \( E\Delta_2 < 0 \) we expect to control the behavior of the term corresponding to \( J_1 \) by selecting \( p(w_1, w_2) \) with a similar behavior (as \( b \to \infty \)) to

\[
\max \left( \frac{\partial w_1 g}{g}, \frac{\partial w_2 g}{g} \right) = \max (\partial w_1 \log g, \partial w_2 \log g) .
\]

The program that we pursue in this section is analogous to that described in the case \( \rho < 1/2 \). The first result establishes the negative drift of the Kiefer-Wolfowitz vector.

**Proposition 8** There exists \( \varepsilon_0 > 0 \) (for example, \( \varepsilon_0 = -EX/2 \)) and \( \kappa' > 0 \) such that if \( b - w_1 \geq \kappa' \) and \( w_1 \geq \kappa'_1 \), then

\[ E(\Delta_1; X \leq a(b - w_1)) \leq -\varepsilon_0 . \]

In addition, we can find \( \kappa'_2 > 0 \) such that if \( w_2 \geq \kappa'_2 \) we also have

\[ E(\Delta_2; X \leq a(b - w_1)) \leq -\varepsilon_0 . \]

The next step is to study \( J_2 \). Ultimately we are interested in estimating the contribution of the ratios \( \partial w_1 g/g \) and \( \partial w_2 g/g \) respectively. A quick analysis (which we shall provide in detail later) of these ratios yields that the dominant contributions come from \( \partial w_1 g_0/g_0 \) and \( \partial w_2 g_0/g_0 \). In particular, \( \partial w_1 g_0/g_0 \) is larger on the region \( w_1 > ab \) and \( \partial w_2 g/g \) dominates on the region \( w_1 \leq ab \). This observation motivates the analysis of \( J_2 \) in two cases, namely, \( w_1 > ab \) and \( w_1 \leq ab \).
First, let us deal with the case $w_1 > ab$. Note that (assuming $g(w_1, w_2) < 1$) we can write

$$E \left( \frac{g_1(W'_2) g_0(W'_1, W'_2)}{g_1(w_2) g_0(w_1, w_2)} ; X \leq a(b - w_1) \right)$$

$$= E \left( \frac{g_1(W'_2) g_0(w_1, W'_2)}{g_1(w_2) g_0(w_1, w_2)} ; X \leq a(b - w_1) \right)$$

$$+ \frac{\partial w_1 g_0(w_1, w_2)}{g_0(w_1, w_2)} E \left( \frac{g_1(W'_2) \partial w_1 g_0(W'_1, W'_2)}{g_1(w_2) \partial w_1 g_0(w_1, w_2)} \Delta_1 ; X \leq a(b - w_1) \right).$$

We shall write $J_{2,1}^A$ and $J_{2,2}^A$ to denote the first and second terms respectively in the right hand side of the equal sign in the previous display. The next result describes the behavior of $J_{2,1}^A$ and $J_{2,2}^A$ when $b - w_1 \geq \kappa'$ and $b - w_2 \geq \kappa''$.

**Lemma 5** Assume that $E|X| < \infty$ and $w_1 \geq ab$. Then, we can compute constants $\kappa'$ and $\kappa''$ such that if $b - w_1 \geq \kappa'$ and $b - w_2 \geq \kappa''$, then

$$J_{2,1}^A + J_{2,2}^A \leq 1 - \frac{\partial w_1 g_0(w_1, w_2)}{g_0(w_1, w_2)} \varepsilon_0.$$

Now, if $w_1 \leq ab$, we write

$$E \left( \frac{g_1(W'_2) g_0(W'_1, W'_2)}{g_1(w_2) g_0(w_1, w_2)} ; X \leq a(b - w_1) \right)$$

$$= E \left( \frac{g_1(W'_2) g_0(W'_1, w_2)}{g_1(w_2) g_0(w_1, w_2)} ; X \leq a(b - w_1) \right)$$

$$+ \frac{\partial w_2 g_0(w_1, w_2)}{g_0(w_1, w_2)} E \left( \frac{g_1(W'_2) \partial w_2 g_0(W'_1, \tilde{W}_2)}{g_1(w_2) \partial w_2 g_0(w_1, w_2)} \Delta_2 ; X \leq a(b - w_1) \right).$$

We use the notation $J_{2,1}^B$ to denote the first expectation in the right hand side of the previous display and $J_{2,2}^B$ for the second expectation respectively. We now estimate $J_{2,1}^B$ and $J_{2,2}^B$ on the region $b - w_1 \geq \kappa'$ and $b - w_2 \geq \kappa''$.

**Lemma 6** Assume that $EX^2 < \infty$, $w_1 < ab$ and $w_2 > \kappa'_2$. Then, we can compute constants $\kappa'$, $\kappa''$ and $d_0$ large enough such that if $b - w_1 \geq \kappa'$ and $b - w_2 \geq \kappa''$, then

$$J_{2,1}^B + J_{2,2}^B \leq 1 - \frac{\partial w_2 g_0(w_1, w_2)}{g_0(w_1, w_2)} \varepsilon_0.$$
Combining the estimates in the previous two lemmas we obtain that if \( b - w_1 \geq \kappa' \)

\[
\frac{J_2}{g} \leq \frac{1}{1 - p(w_1, w_2)} - \frac{\varepsilon_0}{1 - p(w_1, w_2)} \partial_{w_1} g_0(w_1, w_2) I(w_1 \geq ab)
- \frac{\varepsilon_0}{1 - p(w_1, w_2)} \partial_{w_2} g_0(w_1, w_2) I(w_1 < ab, w_2 > \kappa') .
\] (26)

We now turn to the analysis of \( J_1/g \).

