Large deviations and exact asymptotics for perpetuities with small discount rates

Blanchet, J. and Glynn, P.

August, 2005.

Abstract

A perpetuity takes the form \( D = \int_0^\infty e^{-\Gamma(t-)}d\Lambda(t) \). In this paper, we develop large deviation theory for \( D \) in a context in which the accumulated discount process, \( \Gamma \), is small. More precisely, in this paper we provide: 1) logarithmic large deviations that hold in great generality in both discrete and continuous time, and 2) exact asymptotics derived in the iid (independent and identically distributed) case and also in the case in which \( \Lambda \) is a Levy process and \( \dot{\Gamma}(t) \) is a function of a Markov process. This setting is motivated by applications in insurance risk theory with return on investments. Our development of logarithmic asymptotics requires the study of a certain non-standard topology in order to apply contraction principles. Finally, our exact asymptotic results, although analogous to classical results for sums of iid random variables, present important important qualitative differences that we also explain here.

1 Introduction

Our focus in this paper is on the development of large deviations theory and sharp asymptotics for the distribution of general perpetuities - also known as infinite horizon discounted rewards - in a the context of small discount rates.

In order to motivate our theory and to define perpetuities let us discuss, as an illustrative example, a model taken from insurance risk theory with compound assets. Consider an insurance risk model in which the accumulated pure claim process, \( A(t) \), at time \( t \) is a compound Poisson process. If the risk reserve, \( R(t) \), of the company at time \( t \) is invested at rate \( \gamma > 0 \) and the insurance company receives premiums at rate \( p \), then the discounted risk reserve, \( D(t) = e^{-\gamma t}R(t) \), satisfies

\[
D(t) = R_0 + \int_{[0,t]} e^{-\gamma s}d[p\{s-A(s)\}].
\]

This model was studied in Harrison (1977) and it was shown there that the probability of eventual ruin, namely \( P(\max_{t \geq 0} (-D(t)) > R_0) \), can be expressed in terms of the
distribution of
\[ D = \int_0^\infty e^{-\gamma s} d\Lambda(s), \]  \hspace{1cm} (1)
where \( \Gamma(s) = \gamma s \) is the accumulated discount at time \( s \) and \( \Lambda(s) = ps - A(s) \) is the pure surplus process at time \( s \).

A random variable that takes the form of (1) is called a perpetuity in the financial mathematics literature and some researchers refer to perpetuities also as infinite horizon discounted rewards, a term that is common in the performance engineering literature. Although we have taken the insurance context to motivate our investigations, it is important to emphasize that, in addition to insurance risk theory with investments, perpetuities also play a crucial role in a variety of application domains, including mathematical finance, see the book by Embrechts, Klüppelberg, and Mikosch (1997) and time series analysis in the context of ARCH processes, see Campbell, Lo and Mackinlay (1999) and Shephard (1996). Other application areas in which perpetuities arise prominently are discussed in Goldie and Grübel (1996) and Carmona, Petit and Yor (2001), these include theoretical computer science (in the context of complexity of sorting algorithms related to “Quicksort”), analytic number theory and mathematical physics (see Blanchet and Glynn (2005) for a more detailed discussion).

Now, let us go back to our motivating application in the setting of insurance theory. Since insurance companies tend to have large initial reserves (i.e. \( R_0 \) is typically large) it makes sense to study the tail behavior of the distribution of \( D \). Moreover, since insurance companies already experience a significant exposure to random fluctuations due to the nature of the insurance business, we expect the company to invest in low risk assets, which give rise to relatively small returns. Of course, in the example that we discussed we have assumed that the company invests the reserve in a riskless asset. However, there are extensions of the previous model that consider the more realistic situation in which a stochastic economic environment is allowed, see the papers by Paulsen (1998, 2000) and the book by Asmussen (2000). These observations motivate the study large deviations of perpetuities, such as \( D \), in the context of small discount rates.

In order to explain our “low discount rates” regime and the large deviations results that we develop here let us introduce some notions. As we outlined before, a perpetuity can be expressed as
\[ D = \int_{[0,\infty)} \exp(-\Gamma(t-)) d\Lambda(t), \]
where, for each \( t > 0 \), \( \Lambda(t) \) representing the cumulative reward over the time interval \([0, t] \) and \( \Gamma = (\Gamma(t) : t \geq 0) \) is a real-valued process representing the cumulative discount to time \( t \). Suppose that \( \lambda \in \mathbb{R} \) is the asymptotic mean reward rate and let \( \gamma \in (0, \infty) \) be the asymptotic mean discount rate (being somewhat vague \( \Lambda(t) \approx \lambda t \))
and $\Gamma(t) \approx \gamma t$), then we expect

$$D \approx \int_0^\infty \exp(-\gamma s) \lambda ds = \lambda/\gamma.$$  \hspace{1cm} (2)$$

It is possible to provide rigorous support for this approximation in great generality in a context in which $\gamma > 0$ is small. In particular, we introduce a scaling parameter $\alpha > 0$ and consider a family of systems indexed by $\alpha$ such that

$$D(\alpha) = \int_{[0,\infty)} \exp(-\alpha \Gamma(t-)) d\Lambda(t).$$

Rigorous support for (2) is provided under mild assumptions (see, for example, Blanchet and Glynn (2005) and references therein) by showing that $\alpha D(\alpha) \to \lambda/\gamma$ as $\alpha \to \infty$ in some appropriate sense. Our large deviation analysis involves computing the rate of decay to zero of the tail probability $P(\alpha D(\alpha) > x)$ for $x > \lambda/\gamma$.

The theory developed here is parallel to the classical theory developed for sums of iid random variables (see Dembo and Zeitouni (1998)). In order to see why this is perhaps expected, consider the insurance model discussed at the beginning, namely, consider the special case in which $\Gamma(t) = \gamma t$ and $\Lambda(\cdot)$ is, say, a compound Poisson process with drift. Note, using integration by parts, that

$$\alpha D(\alpha) = \int_0^\infty \exp(-u) a\Lambda(u/(\alpha \gamma)) du \hspace{1cm} (3)$$

$$= \int_0^\infty \exp(-u) a\Lambda(\Gamma^{-1}(u/\alpha)) du$$

Since $\Lambda$ has stationary independent increments and we are applying a linear functional to this process, it is natural to expect that much of the large deviations theory that is known for $\Lambda$ extends to $D$.

This idea lies at the core of our investigations. However, it is worth to point out some important observations. First, the infinite horizon nature of perpetuities introduces technical difficulties that prevent us from using standard large deviation tools (such as the contraction principle). Instead, we have to study basic properties, such as exponential tightness in non-standard topologies (see Sections 4 and 5). Second, in the context of sharp asymptotics (discussed in Sections 2 and 3), we obtain prefactors that reflect the role played by the exponential function weighting the process $\Lambda$. Third, at the level of sharp asymptotics, perhaps surprisingly, there are qualitative differences that arise between the discrete time case and the continuous case. These distinctions are clear from the appearance of an additional prefactor term in the discrete case. This prefactor term appears because, as (3) indicates, one can think of a perpetuity as a linear functional of a process $\Lambda$ evaluated at a time change depending on $\Gamma^{-1}$. In the discrete time case one cannot perform such a time change substitution directly without paying a small order price that vanishes in logarithmic
scale, but contributes in sharp scale. Because of this qualitative reason we provide all the details of our sharp large deviations theory both in discrete and continuous time.

There are other approximations that are natural to investigate as refinements of (2). In particular, (2) suggests to study associated Central Limit Theorems (CLT’s) and Edgeworth expansions. These issues (specially the associated CLT) have been the focus of many papers that are motivated by a great variety of disciplines including mathematical statistics and stochastic processes (see, Gerber (1971) and Whitt (1972)), time series analysis (see Nelson (1990)) and machine learning (see Bucklew, Kurtz, and Sethares (1993) among others. Additional refinements in the form of Edgeworth expansions are given in Blanchet and Glynn (2005) where the authors also discuss related results in the literature whose main focus is the analysis of the distribution of $D$ around $\lambda/\gamma$ at a “regular” fluctuations (typically of order $O(1/\gamma^{1/2})$).

The literature on large deviation analysis for $D(\alpha)$ is not as substantial as that of weak convergence analysis. Nevertheless, we should mention that there is a substantial amount of literature on the tail analysis of perpetuities, however, this analysis has been focused mostly on tail asymptotics for fixed interest rates (see, for example, Kesten (1973) and Goldie (1991)).

The main contributions of the paper are the following:

1) Exact asymptotics derived in a discrete time setting in which the discount rate or force of interest at period $k$ and the reward or the benefit paid at time $k$ form a two dimensional iid sequence.

2) Exact asymptotics in the continuous setting assuming that $\Lambda$ follows a Levy process and that the short rate process $\dot{\Gamma}(t)$ (or instantaneous force of interest) is a positive bounded function of a geometrically ergodic Markov process. The general expression of the asymptotics is given formally in greater generality.

3) It is worth to emphasize that the multiplying constants in both of the results described above are not exactly analogous to corresponding multipliers that appear in more standard settings (such as those involving sums of random variables). Our prefactors reflect in a clear way the role played by the exponential function as a weighting function.

4) Logarithmic asymptotics in both discrete in continuous time under general dependence structures.

5) Characterization of exponential tightness for a non-standard weighted topology which is required to develop the results described in 4).

The rest of the paper is organized as follows. In Section 2 we develop exact asymptotics in the iid case. The exact large deviation results for the Markov setting in continuous time are given in Section 3. Some extensions of our sharp asymptotic results are discussed in Section 4. Our logarithmic asymptotics for accumulated reward and discount processes in continuous time are given in Section 5. Finally, the logarithmic large deviations theory for perpetuities in discrete time is given in Section 6.
2 Sharp Asymptotics: The discrete iid case

In this section, we shall consider a sequence \((X_k, Z_k) : k \geq 1\) of iid two dimensional random variables. Define the accumulated discount process \(S = (S_k : k \geq -1)\) as \(S_k = Z_1 + \ldots + Z_k\) for \(k > 0\) and \(S_k = 0\) for \(k \leq 0\). We are interested in the tail behavior of

\[
D = \sum_{k=0}^{\infty} \exp(-S_{k-1}) X_k,
\]

in the case in which the discount process \(S\) is small in some appropriate sense.

The following set of assumptions will hold throughout the rest of the section.

**IID1** Assume that \(0 < \varepsilon \leq Z_k \leq b < \infty\)

**IID2** Suppose that \(X_1\) has non-lattice distribution and that \(E \exp(\theta |X_1|) < \infty\) for \(\theta \in (-\delta, \delta)\).

**IID3** Moreover, let us assume that

\[
\sup_{\varepsilon \leq |\theta| \leq 1/\varepsilon} \sup_{z} \left| E \left( e^{i\theta X_1} \right) \right| < 1
\]

for \(\varepsilon > 0\).

Condition **IID3** may be seen as a uniform non-latticity of \(X_1\) given \(Z_1\). Notice that in the important special case in which the \(X_k\)'s are independent of the \(Z_k\)'s assumption **IID3** is an immediate consequence of **IID2**.

Set \(\psi(\eta, \theta) = \log E \exp(\eta Z_1 + \theta X_1)\) and define \(\chi(\cdot)\) satisfying \(\psi(-\chi(\cdot), \cdot) = 0\).

Under conditions **IID1** to **IID3**, if \(x > \lambda/\gamma\), we shall provide rigorous justification for the approximation

\[
P(D > x) \approx \frac{\sqrt{\gamma} \exp(cx^\theta) \exp(-x^\theta - \int_0^{\theta^*} \frac{\chi(u)}{u} du) / \gamma)}{\theta^* / \sqrt{2\pi \sigma^2(\theta^*)}},
\]

assuming that we can find \(\theta^*\) such that \(x\theta^* = \chi(\theta^*)\), where

\[
\sigma^2(\theta^*) \triangleq \int_0^{\theta^*} (\theta^*)^{-2} \chi''(u) du = \int_0^{\infty} \chi''(\theta^* e^{-s}) e^{-2s} ds,
\]

and \(c = EZ^2 / (2EZ)\).

At this point, it is worth to contrast the previous approximation with the corresponding Bahadur-Rao sharp asymptotic result for sums of iid rv’s (see Dembo and Zeitouni (1998)). In particular, we observe the appearance of the additional prefactor \(\exp(cx^\theta)\) which, as we discussed in the Introduction, arises from a random time change that in discrete time is approximated and not exact. The form of \(\sigma^2(\theta^*)\)
reveals, as we mentioned in the Introduction, the non-stationary behavior induced by the exponential discounting.

