On-Line Supplement: Rare-event simulation for stochastic recurrence equations with heavy-tailed innovations

JOSE BLANCHET, Columbia University
HENRIK HULT, Royal Institute of Technology
KEVIN LEDER, University of Minnesota

ACM Reference Format:
DOI: http://dx.doi.org/10.1145/0000000.0000000

1. THE SCALING ESTIMATOR

One shortcoming of utilizing the conditional mixture algorithm is the necessity to sample from the conditional distribution. A natural way to get around that is to utilize a scaling based approach introduced in [Hult and Svensson 2012]. When simulating $B_{j+1}$ given that $X_j = x$ the following sampling density is used, where $a \in (0, 1)$,

$$g_{B_{j+1}}^{S}(y \mid x) = I\{x \in \Gamma_{j+1}\} p(j + 1, x) \left( f_B(y) I\{y \leq 0\} + I\{y > 0\} \frac{c_{j+1}}{a(b - x)} f_B \left( \frac{c_{j+1}y}{a(b - x)} \right) \right) + I\{x \in \Gamma_{j+1}\}(1 - p(j + 1, x)) f_B(y) + I\{x \notin \Gamma_{j+1}\} f_B(y).$$

Sampling $B_{j+1}$ given $X_j = x$ according to $g_{B_{j+1}}^{S}(y \mid x)$ can be more intuitively be described as follows:

1. If $x \notin \Gamma_{j+1}$ return sample from original density $f_B$ and stop.
2. With probability $1 - p(j + 1, x)$ return sample from original density $f_B$ and stop.
3. Sample $Y$ from $f_B$
   - If $Y \leq 0$, $Y \rightarrow B_{j+1}$
   - Else \( \frac{a(b-x)}{c_{j+1}} Y \rightarrow B_{j+1} \).

The following result establishes bounded relative error for the estimator based on the scaling algorithm, it will be assumed that the density $f_B$ is regularly varying and in addition it is bounded away from 0 on all compact subsets of $[0, \infty)$. Since the proof rests on the application of the same Lyapunov technique behind Theorem 3.1, it is given in the on-line supplement.
THEOREM 1.1. Let \( \{X_k; k = 0, \ldots, n\} \) be the random walk model given in (11). The estimator \( \hat{p}_b^{(1)} \) in (5), obtained from the scaling mixture algorithm with sampling densities given by \( g_{B_j}^S \), has bounded relative error for computing \( p_b = P\{X_n > b\} \). More precisely

\[
\limsup_{b \to \infty} \frac{E[(\hat{p}_b^{(1)})^2]}{p_b^2} \leq \frac{1}{a^{2\alpha} P\{B_1 > 0\}} \prod_{j=0}^{n-1} \max\{\tilde{\beta}_{j+1}^S, 1\},
\]

where

\[
\tilde{\beta}_{j+1}^S = \left( \sum_{k=j+1}^{n} c_k^0 \right)^{-2} \left( \alpha \sqrt{p_0 I_1(\alpha)} c_{j+1}^0 + \sum_{k=j+2}^{n} c_k^0 \right)^2.
\]

The upper bound is achieved by taking \( d_j = a^{-2\alpha}/P\{B_1 > 0\} \) and

\[
p(j + 1, x) = p_{j+1} = \frac{\sqrt{w_1(j + 1)}}{\sqrt{w_1(j + 1) + w_2(j + 1)}},
\]

where

\[
w_1(j + 1) = \frac{\alpha^2 I_1(\alpha)}{d_j} \left( \sum_{k=j+1}^{n} \left( \frac{ac_k}{c_{j+1}} \right)^\alpha \right)^{-2},
\]

\[
w_2(j + 1) = v_2(j + 1),
\]

and

\[
I_1(\alpha) = \int_{1}^{\infty} \frac{1}{y^{2(\alpha+1)} f(y)} dy.
\]

Note that in contrast to the conditional mixture algorithm, we are not able to establish that \( \tilde{\beta}_{j+1}^S \leq 1 \), and therefore it is not possible with our methods to establish that the algorithm has relative error arbitrarily close to 1. This indicates that it is possible that the relative error of the algorithm grows with \( n \) and might not be suitable for the infinite horizon problems considered later.

Let us begin to prove that the estimator has bounded relative error. It follows precisely as in the proof of Theorem 3.1 with Proposition 3.3 replaced by the following Lyapunov inequality.