**Proposition 9** We can find \( m_2 \in (0, \infty) \), \( \kappa' \) and \( \kappa'' \) such that if \( b - w_1 \geq \kappa' \) and \( b - w_2 \geq \kappa'' \), then

\[
\frac{J_1}{g} \leq \frac{F(a(b - w_1))^2 G(b - w_2)^2 m_2}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)^2 g_1(w_2) p(w_1, w_2)} .
\]

Using Proposition 9 together with (26), we now are ready to select \( p(w_1, w_2) \) in order to satisfy the Lyapunov bound (22). In particular, we propose selecting

\[
p(w_1, w_2) = \frac{\theta F(a(b - w_1))}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)}
\]

for some \( \theta > 0 \). We now find conditions on \( \theta \) so that the Lyapunov bound is satisfied. The idea is to use the negative contributions from \( J_2 \) to control the behavior of \( J_1 \). We proceed by considering two regions, namely \( w_1 \leq ab \) and \( w_1 > ab \). In the first case, there exists a constant \( \varepsilon > 0 \) such that

\[
\frac{F(a(b - w_1))}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)} \leq \frac{\partial_{w_2} g_0(w_1, w_2)}{\varepsilon g_0(w_1, w_2)} = \frac{\varepsilon^{-1} F(b)}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)}
\]

\[
\leq \varepsilon^{-1} \partial_{w_2} g_0(w_1, w_2) / g_0(w_1, w_2) .
\] (27)

In the second case, namely \( w_1 > ab \) we note that \( \varepsilon > 0 \) can also be selected such that

\[
\frac{F(a(b - w_1))}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)} \leq \frac{\varepsilon^{-1} (F(b) - w_1 - F(b))}{G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)}
= \varepsilon^{-1} \partial_{w_1} g(w_1, w_2) / g(w_1, w_2) .
\] (28)
Consequently, we obtain that if $\theta \leq \varepsilon$, then

$$p (w_1, w_2) \leq \frac{\partial w_1 g (w_1, w_2)}{g (w_1, w_2)} I (w_1 \geq ab) + \frac{\partial w_2 g_0 (w_1, w_2)}{g_0 (w_1, w_2)} I (w_1 < ab).$$

We then obtain the following Proposition

**Proposition 10** By properly choosing $\theta$, select

$$p (w_1, w_2) \leq \min (\varepsilon_0 / 3, 1 / 2). \quad (29)$$

Then, choose $m_3 > 0$ such that

$$\frac{G (b - w_2)^2 m_2}{(G (b - w_2) - G (2 (b - w_2) + d_1))^2} \leq m_3. \quad (30)$$

If $c_1 \geq 2m_3 / (\theta^2 \varepsilon_0^2)$,

$$\frac{J_1}{g} + \frac{J_2}{g} \leq 1,$$

on $b - w_1 \geq \kappa'$ and $b - w_2 \geq \kappa''$.

**Remark 6** Note that by possibly increasing the constant $c_1 > 0$ we can assume that $g_1 (w_2) = 1$ if $b - w_2 \leq \kappa''$.

So far we have been able to construct a Lyapunov function that allows us to control the algorithm up to the time when the first of the two rare events occurs (i.e. by the time when one of the queues is blocked). Now, we consider the second rare event. In other words, we now assume that $b - w_2 < \kappa''$. As we indicated in the previous remark $c_1$ large enough guarantees $g_1 (w_2) = 1$ if $b - w_2 < \kappa''$. We then obtain

$$\frac{J_1}{g} \leq \frac{F (a (b - w_1))^2}{p (w_1, w_2)} E \left( \frac{g_0 (W_1, W_2)}{g_0 (w_1, w_2)}; X \leq a (b - w_1) \right)$$

and

$$\frac{J_2}{g} \leq \frac{1 - p (w_1, w_2)}{1 - p (w_1, w_2)} E \left( \frac{g_0 (W_1', W_2')}{g_0 (w_1, w_2)}; X \leq a (b - w_1) \right).$$

A completely analogous strategy as before allows us to bound $J_1 / g + J_2 / g$. In this case is even easier because we do not have to deal with the contribution of the ratios involving $g_1$. Then, as the next proposition indicates, we obtain that the validity of the Lyapunov bound can be further extended to a larger domain. The proof is completely analogous to that of Proposition 10 and therefore is omitted.
Proposition 11 Select \( \theta \) and \( p(w_1, w_2) \) satisfying (29). Then, \( \kappa' \) and \( \kappa'_2 \) can be computed such that

\[
\frac{J_1}{g} + \frac{J_2}{g} \leq 1,
\]

on \( b - w_1 \geq \kappa' \) and \( \kappa'_2 \leq w_2 \) as long as \( c_0 \geq 2m_3 / (\theta^2 \varepsilon_0^2) \).

Remark 7 Once again note that by possibly increasing \( c_0 \) we can assume that \( g_0(w_2, w_1) = 1 \) if \( b - w_1 \leq \kappa' \).

To complete the program and extend the validity of the Lyapunov function (and thereby of the importance sampling scheme) throughout the region \( w_1 \leq b \), we need to deal with the case in which \( w_2 \leq \kappa'_2 \). To this end, we can proceed in a completely analogous way as we did for Step 5) in the previous section. In particular, if \( w_2 \leq \kappa'_2 \) we sample from a transition kernel \( Q_{a,p}^* (\cdot) \) such that if \( h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is any bounded (measurable) function then

\[
E_{(w_1, w_2)}^{Q_{a,p}^*} h \left( W_1^{(1)}, W_2^{(2)} \right) = p(w_1, w_2) \frac{E(h(W_1', W_2'); X > a(b - w_1))}{\mathcal{F}(a(2b - w_1 - w_2))}
+ (1 - p(w_1, w_2)) \frac{E(h(W_1', W_2'); X \leq a(b - w_1), W_2' > 0)}{P(X \leq a(2b - w_1 - w_2), W_2' > 0)}.
\]

The we can select \( p(w_1, w_2) \) just as we indicated in Proposition 11. Then, just as in the case \( \rho > 1/2 \), this step yields lower bounds on the values of \( d_0 \) and \( d_1 \) so that we satisfy the Lyapunov bound (note here that the selection of \( c_0 \) and \( c_1 \) does not depend on \( d_0 \) and \( d_1 \)). We conclude this section with the description of our algorithm.

Algorithm 2

The objective of this algorithm is to estimate \( P_{0,0}(T_{b^*} < K) \). Set \( b^* \geq 0 \) and fix \( a \in (0, 1) \) (say \( a = 1/2 \)). Initialize \( w_1 = 0 = w_2 \), \( REACH = 0 \) and \( L = 1 \). Pick \( \theta \) as indicated in Proposition 7 and the parameters \( c_0, c_1, d_0, d, \theta, \kappa_0 \) and \( \zeta \) in order to specify \( g \), and also, set \( \varepsilon_0 \) and \( \kappa_2 > 0 \).

**STEP 1**

While \( REACH = 0 \)

If \( g(w_1, w_2) = 1 \) then sample \( (V, \tau) \) according to the nominal distribution and let \( X = V - \tau \).