Approximation (4) will be shown to hold in a regime of small discount rates. That is, in our next theorem, we shall provide rigorous support to the previous approximation via an asymptotic analysis as $\alpha \searrow 0$ of the discrete time perpetuity

$$D(\alpha) = \sum_{k=0}^{\infty} \exp \left( -\alpha \sum_{j=0}^{k-1} Z_j \right) X_k.$$ 

**Theorem 1** Assume conditions $IID1$ to $IID3$. Let $\chi(\theta)$ be defined as the solution to

$$\psi(-\chi(\theta), \theta) = 0,$$

where $\psi(\eta, \theta) = \log E \exp (\eta Z_1 + \theta X_1)$. Suppose that $x > \lambda / \gamma$ and assume that there exists $\theta^* > 0$ such that $x\theta^* = \chi(\theta^*)$. Then,

$$\exp \left( I(x) / \alpha \right) P(\alpha D(\alpha) > x) \sim \alpha^{1/2} \frac{\exp(cx\theta^*)}{\theta^* \sqrt{2\pi \sigma^2(\theta^*)}} \text{ as } \alpha \searrow 0,$$

where $I(x) = x\theta^* - \int_{0}^{\theta^*} u^{-1} \chi(u) \, du$ and $c, \sigma^2(\theta^*)$ are defined in (5).

The standard strategy in the development of sharp asymptotics is first to use a suitable change-of-measure to center the distribution of interest (in this case that of $\alpha D(\alpha)$) around the large deviations point and then the application of a local central limit type-theorem under the exponentially twisted probability measure. Let us take a precise look to this program in our current context.

**Proof of Theorem 1.** In the following steps we will refer to some lemmas that are given at the end of the present section.

**STEP 1** We first introduce a convenient change-of-measure. Consider the family of probability measures $P_{\alpha}^*$ defined as

$$dP_{\alpha}^* = \exp (\theta^* D(\alpha) - \psi(\theta^*, \alpha)) \, dP,$$

where $\psi(\theta^*, \alpha) = \log E \exp (\theta^* D(\alpha))$ and $\theta^*$ is chosen as in Theorem 1. Note that

$$\exp \left( I(x) / \alpha \right) P(\alpha D(\alpha) > x) = \exp \left( I(x) / \alpha \right) E_{\alpha}^* \left( 1(\alpha D(\alpha) > x) \exp (\psi(\theta^*, \alpha) - \theta^* D(\alpha)) \right). \quad (6)$$

**STEP 2** Analyze the asymptotic behavior of (6). Let us introduce an appropriate exponential Martingale. Note that the process

$$M_n = \exp \left( \sum_{k=1}^{n} \theta e^{-\alpha S_{k-1}} X_k - \sum_{k=1}^{n} \chi(\theta e^{-\alpha S_{k-1}}) Z_k \right)$$

6
is a Martingale (take, say, the filtration generated by the process \((X_k, Z_k)_{k \geq 1}\)). Moreover, using assumption \textbf{IID1}, if \(\theta\) is sufficiently close to zero, we can find constants \(b_2, \varepsilon_1 > 0\) such that

\[
(M_n)^{1+\varepsilon_1} \leq b_2 \exp \left( \sum_{k=1}^{\infty} \theta (1 + \varepsilon_1) e^{-\alpha \varepsilon (k-1)} \right).
\]

This estimate implies, using assumption \textbf{IID2}, that \((M_n : n \geq 0)\) is a uniformly integrable Martingale and thus we obtain

\[
1 = EM_\infty = E \exp \left( \theta D(\alpha) - \sum_{k=1}^{\infty} \chi(\theta e^{-\alpha S_{k-1}}) Z_k \right).
\tag{7}
\]

Note that (making use of assumption \textbf{IID1})

\[
\Delta = \sum_{k=1}^{\infty} \chi(\theta e^{-\alpha S_{k-1}}) Z_k - \int_{0}^{\infty} \chi(\theta e^{-s}) \, ds
\]

\[
= \sum_{k=1}^{\infty} \int_{S_{k-1}}^{S_k} (\chi(\theta e^{-\alpha S_{k-1}}) - \chi(\theta e^{-s})) \, ds
\]

\[
= -\sum_{k=1}^{\infty} \theta \int_{S_{k-1}}^{S_k} \chi(\theta e^{-\alpha S_{k-1}}) (e^{-s} - e^{-\alpha S_{k-1}}) \, ds + O(\alpha)
\]

\[
= \alpha \sum_{k=1}^{\infty} \theta \chi(\theta e^{-\alpha S_{k-1}}) e^{-\alpha S_{k-1}} Z_k^2/2 + O(\alpha) \xrightarrow{P} \frac{EZ^2}{2EZ} \int_{0}^{\infty} \theta \chi(\theta e^{-s}) e^{-s} \, ds
\]

\[
= -\frac{EZ^2}{2EZ} \int_{0}^{\infty} \frac{d\chi(\theta e^{-s})}{ds} \, ds
= \frac{EZ^2}{2EZ} \chi(\theta).
\tag{8}
\]

Therefore, combining (8) and (7), using assumption \textbf{IID1} which allows to invoke the bounded convergence theorem, and the asymptotic independence between \(\Delta\) and \(D(\alpha)\) we obtain

\[
\exp \left( \psi(\theta, \alpha) - \int_{0}^{\infty} \chi(\theta e^{-s}) \, ds \right) \sim \exp \left( EZ^2 \chi(\theta) / (2EZ) \right),
\tag{9}
\]

as \(\alpha \searrow 0\). Now, observe that

\[
I(x) / \alpha - x \theta^* / \alpha = -\int_{0}^{\theta^*} \frac{\chi(u)}{u} \, du = -\int_{0}^{\infty} \chi(\theta^* e^{-s}) \, ds.
\tag{10}
\]

Whence, using (10) and the fact that \(\chi(\theta^*) = x \theta^*\), we have that

\[
\exp \left( I(x) / \alpha \right) P(\alpha D(\alpha) > x)
\sim \exp \left( cx \theta^* \right) E^*_a[1(D(\alpha) - x/\alpha > 0) \exp \left( -\theta^* (D(\alpha) - x/\alpha) \right)].
\tag{11}
\]
STEP 3 The final step is to develop a local CLT for $\alpha^{1/2}(D(\alpha) - x/\alpha)$ under $E_\alpha^\ast$. We shall use results by Chaganty and Sethuraman (1993) (see Lemma 2). In particular, in Lemma 3 below, we analyze the asymptotic behavior of $\psi_\alpha^\ast(i\theta) = \log E_\alpha^\ast \exp(i\theta\alpha^{1/2}(D(\alpha) - x/\alpha))$ as $\alpha \searrow 0$ and verify the conditions required to apply local central limit theorems. These results strengthen the CLT.

$$P_\alpha^\ast(\alpha^{1/2}(D(\alpha) - x/\alpha) \leq y) \longrightarrow P(N(0, \sigma^2(\theta^*)) \leq y).$$

Consequently, we obtain that

$$E_\alpha^\ast[1(D(\alpha) - x/\alpha > 0) \exp(-\theta^* (D(\alpha) - x/\alpha))] = \alpha^{1/2} \int_0^\infty \exp(-x) \exp(-\alpha x^2/\sigma^2(\theta^*) \theta^*) \frac{dx}{\theta^* \sqrt{2\pi \sigma^2(\theta^*)}} + o(\alpha^{1/2}),$$

which yields (by virtue of dominated convergence)

$$\alpha^{-1/2} E_\alpha^\ast[1(D(\alpha) - x/\alpha > 0) \exp(-\theta^* (D(\alpha) - x/\alpha))] \rightarrow \frac{1}{\theta^* \sqrt{2\pi \sigma^2(\theta^*)}}.$$

Combining these estimates with (11) yields the conclusion of the theorem.

The last step in the proof of the previous theorem requires the verification of some conditions that guarantee the application of a suitable local central limit theorem (CLT). The next result, due to Chaganty and Sethuraman (1993) provides necessary conditions for the application of a local CLT.

**Lemma 2** Define $\psi_\alpha^\ast(i\theta) \triangleq \log E_\alpha^\ast \exp(i\theta\sqrt{\alpha}(D(\alpha) - x/\alpha))$ and assume that there exists $\delta_0 > 0$ such that:

1. if $\theta \in \mathbb{R}$, then $\sup_{|\theta|, |\theta| < \delta_0} |\psi_\alpha^\ast(i\theta)| < \beta < \infty$ for some $\delta_0 > 0$,
2. if $\sigma^2(\theta^*) \triangleq \int^\theta_0 (\theta^*)^{-2} \chi''(u) du$, then $\psi_\alpha''(i\theta) = -2^{-1} \theta^2 \sigma^2(\theta^*) + o(1)$ uniformly for $\theta \in \mathbb{R}$ with $|\theta| < \varepsilon$ for some $\varepsilon > 0$, and
3. if $\theta \in \mathbb{R}$ and $\phi_\alpha^\ast(i\theta) = E_\alpha^\ast \exp(i\theta D(\alpha))$, then $|\phi_\alpha^\ast(i\theta)| = o(1/\alpha^{1/2})$ uniformly on $a_0 < |\theta| < a_1$ for $0 < a_0 < a_1 < \infty$.

Then, as $\alpha \searrow 0$

$$\frac{1}{\alpha^{1/2}} E_\alpha^\ast[1(D(\alpha) - x/\alpha > 0) \exp(-\theta^* (D(\alpha) - x/\alpha))] \frac{1}{\theta^* \sqrt{2\pi \sigma^2(\theta^*)}}.$$

**Proof.** This result follows immediately from Lemma 3.2 of Chaganty and Sethuraman (1993). This lemma allows to apply a local central limit theorem expansion in a very similar way as in the standard case of sums of iid rv’s.

The conclusion of **STEP 3** requires verification of the conditions in given in Lemma 2.
Lemma 3 In the setting of Lemma 2, conditions i), ii) and iii) hold.

Proof. The verification of i) and ii) of Lemma 2 follows by noting that
\[ E_α^* \exp \left( iθα^{1/2} (D(α) - x/α) \right) \]
\[ = E \exp \left( iθα^{1/2} (D(α) - x/α) + θ^* D(α) - ψ(θ^*, α) \right) \]
\[ = \exp \left( ψ(θ^* + iθα^{1/2}, α) - ψ(θ^*, α) - iθx/α^{1/2} \right) \]
\[ = \int_0^∞ \left[ \chi \left( (θ^* + iθα^{1/2}) e^{-αs} \right) - \chi \left( θ^* e^{-αs} \right) - iθxα^{1/2} e^{-αs} \right] ds \]
\[ + O \left( iθα^{1/2} \right) , \]
in the last equality we have used (9). We conclude that
\[ E_α^* \exp \left( iθα^{1/2} (D(α) - x/α) \right) \]
\[ = \int_0^∞ \left[ \chi \left( (θ^* + iθα^{1/2}) e^{-αs} \right) - \chi \left( θ^* e^{-αs} \right) - iθxα^{1/2} e^{-αs} \right] ds \]
\[ + O \left( iθα^{1/2} \right) \]
\[ = iθα^{1/2} \int_0^∞ \left[ \chi \left( θ^* e^{-αs} \right) e^{-αs} - x e^{-αs} \right] ds \]
\[ - 2^{-1} θ^2 \int_0^∞ \chi'' \left( θ^* e^{-αs} \right) α e^{-2αs} ds + O \left( iθα^{1/2} \right) . \]

Note that
\[ \int_0^∞ \chi \left( θ^* e^{-αs} \right) e^{-αs} ds = - (θ^* α)^{-1} \int_0^∞ d\chi \left( θ^* e^{-αs} \right) \]
\[ = \chi \left( θ^* \right) (θ^* α)^{-1} = xθ^* (θ^* α)^{-1} = x/α. \]

Therefore,
\[ E_α^* \exp \left( iθα^{1/2} (D(α) - x/α) \right) \]
\[ = - 2^{-1} θ^2 \int_0^∞ \chi'' \left( θ^* e^{-s} \right) e^{-2s} ds + O \left( iθα^{1/2} \right) . \]

A simple change of variables shows that
\[ \int_0^∞ \chi'' \left( θ^* e^{-s} \right) e^{-2s} ds = \int_0^{θ^*} (θ^*)^{-2} \chi'' (u) du = σ^2 (θ^*). \]