PROPOSITION 1.2. If

\[
\sup\{p(j, x) : x \in \mathbb{R}, j < n\} < 1
\]

\[
\inf\{p(j, x) : x \in \mathbb{R}, j \leq n\} > 0
\]

then there exists non-negative constants \( \{\gamma_j^S\}_{j=1}^{n} \), greater than or equal to 1 and bounded in \( b \), such that, for \( j = 0, \ldots, n-1 \) and \( x \in \mathbb{R} \),

\[
E \left[ \frac{\tilde{V}_b(j + 1, X_{j+1})}{\tilde{V}_b(j, x)} \frac{f_B((X_{j+1} - x)/c_{j+1})}{g_{B_{j+1}}^S((X_{j+1} - x)/c_{j+1} | x)} \bigg| X_j = x \right] \leq \gamma_{j+1}^S.
\]
PROOF. We consider only the case $\hat{V}_b(j, x) < 1$, since the other case is treated in the same manner as in the proof of Proposition 3.3. Again consider

$$J_1(j, x) = E \left[ \frac{\hat{V}_b(j + 1, x + c_j + 1 B_{j+1})}{\hat{b}(j, x)} \frac{f_B(B_{j+1})}{g^S_{B_{j+1}} (B_{j+1} \mid x)} I\{c_j + 1 B_{j+1} > a(b - x)\} \right]$$

$$J_2(j, x) = E \left[ \frac{\hat{V}_b(j + 1, x + c_j + 1 B_{j+1})}{\hat{b}(j, x)} \frac{f_B(B_{j+1})}{g^S_{B_{j+1}} (B_{j+1} \mid x)} I\{c_j + 1 B_{j+1} \leq a(b - x)\} \right],$$

for $j = 0, \ldots, n - 1$.

Consider first the $J_1$-term. Recall that $V_b(j, x) < 1$ implies $b - x > 0$, so $x \in \Gamma_{j+1}$ implies that

$$\frac{f_B(y)}{g^S_{B_{j+1}} (y \mid x)} I\{c_{j+1} y > a(b - x)\} = \frac{f_B(y) I\{c_{j+1} y > a(b - x)\}}{(1 - p(j + 1, x)) f_B(y) + p(j + 1, x) \frac{c_{j+1}}{a(b - x)} f_B \left( \frac{c_{j+1} y}{a(b - x)} \right)} \leq \frac{a(b - x)}{c_{j+1}} \frac{f_B(y) I\{c_{j+1} y > a(b - x)\}}{p(j + 1, x) f_B \left( \frac{c_{j+1} y}{a(b - x)} \right)}.$$

Since $\hat{V}_b \leq 1$, it follows that for $j = 0, \ldots, n - 1$

$$J_1(j, x) \leq \frac{a(b - x)}{c_{j+1} d_j p(j + 1, x)} \left( \sum_{k=j+1}^n P\{c_k B_{j+1} > b - x\} \right)^{-2} \times E \left[ \frac{f_B(B_{j+1})}{f_B \left( \frac{c_{j+1} B_{j+1}}{a(b - x)} \right)} I\{c_{j+1} B_{j+1} > a(b - x)\} \right] \leq \frac{a(b - x)}{c_{j+1} d_j p(j + 1, x) \left( P\{c_{j+1} B > b - x\} \right)^2} \times E \left[ \frac{f_B(B_{j+1})}{f_B \left( \frac{c_{j+1} B_{j+1}}{a(b - x)} \right)} I\{c_{j+1} B_{j+1} > a(b - x)\} \right] \leq \frac{a(b - x)}{c_{j+1} d_j p(j + 1, x) \left( P\{c_{j+1} B > b - x\} \right)^2} \int_{a(b - x)/c_{j+1}}^\infty \frac{f_B(y)^2}{f_B \left( \frac{c_{j+1} y}{a(b - x)} \right)} dy \int_{1}^\infty f_B \left( \frac{a(b - x)y}{c_{j+1}} \right)^2 \int_{1}^\infty f_B(y) dy. \quad (2)$$