Elseif set

\[
p = \frac{\theta \mathcal{F}(a(b - w_1))}{[G(b - w_1) - G(b + d_0) + (w_2 - w_1) \mathcal{F}(b)]} \land 1/2 \land \varepsilon_0/3.
\]

If \( w_2 \leq \kappa_2 \), sample \( (V, \tau) \) as follows. With probability \( p \) generate \( (V, \tau) \) with law \( \mathcal{L}( (V, \tau) | V - \tau \geq a(b - w_1)) \), with probability \( 1 - p \) sample \( (V, \tau) \) with
law $\mathcal{L}( (V, \tau)|W_2' > 0, X \leq a(b - w_1))$, where $W_2' = (w_1 + X)^+ \lor (w_2 - \tau)^+$ and $X = V - \tau$. Then, update
\[
L \leftarrow L \cdot [p(w_1, w_2)^{-1} F(a(b - w_1)) I(X \geq a(b - w_1))
+ (1 - p(w_1, w_2))^{-1} P(W_2' > 0, X \leq a(b - w_1))].
\]

Else, sample $(V, \tau)$ according to the following mixture. With probability $p$ generate $(V, \tau)$ with law $\mathcal{L}( (V, \tau)|V - \tau \geq a(b - w_1))$, with probability $1 - p$ sample $(V, \tau)$ with law $\mathcal{L}( (V, \tau)|V - \tau < a(b - w_1))$. Let $X = V - \tau$ and update
\[
L \leftarrow L \cdot [p(w_1, w_2)^{-1} F(a(b - w_1)) I(X \geq a(b - w_1))
+ (1 - p(w_1, w_2))^{-1} F(b - w_1)].
\]

Endif

Update
\[
\begin{align*}
w_1 & \leftarrow \min ((w_1 + X)^+, (w_2 - \tau)^+), \\
w_2 & \leftarrow \max ((w_1 + X)^+, (w_2 - \tau)^+).
\end{align*}
\]

Endif

If $w_1 \notin (0, b]$ then $REACH \leftarrow 1$

Endif

Loop

**STEP 2** Set $L \leftarrow L \cdot I(w_1 > b)$ and RETURN $L$.

**Theorem 2** If $s_b^{(2)}(0, 0) = \tilde{E}_{(0,0)}(L^2)$, where $\tilde{E}(\cdot)$ is the probability measure induced by the importance sampling scheme indicated in Algorithm 2 and $L$ is the final output indicated in **STEP 2**. Then,
\[
\sup_{b > 0} \frac{\tilde{E}_{(0,0)}(L^2)}{P_{(0,0)}(T_b < T_K)^2} \leq \infty.
\]

**Proof.** Our previous analysis combined with Proposition 3 yields
\[
s_b^{(2)}(0, 0) \leq g(0, 0)
\]
On the other hand, it follows (isolating two large events that occur right at the beginning) that
\[
\lim_{b \to \infty} \frac{P_{(0,0)}(T_b < T_K)}{bP(X > b)^2} > 0.
\]
Consequently, using Corollary 1, we obtain that there exists $\delta > 0$ such that
\[
\delta \bar{F}(b) \leq P_{(0,0)}(T_b < T_K) \leq g(0, 0)^{1/2}.
\]
Since
\[
g(0, 0)^{1/2} \sim c_0^{1/2} c_1^{1/2} (G(b) - G(b + d_0)) (G(b) - G(2b + d_1)) \leq c_0^{1/2} c_1^{1/2} \bar{F}(b) G(b) d_0
\]
as $b \nearrow \infty$. These estimates imply the statement of the Theorem.
5.1 Technical proofs $\rho < 1/2$

Proof of Proposition 8. This result follows easily from the fact that

\[
E(\Delta_1; X \leq a(b - w_1)) \\
\leq E((w_1 + X)^+ - w_1); X \leq a(b - w_1)) \\
= E(\max(-w_1, X); X \leq a(b - w_1)).
\]

Similarly we obtain

\[
E(\Delta_2; X \leq a(b - w_1)) \\
= E(X; \tau \leq w_2, X \leq a(b - w_1)) + E(X^+; \tau > w_2, X \leq a(b - w_1)).
\]

Since $EX < 0$, the previous two estimates immediately imply the conclusion of the proposition. 

Proof of Lemma 5. We first argue that $\kappa'$ can be selected so that $J_{2,1}^A \leq 1$. In order to see this, observe that

\[
J_{2,1}^A = E\left(\frac{g_1(W_2') g_0(w_1, W_2')}{g_1(w_2)} \frac{g_0(w_1, w_2)}{g_0(w_1, w_2)}; X \leq a(b - w_1)\right) \\
= E\left(\frac{g_1(W_2')}{g_1(w_2)} \left(1 + \frac{\partial w_2 g_0(w_1, \tilde{W}_2)}{g_0(w_1, w_2)} \Delta_2\right); X \leq a(b - w_1)\right) \\
= E\left(\frac{g_1(W_2')}{g_1(w_2)}; X \leq a(b - w_1)\right) \\
+ E\left(\frac{g_1(W_2') \partial w_2 g_0(w_1, \tilde{W}_2)}{g_1(w_2) g_0(w_1, w_2)} \Delta_2; X \leq a(b - w_1)\right).
\]

First, note that

\[
E\left(\frac{g_1(W_2')}{g_1(w_2)}; X \leq a(b - w_1)\right) = P(X \leq a(b - w_1)) + E\left(\frac{\partial w_2 g_1(\tilde{W}_2)}{g_1(w_2)} \Delta_2; X \leq a(b - w_1)\right).
\]

We now analyze the family of rv's

\[
\frac{\partial w_2 g_1(\tilde{W}_2)}{\partial w_2 g_1(w_2)} \Delta_2 I(X \leq a(b - w_1))
\]
First we consider \( h_1 (\overline{W}_2) / h_1 (w_2) I (X \leq a (b - w_1)) \). Note that

\[
    b - \overline{W}_2 \geq b - \max (w_2, W_2') \\
    = b - \max (w_2, w_1 + X, w_2 - \tau) \\
    \geq b - \max (w_2, w_1 + (b - w_1) (1 - a)) \\
    \geq \min (b - w_2, (b - w_1) (1 - a)) \\
    \geq (b - w_2) (1 - a).
\]

This yields that

\[
    0 \leq \frac{G (b - \overline{W}_2) - G (2 (b - \overline{W}_2) + d_1)}{G (b - w_2) - G (2 (b - w_2) + d_1)} \\
    \leq \frac{G ((1 - a) (b - w_2)) - G (2 (1 - a) (b - w_2) + d_1)}{G (b - w_2) - G (2 (b - w_2) + d_1)}.
\]