Now we study condition iii) of Lemma 2. Assume 0 < a_0 < |θ| < a_1 < ∞ and note that there exists c > 0 such that
\[ |E_α^* \exp (iθ D(α))| = |E \exp ((iθ + θ^*) D(α) - ψ(θ^*, α))| \]
\[ \leq c \left| E \exp \left( (iθ + θ^*) D(α) - \int_0^∞ \chi \left( θ^* e^{-αs} \right) ds \right) \right| . \]
Also, recall from the calculations in STEP 2 of the proof of Theorem 1 that (using the notation $S_k = Z_1 + \ldots + Z_k$)

$$
\int_0^\infty \chi(\theta e^{-\alpha s}) \, ds = \sum_{k=1}^\infty \chi(\theta e^{-\alpha S_{k-1}}) Z_k + R(\alpha),
$$

for some random variable $|R(\alpha)| < c_1 < \infty$ (uniformly in $\alpha > 0$ small, here we are using IID1). Therefore, we obtain that there exists $c_2 > 0$ such that

$$
|E_* \exp(i\theta D(\alpha))| 
\leq c_2 \left| E \exp \left( (i\theta + \theta^*) \sum_{k=1}^\infty e^{-\alpha S_{k-1}} X_k - \sum_{k=1}^\infty \chi(\theta e^{-\alpha S_{k-1}}) Z_k \right) \right|.
$$

Now, observe that if $\eta \in \mathbb{R}$ is sufficiently close to zero, we have that

$$
P_{\eta}(X_1 \in A, Z_1 \in B) = E[\exp(\eta X_1 - \chi(\eta) Z_1) 1(X_1 \in A, Z_1 \in B)]
$$

is a well defined probability measure. By virtue of assumptions IID2 and IID3 we have that if $0 < a_0 < a_1 < \infty$, then

$$
\sup_{|\theta| \leq [a_0, a_1]} \sup_{e^{-1} \leq \delta \leq 1} |E_{\theta + \delta} \exp(i\theta \delta X_1)| = \Delta < 1,
$$

where $E_{\theta + \delta} (\cdot)$ denotes the expectation operator corresponding to $P_{\eta} (\cdot)$ with $\eta = \theta + \delta$. We then conclude (using assumption IID1 which states that $Z \in [\varepsilon, b]$) that

$$
\sup_{|\theta| \in [a_0, a_1]} |E_* \exp(i\theta D(\alpha))| 
\leq \Delta^{1/(\beta \alpha)},
$$

which goes to zero exponentially fast as $\alpha \searrow 0$ which is more than we need to reach the conclusion of our lemma. ■

3 Sharp Asymptotics: The Markov case

In this section we shall develop exact asymptotics for continuous time processes with a Markovian structure. A family of models that has been used often in the risk theory context with stochastic return on investments discussed in the Introduction postulates the processes $\Gamma$ as $\Lambda$ two independent Levy processes (i.e. two stationary independent increment processes, see Gjessing and Paulsen (1997)). Since $\Lambda$ represents the pure claim process, the stationary and independent increments property
imposed by the definition of a Levy process seems reasonable. On the other hand, in finance, short rate processes usually are modelled as functions of a Markov process (typically diffusions with mean reverting characteristics). Motivated by these applications we develop our exact large deviation results in this setting. However, we shall also provide, formally, sharp asymptotics for more general Markov driven perpetuities. We shall indicate what are the issues that are involved in making those formal asymptotics rigorous.

Let us be precise in the description of the framework under which we develop the results of this section.

Suppose that $\Lambda = (\Lambda (t) : t \geq 0)$ is a Levy process and let $Y = (Y (s) : s \geq 0)$ be a (RCLL) time-homogeneous Markov process taking values in a compact Polish space $\Xi$ and let $\mathcal{B} (\Xi)$ be the Borel sigma-field in $\Xi$.

We adopt the Markovian framework described in Kontoyiannis and Meyn (2003). In particular, we define the generator of the process $Y$ by letting the corresponding transition semigroup, which we denote by $P (t)$, act on the space $L_\infty$ of measurable and bounded function taking values on $\Xi$. That is, for each $f \in L_\infty$ we define $(P (t) f) (y) = E_y f (Y (t))$. We shall assume that $(P (t) ; t \geq 0)$ is a strongly continuous contraction semigroup on $L_\infty$ (see Ethier and Kurtz (1986) pag. 6). We define the generator $A$ of $Y (\cdot)$ as follows. Given $f \in L_\infty$ we say that $f \in D (A)$ and that $Af = g$ if there exists $g \in L_\infty$ for which

$$f (Y (t)) - \int_0^t g (Y (s)) \, ds$$

is a Martingale. The form of the generator that we defined is called the full generator of $Y$ and the corresponding Hille-Yosida theory is explained in detail in Ethier and Kurtz (1986) Sections 1.5, see also Kontoyiannis and Meyn (2003), equation (22).

In addition, suppose that the following two conditions are in force.

**ME1** $\Lambda$ and $Y$ are independent

**ME2** The distribution of $\Lambda (1)$ is non-lattice and for $\theta \in (-\delta, \delta)$

$$\psi_\Lambda (\theta) = \log E \exp (\theta \Lambda (1)) < \infty; \quad \theta \in (-\delta, \delta).$$

**ME3** The process $Y$ is geometrically ergodic in the sense of Kontoyiannis and Meyn (2003) pag. 9.

For convenience, we state the characterization of geometric ergodicity given in Kontoyiannis and Meyn (2003), but in order to do this we have to recall a couple of concepts.

**Definition** Let $T$ be exponentially distributed with mean one and independent of $Y (\cdot)$. We say that $Y (\cdot)$ is $\psi$-irreducible if there exists a $\sigma-$finite measure on $\psi$
defined on \((\Xi, \mathcal{B}(\Xi))\) such that \(E_{y}s(Y(T)) > 0\) for each \(s : \Xi \to [0, \infty)\) such that \(\int s(y)\psi(dy) > 0\).

**Definition** We say that the \(\psi\)- irreducible Markov process \(Y(\cdot)\) is aperiodic if for any \(y \in \Xi\) and any \(s : \Xi \to [0, \infty)\) such that \(\int s(y)\psi(dy) > 0\) we have that
\[
\lim_{t \to \infty} E_{y}s(Y(t)) > 0.
\]

With these two definitions at hand we now define geometric ergodicity.

**GEOMETRIC ERGODICITY** [Kontoyiannis and Meyn (2003)]. A Markov process \(Y(\cdot)\) is said to be geometrically ergodic with Lyapunov function \(V\) if

**GE1)** \(Y\) is \(\psi\)- irreducible, aperiodic, and

**GE2)** for a probability measure \(v(\cdot)\) on \(\mathcal{B}(\Xi)\), we can find constants \(b < \infty\), \(\delta > 0\) and functions \(s : \Xi \to (0, 1]\) and \(V : \Xi \to [1, \infty)\), such that for every \(f \in L_{\infty}\) we have
\[
AV \leq -\delta V + bs, \quad (12)
\]
\[
E_{y}f(Y(T)) \geq s(y)\int f(x)v(dx), \quad (13)
\]
where \(T\) is exponentially distributed with mean 1 and independent of \(Y(\cdot)\).

We assume that \(\bar{\gamma}(\cdot) : \Xi \to \mathbb{R}\) is a continuous mapping such that \(\bar{\gamma}(y) > 0\) for all \(y \in \Xi\) and we define \(\Gamma\) as
\[
\Gamma(t) = \int_{0}^{t} \bar{\gamma}(Y(s)) ds.
\]
Assumption **ME3** guarantees the existence of a \(\psi_{\Gamma}(\theta)\) satisfying
\[
\psi_{\Gamma}(\theta) = \lim_{t \to \infty} \frac{1}{t} \log E \exp(\theta \Gamma(t)).
\]
Moreover, Theorem 4.2 of Kontoyiannis and Meyn (2003) implies that, for \(\theta\) in a neighborhood of the origin in the complex plane, there exists a unique solution pair \((u(y, \theta), \psi_{\Gamma}(\theta))\) satisfying
\[
(Au)(y, \theta) = (\psi_{\Gamma}(\theta) - \theta \bar{\gamma}(y)) u(y, \theta), \quad (14)
\]
\[
u(y, 0) = 1.
\]

Our exact large deviations result for
\[
\alpha D(\alpha) = \alpha \int_{0}^{\infty} \exp(-\alpha \Gamma(t)) d\Lambda(t),
\]
developed in Theorem 4 below, provides rigorous support for the following approximation (valid when \(x >> \lambda/\gamma\))
\[
P_{y}(D > x) \approx \gamma^{1/2} u(y_{0}, -\chi(\theta^{*})) \frac{\exp(c(\theta^{*}))}{\theta^{*}\sqrt{\pi \chi''(\theta^{*})}} \exp(-I(x)/\gamma). \quad (15)
\]
The elements appearing in this approximations are defined as follows. The function 
\( u(y, \cdot) \) is obtained from the solution of (14). For \( \theta \) sufficiently close to zero, \( \chi(\theta) \) is defined as 
\[ \chi(\theta) \triangleq -\psi_\Lambda(-\psi_\Gamma^{-1}(\theta)). \]
We assume that there exists \( \theta^* > 0 \) such 
\[ x\theta^* = \chi(\theta^*) \] and we define
\[ I(x) = x\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} du. \]

The constant \( \sigma^2(\theta^*) \) satisfies
\[ \sigma^2(\theta^*) = \int_0^\infty \chi''(\theta^* e^{-s}) e^{-2s} ds = \int_0^{\theta^*} (\theta^*)^{-2} \chi''(u) du, \]
and, finally, \( c(\theta^*) \) satisfies
\[ c(\theta^*) = -E_{\pi} \log (u(Y, -\chi(\theta^*))), \] (16)
where \( \tilde{\pi}(dy) = \tilde{\gamma}(y) \pi(dy) / E_{\pi} \tilde{\gamma}(Y) \) and \( \pi(dy) \) denotes the stationary distribution of the chain \( Y \).

**Remark** As an important difference between the terms appearing in approximation (15) and the discrete time analogue (4), we note that the term \( c(\theta^*) \) defined in (5) is completely different in nature to the term involving \( c = EZ^2 / (2EZ) \) appearing in (4). The term defined in (5) is due to the Markovian nature of the process \( Y \) and the time inhomogeneous structure induced by the exponential discounting. The nature of the term involving \( c = EZ^2 / (2EZ) \) was explained in the previous section.

**Theorem 4** Under the assumptions described at the beginning of this section and conditions EC1 to EC3. If \( x > \lambda / \gamma \), then
\[
\exp \left( \frac{I(x)}{\alpha} \right) P_y(\alpha D(\alpha) > x) \sim \frac{\alpha^{1/2}}{u(y, -\chi(\theta^*))} \frac{\exp (\theta^* \chi(\theta^*) c(\theta^*))}{\theta^* \sqrt{2\pi \sigma^2(\theta^*)}} \quad \text{as } \alpha \searrow 0.
\]
where \( \chi(\theta) = -\psi_\Gamma^{-1}(-\psi_\Lambda(\theta)) \), \( \theta^* \) satisfies \( \chi(\theta^*) = \theta^* x, \ u(y, \theta) \) (with \( u(y, 0) = 1 \)) solves the eigenvalue problem
\[
\frac{1}{\tilde{\gamma}(y)} (Au)(y, \theta) = \left( \frac{-\psi_\Lambda(\theta)}{\tilde{\gamma}(y)} + \chi(\theta) \right) u(y, \theta),
\]
and \( c(\theta^*) \) is defined in (16).

