The Consistency of the Continuum Hypothesis[1]

- *Gödel's Theorems* supply concrete examples of statements that are not provable from standard axioms, namely their Gödel and consistency sentences. Moreover, by the *MRDP Theorem*, if *G* is some Π_1 Gödel sentence of *PA*, for example, then the claim that a corresponding diophantine equation has no solutions in the integers is undecidable in *PA*.

- However, some deny that these count as 'ordinary' mathematical claims of 'intrinsic mathematical interest'. Even if that were so (which is doubtful in the *MRDP* case), we think we *know* – or can find out – their truth-values. The fact that *ZF* neither proves *Con(ZF)* nor *~Con(ZF)* [given that *Con(ZF)*] just shows that *ZF* – and any consistent recursively axiomatizable extension of it – is weak. We still know that *ZF* is consistent!

- There are other claims that are independent of standard mathematics, *ZFC*, whose intrinsic interest is universally conceded but whose truth-values are not known, and not readily knowable, even by everyday, non-philosophical, standards. The *Continuum Hypothesis* (*CH*) is the most celebrated of these. *CH* says that, for any uncountable subset of the real numbers, there is a bijection between it and the set of real numbers.

    - *Note*: $|R| = 2^{\aleph_0}$. Hence, if every set has a well-order, *CH* says that $2^{\aleph_0} = \aleph_1$. The so-called *Generalized Continuum Hypothesis* (*GCH*) then says that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

- The conjecture that every set has a well-order is actually equivalent to the *Axiom of Choice* (*AC*). Recall that *AC* says that if t is a disjointed set not containing the empty set, ∅, then there is a subset of $\cup$t whose intersection with each member of t is a singleton.

- Although *AC* has become a mainstay of mathematics, this should not be confused with consensus among those (few!) who concern themselves with foundational questions. It would, thus, be desirable if *AC* could be proved relatively consistent with *ZF*, i.e., if *Con(ZF)* → *Con(ZFC)*. Even better, we would like to prove that *ZF* |- *AC*. But while we can prove *Con(ZF)* → *Con(ZFC)*, we can also prove that *Con(ZF)* → *Con(ZF + ~AC)*. So, AC is *independent* of ZF [if Con(ZF)] as *CH* is independent of *ZFC* [if *Con(ZF)*].

- *Note*: One upshot of the this is that belief in *ZFC* + *CH* and *ZFC* + *~CH* (or, indeed, *ZF* + *~AC* + *CH* and *ZF* + *~AC* + *CH*) is no more 'risky' than belief in ZF *as far as the danger of believing a contradiction goes*. This contrasts with belief in ZF(C) + Large Cardinals.

- These varied facts have related proofs. So, we focus on the proof that *Con(ZFC)* → *Con(ZFC + CH)* and that *Con(ZFC)* → *Con(ZF + ~CH)*. Let us begin with the first.

The Constructible Universe

- The *Soundness Theorem* says that if *~Con(T)*, then there is no model, *M*, such that *M* |= *φ*, for every *φ* ∈ *T*, or, equivalently, that if ∃*M* : *M* |= *φ*, for each *φ* ∈ *T*, then *Con(T)*.

[1] Thanks to Tim Button for helpful comments.

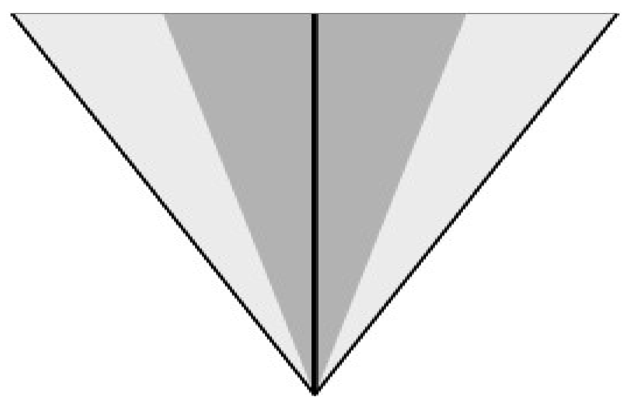
- *Problem*: *Gödel's Second Incompleteness Theorem* guarantees that $ZFC \vdash Con(ZFC)$ only if $\sim Con(ZFC)$. Thus, hopefully (!), $ZFC \nvdash \ulcorner \exists M : M \models \varphi$, *for each* $\varphi \in ZFC + CH\urcorner$.