The previous expression implies the ratio corresponding to \( h_1 \) is bounded. A very similar argument implies that

\[
    \frac{\partial_{w_2} h_1 (\overline{W}_2)}{\partial_{w_2} h_1 (w_2)} I (X \leq a (b - w_1))
\]

is also bounded uniformly over \( w_1, w_2 \) and \( b \). Since \( E |X| < \infty \) we conclude that the ratio (32) is bounded. Since \( w_j - \tau \leq \overline{W}_j \leq w_j + V \) for \( j \in \{1, 2\} \) we conclude (using uniform integrability and immediate properties of regularly varying distributions that) that

\[
    E \left( \frac{\partial_{w_2} g_1 (\overline{W}_2)}{g_1 (w_2)} \Delta_2; X \leq a (b - w_1) \right) \\
    \sim \frac{\partial_{w_2} g_1 (w_2)}{g_1 (w_2)} E (\Delta_2)
\]

uniformly over \( w_1 \leq w_2 \) as \( b - w_2 \nearrow \infty \). Since \( w_1 \geq ab \), we conclude by virtue of Proposition 8 that \( E \Delta_2 < 0 \) and therefore the previous quantity is negative (it converges to zero from below). We also have that

\[
    E \left( \frac{g_1 (W'_2) \partial_{w_2} g_0 (w_1, \overline{W}_2)}{g_1 (w_2) g_0 (w_1, w_2)} \Delta_2; X \leq a (b - w_1) \right) \\
    = \frac{F (b)}{g_0 (w_1, w_2)} E \left( \frac{g_1 (W'_2)}{g_1 (w_2)} \Delta_2; X \leq a (b - w_1) \right),
\]

41
which again by the previous analysis can be shown to be negative as \( b - w_2 \not\to \infty \).

Now we turn to the analysis of the term \( J_{2,2}^A \). We first show that there exists a constant \( m \in (0, \infty) \) such that (uniformly over \( b - w_1 \geq \kappa', b - w_2 \geq \kappa'' \) and \( w_1 \geq ab \) and \( b > 0 \)) we have

\[
\frac{h_0 \left( \widetilde{W}_1, W'_2 \right)}{h_0 (w_1, w_2)} I (X \leq a (b - w_1)) \leq m (1 + \tau/b) .
\]

In order to see this just note that

\[
\frac{G \left( b - \widetilde{W}_1 \right) - G \left( b + d_0 \right) + \left( W'_2 - \widetilde{W}_1 \right) F (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)} \leq \frac{G \left( b - w_1 - X^+ \right) - G \left( b + d_0 \right) + (w_2 - w_1 + V) F (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)} \tag{33}
\]

Since we assume \( X \leq a (b - w_1) \) we have that (33) is less or equal than

\[
\frac{G \left( (b - w_1) (1 - a) \right) + (w_2 - w_1 + V) F (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)} \leq \frac{G \left( (b - w_1) (1 - a) \right) + (w_2 - w_1) F (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)} + \frac{b F (b) V}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)} .
\]

Also, note that we are assuming \( w_1 \geq ab \), which implies that previous two quantities are bounded by \( m (1 + V/b) \) for some positive constant \( m > 0 \). A similar computation can be done to for the ratio

\[
\frac{\partial_{w_1} h_0 \left( \widetilde{W}_1, W'_2 \right)}{\partial_{w_1} h_0 (w_1, w_2)} I (X \leq a (b - w_1)) .
\]

In particular, we have that there exists \( m \in (0, \infty) \) such that if \( w_1 \geq ab \), then

\[
\frac{F \left( b - \widetilde{W}_1 \right) - F (b)}{F (b - w_1) - F (b)} I (X \leq a (b - w_1)) \leq \frac{F \left( (b - w_1) (1 - a) \right)}{F (b - w_1) - F (b)} \leq m.
\]

As a consequence, we obtain that if \( E |X| < \infty \) the family of random variables

\[
0 \leq \frac{g_1 (W'_2) \partial_{w_1} g_0 \left( \widetilde{W}_1, W'_2 \right)}{g_1 (w_2) \partial_{w_1} g_0 (w_1, w_2)} \Delta_1 I (X \leq a (b - w_1)) \leq m X (1 + V/b) I (0 \leq X \leq a (b - w_1))
\]

42
is uniformly integrable (over the space $w_1 \geq ab$, $w_2 \geq w_1$ and $b > 0$). Moreover, it follows that almost surely

$$g_1 (W_2') \frac{\partial \omega_0 (\tilde{W}_1, W_2)}{g_1 (w_2) \partial \omega_0 (w_1, w_2)} \to 1$$

as $b - w_1$, $b - w_2 \to \infty$ uniformly over $w_1 \geq ab$. Therefore, we obtain that if $\kappa'$ and $\kappa''$ are large enough (if $b$ is large enough so that $w_1$ is larger than $\kappa'$ from Proposition 8)

$$J_{2, 2}^A \leq -\varepsilon_0 \frac{\partial \omega_0 (w_1, w_2)}{g_0 (w_1, w_2)}.$$

Combining the previous estimates we obtain the conclusion of the lemma. \qed

**Proof of Lemma 6.** We first analyze the term $J_{2, 2}^B$. Note that

$$\partial \omega_0 (W_1', \tilde{W}_2) = 2c_0 h_0 (W_1', \tilde{W}_2) \bar{F} (b).$$

Therefore,

$$\frac{\partial \omega_0 (W_1', \tilde{W}_2)}{\partial \omega_0 (w_1, w_2)} I (X \leq a (b - w_1))$$

$$= \frac{h_0 (W_1', \tilde{W}_2)}{h_0 (w_1, w_2)} I (X \leq a (b - w_1))$$

$$\leq \frac{G (b - w_1 - X^+) - G (b + d_0) + (w_2 - w_1 + V) \bar{F} (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) \bar{F} (b)} \cdot I (X \leq a (b - w_1)).$$

Since $X \leq a (b - w_1)$ we have that there exists $m \in (0, \infty)$ such that

$$G (b - w_1 - X^+) \leq G (b - w_1) + m \bar{F} (b - w_1) X^+.$$

Therefore,

$$0 \leq \frac{\partial \omega_0 (W_1', \tilde{W}_2)}{\partial \omega_0 (w_1, w_2)} I (X \leq a (b - w_1))$$

$$\leq \left( 1 + \frac{m \bar{F} (b - w_1) X^+ + V \bar{F} (b)}{G (b - w_1) - G (b + d_0) + (w_2 - w_1) \bar{F} (b)} \right) \cdot I (X \leq a (b - w_1)) \tag{34}$$
Now, we can pick \( m' \in (0, 1) \) such that uniformly over \( w_1 < ab \) we have that
\[
G(b - w_1) - G(b + d_0) \geq m' \bar{F}(b - w_1)(w_1 + d_0)
\]
This yields that by possibly increasing \( m \), since \( w_1 < ab \), we obtain that (34) is bounded by
\[
\left( 1 + \frac{mX^+ + V}{(w_1 + d_0) m' + (w_2 - w_1)} \right) I(X \leq a(b - w_1)).
\]  \( (35) \)