**Proof of Theorem 4.** We will follow the same strategy as in the proof of Theorem 1.

**STEP 1** Consider the family of probability measures \( P_{y,\alpha}^* \) defined as
\[
dP_{y,\alpha}^* = \exp (\theta^* D(\alpha) - \psi(\theta^*, \alpha)) dP_y,
\]
where \( \psi(\theta^*, \alpha) = \log E_{y_0} \exp(\theta^* D(\alpha)) \) and \( \theta^* \) is chosen as in Theorem 4. Note that

\[
\exp(I(x)/\alpha) P_{y_0}(\alpha D(\alpha) > x) = \exp(I(x)/\alpha) E_{y_0}^{*, \alpha}(1(\alpha D(\alpha) > x) \exp(\psi(\theta^*, \alpha) - \theta^* D(\alpha))). \tag{17}
\]

**STEP 2** Analyze the asymptotic behavior of (17). As in the discrete time case

\[
I(x)/\alpha - x\theta^*/\alpha = -\int_0^\theta \frac{\chi(u)}{\log E_{\gamma} \exp(y)} \frac{u}{\gamma} \frac{\gamma(y)}{\gamma(\theta)} \frac{d\gamma(y)}{d\gamma(\theta)} \Rightarrow \chi(\theta). \tag{18}
\]

On the other hand, a useful identity that we shall prove later in Lemma 5 states that

\[
\exp\left(\int_0^\infty \chi(\theta e^{-\alpha t}) \frac{d\gamma(t)}{d\gamma(\theta)} \right) u(y_0, \gamma(\theta)) = E_{y_0} \exp\left(\int_0^\infty \psi_\Lambda(\theta e^{-\alpha \Gamma(\theta)}) \frac{d\gamma(\theta)}{d\gamma(\theta)} \right) \tag{19}
\]

The previous identity implies, using the representation

\[
E_{y_0} \exp(\theta D(\alpha)) = E_{y_0} \exp\left(\int_0^\infty \psi_\Lambda(\theta e^{-\alpha \Gamma(\theta)}) \frac{d\gamma(\theta)}{d\gamma(\theta)} \right),
\]

that

\[
\exp\left(\psi(\theta, \alpha) - \int_0^\infty \chi(\theta e^{-\alpha \gamma}) \frac{d\gamma(\theta)}{d\gamma(\theta)} \right) \sim u(y_0, \gamma(\theta)) \exp(c(\theta)) = \zeta(y_0, \theta).
\tag{20}
\]

as \( \alpha \searrow 0 \) (recall the definition of \( c(\theta) \) given in the statement of the theorem and note that here we have used the ergodicity of the chain \( \tilde{\gamma} \)). This follows because

\[
\alpha \int_0^\infty \frac{e^{-\alpha s} u_\theta(\tilde{Y}(s), -\chi(\theta e^{-\alpha s})) \chi(\theta e^{-\alpha s})}{u(\tilde{Y}(s), -\chi(\theta e^{-\alpha s}))} ds
\]

\[
= \int_0^\infty \frac{e^{-s} u_\theta(\tilde{Y}(s/\alpha), -\chi(\theta e^{-s})) \chi(\theta e^{-s})}{u(\tilde{Y}(s/\alpha), -\chi(\theta e^{-s}))} ds
\]

\[
\overset{P}{\to} \int_0^\infty \frac{\theta e^{-s} E_{\theta} u_\theta(Y, -\chi(\theta e^{-s})) \chi(\theta e^{-s})}{u(Y, -\chi(\theta e^{-s}))} ds = E_{\theta} \frac{d\gamma(u)}{ds} = -E_{\theta} \log u(Y, -\chi(\theta)).
\]
Therefore, using (18), we have that
\[
\exp \left( I \left( x / \alpha \right) P_{y_0} (\alpha D (\alpha) > x) \right) \\
\sim \zeta (y_0, \theta^*) E_{y_0}^{\star, \alpha} \left( 1 (D (\alpha) - x / \alpha > 0) \exp (-\theta^* (D (\alpha) - x / \alpha)) \right).
\]

**STEP 3** Next we apply a local CLT to \( \alpha^{1/2} (D (\alpha) - x / \alpha) \) under \( E_{y_0}^{\star, \alpha} \). As in the discrete time case we use the results by Chaganty and Sethuraman (1993) given in Lemma 2. In particular, in Lemma 6 below, we analyze the asymptotic behavior of \( \psi^*_\alpha (i\theta) = \log E_{y_0}^{\star, \alpha} (i\theta \alpha^{1/2} (D (\alpha) - x / \alpha)) \) as \( \alpha \searrow 0 \) and verify (just as in the discrete time case) the conditions required to apply a local limit theorem companion of the CLT
\[
P_{y_0}^{\star, \alpha} \left( \alpha^{1/2} (D (\alpha) - x / \alpha) \leq y \right) \longrightarrow P \left( N \left( 0, \sigma^2 (\theta^*) \right) \leq y \right).
\]

Therefore, we obtain
\[
\frac{1}{\alpha^{1/2}} E_{y_0}^{\star, \alpha} \left[ 1 (D (\alpha) - x / \alpha > 0) \exp (-\theta^* (D (\alpha) - x / \alpha)) \right] \rightarrow \frac{1}{\theta^* \sqrt{2\pi \sigma^2 (\theta^*)}}.
\]
Combining these estimates with (21) yields the conclusion of the theorem.

**Remark:** In **STEP 3** of the previous argument we can alternatively use Edgeworth expansions as those developed by Blanchet and Glynn (2005). A direct application of the results developed in Blanchet and Glynn (2005) holds if, for example, \( \Lambda (1) \) has a density. The more general non-lattice case would then follow by a smoothing technique via pseudo-densities.

**Lemma 5** Under the assumptions of Theorem 4, for \( z \) in a neighborhood of the origin in the complex plane,
\[
\exp \left( \int_0^\infty \chi \left( ze^{-\alpha t} \right) dt \right) u (y_0, -\chi (z)) = E \exp \left( \int_0^\infty \psi_{\Lambda} \left( ze^{-\alpha \Gamma (t)} \right) dt - \alpha \int_0^\infty z \frac{e^{-au} u \theta (\tilde{Y} (u), -\chi (ze^{-au})) \check{\chi} (ze^{-au})}{u (\tilde{Y} (u), -\chi (ze^{-au}))} du \right).
\]

**Proof.** It is known that (using that \( \Xi \) is compact and the Feller condition) for every bounded away from zero function \( u \in D (A) \) we have that
\[
M_t (z) = \frac{u (Y (t), z)}{u (Y_0, z)} \exp \left( -\int_0^t \frac{A u (Y (s), z)}{u (Y (s), z)} ds \right)
\]
(22)
Observe that the solution to (23) automatically provides the solution to the problem

\begin{equation}
(Au) (y, \theta) = (\psi_\Gamma (\theta) - \theta \bar{\gamma} (y)) u (y, \theta), \quad u (y, 0) = 1 \tag{23}
\end{equation}

has a unique solution pair \((u (y, \theta), \psi_\Gamma (\theta))\) for every \(\theta \in \mathbb{R}\). In addition, \(\inf_{\theta \in \Xi} u (y, \theta) > 0\) for all \(\theta \in \mathbb{R}\) and \(\psi_\Gamma (\cdot)\) is a strictly increasing function (since

\[\psi_\Gamma (\theta) = \lim_{t \to \infty} \frac{1}{t} \log E \exp (\theta \Gamma (t)) .\]

Observe that the solution to (23) automatically provides the solution to the problem

\[\frac{1}{\bar{\gamma} (y)} (Au) (y, \theta) = \left( \frac{\psi_\Gamma (\theta)}{\bar{\gamma} (y)} - \theta \right) u (y, \theta) = \left( -\psi_\Gamma^{-1} (-\nu) - \frac{\nu}{\bar{\gamma} (y)} \right) u (y, \psi_\Gamma^{-1} (-\nu)) ,\]

(where \(\nu = -\psi_\Gamma (\theta)\)). In addition, Proposition 4.8 of Kontoyiannis and Meyn (2003) states that for each \(y \in \Xi\), both \(u (y, \cdot)\) and \(\psi_\Gamma (\cdot)\) are analytic in \(\mathcal{N} = \{z \in \mathbb{C} : |z| \leq \delta_0\}\) for some \(\delta_0 > 0\) (which immediately implies the analyticity of \(\zeta (\cdot) = -\psi_\Gamma^{-1} (-\cdot)\)) and that \(\inf_{x \in \Xi, x \in \mathcal{N}} |u (x, z)| > 0\). Note that the Markov process \(\tilde{Y} = (\tilde{Y} (t) : t \geq 0)\) defined as \(\tilde{Y} (t) = Y (\Gamma^{-1} (t))\) is also a geometrically ergodic Markov process with generator \(\tilde{A} = \frac{1}{\tilde{\gamma}} A\). The reason is that \(\tilde{\gamma}\) being continuous and positive implies \(\inf_{x \in \Xi} \tilde{\gamma} (x) > 0\), which yields that the Lyapunov bound needed in the definition of geometric ergodicity for \(\tilde{Y}\) obtained in Kontoyiannis and Meyn (2003) p. 9 follows easily from the geometric ergodicity imposed on \(Y\). Therefore, by considering the Markov generator \(\partial_t + \tilde{A}\) and the function \(u (y, \psi_\Lambda (i \theta e^{-\alpha t}))\), in the relation (22) we can build the Martingales

\[M_t (\theta) = \frac{u (\tilde{Y} (t), -\chi (i \theta e^{-\alpha t}))}{u (Y_0, -\chi (i \theta e^{-\alpha t}))} \exp \left( \int_0^t \psi_\Lambda (i \theta e^{-\alpha t}) \frac{\tilde{\gamma} (\tilde{Y} (t))}{\bar{\gamma} (\tilde{Y} (t))} dt - \int_0^t \chi (i \theta e^{-\alpha t}) dt \right) \]

\[\exp \left( -\alpha \int_0^t i \theta e^{-\alpha t} u_{\alpha} (\tilde{Y} (t), -\chi (i \theta e^{-\alpha t})) \frac{\tilde{\gamma} (\tilde{Y} (t))}{u (\tilde{Y} (t), -\chi (i \theta e^{-\alpha t}))} \chi (i \theta e^{-\alpha t}) dt \right).\]

Note that \(M_t (i \theta)\) is a bounded martingale (in particular, uniformly integrable). Thus, it possesses a last element \(M_\infty (i \theta)\), which implies that

\[\exp \left( \int_0^\infty \chi (i \theta e^{-\alpha t}) dt \right) u (Y_0, i \theta) = E \exp \left( \int_0^\infty \psi_\Lambda (i \theta e^{-\alpha t}) \frac{\tilde{\gamma} (\tilde{Y} (t))}{\bar{\gamma} (\tilde{Y} (t))} dt - \xi (\alpha, i \theta) \right) , \tag{24}\]
where
\[
\xi(\alpha, i\theta) = \alpha \int_0^t i\theta e^{-\alpha t} \frac{u_0(\overline{Y}(t), -\chi(i\theta e^{-\alpha t}))}{u(\overline{Y}(t), -\chi(i\theta e^{-\alpha t}))} \chi(i\theta e^{-\alpha t}) \, dt.
\]

Finally, note that (by introducing the change of variables \(t = \Gamma(s)\))
\[
\int_0^\infty \frac{\psi_\Lambda(i\theta e^{-\alpha t})}{\tilde{\gamma}(-\chi(i\theta e^{-\alpha t}))} \, dt = \int_0^\infty \psi_\Lambda(i\theta e^{-\alpha \Gamma(s)}) \, ds,
\]
this combined with (24) yields the conclusion of the lemma. □

The conclusion of **STEP 3** requires verification of the conditions in given in Lemma 2.

**Lemma 6** *In the setting of Lemma 2, conditions i), ii) and iii) hold.*

**Proof.** Parts i) and ii) follow from a precise description of the local behavior of \(\psi_\alpha^*(i\theta) = \log E_{y_0}^* \exp (i\theta \sqrt{\alpha} (D(\alpha) - x/\alpha))\). In fact, we can obtain (essentially from representation given in Lemma 5 as in Blanchet and Glynn (2005))
\[
\psi_\alpha^*(i\theta) = -\theta^2 \chi''(\theta^*) / 4 + \sqrt{\alpha} (c_1 i\theta + c_2 (i\theta)^3) + o(\sqrt{\alpha})
\]
(uniformly on \(\theta \in (-\delta, \delta)\) for some \(\delta > 0\)). The coefficients \(c_1\) and \(c_2\) can actually be computed but their values are not relevant for purposes of developing sharp large deviations. The coefficient \(\chi''(\theta^*) / 4\) comes from the development of
\[
\int_0^\infty \left( \chi\left(\sqrt{\alpha} \theta + \theta^*\right) e^{-\alpha u} - \chi\left(\theta^* e^{-\alpha u}\right) - \sqrt{\alpha} e^{-\alpha u} x \theta \right) \, du
\]
\[
= \frac{1}{\sqrt{\alpha}} \int_0^\infty \left( \chi\left(\theta^* e^{-\alpha u}\right) - x \right) e^{-\alpha u} \, du + \theta^2 \int_0^\infty \frac{\alpha \chi''\left(\theta^* e^{-\alpha u}\right) e^{-2\alpha u}}{2} \, du + o(\sqrt{\alpha}).
\]

Indeed, since
\[
\int_0^\infty \chi\left(\theta^* e^{-\alpha u}\right) e^{-\alpha u} \, du = -\frac{1}{\theta^* \alpha} \int_0^\infty d\chi\left(\theta^* e^{-\alpha u}\right) = \frac{\chi(\theta^*)}{\alpha \theta^*} = \frac{x}{\alpha},
\]
we obtain that the coefficient multiplying \(\theta\) in (25) vanishes and, thus, \(\psi_\alpha^*(\theta) \sim -\theta^2 \int_0^\infty \chi''(\theta^* e^{-s}) e^{-2s} \, ds/2\) as stated.