For ease of notation we write $\phi_b = a(b - x)/c_{j+1}$. Then the inequality in the previous display can be rewritten as

$$J_1(j, x) \leq \frac{\phi_b^2}{d_j p(j + 1, x) P\{B > \phi_b / a\}^2} \int_{1}^\infty \frac{f_B(\phi_b y)^2}{f_B(y)} dy \leq \frac{1}{d_j p(j + 1, x)} \int_{1}^\infty \left( y \phi_b f_B(\phi_b y) \right)^2 \left( P\{B > y \phi_b\} / P\{B > \phi_b / a\} \right)^2 \frac{1}{y^2 f(y)} dy.$$
Using Potter’s bounds again it follows that for \( \phi_b > 0 \), and \( y \geq 1 \), and any \( \varepsilon_1 > 0 \) there exists \( K_1 > 1 \) such that

\[
P \{ B > y \phi_b \} \leq \frac{1}{y \phi_b f_B(y \phi_b)} \int_{y \phi_b}^{\infty} f(z) dz = \int_{1}^{\infty} \frac{f(z \phi_b y)}{f(y \phi_b)} \frac{dz}{y \phi_b} \geq K_1'' \int_{1}^{\infty} \frac{z^{-\alpha-\varepsilon_2} dz}{\alpha + \varepsilon_2}.
\]

(3)

Note that we choose \( \varepsilon_1 << \alpha \). Since we are assuming that \( f \) is regularly varying and bounded away from 0 on compact subsets of \([0, \infty)\) we have that for \( y \geq 1, z \geq 1 \), and \( \varepsilon_2 > 0 \) there exists a \( K_1'' > 1 \) such that

\[
\frac{f(bz)}{f(b)} \geq K_1''' z^{-\alpha-\varepsilon_2},
\]

we again choose \( \varepsilon_2 << \alpha \). Applying this bound we get

\[
P \{ B > y \phi_b \} \leq \frac{1}{y \phi_b f_B(y \phi_b)} \int_{y \phi_b}^{\infty} f(z) dz = \int_{1}^{\infty} \frac{f(z \phi_b y)}{f(y \phi_b)} \frac{dz}{y \phi_b} \geq K_1'' \int_{1}^{\infty} \frac{z^{-\alpha-\varepsilon_2} dz}{\alpha + \varepsilon_2}.
\]

(4)

Let \( K_1 = K_1'/K_1'' \), and combine the bounds in (3) and (4) to get

\[
J_1(j, x) \leq \frac{K_1^2 a^{-2\alpha} \alpha^2}{d_j} \frac{1}{p(j + 1, x)} \int_{1}^{\infty} \frac{y^{-2(\alpha+1)+\varepsilon_1} dy}{f(y)},
\]

for simplicity we replaced \( \alpha + \varepsilon_2 \) with \( \alpha \), and \( a^{-2(\alpha-\varepsilon_1)} \) with \( a^{-2\alpha} \).

The density can be written as \( f(y) = L_0(y)/(1 + |y|)^{\alpha+1} \), based on this we see that the integrability of \( y^{-2(\alpha+1)+\varepsilon} f(y) \) is equivalent to the integrability of \( y^{-(\alpha+1)+\varepsilon}/L_0(y) \). Since \( 1/L_0 \) is again a slowly varying function, the integrability of \( y^{-(\alpha+1)+\varepsilon}/L_0(y) \) on \([1, \infty)\) follows from the representation theorem ([Bingham et al. 1987], Theorem 1.3.1).

Setting

\[
I_1^B(\alpha) = \int_{1}^{\infty} \frac{y^{-2(\alpha+1)+\varepsilon}}{f(y)} dy,
\]

we see that

\[
J_1(j, x) \leq \frac{K_1^2 a^{-2\alpha} \alpha^2}{p(j + 1, x)} I_1^B(\alpha).
\]

Following the proof in Proposition 3.3 we can arrive at

\[
J_2(j, x) \leq \frac{d_{j+1}}{d_j (1 - p(j + 1, x))} \frac{K_1^2}{(1 - \alpha)^{-2(\alpha+\varepsilon)}},
\]

since the proof is nearly identical we leave it out.