In Lemma 5 we proved that the ratio involving \( g_1 \) is uniformly bounded. Since \( \Delta_2 \leq V \) we obtain, assuming \( \text{Var}(X) < \infty \), that the random variables
\[
\frac{g_1(W_2')}{g_1(w_2)} \frac{\partial w_2 g_0(W_1', \tilde{W}_2)}{\partial w_2 g_0(w_1, w_2)} \Delta_2 I(X \leq a(b - w_1)),
\]
indexed by \( w_1 < ab, w_2 \geq w_1 \) and \( b > 0 \) are uniformly integrable. Moreover, it follows that by selecting \( \kappa', \kappa'', \kappa'_2 \) and \( d_0 \) large enough and \( \varepsilon_0 \) from Proposition 8 appropriately, then
\[
E \left( \frac{g_1(W_2')}{g_1(w_2)} \frac{\partial w_2 g_0(W_1', \tilde{W}_2)}{\partial w_2 g_0(w_1, w_2)} \Delta_2 I(X \leq a(b - w_1)) \right) \leq -\varepsilon_0
\]  \( (36) \)

if \( b - w_1 \geq \kappa', b - w_2 \geq \kappa'' \) and \( w_2 \geq \kappa'_2 \).

Now, we proceed to analyze the term \( J_{2,1}^B \). First note that
\[
E \left( \frac{g_1(W_2')}{g_1(w_2)} \frac{g_0(W_1', w_2)}{g_0(w_1, w_2)} \frac{\partial w_2 g_0(W_1', \tilde{W}_2)}{\partial w_2 g_0(w_1, w_2)} \Delta_1; X \leq a(b - w_1) \right) = E \left( \frac{g_1(W_2')}{g_1(w_2)} ; X \leq a(b - w_1) \right) + E \left( \frac{g_1(W_2')}{g_1(w_2)} \frac{\partial w_2 g_0(\tilde{W}_1, w_2)}{\partial w_2 g_0(w_1, w_2)} \Delta_1; X \leq a(b - w_1) \right). \]  \( (37) \)

Just as in Lemma 5, if assume that \( b - w_1 \geq \kappa' \) and \( w_3 \geq \kappa'_2 \), we conclude that
\[
E \left( \frac{g_1(W_2'')}{g_1(w_2)} ; X \leq a(b - w_1) \right) \leq 1
\]  \( (38) \)

can be handled. This allows to conclude that (38) is bounded by 1. On the other hand, for the second expectation, namely (37), we need to consider two cases. First, we analyze the case \( w_1 \leq \kappa'_1 \) (\( \kappa'_1 \) selected in Proposition 8). After showing uniform
integrability, just as we did for (35), we conclude that as \( b - w_1 \not\to \infty \) uniformly over \( w_1 \leq \kappa'_1 \) and \( b - w_2 \geq \kappa'' \)

\[
J_{2,2}^B = \frac{\partial_{w_1} g_0 (w_1, w_2)}{g_0 (w_1, w_2)} O(1).
\]

However, we have that uniformly over \( w_1 \leq \kappa'_1 \),

\[
\partial_{w_1} g_0 (w_1, w_2) = F (b - w_1) - F (b) = o (F (b))
\]

as \( b \not\to \infty \). The analysis of the case \( w_1 > \kappa'_1 \) is analogous to that of the expectation term \( J_{2,1}^A \) in Lemma 5, the key point is that if \( w_1 > \kappa'_1 \) then \( E (\Delta_1) < 0 \). Thus, combining the estimates for (36) and (37) we obtain the conclusion of the Lemma. ■

**Proof of Proposition 9.** If \( b - w_2 \leq \kappa'' \) then we have that

\[
\frac{J_1}{g} = \frac{E [g (W_1', W_2') \mid X > a (b - w_1)]}{p (w_1, w_2) g (w_1, w_2)} F (a (b - w_1))^2
\]

\[
\leq \frac{F (a (b - w_1))^2 E (g_0 (W_1', W_2') \mid X \geq a (b - w_1))}{g_0 (w_1, w_2) g_1 (w_1)}
\]

\[
\leq \frac{F (a (b - w_1))^2}{g_0 (w_1, w_2)} \cdot \frac{2c_0 [G (b - w_2)^2 + F (b)^2 E (((w_2 - w_1) \lor X)^2 \mid X \geq a (b - w_1))]}{g_1 (w_2) p (w_1, w_2)}
\]

\[
= \frac{[G (b - w_1) - G (b + d_0) + (w_2 - w_1) F (b)]^2}{2 [G (b - w_2)^2 + F (b)^2 E (((w_2 - w_1) \lor X)^2 \mid X \geq a (b - w_1))]}
\]

Note that \( EX^2 < \infty \) implies

\[
E \left( (w_2 - w_1)^2 \lor X^2 \mid X \geq a (b - w_1) \right)
\]

\[
\sim E \left( (w_2 - w_1)^2 \lor (a^2 (b - w_1)^2 Y^2) \right)
\]

as \( b - w_1 \not\to \infty \), where \( Y \) is a Pareto rv with index \( \alpha > 2 \) (i.e. \( P (Y > t) = (1 + t)^{-\alpha} \)). Since we are assuming \( w_2 \leq b \) we then have that if \( b - w_1 \geq \kappa' \) and \( \kappa' \) is chosen large enough then

\[
E \left( ((w_2 - w_1) \lor X)^2 \mid X \geq a (b - w_1) \right)
\]

\[
\leq (b - w_1)^2 \left( 1 + EY^2 \right) = (b - w_1)^2 \frac{\alpha + 2}{\alpha - 1}.
\]
On the other hand, if \( w_2 + \kappa'' \leq b \) and \( \kappa'' \) is large enough, then there exists \( m_2 > 0 \) such that
\[
\frac{(m_2 - 1)}{2} G(b - w_2)^2 \geq \frac{\alpha + 2}{\alpha - 1} F(b)^2 (b - w_1)^2.
\]
Therefore, we have that
\[
\frac{J_1}{g} \leq \frac{F(a (b - w_1))^2 G(b - w_2)^2 m_2}{[G(b - w_1) - G(b + d_0) + (w_2 - w_1) F(b)]^2 g_1(w_2) p(w_1, w_2)}.
\]