For part iii) we must show that \(|\phi_\alpha^*(\theta, \alpha)| = |E_{y_0}^* \exp (i\theta D(\alpha))| = o(\sqrt{\alpha})\) uniformly for \(\theta\) in compact sets not containing the origin. The key observation to prove this condition is to note that
\[
\phi_\alpha^*(\theta, \alpha) = \frac{E \exp \left( \int_0^\infty \psi_\Lambda((i\theta + \theta^*) \exp (-\alpha \Gamma(t))) \, dt \right)}{E \exp \left( \int_0^\infty \psi_\Lambda(\theta^* \exp (-\alpha \Gamma(t))) \, dt \right)}.
\]

17
Now consider the (geometrically ergodic) Markov process \( \bar{Y} = (\bar{Y}(s) : s \geq 0) \) defined as \( \bar{Y}(t) = Y(\Gamma^{-1}(t)) \). This process has generator \( \bar{A} = (1/\gamma) A \) and was introduced in the proof of Lemma 5. Let us define the probability measure \( \bar{P} \) acting on the sigma-field generated by \( \bar{Y} \) as

\[
d\bar{P} = M_\infty(\theta^*) dP,
\]

where \( M_\infty(\theta^*) \) is the last element of the bounded martingale \( M^* = (M_t(\theta^*) : 0 \leq t \leq \infty) \) defined as

\[
M_t(\theta^*) = \frac{u(\bar{Y}(t), -\chi(\theta^* e^{-\alpha t}))}{u(0, -\chi(\theta^* e^{-\alpha}))} \exp \left( \int_0^t \psi_{\Lambda}(\theta^* e^{-\alpha t}) \left( \frac{\bar{Y}(t)}{\gamma} \right) dt - \int_0^t \chi(\theta^* e^{-\alpha t}) dt \right)
\]

\[
\exp \left( -\alpha t \theta^* e^{-\alpha t} \right) \frac{u(\bar{Y}(t), -\chi(\theta^* e^{-\alpha t}))}{u(0, -\chi(\theta^* e^{-\alpha}))} \hat{\chi}(\theta^* e^{-\alpha t}) dt \right).
\]

(This martingale was also introduced in the proof of Lemma 5, where it has been indicated how the martingale property follows from Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001)). Note, therefore, that

\[
E \exp \left( \int_0^\infty \psi_{\Lambda}((i\theta + \theta^*) e^{-\alpha \Gamma(t)}) - \chi(\theta^* e^{-\alpha t}) dt \right) = \bar{E} \left( \exp \left( \int_0^\infty \psi_{\Lambda}((i\theta + \theta^*) e^{-\alpha \Gamma(t)}) - \int_0^\infty \psi_{\Lambda}(\theta^* e^{-\alpha \Gamma(t)}) dt \right) Z(\alpha) \right),
\]

where \( B_1 < |Z(\alpha)| < B_2 \) for some constants \( 0 < B_1 < B_2 < \infty \). This implies, using the bound (26), that

\[
|\phi^*(\theta, \alpha)| \leq CB_2 \bar{E} \left( \exp \left( \int_0^\infty \left( \psi_{\Lambda}((i\theta + \theta^*) e^{-\alpha \Gamma(t)}) - \psi_{\Lambda}(\theta^* e^{-\alpha \Gamma(t)}) \right) dt \right) \right) |.
\]

From this bound it is easy to see that \( |\phi^*(\theta, \alpha)| = o(\sqrt{\alpha}) \) by noting that for each small enough \( \eta_2 \in \mathbb{R}, \exp(\psi_{\Lambda}(i \cdot + \eta_2) - \psi_{\Lambda}(\eta_2)) \) is the characteristic function of \( \Lambda(1) \) under the exponential change of measure induced by \( \eta_2 \). Also, observe that \( \delta t \leq \Gamma(t) \leq b_1 t \) for positive finite constants \( \delta \) and \( b \). Hence, it follows that there exist constants \( b_2 \in [\delta, b_1] \) and \( B_3 \) such that

\[
|\phi^*(\theta, \alpha)| \leq B_3 \left( \exp \left( \int_0^\infty \left( \psi_{\Lambda}((i\theta + \theta^*) e^{-\alpha \Gamma(t)}) - \psi_{\Lambda}(\theta^* e^{-\alpha \Gamma(t)}) \right) dt \right) \right) |.
\]

\[
|\phi^*(\theta, \alpha)| \leq B_3 |E \exp \left( \left( \Lambda(1) (i\theta + \theta^*) \delta_3 \right) - \psi_{\Lambda}(\theta^* \delta_3) \right)^{1/\alpha} \right),
\]

(27)
where $\delta_3 \in (0, 1)$. Since $\Lambda (1)$ is non-lattice (and the non-lattice property is preserved under exponential changes of measure) we conclude, from (27), that $|\phi^\ast (\theta, \alpha)|$ actually decreases exponentially fast to zero uniformly in $\theta$ on compact intervals not containing the origin, which is more than we need no verify $\text{iii}$) and conclude the proof of the lemma. \hfill \blacksquare

### 3.1 Extensions of Sharp Asymptotics

Note that the assumption that $\Xi$ is compact, although convenient to develop the mathematical theory, by no means plays an essential role. This assumption can be replaced by the assumption that $0 < \varepsilon \leq \tilde{\gamma} (y) \leq b < \infty$ for all $y \in \Xi$, provided that the process $Y$ is geometrically ergodic. The spectral theory developed by Kontoyiannis and Meyn (2003), which was crucial in finding a unique solution for (14) and the arguments given proceed virtually unchanged.

In fact, it may even be possible to extend this theory to some cases in which $\tilde{\gamma}$ is only assumed to be positive, under some conditions, using the theory recently developed by Kontoyiannis and Meyn (2005). We shall explore this potential extensions elsewhere.

The independence between $\Gamma$ and $\Lambda$ can also be relaxed. For example, one could have assumed that both processes are conditionally independent given another Markov process, say $Z$, provided that $\Lambda$ remains a possibly non-time homogeneous Levy process with a suitably non-lattice conditional distribution type assumption analogous to condition $\text{IID3}$ in section 2.

Following the same ideas as in Lemma $\text{??}$, a local expansion for $\psi_\alpha (\theta)$ can be obtained for the case in which

$$D (a) = \int_0^\infty \exp \left( - \alpha \int_0^t \tilde{\gamma} (Y (s)) \, ds \right) \tilde{\lambda} (Y (s)) \, ds.$$  

(where $\tilde{\lambda}$ is, say, continuous on the compact Polish space $\Xi$). In this case, the corresponding generalized eigenvalue problem takes the form

$$\frac{1}{\tilde{\gamma}} (Au) (y, \theta) = \left( \chi (\theta) - \frac{\tilde{\lambda} (y)}{\tilde{\gamma} (y)} \right) u (y, \theta), \quad u (y, 0) = 1,$$

and a formal corrected approximation can be written just as in (15). The crucial steps required to make approximation (15) rigorous in this case involve showing that there is a unique solution to the previous generalized eigenvalue problem and verifying condition $\text{iii}$) in Lemma 2.
4 Large Deviations

As we indicated in the introduction, it is natural to expect that a Law of Large Numbers (LLN) type approximation such as (2) holds in great generality. As we discussed previously, in a number of applications (for instance in the contexts of risk theory, finance and time series analysis), one is often interested in computing $P(D > x)$ for $x$ suitably large. In particular, these types of applications motivate interest in the analysis of the tail probability $P(D > x)$ for $x \gg \lambda / \gamma$. Under suitable exponential moment conditions on $(\Gamma, \Lambda)$, the approximation proposed here will take the form

$$P(D > x) \approx \exp \left( -I(x) / \gamma \right),$$

where $I(x) > 0$ corresponds to the so-called rate function and will typically take the form $I(x) = x\theta^* - \int_0^\infty \chi(\theta^* e^{-s}) \, ds$, where $\theta^*$ satisfies $\theta^* x = \chi(\theta^*)$ and $\chi(\cdot)$ is a suitably defined convex function. The goal of this section is to provide, under general conditions, rigorous justification (at least in a rough logarithmic sense) for the previous approximation.

Applications in finance and risk theory motivate study of continuous time processes, including the case in which the processes $\Gamma$ and $\Lambda$ take the form

$$\Gamma(t) = \int_0^t \tilde{\gamma}(s) \, ds \quad \text{and} \quad \Lambda(t) = \int_0^t \tilde{\lambda}(s) \, ds,$$

where, for all $s$, $\tilde{\gamma}(s) > 0$ represents the “short rate” process and $\tilde{\lambda}(s)$ represent the reward rate. Also, other applied contexts such as the analysis of ARCH processes in time series motivate study of the discrete time setting, in which

$$\Gamma(t) \triangleq \sum_{k=1}^{|t|} Z_k \quad \text{and} \quad \Lambda(t) \triangleq \sum_{k=1}^{|t|} X_k,$$

and $(X_k, Z_k)_{k \geq 0}$ is a (typically stationary) sequence of two dimensional random vectors with the property that $Z_k > 0$ for all $k \geq 0$.

In order to provide rigorous justification for the approximation (28), we shall consider

$$aD(\alpha) = \int_{[0, \infty)} \exp \left( -a \Gamma(t-) \right) \, d\Lambda(t) = \alpha \int_{[0, \infty)} \exp \left( -u \right) \Lambda \left( \Gamma^{-1}(u/\alpha) \right) \, du,$$

and study $P(aD(\alpha) > x)$ for $x > \lambda$. Note that the previous identity holds in general provided that $\Gamma(\cdot)$ is non-decreasing and $\Lambda(\cdot)$ has RCLL sample paths. In other words, (29) may hold even if $\Lambda$ does not have bounded variation. Expression (29) suggests a natural strategy to derive a LDP for $\{aD(\alpha)\}_{\alpha > 0}$ as $\alpha \searrow 0$; namely,
to apply the contraction principle (under appropriate sample path large deviations assumptions on \( \alpha \left( \Gamma \left( \cdot / \alpha \right), \Lambda \left( \cdot / \alpha \right) \right) \)), to the mapping \( \Psi : D[0, \infty) \times D[0, \infty) \to \mathbb{R} \) defined as

\[
\Psi(x, y) = \int_0^{\infty} \exp(-t) y \left( x^{-1}(t) \right) dt.
\]

Actually, we will follow more or less this idea, although with important modifications arising due to the fact that \( \Psi \) is not continuous. Indeed, if we consider the map \( \Psi_1 \), acting on \( D[0, \infty) \) endowed with the Skorohod \( J_1 \) topology (see Whitt (2001)) and defined as \( \Psi_1(x) = \int_0^{\infty} \exp(-t) x(t) dt \), then, we can see, aside from the fact that \( \Psi_1 \) is not well defined for every element in \( D[0, \infty) \), that \( \Psi_1 \) is discontinuous at every single point. In order to see this, just consider the sequence of functions \( (x_n : n \geq 1) \), defined as \( x_n(t) = e^n I(n \leq t < n+1) \), and note that \( x_n \to 0 \) while \( \Psi_1(x_n) = 1 - e^{-1} \). (This example was given by Whitt (1972); that \( \Psi_1(\cdot) \) is discontinuous at every element of \( D[0, \infty) \) follows by linearity of \( \Psi_1 \).)

The idea is to restrict the domain of \( \Psi_1 \) to a proper subspace of \( D[0, \infty) \), endowed with a finer topology under which \( \Psi_1(\cdot) \) is continuous. This idea will be studied in detail in the next subsection, in which we treat the continuous setting. Later, we will return to the discrete setting.

4.1 The continuous time setting

We will restrict the domain of \( \Psi_1 \) to the subspace

\[
L_\beta[0, \infty) \triangleq \{ x \in C[0, \infty) : \lim_{t \to \infty} \left| \frac{x(t)}{t^\beta} \right| = 0 \},
\]

for some \( \beta > 0 \), with the topology generated by the weighted norm

\[
\| x \|_\beta = \sup_{t \geq 0} \left| \frac{x(t)}{1 + t^\beta} \right|.
\]

Whitt (1972) proved that \( \Psi_1 \) is continuous on \( \left( L_\beta[0, \infty), \| \cdot \|_\beta \right) \), which suggests using the contraction principle on this space. The following proposition constitutes an intermediate step in this direction.

**Proposition 7** Suppose that the family of processes \( \alpha \left( \Gamma \left( \cdot / \alpha \right), \Lambda \left( \cdot / \alpha \right) \right) \alpha > 0 \) satisfies a LDP on \( C[0, \infty) \times C[0, \infty) \) (endowed with the product topology generated by the uniform convergence on compact sets also known as Stone’s topology) with a good rate function \( I(x, y) \). Then, \( R_\alpha(\cdot) = \alpha \Lambda \left( \Gamma^{-1}(\cdot / \alpha) \right) \) satisfies a LDP on \( C[0, \infty) \) (endowed with Stone’s topology) with good rate function \( I'(z) = \inf \{ I(x, y) : z = y \circ x^{-1} \} \).

**Proof.** This is just a direct consequence of the contraction principle (see Theorem 4.2.1, p. 126 of Dembo and Zeitouni (1999)) and the fact that the mapping \( (x, y) \to \)
$y \circ x^{-1}$ in the topological spaces described (see Whitt (2001), Theorem 13.2.2., p. 430).