To summarize we have that for \( x \notin \Gamma_{j+1} \)

\[
E \left[ \frac{\hat{V}_b(j + 1, X_j + 1)}{V_b(j, x)} \frac{f_B((X_j + 1 - x)/c_{j+1})}{g_{B_{j+1}}((X_j + 1 - x)/c_{j+1} | x)} \right] X_j = x \leq 1,
\]
and for \( x \in \Gamma_{j+1} \)
\[
E \left[ \frac{\hat{V}_b(j + 1, X_{j+1})}{\hat{V}_b(j, x)} \frac{f_B((X_{j+1} - x)/c_{j+1})}{g_{B_{j+1}}((X_{j+1} - x)/c_{j+1})} \mid X_j = x \right] 
\leq J_1(j, x) + J_2(j, x) 
\leq \frac{K_2^2 a^{-2\alpha}}{d_j p(j + 1, x)} I_1^2(\alpha) + \frac{d_{j+1}}{d_j (1 - p(j + 1, x)) K_2^2 (1 - a)^{-2(\alpha + \varepsilon)}} I(j < n - 1).
\]

Since we assumed (1) we can define
\[
\gamma_{j+1}^S = \frac{K_2^2 a^{-2\alpha}}{d_j p(j + 1, x)} I_1^2(\alpha) + \frac{d_{j+1}}{d_j (1 - p(j + 1, x)) K_2^2 (1 - a)^{-2(\alpha + \varepsilon)}} I(j < n - 1)
\]
which is finite and independent of \( b \). \( \square \)

As in the conditional mixture algorithm, we will make optimal parameter choices by looking at the asymptotic upper bounds on the second moment. Therefore consider
\[
\limsup_{b \to \infty} E \left[ \frac{\hat{V}_b(j + 1, x + c_{j+1} B_{j+1}) f_B(B_{j+1})}{\hat{V}_b(j, x) g_{B_{j+1}}^S(B_{j+1} | x)} I\{c_{j+1} B_{j+1} > a(b - x)\} \right] 
\]
and
\[
\limsup_{b \to \infty} E \left[ \frac{\hat{V}_b(j + 1, x + c_{j+1} B_{j+1}) f_B(B_{j+1})}{\hat{V}_b(j, x) g_{B_{j+1}}^S(B_{j+1} | x)} I\{c_{j+1} B_{j+1} \leq a(b - x)\} \right].
\]

Starting with (5), we see that based on the display of equations (2) it suffices to find
\[
\lim_{b \to \infty} \frac{1}{d_j p(j + 1)} \left( \sum_{k=j+1}^{n} \frac{P\{c_k B > c_{j+1} \phi_b / a\}}{P\{B > \phi_b / a\}} \right)^{-2} \left( \frac{\phi_b}{P\{B > \phi_b / a\}} \right)^2 \int_1^\infty \left( \frac{f_B(\phi_b y)}{f_B(y)} \right)^2 dy 
= \frac{2 \alpha}{d_j p(j + 1)} \lim_{b \to \infty} \int_1^\infty \left( \frac{y \phi_b f_B(\phi_b y)}{P\{B > \phi_b y\}} \right)^2 \left( \frac{P\{B > \phi_b y\}}{P\{B > \phi_b / a\}} \right)^2 \frac{dy}{y^2 f(y)}.
\]