**Proof of Proposition 10.** If (30) is satisfied, then Proposition 9, together with (27) and (28) imply that if \( \theta \leq \varepsilon \), then
\[
\frac{J_1}{g} \leq \frac{p(w_1, w_2) m_3}{\theta^2 c_1} \leq \frac{m_3}{\theta^2 c_1} \frac{\partial w_1 g(w_1, w_2)}{g(w_1, w_2)} I(w_1 \geq ab) + \frac{m_3}{\theta^2 c_1} \frac{\partial w_2 g_0(w_1, w_2)}{g_0(w_1, w_2)} I(w_1 < ab).
\]
Now we combine the previous estimate with (26) we obtain that if (29) holds and \( \theta \leq \varepsilon \) then
\[
\frac{J_1}{g} + \frac{J_2}{g} \leq 1 - \left( \frac{\varepsilon_0}{2} - \frac{m_3}{\theta^2 c_1} \right) \frac{\partial w_1 g(w_1, w_2)}{g(w_1, w_2)} I(w_1 \geq ab) - \left( \frac{\varepsilon_0}{2} - \frac{m_3}{\theta^2 c_1} \right) \frac{\partial w_1 g_0(w_1, w_2)}{g_0(w_1, w_2)} I(w_1 \geq ab).
\]
In addition, if \( c_1 \geq 2m_3/(\theta^2 \varepsilon_0) \) we conclude that \( J_1/g + J_2/g < 1 \). □

6 **Efficiency for Steady-state Waiting Time**

We impose the same assumptions indicated in Section 5. Our goal is to show that the algorithms developed in the previous two sections provide efficient estimators for the tail of the steady-state waiting time, namely, \( P \left( W_{\infty}^{(1)} > b \right) \) when \( b \) is large. We shall focus on the case \( \rho < 1/2 \) only – the main ideas can be easily adapted to cover the case \( \rho > 1/2 \).
We use the notation introduced in Section 2, namely, we define

\[ N_b = \sum_{j=0}^{T_K-1} I \left( W_j^{(1)} > b \right) \]

and let \( N_b(x) \) be a rv with the distribution \( P_x (N_b \in \cdot) \). Finally, we set \( \iota_{b,2}(x) = E_x N_b^2 \).

We are interested in studying the performance of the estimator

\[ Z_b = L_b N_b (W_{T_b}) I (T_b < T_K), \]

where \( L_b \) is the likelihood ratio obtained by running the importance sampling algorithm described in Section 5. We want to prove establish efficiency, which, as explained in Section 2, involves proving

\[
\sup_{b > 0} \frac{\tilde{E}_{(0,0)} \left( L_b^2 N_b^2 \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right)}{E (N_b)^2} < \infty,
\]

where \( \tilde{E}_{(0,0)} (\cdot) \) denotes the expectation operator induced by the importance sampler selected in Section 5. Now, we have that

\[
\tilde{E}_{(0,0)} \left( L_b^2 N_b^2 \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right) = E_{(0,0)} \left( L_b N_b \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right).
\]

Our strategy is to study

\[
E_{(w_1,w_2)} \left( L_b N_b^2 \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right)
\]

again using Lyapunov-type arguments in order to verify conditions that basically correspond to those of Proposition 1.

Note that

\[
E_{(w_1,w_2)} \left( L_b N_b^2 \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right) = E_{(w_1,w_2)} \left( L_b \cdot \iota_{b,2} \left( W_{T_b}^{(1)} \right) I (T_b < T_K) \right).
\]

We will complete our program in thee steps. First, we will establish the following proposition (the proof of which is given at the end of the section).

**Proposition 12** There exists a constant \( m > 0 \) such that

\[
\iota_{b,2} \left( W_{T_b}^{(1)} \right) \leq m \left( W_{T_b}^{(1)} \right)^2.
\]
This implies that
\[
E_{(w_1,w_2)} \left( L_b N_b^2 \left( W_{T_b}^{(1)} \right) I \left( T_b < T_K \right) \right) \\
\leq m E_{(w_1,w_2)} \left( L_b \cdot \left( W_{T_b}^{(1)} \right)^2 I \left( T_b < T_K \right) \right),
\]
(40)
for some \( m > 0 \). The key issue then becomes finding a convenient bound for
\[
e_b(w_1,w_2) = E_{(w_1,w_2)} \left( L_b \cdot \left( W_{T_b}^{(1)} \right)^2 I \left( T_b < T_K \right) \right),
\]
which is the content of the following result.

**Proposition 13** Define we \( \tilde{e}_b(\cdot) \) via
\[
\tilde{e}_b(w_1,w_2) = g(w_1,w_2) \left( b^2 I \left( b > w_1 \right) + \delta w_1^2 I \left( b \leq w_1 \right) \right).
\]
Then, we can find \( \delta \in (0,1) \) (independent of \( b \)) such that
\[
\tilde{e}_b(w_1,w_2) \geq E_{(w_1,w_2)} \left( L_b \cdot \left( W_{T_b}^{(1)} \right)^2 I \left( T_b < T_K \right) \right) / \delta.
\]

Using the previous two propositions we arrive at the last step of our program, which yields the main result of this section, namely

**Theorem 3**
\[
\sup_{b>0} \frac{\tilde{E}_{(0,0)} \left( L_b^2 N_b^2 \left( W_{T_b}^{(1)} \right) I \left( T_b < T_K \right) \right)}{P \left( W_{\infty}^{(1)} > b \right)^2} < \infty
\]

**Proof.** Proposition 13 together with (40) imply that
\[
\tilde{E}_{(0,0)} \left( L_b^2 N_b^2 \left( W_{T_b}^{(1)} \right) I \left( T_b < T_K \right) \right) \leq mg(0,0) b^2.
\]
On the other hand, Theorem 3 of Foss and Korshunov (2006) provides a lower bound that implies the existence of \( \delta > 0 \) such that
\[
P \left( W_{\infty}^{(1)} > b \right) \geq \delta b^2 P \left( X > b \right)^2.
\]
We conclude that
\[
\frac{\tilde{E}_{(0,0)} \left( L_b^2 N_b^2 \left( W_{T_b}^{(1)} \right) I \left( T_b < T_K \right) \right)}{P \left( W_{\infty}^{(1)} > b \right)^2} \leq \frac{m \left( G(b) - G(b+d) \right)}{\delta P \left( X > b \right)}.
\]
The previous quantity is clearly bounded uniformly over \( b > 0 \), so the conclusion of the theorem follows. ■
6.1 Technical Proofs for the Steady-state Waiting Time

In order to provide a proof of Proposition 12, we shall use the following result.