At this point, one may be tempted to invoke, once again, the contraction principle in combination with Proposition 7 to obtain the desired LDP. However, in order to proceed with this program, we must show that the LDP developed in Proposition 7 actually holds on $(L_\beta[0, \infty), \|\cdot\|_\beta)$ (since, in order to apply the contraction principle, the continuity of $\Psi_1$ must be compatible with the topology under which the original LDP was derived). In order to show the LDP on $(L_\beta[0, \infty), \|\cdot\|_\beta)$ we will need to show that the random elements $(\alpha \Lambda(\Gamma^{-1} \cdot/\alpha))_{\alpha > 0}$ are exponentially tight (see Dembo and Zeitouni (1998)). (This type of reasoning parallels similar arguments in the context of weak convergence theory and the important role that tightness plays in this theory). Recall that a sequence of probability measures $P_n$ is said to be exponentially tight if for every $a > 0$ there exist compact sets $K_a$, such that

$$
\lim_{n \to \infty} \frac{1}{n} \log P_n(K_a) \leq -a,
$$
or, if the $P_n$’s take values on subsets of a Polish space, then the $P_n$’s are exponentially tight if for $\varepsilon > 0$, there exists a compact set $K_\varepsilon$ such that, for all $n \geq 1$,

$$
\varepsilon^n > 1 - P_n(K_\varepsilon),
$$
(see Zajic (1993) p. 11). In view of these observations, we must characterize exponential tightness in $L_\beta[0, \infty)$. This is the aim of the following theorem.

**Lemma 8** Consider a sequence of probability measures $(P_n : n \geq 1)$ on $L_\beta[0, \infty)$ (such that $P_n\{x : x(0) = 0\} = 1$) and acting on the Borel sigma-field corresponding to the topology generated by the norm $\|\cdot\|_\beta$. Then, $(P_n : n \geq 1)$ is exponentially tight if and only if $(P_n : n \geq 1)$ is exponentially tight under the (relative) Stone topology, and that for each $\delta > 0$

$$
\lim_{n \to \infty} \frac{1}{n} \log P_n \left( x : \sup_{t \geq t_0} \frac{|x(t)|}{t^\beta} > \delta \right) \to -\infty \text{ as } t_0 \to \infty.
$$

**Proof.** Lemma 3.3 of Whitt (1972) establishes that relatively compact sets in $(L_\beta[0, \infty), \|\cdot\|_\beta)$ are those sets $B$ with compact closure under the relative Stone topology, and satisfying

$$
\lim_{t \to \infty} \sup_{x \in B} \frac{|x(t)|}{t^\beta} = 0.
$$
Also, recall that (if $x(0) = 0$ a.s. with respect to each $P_n$) for exponential tightness under Stone’s topology, it is necessary and sufficient (see Feng and Kurtz (2000), p. 30) that, for each $\varepsilon, T > 0$,

$$
\lim_{n \to \infty} \frac{1}{n} \log P_n(x : \omega(x, \delta, T) > \varepsilon) \to -\infty \text{ as } \delta \searrow 0,
$$

22
where $\omega(x, \delta, T)$ is the modulus of continuity of $x$, on the interval $[0, T]$, evaluated at $\delta$. We now show that if conditions (30) and (31) are satisfied, then the sequence $(P_n : n \geq 1)$ is exponentially tight. Pick $\lambda > 0$, choose $\delta_k$ so that

$$P_n(x : \omega(x, \delta_k, T) > 1/k) \leq e^{-n\lambda/2^{k+1}},$$

and let $B_k = \{x : \omega(x, \delta_k, T) \leq 1/k\}$. Also, pick $t_k$ so that

$$P_n\left(x : \sup_{t \geq t_k} \frac{|x(t)|}{t^\beta} > 1/k\right) \leq e^{-n\lambda/2^{k+1}},$$

and let $C_k = \{x : \sup_{t > t_k} |x(t)|/t^\beta \leq 1/k\}$. Consider the closure, $\bar{A}_\lambda$, of $A_\lambda = \cap_k (B_k \cap C_k)$. Note that

$$1 - P(\bar{A}_\lambda) \leq 1 - P(A_\lambda) = P(\cup_k (B_k \cap C_k)) \leq e^{-n\lambda}$$

We claim that $A_\lambda$ is relatively compact (i.e. that $\bar{A}_\lambda$ is compact), to see this, choose $\varepsilon > 0$ and let $k_0 > 1/\varepsilon$. Then, for all $\delta < \delta_{k_0}$ we have that

$$\sup_{x \in A} \omega(x, \delta, T) < \varepsilon.$$  

Similarly, for every $T > t_{k_0}$ we have that

$$\varepsilon > \sup_{x \in A} \sup_{t > T} \frac{|x(t)|}{t^\beta},$$

which implies that

$$\lim_{t \to \infty} \sup_{x \in A} \frac{|x(t)|}{t^\beta} \leq \varepsilon$$

for all $\varepsilon > 0$. Thus, by virtue of the Arzela-Ascoli theorem (see Billingsley (1999) p.81) and Lemma 3.3 of Whitt (1972), which concludes the argument for sufficiency. The necessity part is easier and follows just as in Feng and Kurtz (2000) p. 30. Therefore, it is omitted.

With the aid of the previous lemma, the exponential tightness of $(\alpha \Lambda(\Gamma^{-1}(\cdot/\alpha)))_{\alpha > 0}$ follows easily.

**Lemma 9** Suppose that $\alpha (\Gamma (\cdot/\alpha), \Lambda (\cdot/\alpha))_{\alpha > 0}$ satisfies a full LDP with rate function $I(x, y)$ (under Stone’s topology). (Recall that a full LDP means an LDP with convex good rate function). Then,

a) The family $(\alpha \Gamma (\cdot/\alpha) - \gamma ; \alpha \Lambda (\cdot/\alpha) - \lambda \cdot)_{\alpha > 0}$ is exponentially tight in $L_1[0, \infty) \times L_1[0, \infty)$ with the product topology generated by the norm $\| \cdot \|_1$.
b) The class of random elements \( \alpha \Lambda \left( \Gamma^{-1} \left( \cdot / \alpha \right) \right) - \lambda \cdot / \gamma)_{\alpha > 0} \) is exponentially tight in \( (L_1[0, \infty), \| \cdot \|_1) \).

**Remark** The convexity of the rate function does not really play a role in this lemma, but only the goodness of the rate function is required.

**Proof.** For part a), it suffices to show that \( \alpha \Gamma (\cdot / \alpha) - \gamma \cdot \) and \( \alpha \Lambda (\cdot / \alpha) - \gamma \cdot \) are both exponentially tight in \( (L_1[0, \infty), \| \cdot \|_1) \). Since \( \alpha \Lambda (\cdot / \alpha) \) satisfies a full LDP in \( C[0, \infty) \) (under Stone’s topology), which is a topological group (which implies the addition is a continuous operation), it follows from the contraction principle that \( \alpha \Lambda (\cdot / \alpha) - \gamma \cdot \) also satisfies a full LDP. Note that \( C[0, \infty) \), endowed with Stone’s topology, is a Polish space. Thus, the existence of a full LDP guarantees the exponential tightness of \( \alpha \Lambda (\cdot / \alpha) - \gamma \cdot \) (see Dembo and Zeitouni (1999), p. 120 (c)). Therefore, we just have to prove condition (30). Note that for any \( 0 < a < b < \infty \), the mapping \( x \to \sup_{t \in [a, b]} \left| x(t)/t \right| \) is continuous (under Stone’s topology), which implies that the family \( V_{\alpha} = \sup_{t \in [a, b]} \left| \alpha \Lambda (t/\alpha) / t - \gamma \right| \) satisfies an LDP with good rate function \( J \), say. Hence, we can write

\[
P \left( \sup_{t > t_0} \left| \frac{\alpha \Lambda (t/\alpha) - \gamma t}{t} \right| \geq \delta \right)
\leq \sum_{k=1}^{\infty} P \left( \sup_{t > t_0 [k,k+1]} \left| \frac{\alpha \Lambda (t/\alpha) - \gamma t}{t} \right| \geq \delta \right)
\leq \sum_{k=1}^{\infty} P \left( \sup_{u \in \frac{t_0}{k} > [1,2]} \left| \frac{\alpha \Lambda (ukt_0/\alpha) - \gamma ukt_0}{ukt_0} \right| \geq \delta \right)
= \sum_{k=1}^{\infty} \exp \left( - \left( J(\delta) + o_{kt_0/\alpha} (1) \right) \right) \cdot \exp \left( \frac{\delta}{kt_0} \right),
\]

where the subindex in \( o_{kt_0/\alpha} (1) \) has been used just to emphasize that \( o_{kt_0/\alpha} (1) \to 0 \) as \( kt_0/\alpha \to \infty \). So we can choose \( k_0 \) big enough so that for every \( k > k_0 \) we have \( J(\delta) + o_{kt_0/\alpha} (1) > J(\delta)/2 > 0 \). From these estimates it is easy to conclude that

\[
\lim_{\alpha \to 0} \log P \left( \sup_{t > t_0} \left| \frac{\alpha \Lambda (t/\alpha) - \gamma t}{t} \right| \geq \delta \right) \to -\infty \quad \text{as} \quad t_0 \nearrow \infty,
\]

which yields, by virtue of Lemma 8, the corresponding exponential tightness for \( \alpha \Lambda (\cdot / \alpha) - \gamma \cdot \). The argument for \( \alpha \Gamma (\cdot / \alpha) - \gamma \cdot \) is exactly the same and therefore has been omitted. Part b) also proceeds along the same lines as the previous argument, since it follows from Proposition 7 that \( \alpha \Lambda \left( \Gamma^{-1} (\cdot / \alpha) \right) \) satisfies a full LDP under Stone’s topology. \( \blacksquare \)

We are ready to derive the LDP for \( (\alpha D (\alpha))_{\alpha > 0} \) in the continuous setting.
**Theorem 10** Suppose that the family of processes $\alpha (\Gamma (\cdot /\alpha), \Lambda (\cdot /\alpha))_{\alpha > 0}$ satisfies a full LDP on $C[0, \infty] \times C[0, \infty]$ (endowed with the corresponding product Stone’s topology) with a good rate function $I (x, y)$. Then, $\{\alpha D (\alpha)\}_{\alpha > 0}$ satisfies an LDP on $\mathbb{R}$ with good rate function

$$I (z) = \inf \{I (x, y) : z = \int_0^\infty e^{-t} (y \circ x^{-1}) (t) dt\}.$$

**Proof.** Proposition 7 combined with the contraction principle tells us that the family of random variables $(\alpha \Lambda (\Gamma^{-1} (\cdot /\alpha)) - \lambda /\gamma)_{\alpha > 0}$ satisfies a full LDP on $C[0, \infty]$. Since the product topology generated by the norm $\|\cdot\|_1$ in the subspace $L_1 [0, \infty]$ is finer than Stone’s topology, Corollary 4.2.6 of Dembo and Zeitouni (1999) (which is a simple consequence of the inverse contraction principle applied with the identity mapping) applies yielding that $(\alpha \Lambda (\Gamma^{-1} (\cdot /\alpha)) - \lambda /\gamma)_{\alpha > 0}$ satisfies a full LDP on $(L_1 [0, \infty], \|\cdot\|_1)$. Since the mapping $\Psi_1$ is continuous on $(L_1 [0, \infty], \|\cdot\|_1)$, we can apply the contraction principle once again here thereby yielding the conclusion of the theorem. 

The previous theorem provides rigorous justification for approximation (28) in very general setting (essentially all those in which functional LDPs for $(\Gamma, \Lambda)$ exist in the space of continuous functions). This includes, for example, the setting in which $\Lambda$ and $\Gamma$ are diffusion processes (see Dembo and Zeitouni (1999) Section 5.6). It is desirable to provide sufficient conditions that guarantee the validity of an LDP with good rate function on $C[0, \infty] \times C[0, \infty]$. Fortunately, these types of conditions have been well studied in the literature. In particular, the following set of assumptions taken from Zajic (1993) are useful to guarantee the existence of a full LDP (more than we actually need) and their validity has been shown in many different settings (see Zajic (1993) Ch. 3 and Ch. 4 and Dembo and Zajic (1995)).

**ACL1** For all nonnegative $\theta, \eta, r < \infty$ suppose that

$$g_r (\eta, \theta) \triangleq \sup_{s, t, r \in \mathbb{R}, t \in [0, r]} \frac{1}{t} \log E \exp \left( \eta \int_s^{s+1} \gamma (u) du + \theta \int_s^{s+1} |\lambda (u)| du \right) < \infty,$$

and

$$\sup_{\theta, \eta} \lim_{r \to \infty} r^{-1} g_r (\eta, \theta) < \infty.$$

**ACL2** If $0 = t_0 < t_1 < \ldots < t_m < \infty$ then

$$W_{\alpha, m} \triangleq \alpha ((\Gamma (t_i /\alpha) - \Gamma (t_{i-1} /\alpha)), (\Lambda (t_i /\alpha) - \Lambda (t_{i-1} /\alpha)))_{i=1}^m$$

satisfies a Large Deviations Principle (LDP) on $\mathbb{R}^m$ with good rate function

$$I_m (z) = \sum_{i=1}^m (t_i - t_{i-1}) I \left( \frac{z_i}{t_i - t_{i-1}} \right),$$

where $I (x, y)$ is the rate function governing the LDP of $n^{-1} (\Gamma (n), \Gamma (n))$. 