From the proof of proposition 1.2 we know that we can dominate the integrand in the previous display. In addition, it follows from the regularly varying tails property that for each \( y \geq 1 \)
\[
\lim_{b \to \infty} \frac{P\{B > \phi_b y\}}{P\{B > \phi_b / a\}} = (ya)^{-\alpha}
\]
\[
\lim_{b \to \infty} \frac{y \phi_b f_B(\phi_b y)}{P\{B > \phi_b y\}} = \alpha,
\]
and therefore
\[
\limsup_{b \to \infty} E \left[ \frac{\hat{V}_b(j + 1, x + c_{j+1} B_{j+1}) f_B(B_{j+1})}{\hat{V}_b(j, x) g_{B_{j+1}}^S(B_{j+1} | x)} I\{c_{j+1} B_{j+1} > a(b - x)\} \right] 
\leq \frac{c_{j+1}^2 a^{-2\alpha}}{d_j p(j + 1) (\sum_{k=j+1}^{n} c_k^2)^2} I_1(\alpha),
\]
where
\[
I_1(\alpha) = \int_1^\infty \frac{1}{y^{2(\alpha + 1)} f(y)} dy.
\]
Following the same analysis as for the conditional mixture algorithm we see that
\[
\limsup_{b \to \infty} E \left[ \frac{\tilde{V}_b(j + 1, x + c_{j+1}B_{j+1})f_B(B_{j+1})}{\tilde{V}_b(j, x)g^B_{j+1}(B_{j+1} | x)} I\{c_{j+1}B_{j+1} \leq a(b - x)\} \right]
\]
\[\leq \frac{d_{j+1}}{d_j (1 - p(j + 1, x))} \left( \frac{\sum_{k=j+1}^{n} \left( \frac{c_k}{c_{j+1}} \right) \alpha - 1}{\sum_{k=j+1}^{n} \left( \frac{c_k}{c_{j+1}} \right) \alpha} \right)^2.\]
Thus for \(x \in \Gamma_{j+1}\) and \(j \leq n - 2\)
\[
\limsup_{b \to \infty} E \left[ \frac{V_b(j + 1, X_{j+1}) f_{j+1|j}(X_{j+1} | x)}{V_b(j, x) g^B_{j+1}(X_{j+1} | x)} \Bigg| X_j = x \right] \quad (7)
\]
\[\leq \frac{1}{d_j p(j + 1, x)} \left( \sum_{k=j+1}^{n} \left( \frac{ac_k}{c_{j+1}} \right)^\alpha \right) \alpha^2 I_1(\alpha)\]
\[+ \frac{d_{j+1}}{d_j (1 - p(j + 1, x))} \left( \frac{\sum_{k=j+1}^{n} \left( \frac{c_k}{c_{j+1}} \right) \alpha - 1}{\sum_{k=j+1}^{n} \left( \frac{c_k}{c_{j+1}} \right) \alpha} \right)^2\]
\[= \tilde{\beta}_{j+1}^S.\]

It is clear that the quantities \(\tilde{\beta}_{j+1}^S\) and \(\tilde{\beta}_{j+1}^C\) are nearly the same, except for the term \(\alpha^2 I_1(\alpha)\). It is easily established that this term is greater than 1 and therefore \(\tilde{\beta}_{j+1}^S \geq \tilde{\beta}_{j+1}^C\):
\[
\frac{1}{\alpha^2} = \left( \int_1^{\infty} y^{-\alpha-1} dy \right)^2 = \left( \int_1^{\infty} \left( \frac{y^{2(\alpha+1)}}{f(y)} \right)^{1/2} f(y)^{1/2} dy \right)^2
\]
\[\leq \int_1^{\infty} \frac{dy}{y^{2(\alpha+1)}f(y)} \int_1^{\infty} f(y) dy \leq I_1(\alpha),\]
where the final inequality follows from the fact that \(f\) is probability density on \(\mathbb{R}\). Returning to (7) we can find the optimal probabilities to be of the form
\[
p(j + 1) = \frac{\sqrt{w_1(j + 1)}}{\sqrt{w_1(j + 1)} + \sqrt{w_2(j + 1)}}.
\]
where
\[
w_1(j + 1) = \frac{\alpha^2 I_1(\alpha)}{d_j} \left( \sum_{k=j+1}^{n} \left( \frac{ac_k}{c_{j+1}} \right)^\alpha \right)^{-2},
\]
and \(w_2(j + 1) = v_2(j + 1)\). Using these optimal probabilities and setting \(d_j = \alpha^{-2\alpha}/p_0\), we see that for \(j \leq n - 2\),
\[
\tilde{\beta}_{j+1}^S = \left( \sum_{k=j+1}^{n} c_k^\alpha \right)^{-2} \left( \alpha \sqrt{p_0 I_1(\alpha)} c_{j+1}^\alpha + \sum_{k=j+2}^{n} c_k^\alpha \right)^2,
\]
and \( \beta_n = \max(1, p_0\alpha^2 I_1(\alpha)) \). Since \( \beta_j \geq 1 \) for \( 0 \leq j \leq n - 2 \) we define \( \beta_{j+1} = \max(1, \beta_j + 1) \). Using Proposition 3.3 then gives that

\[
\limsup_{b \to \infty} \frac{1}{p_0} E \left[ I\{X_n > b\} \prod_{j=0}^{n-1} \frac{f_B((X_{j+1} - X_j)/c_j + 1)}{\beta_{j+1}((X_{j+1} - X_j)/c_j + 1)X_j} \right] \\
\leq \frac{a^{-2\alpha}}{p_0} \prod_{j=0}^{n-1} \beta_j + 1.
\]

REFERENCES