**Proposition 14** Let $W = (W_n : n \geq 0)$ be a Markov chain taking values on a general state-space $S$. Let $T_C = \inf\{n \geq 0 : W_n \in C\}$ and define $(\mu_1 (w), \mu_2 (w) : w \in S)$ via $\mu_1 (w) = E_w T_C$, $\mu_2 (w) = E_w T_C^2$. Suppose that there exist function non-negative functions $\tilde{\mu}_1 (\cdot)$ and $\tilde{\mu}_2 (\cdot)$ such that for all $w \notin C$,

\[ E_w \tilde{\mu}_1 (W_1) \leq \tilde{\mu}_1 (w) - \varepsilon, \]
\[ E_w \tilde{\mu}_2 (W_2) \leq \tilde{\mu}_2 (w) - \varepsilon \tilde{\mu}_1 (w). \]  

Then,

\[ \mu_2 (w) \leq \left( 1 + \sup_{w \in C} E_w \tilde{\mu}_1 (W_1) / \varepsilon \right) I (w \in C) + I (w \notin C) \tilde{\mu}_2 (w) / \varepsilon. \]

**Proof.** This result is a direct consequence of Theorem 3.2 of Jarner and Roberts (2002).

**Proof of Proposition 12.** We shall apply Proposition 14. In our case, the chain $W$ corresponds to the Kiefer-Wolfowitz process. Note that $N_b \leq T_K$, so in order to upper bound $\left( \mu_1 (w_1, w_2) \right)$ it suffices to upper bound the second moment of $T_K$. However, we shall consider a set $C \supseteq K$, in particular $C = \{ w_1 \leq c' \}$ for some $c' > 0$ (to be specified). Using a geometric trial argument (assuming for simplicity that $\tau$ has unbounded support), it follows that if $w_1 > 0$, then there exists a constant $m' \in (0, \infty)$ such

\[ E_{(w_1, w_2)} (T_K^2) \leq m' (E_{(w_1, w_2)} (T_C^2) + 1). \]

Now, we claim that if $\tilde{\mu}_1 (w_1, w_2) = w_1$ and $\tilde{\mu}_2 (w_1, w_2) = w_1^2$, then there exists a $\varepsilon > 0$ and $c' > 0$ such that (41) is satisfied. In order to see this, note that we need to show that there exists $\varepsilon > 0$ and $c' > 0$ such that if $w_1 \geq c'$, then

\[ E_{(w_1, w_2)} \tilde{\mu}_1 \left( W_1^{(1)} \right) - w_1 = E \Delta_1 \leq -\varepsilon. \]

However, the existence of such $\varepsilon, c' > 0$ was established in Proposition 8. Similarly, we need to show that if selects $\tilde{\mu}_2 (w_2) = w_2^2$ then

\[ E \left( (W_1')^2 \right) - w_1^2 \leq -\varepsilon w_1. \]  

for some $\varepsilon > 0$ as long as $w_1 \geq c'$. Now, we have

\[ E \left( (W_1')^2 \right) - w_1^2 \leq E \left( (w_1 + X)^2 ; X \geq -w_1 \right) - w_1^2 \]
\[ \leq -2w_1 E (X ; X \geq -w_1) + E (X^2). \]
Clearly (42) is satisfied if one chooses \( \varepsilon < -EX \) and \( \epsilon' \) large enough. We conclude that there exists a constant \( m \in (0, \infty) \) such that (for all \( b \geq 1 \))

\[
\nu_{b,2}(w_1, w_2) \leq (w_1^2 + 1) m,
\]

which implies (by possibly increasing the value of \( m \) (39)). ■

**Proof of Proposition 13.** We will apply Proposition 3 using as our Lyapunov function \( \tilde{e}_b(\cdot) \). The strategy proceeds using similar steps as we did for Section 5. First we define

\[
\tilde{J}_1 = \frac{E(\tilde{e}_b(W'_1, W'_2); X \geq a(b - w_1)) F(a(b - w_1))}{p(w_1, w_2)},
\]

\[
\tilde{J}_2 = \frac{E(\tilde{e}_b(W'_1, W'_2); X \leq a(b - w_1)) F(a(b - w_1))}{p(w_1, w_2)}.
\]

Note that

\[
\tilde{J}_2 = \frac{b^2 E(g(W'_1, W'_2); X \leq a(b - w_1)) F(a(b - w_1))}{p(w_1, w_2)}.
\]

So, the analysis of \( \tilde{J}_2 \) is completely analogous to that of \( J_2 \) in Section 5. We just need to analyze \( \tilde{J}_1 \) on \( w_1 < b \). Note that

\[
\tilde{J}_1 \leq b^2 E \left( \frac{g(W'_1, W'_2)}{g(w_1, w_2)}, X \geq a(b - w_1), W'_1 < b \right) \frac{F(a(b - w_1))}{p(w_1, w_2)}
\]

\[
+ \delta E \left( \frac{(W'_1)^2}{b^2} \right) X \geq (b - w_1) \frac{F(a(b - w_1))^2}{g(w_1, w_2) p(w_1, w_2)}.
\]

As in the proof of Proposition 9, since \( W'_1 \leq w_1 + X^+ \), we conclude (using the fact that \( EX^2 < \infty \)) that there exists a constant \( m \in (0, \infty) \) such that for all \( b \geq 1 \)

\[
E \left( \frac{(W'_1)^2}{b^2} \right) X \geq (b - w_1) \leq m.
\]

Therefore, we obtain that if \( w_1 < b \)

\[
\frac{\tilde{J}_1}{\tilde{e}_b} + \frac{\tilde{J}_2}{\tilde{e}_b} \leq \frac{J_1}{g} + \frac{J_2}{g} + \delta m \frac{J_1}{g},
\]

Given our analysis of \( J_1 \) and \( J_2 \) it is clear then that \( \delta > 0 \) can be chosen so that

\[
\frac{\tilde{J}_1}{\tilde{e}_b} + \frac{\tilde{J}_2}{\tilde{e}_b} \leq 1
\]

on \( w_1 \leq b \). The result then follows by applying Proposition 3. ■
7 Illustration of an $M/G/2$ Implementation