25
The following theorem provides a form of the LDP that is well suited for applications. Define (as in Zajic (1993) p. 9)

\[ \psi(\eta, \theta) = \lim_{n \to \infty} \frac{1}{n} \log E \exp(\eta \Gamma(n) + \theta \Lambda(n)) < \infty. \]

**Theorem 11** Suppose that assumptions ACL1 and ACL2 are in force. Let \( AC_0 \) be the set of absolutely continuous functions, defined on \([0, \infty)\), taking values on the real line and vanishing at the origin. Then, if \( y > \lambda / \gamma \), we have that

\[
\lim_{\alpha \to \infty} \alpha \log P(\alpha D(\alpha) \geq y) = -I(y) \triangleq -\inf_{x \in AC_0} \left\{ \int_0^\infty \sup_{\theta} (\theta \dot{x}(s) - \chi(\theta)) \, ds : y = \int_0^\infty e^{-s} x(s) \, ds \right\},
\]

where \( \chi(\cdot) \) is defined via \( \psi(-\chi(\cdot), \cdot) = 0 \). In addition, if there exists \( \theta^* = \theta^*(y) \) such that \( y\theta^* = \chi(\theta^*) \), then we have

\[
\lim_{\alpha \to \infty} \alpha \log P(\alpha D(\alpha) \geq y) = \sup_\theta \left( y\theta - \int_0^\infty \chi(\theta e^{-s}) \, ds \right) = y\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} \, du.
\]

**Proof.** All what we have to do is to identify the rate function. Theorem 2.2.2., p. 25, of Zajic (1993) indicates that \( \alpha (\Gamma(\cdot)/\alpha), \Lambda(\cdot)/\alpha)_{\alpha > 0} \) satisfies a LDP with good rate function

\[
I(x, y) \triangleq \left\{ \int_0^\infty \sup_{\eta, \theta} (\dot{x}(s) \eta + \dot{y}(s) \eta - \psi(\eta, \theta)) \, ds \text{ if } x, y \in AC_0 \right\} \text{ otherwise}
\]

This implies (combining the results of Puhalskii and Whitt (1997) and Russell (1998)) that \( (\alpha \Lambda(\Gamma^{-1}(\cdot)/\alpha))_{\alpha > 0} \) satisfies a full LDP with good rate function

\[
J(x) \triangleq \left\{ \int_0^\infty \sup_\theta (\dot{x}(s) \theta - \chi(\theta)) \, ds \text{ if } x \in AC_0 \right\} \text{ otherwise}
\]

This expression, combined with the contraction principle, yields the first part of the theorem. Hence, we only need to show that if \( y > \lambda / \gamma \) and \( y\theta^* = \chi(\theta^*) \), then

\[
I(y) = \sup_\theta \left( y\theta - \int_0^\infty \chi(\theta e^{-s}) \, ds \right) = y\theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} \, du.
\]
First, observe that integration by parts yields
\[
\inf_{x \in AC_0; \ y = \int_0^\infty e^{-s}x(s)ds} \left( \int_0^\infty \sup_{\theta} (\theta \dot{x} (s) - \chi (\theta)) ds \right) = \inf_{x \in AC_0; \ y = \int_0^\infty e^{-s} \dot{x}(s)ds} \left( \int_0^\infty \sup_{\theta} (\theta \dot{x} (s) - \chi (\theta)) ds \right).
\]

Also, note that for every \( s \in \mathbb{R} \)
\[
\sup_{\theta} (\theta \dot{x} (s) - \chi (\theta)) = \sup_{\theta} (\theta e^{-s} \dot{x} (s) - \chi (\theta e^{-s})).
\]

In particular, we have that for \( x \in AC_0 \) and \( y = \int_0^\infty e^{-s}x(s)ds \)
\[
\int_0^\infty \sup_{\theta} (\theta \dot{x} (s) - \chi (\theta)) ds = \int_0^\infty \sup_{\theta} (\theta e^{-s} \dot{x} (s) - \chi (\theta e^{-s})) ds \geq \sup_{\theta} \int_0^\infty (\theta e^{-s} \dot{x} (s) - \chi (\theta e^{-s})) ds = \sup_{\theta} \left( y \theta - \int_0^\infty \chi (\theta e^{-s}) ds \right).
\]

Consequently,
\[
I (y) \geq \sup_{\theta} \left( y \theta - \int_0^\infty \chi (\theta e^{-s}) ds \right).
\]

Now, if \( y > \lambda / \gamma = \dot{\chi} (0) \), then
\[
\sup_{\theta \geq 0} \left( y \theta - \int_0^\infty \chi (\theta e^{-s}) ds \right) \geq 0.
\]

On the other hand, for every \( \theta > 0 \), we have (by making the change of variables \( \theta e^{-s} = u \))
\[
\int_0^\infty \chi (\theta e^{-s}) ds = \int_0^{\theta} \frac{\chi (u)}{u} du.
\]

Therefore, by first order optimality conditions we have that (using the convexity of the rate function)
\[
\sup_{\theta \geq 0} \left( y \theta - \int_0^\infty \chi (\theta e^{-s}) ds \right) = y \theta^* - \int_0^{\theta^*} \frac{\chi (u)}{u} du > 0.
\]

Finally consider the function \( x^*_s (s) \) such that \( \dot{\chi} (\theta^* e^{-s}) = \dot{x}_s (s) \) and \( x (0) = 0 \). Note that
\[
\int_0^\infty e^{-s} \dot{x}_s (s) ds = \int_0^\infty e^{-s} \dot{\chi} (\theta^* e^{-s}) ds = \int_0^{\theta^*} d\chi (\theta^* e^{-s}) = \frac{\chi (\theta^*)}{\theta^*} = y.
\]

27
Hence, we have that
\[
I(y) = \inf_{x \in AC_0; y \in o \infty} e^{-x(s)} \left( \int_0^\infty \sup_{\theta} (\theta \dot{x}(s) - \chi(\theta)) \, ds \right)
\leq \int_0^\infty \sup_{\theta} (\theta e^{-s} \chi(\theta e^{-s}) - \chi(\theta e^{-s})) \, ds
= y \theta^* - \int_0^{\theta^*} \frac{\chi(u)}{u} \, du = \sup_{\theta} \left( y \theta - \int_0^\infty \chi(\theta e^{-s}) \, ds \right),
\]
which yields the conclusion of the theorem. ■

### 4.2 The discrete time setting

The goal now is to obtain the LDP for the discrete time case. The following set of assumptions are completely analogous to those stated at the end of the previous section, and their validity has been verified in a variety of contexts (including under Markovian and strong mixing assumptions; see Zajic (1993), chapters 3 and 4 and Dembo and Zajic (1995)).

**ADL1** For all nonnegative \( \theta, \eta, r < \infty \), suppose that
\[
g_r(\eta, \theta) \triangleq \sup_{n, k \in \mathbb{N}, k \in [0, rm]} \frac{1}{n} \log E \exp \left( \eta \sum_{j=k}^{n+k} Z_j + \theta \sum_{j=k}^{n+k} |X_j| \right) < \infty,
\]
and
\[
\sup_{\theta, \eta} \lim_{r \to \infty} r^{-1} g_r(\gamma) < \infty
\]

**ADL2** If \( 0 = t_0 < t_1 < \ldots < t_m < \infty \) then
\[
W_{\alpha, m} \triangleq \alpha \left( (\Gamma(t_i/\alpha) - \Gamma(t_{i-1}/\alpha)), (\Lambda(t_i/\alpha) - \Lambda(t_{i-1}/\alpha)) \right)_{i=1}^m
\]
satisfies an LDP on \( \mathbb{R}^{2m} \) with good rate function
\[
I_m(z) = \sum_{i=1}^m (t_i - t_{i-1}) I \left( \frac{z_i}{t_i - t_{i-1}} \right),
\]
where \( I(x_1, x_2) \) is the rate function governing the LDP of \( n^{-1} (\Gamma(n), \Lambda(n)) \)

A natural strategy here is first to consider a related family of approximating processes \( \left( \tilde{\Gamma}, \tilde{\Lambda} \right) \) defined via
\[
\tilde{\Gamma}(t) \triangleq \sum_{k=1}^{[t]} Z_k + (t - [t]) Z_{[t]+1},
\]
\[
\tilde{\Lambda}(t) \triangleq \sum_{k=1}^{[t]} X_k + (t - [t]) X_{[t]+1}.
\]
Theorem 2.1.1., p. 19, of Zajic (1993) establishes that \( \alpha \left( \tilde{\Gamma} (\cdot/\alpha), \tilde{\Lambda} (\cdot/\alpha) \right) \) satisfies a full LDP under Stone’s topology. Note that \([\cdot] \) is being used here instead of \([\cdot] \) in the definition of \( \tilde{\Lambda} \), but it is straightforward to adapt Zajic’s estimates in this setting. Also, recall that a full LDP is one that holds with a good and convex rate function. See Dembo and Zeitouni (1999) for the definition of good rate function. Thus, Theorem 10 applies here yielding the full LDP for the corresponding normalized infinite horizon discounted reward

\[
\alpha \tilde{D} (\alpha) \triangleq \alpha \Psi \left( \tilde{\Gamma}_\alpha^{-1}, \tilde{\Lambda}_\alpha \right) = \alpha \int_{[0,\infty)} \exp (-u) \tilde{\Lambda}_\alpha (\tilde{\Gamma}_\alpha^{-1} (u)) \, du.
\]

In view of this observation, the natural step is to show that \( \alpha \tilde{D} (\alpha) \) is suitably close to \( \alpha D (\alpha) \) (in exponential scale) as \( \alpha \searrow 0 \). In other words, we would like to show that the families of random variables \( \{ \alpha \tilde{D} (\alpha) \}_{\alpha > 0} \) and \( \{ \alpha D (\alpha) \}_{\alpha > 0} \) are exponentially equivalent (i.e. that for each \( \delta > 0 \))

\[
\lim_{\alpha \to \infty} \alpha \log P \left( \left| \alpha \tilde{D} (\alpha) - \alpha D (\alpha) \right| > \delta \right) = -\infty,
\]

see Dembo and Zeitouni (1999), p. 130). With exponential equivalence on hand we would be able to conclude, by virtue of Theorem 4.2.13 of Dembo and Zeitouni (1999), that a full LDP also holds for \( \alpha D (\alpha) \) as \( \alpha \searrow 0 \).

We will actually follow this program but we will utilize a different family of approximating processes. The reason is that the integral structure in the definition of \( \tilde{D} (\alpha) = \Psi \left( \tilde{\Gamma}_\alpha^{-1}, \tilde{\Lambda}_\alpha \right) \) allows us to take advantage of the structure of the Lebesgue measure to construct a family of approximating processes \( \{ \tilde{\Lambda}_\alpha \}_{\alpha > 0} \) which is more convenient for purposes of proving the exponential equivalence required. We, thus, define for each \( \alpha > 0 \), the continuous process \( \tilde{\Lambda}_\alpha \) as

\[
\tilde{\Lambda}_\alpha (t) \triangleq \sum_{k=1}^{[t]} X_k + U_\alpha (t),
\]

where

\[
U_\alpha (t) = \left[ \frac{t - ([t] - \alpha)}{\alpha} \right] X_{[t] + 1} (t \in ([t] - \alpha, [t]).)
\]

We now show that \( \left( \alpha \tilde{\Gamma} (\cdot/\alpha), \alpha \tilde{\Lambda}_\alpha (\cdot/\alpha) \right) \) and \( \left( \alpha \tilde{\Gamma} (\cdot/\alpha), \alpha \tilde{\Lambda} (\cdot/\alpha) \right) \) are equivalent from a large deviations standpoint.