We carried over a numerical experiment to illustrate the implementation issues and performance of our algorithm. We consider an $M/G/2$ queue and we focus on the most challenging case in which $\rho < 1/2$. We assume that the service times, i.e the $V_n$'s, are Pareto distributed with index $\alpha > 0$. In particular, for $t > 0$,

$$P(V > t) = \frac{1}{(1+t)^\alpha}.$$

Moreover, suppose that $\alpha = 5/2$ so that $EV = 1/(\alpha - 1) = 2/3$ and $EV^2 < \infty$. The inter-arrival times (the $\tau_n$'s) follow an exponential distribution mean $2$. Consequently, we have that $\rho = 1/3$. The tail of $X = V - \tau$, namely $\Phi(t) = P(X > t)$, is can be computed via

$$\Phi(x) = P(V > x + \tau) = \int_0^\infty \frac{1}{2} e^{-s/2} P(V > x + s) \, ds.$$

$$= I(x < 0) \int_0^{-x} \frac{1}{2} e^{-s/2} ds + \int_{(-k)\cap 0}^\infty \frac{1}{2} e^{-s/2} \frac{1}{(s + x + 1)^{5/2}} ds$$

$$= I(x < 0) (1 - e^{x/2}) + \int_{(-k)\cap 0}^\infty \frac{1}{2} e^{-s/2} \frac{1}{(s + x + 1)^{5/2}} ds.$$

where

$$\int_y^\infty \frac{1}{2} e^{-s/2} \frac{1}{(s + x + 1)^{5/2}} ds$$

$$= e^{(x+1)/2} 2^{5/2} \left[ \frac{2}{3} e^{-(x+y+1)/2} - \frac{4}{3} e^{-(x+y+1)/2} \right]$$

$$+ e^{(x+1)/2} 2^{-5/2} \frac{4}{3} \int_{(x+y+1)/2}^\infty \frac{e^{-t}}{\sqrt{t}} dt.$$

In implementing the algorithm, it is required to evaluate the term

$$\int_{(x+y+1)/2}^\infty \frac{e^{-t}}{\sqrt{t}} dt$$

numerically. These integral expression corresponds to an incomplete Gamma function and there are many methods to evaluate this function efficiently. In general, in the implementation of the proposed algorithms, it will typically be the case that one would need to numerically evaluate one dimensional integrals – which in most cases can be evaluated with high accuracy in relative terms using routine methods.

We are interested in estimating $P(T_b < T_K)$. Our analysis from the previous section provided a parametric family of transition kernels (based on mixtures) that
(after tuning using a convenient Lyapunov function) provide a strongly efficient estimator. When \( w_2 \leq \kappa'_2 \), our theoretical analysis suggested to use one of our mixtures conditioning on the event \( \{ W'_2 > 0 \} \). However, such conditioning strategy is somewhat burdensome to implement. Therefore, if \( w_2 \leq \kappa'_2 \) we propose sampling from

\[
P_{(w_1,w_2)} \left( W^{(1)}_1 \in A_1, W^{(2)}_1 \in A_2 \right)
\]

\[
= p(w_1,w_2) P(W'_1 \in A_1, W'_2 \in A_2 | X > a (b - w_1))
+ (1 - p(w_1,w_2)) P(W'_1 \in A_1, W'_2 \in A_2 | X \leq a (b - w_1), X > -w_2)
\]

Note that \( \{ W'_2 > 0 \} \subseteq \{ X > -w_2 \} \), which implies that the previous sampling strategy is feasible. The adaptation of the analysis of the Lyapunov bounds using this transition kernel is direct (because \( P(X > -\kappa'_2) < 1 \)). Moreover, because the family of Lyapunov functions that we proposed in our theoretical analysis would involve evaluating \( G(\cdot) \) (which in this case implies using a numerical integration routine to deal with the incomplete Gamma function), we adapted our theoretical analysis to the Lyapunov function

\[
g(w_1,w_2) = g_0(w_1,w_2) \cdot g_1(w_2)
\]

where

\[
g_0(w_1,w_2) = (50 (w_2 + 5)^2 \bar{F} (b - w_1)^2) \wedge 1,
\]

\[
g_1(w_2) = (32 (b + 1 - w_2)^2 \bar{F} (b - w_2)^2) \wedge 1.
\]

For the parameter \( a \), we fixed \( a = .8 \). The selection of \( p(w_1,w_2) \) guarantees that the Lyapunov bound is satisfied. This could be done numerically at each step, however, since this may require a fast numerical integration routine, we obtained using expressions that are simpler to evaluate that those given in our theoretical analysis and are asymptotically equivalent. In particular, we define \( p(w_1,w_2) \) as follows:

- When \( w_2 < b, g_0 < 1 \) and \( g_1 < 1 \),

\[
p = \max \left( \frac{2}{\min (w_2 + 5, b + 1 - w_1)}, (1 - \bar{F} (-w_2)) / 2 \right)
\]

- When \( w_2 < b, g_0 < 1 \) and \( g_1 \geq 1 \), \( p = 2 / (w_2 + 5) \).

- When \( w_2 < b, g_0 \geq 1 \) and \( g_1 < 1 \), \( p = 2 / (b + 1 - w_2) \)

- When \( w_2 \geq b, g_0 < 1 \), \( p = 2 / (w_2 + 5) \).

- Otherwise, that is, if \( g_0 \geq 1 \) and \( g_1 \geq 1 \), then sample according to the nominal/original dynamics.
<table>
<thead>
<tr>
<th>b</th>
<th>IS</th>
<th>SD</th>
<th># of people</th>
<th>sd</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>6.43E-06</td>
<td>4.18E-05</td>
<td>3.61E-05</td>
<td>4.88E-04</td>
</tr>
<tr>
<td>50</td>
<td>6.84E-08</td>
<td>4.47E-07</td>
<td>9.67E-07</td>
<td>1.03E-05</td>
</tr>
<tr>
<td>100</td>
<td>4.50E-09</td>
<td>2.69E-08</td>
<td>1.20E-07</td>
<td>1.64E-06</td>
</tr>
</tbody>
</table>

Table 1: Numerical Example – based on $10^5$ simulations

The algorithm starts at $w_1 = 0$, $w_2 = 0$, evolves according to the indicated rules and stops when $w_1 > b$ or $w_2 = 0$. If it stops because of $w_2 = 0$, then we report 0; otherwise, we report the corresponding likelihood ratio.

For $b = 15$, we did crude Monte carlo ($10^5$ simulations) to compare with our importance sampling result, the estimated probability and number of people are $6.57E - 06 (2.56E - 07)$ and $3.97E - 05 (2.37E - 06)$.

References