**Lemma 12** The families \( \{ \left( \alpha \tilde{\Gamma} (\cdot/\alpha), \alpha \tilde{\Lambda}_\alpha (\cdot/\alpha) \right) \}_{\alpha > 0} \) and \( \{ \left( \alpha \tilde{\Gamma} (\cdot/\alpha), \alpha \tilde{\Lambda} (\cdot/\alpha) \right) \}_{\alpha > 0} \) are exponentially equivalent in \( C[0,\infty) \times C[0,\infty) \) under Stone’s topology.
Proof. It suffices to show the corresponding exponential equivalence for $\{\bar{\Lambda}_\alpha\}_{\alpha>0}$ and $\{\bar{\Lambda}_\alpha\}_{\alpha>0}$. Recall that Stone’s topology is generated by the metric

$$d_\infty (x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k (x, y)}{1 + d_k (x, y)},$$

where

$$d_T (x, y) = \sup_{0 \leq t \leq T} |x(t) - y(t)|$$

(see Zajic (1993) p. 20). Fix $\delta > 0$ small and choose $k_0 > \left\lfloor -\log (\delta/2) / \log (2) \right\rfloor$. Then, $\sum_{k=k_0}^{\infty} 2^{-k} < \delta/2$ and, noting that $d_k (\bar{\Lambda}_\alpha, \bar{\Lambda}_\alpha) \leq \alpha \max_{1 \leq k \leq [t/\alpha]} |X_k|$, we can write

$$P \left( d_\infty (\bar{\Lambda}_\alpha, \bar{\Lambda}_\alpha) > \delta \right) \leq P \left( d_{k_0} (\bar{\Lambda}_\alpha, \bar{\Lambda}_\alpha) > \delta/2 \right) \leq \left\lceil k_0/\alpha \right\rceil \max_{1 \leq k \leq [k_0/\alpha]} P (|X_k| > 2^{-1} \delta/\alpha) \leq \left\lceil k_0/\alpha \right\rceil \exp (-A 2^{-1} \delta/\alpha) \max_{k \in \mathbb{N}} E \left( \exp \left( A 2^{-1} \delta/\alpha |X_k| \right) \right),$$

for every $A > 0$ (by virtue of assumption ADL2). Therefore, we conclude that

$$\lim_{\alpha \to \infty} \alpha \log P \left( d_\infty (\bar{\Lambda}_\alpha, \bar{\Lambda}_\alpha) > \delta \right) = -A 2^{-1} \delta.$$ 

Letting $A \nearrow \infty$ yields the conclusion of the lemma. \[\Box\]

The same strategy followed in the continuous case can now be applied to the pair $(\bar{\Gamma}_\alpha (\cdot), \bar{\Lambda}_\alpha (\cdot))$ as the next proposition summarizes.

**Proposition 13** Under assumptions ADL1 and ADL2, the family of random elements $\bar{\Lambda}_\alpha (\bar{\Gamma}_\alpha^{-1} (\cdot))$ satisfies a full LDP on the space of continuous function $C[0, \infty)$ endowed with Stone’s topology. Moreover, the corresponding normalized infinite horizon discounted reward

$$\bar{\alpha \bar{D}} (\alpha) = \alpha \int_{[0, \infty)} \exp (-u) \bar{\Lambda}_\alpha \left( \bar{\Gamma}_\alpha^{-1} (u) \right) du$$

satisfies a LDP with good rate function

$$I (y) = \inf_{x \in AC_0} \left\{ \int_0^\infty \sup_{\theta} (\theta \dot{x} (s) - \chi (\theta)) ds : y = \int_0^\infty e^{-s} x (s) ds \right\}$$

where $\chi (\cdot)$ is defined via $\psi (-\chi (\cdot), \cdot) = 0$. 

30
Proof. It follows just as Theorem 10. ■

We now are ready to show that \( \alpha \bar{D}(a) \) is suitably close to \( \alpha D(a) \) in exponential scale.

**Lemma 14** The families \( \{\alpha D(\alpha)\}_{\alpha > 0} \) and \( \{\alpha \bar{D}(\alpha)\}_{\alpha > 0} \) are exponentially equivalent. In other words, for each \( \delta > 0 \),
\[
\lim_{\alpha \to 0} \alpha \log P \left( |\alpha D(\alpha) - \alpha \bar{D}(\alpha)| > \delta \right) = -\infty.
\]

**Proof.** Note that \( \Gamma(t) = \Gamma(t/\alpha) \) for almost every \( t \) with respect to Lebesgue measure. Therefore, it follows that, for almost every \( t \),
\[
\alpha \Lambda \left( \Gamma(t/\alpha) \right) - \alpha \bar{\Lambda} \left( \Gamma(t/\alpha) \right) = -\alpha U_\alpha \left( \Gamma(t/\alpha) \right).
\]
As a result, we have (making the change of variables \( \Gamma(t/\alpha) = u/\alpha \)) that
\[
|\alpha D(\alpha) - \alpha \bar{D}(\alpha)| \leq \int_0^\infty W(u/\alpha) du,
\]
where
\[
W(u/\alpha) = \exp \left( -\alpha \Gamma(u/\alpha) \right) |U_\alpha(u/\alpha)| Z([u/\alpha] + 1).
\]
Let us define
\[
V_1 = \alpha \int_0^t W(u/\alpha) du \quad \text{and} \quad V_2 = \alpha \int_t^\infty W(u/\alpha) du,
\]
and consider the sets
\[
A_1(t_0, \alpha, \varepsilon) \triangleq \{ \omega : \sup_{t > t_0} \left| \left( \alpha \bar{\Gamma}(t/\alpha) - \gamma t \right) t^{-1} \right| \leq \varepsilon \},
\]
\[
A_2(t_0, \alpha, \varepsilon) \triangleq \{ \omega : \sup_{t > t_0} \left| \left( \alpha \Gamma(t/\alpha) - \gamma t \right) t^{-1} \right| \leq \varepsilon \},
\]
\[
A_3(t_0, \alpha, m) \triangleq \{ \omega : \sup_{0 \leq t \leq t_0} \left| \alpha \bar{\Gamma}(t/\alpha) - \gamma t \right| \leq m \},
\]
\[
A_4(t_0, \alpha, M) \triangleq \{ \omega : \alpha \sum_{k=1}^{[t_0/\alpha]} |X_k| \leq M \},
\]
\[
A_5(t_0, \alpha, \varepsilon) \triangleq \{ \omega : \sup_{k > t_0/\alpha} \frac{|X_k|}{k} \leq \varepsilon \}.
\]
For notational convenience, we will drop the arguments in the definitions of \( A_j \), \( 1 \leq j \leq 5 \). Using these definitions, we can write
\[
P \left( |\alpha D(\alpha) - \alpha \bar{D}(\alpha)| > \delta \right)
\leq P \left( \alpha \int_0^\infty W(u/\alpha) du > \delta; \cap_k A_k \right) + \sum_{k=1}^5 P(A_k).
\]
Observe that if we write \( K_1 = \exp\left(-m + \gamma\right) \) then, on \( \cap_{k=1}^5 A_k \), we have
\[
V_1 \leq \alpha K_1 \int_0^{t_0} \left|X([u/\alpha] + 1)\right| Z\left([u/\alpha] + 1\right) 1\left(u/\alpha \in [\lfloor u/\alpha \rfloor - \alpha, [u/\alpha])\right) du
\]
\[
\leq \alpha^3 K_1 \sum_{k=1}^{[t_0/\alpha]} X_{k+1} |Z_k| \leq \alpha^3 K_1 \sum_{k=1}^{[t_0/\alpha]} |X_k| \sum_{k=1}^{[t_0/\alpha]} Z_k \leq \alpha^2 K_1 M \sum_{k=1}^{[t_0/\alpha]} Z_k.
\]
On the other hand, also on \( \cap_{k=1}^5 A_k \), and for \( t_0 (\varepsilon, \gamma) \) suitably large, there exists a positive constant \( K(\varepsilon, \gamma) < \infty \) such that
\[
V_2 \leq \alpha K(\varepsilon, \gamma).
\]
Thus, if \( \alpha < \delta / (2K(\varepsilon, \gamma)) \), we have that
\[
P\left(\alpha \int_0^\infty W(u/\alpha) du > \delta; \cap_{k=1}^5 A_k\right) \leq P\left(\alpha \sum_{k=1}^{[t_0/\alpha]} Z_k > \delta / (\alpha 2K_1 M)\right).
\]
But we know that \( \alpha \sum_{k=1}^{[t_0/\alpha]} Z_k \) satisfies an LDP (as \( \alpha \to 0 \)) therefore, we must have that (for fixed \( \varepsilon, \gamma \) and large but fixed \( t_0 \))
\[
\alpha \log P\left(\alpha \int_0^\infty W(u/\alpha) du > \delta\right) \to -\infty \text{ as } \alpha \searrow 0.
\]
Now we analyze each \( P(A_{c_k}^t) \) for each \( 1 \leq k \leq 5 \). First, note that (by Lemma 8), \( t_0 \) can be chosen so that
\[
\lim_{\alpha \to 0} \alpha \log P\left(A_{c_k}^t \left((t_0, \alpha, \varepsilon)\right)\right) \leq -t_0.
\]
Because \( (\alpha \Gamma(\cdot/\alpha) - \gamma)_{\alpha > 0} \) satisfies a full LDP on \( D[0, \infty) \) endowed with the topology generated by the uniform convergence on compact sets (see Theorem 2.2.1 of Zajic (1993)), it follows that the same argument provided for the proof of condition (30), applies in this case as well. This implies that a bound such as (32) also applies for the set \( A_{c_k}^t \). Observe that
\[
\alpha \log P\left(A_{c_k}^t (t_0, \alpha, m)\right) \to -J(m),
\]
for some convex good rate function \( J(\cdot) \) (by definition of full LDP, see Dembo and Zeitouni (1999)). Now, for \( A_4 \), we can use Chebyshev’s bound to obtain
\[
\alpha \log P\left(A_{c_k}^t (t_0, \alpha, M)\right) \leq \alpha \left[\frac{t_0}{\alpha}\right] \log E \exp\left(\sum_{k=1}^{[t_0/\alpha]} |X_k|\right) - M
\]
\[
\leq \alpha g(0, 1) - M \to -M \text{ as } \alpha \searrow 0.
\]
Finally, for $A_5$, we have

$$
P(A^c_5(t_0, \alpha, \varepsilon)) \leq \sum_{k>t_0/\alpha}^{\infty} P\left(\frac{|X_k|}{k} > \varepsilon\right)$$

$$\leq \sum_{k>t_0/\alpha}^{\infty} \exp(-\varepsilon k) \exp(|X_k|)$$

$$\leq \exp(g(0,1)) \exp(-\varepsilon [t_0/\alpha]) \frac{1}{1 - \exp(\varepsilon)}.$$

thus, for $\varepsilon > 0$ small but fixed,

$$\alpha \log P(A^c_5(t_0, \alpha, m)) \rightarrow -\varepsilon t_0.$$

Combining the previous estimates, we conclude that

$$\lim_{\alpha \rightarrow 0} \alpha \log P(\alpha D(\alpha) - \alpha \bar{D}(a) \geq y) \leq -\min((2 + \varepsilon) t_0, M, J(m)).$$

since $J(\cdot)$ is a convex good rate function, the previous quantity in the right hand side tends to infinity as $m, t_0, M \rightarrow \infty$, which yields the proof of the lemma.

We now can identify the rate function required to make practical use of approximation (28) and under which the LDP for $\alpha D(\alpha)$ holds. Define (as in Zajic (1993) p. 9)

$$\psi(\eta, \theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E \exp(\eta \Gamma(n) + \theta \Lambda(n)) < \infty.$$

**Theorem 15** Suppose that ADL1 and ADL2 hold. Then, if $y > \lambda/\gamma$,

$$\lim_{\alpha \rightarrow \infty} \alpha \log P(\alpha D(\alpha) \geq y) = \inf_{x \in AC_0} \left\{ \int_0^\infty \sup_{\theta} (\theta \dot{x}(s) - \chi(\theta)) ds : y = \int_0^\infty e^{-s} x(s) ds \right\},$$

where $AC_0$ is the space of absolutely continuous functions, defined on the interval $[0, \infty)$, that vanish at the origin. In addition, if there exists $\theta^* = \theta^*(y)$ such that $y\theta^* = \chi(\theta^*)$, then

$$\lim_{\alpha \rightarrow \infty} \alpha \log P(\alpha D(\alpha) \geq y) = \sup_{\theta} \left( y\theta - \int_0^\infty \chi(\theta e^{-s}) ds \right)$$

$$= y\theta^* - \int_{\theta^*}^{\infty} \frac{\chi(u)}{u} du.$$

**Proof.** We know that $\{\alpha D(\alpha)\}_{\alpha > 0}$ and $\{\alpha \bar{D}(a)\}_{\alpha > 0}$ are exponentially equivalent. On the other hand, Proposition 13 indicates that $\alpha \bar{D}(a)_{\alpha > 0}$ satisfies a full LDP. Thus, By Theorem 4.2.13, p. 130, of Dembo and Zeitouni (1999) $\{\alpha D(\alpha)\}_{\alpha > 0}$ must also satisfy a full LDP with the same rate function. The identification of the rate function follows as in Theorem 11. ■
References


[30] Duffie...


35