- What Gödel's theorem *allows* is that, for any *finite subset* of axioms of $ZFC + CH$, $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n\}$, $ZFC \vdash \ulcorner \exists M : M \models \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \urcorner$. (Recall the Reflection Principle.) But this must *not* be confused with the claim that $ZFC \vdash \ulcorner$*For any finite subset of* $ZFC + CH$, $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n,\}$, $\exists M : M \models \tau\urcorner$, which contravenes Gödel's theorem by *Compactness*.

    - Theorem 1: Suppose that $ZFC \subseteq P$ and $ZFC \subseteq Q$ and that for any *finite* $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n\} \subseteq P$, $Q \vdash \ulcorner \exists M : M \models \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n\urcorner$. Then $Con(Q) \rightarrow Con(P)$.

    - *Proof*: Q proves the *Soundness Theorem* for first-order logic. So, for any finite $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n,\} \subseteq P$, $Q \vdash Con(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n,)$ since $Q \vdash \ulcorner \exists M : M \models \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \urcorner$. So, suppose that $\sim Con(P)$. By *Compactness*, there is a finite $\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n,\} \subseteq P$ such that $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \vdash (\varphi \;\&\; \sim\varphi)$. Moreover, as an extension of ZFC, Q is certainly *Σ_1 complete*. Thus, Q *proves* the Σ_1 sentence $\ulcorner \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n, \vdash (\varphi \;\&\; \sim\varphi)\urcorner$. But, again, as Q proves *Soundness*, $Q \vdash \ulcorner \forall M\, [M \models \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \rightarrow M \models (\varphi \;\&\; \sim\varphi)]\urcorner$. Because $Q \vdash \ulcorner \exists M : M \models \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n\urcorner$, $Q \vdash \ulcorner \exists M : M \models (\varphi \;\&\; \sim\varphi)\urcorner$. This contradicts the *ZFC*-provable fact that no model satisfies $\varphi \;\&\; \sim\varphi$.

- What kind of model would make *CH* true? A model that is *thin*. We want to add the minimum number of subsets at each stage of the cumulative construction. But our model will contain all ordinals (see below). So, it will not really be a model, because it will not be a set. We will define a class model, *L*, in analogy with the cumulative hierarchy, *V*.



- A way to execute this idea is to add only subsets that are *definable* at each stage.

- *Problem*: Since *L* is not a (set) model, we cannot define satisfaction in *L*. So, what do we mean in claiming that a sentence is 'true in L'?

- *Answer*: We mean that the sentence *with its quantifiers restricted to L* is provable in *ZFC*.

- Definition 1: If $\mathcal{L}$ is a language and M a model for it, then $S \subseteq M$ (the domain of M) is definable over M just in case there is a formula $\varphi(x_1, x_2, \ldots x_n)$ of $\mathcal{L}$ and $m_1, m_2, \ldots m_n \in M$ such that: $S = \{m \in M : M \models \varphi(m, m_1, \ldots m_n)\}$.

    - *Example*: If $a_1, a_2 \in M$, then $\{a_1, a_2\}$ is definable in M by '$z = x \vee z = y$'.

- Definition 2: If $\mathcal{L}$ is a language and M a model for $\mathcal{L}$ then the definable powerset over M, written $Def(M)$, is $\{X \subseteq M : X$ is definable over $M\}$.
  - *Intuition*: These are the sets that someone 'living in M' could name.
- We can now characterize the class model, L, Gödel's Constructible Universe, as follows.
  - $L_0 = \emptyset$
  - $L_{\alpha+1} = Def(L_\alpha)$
  - $L_u = \cup L_\alpha : \alpha < u$, for limit, u.
  - *Note*: This is a *class function* from $On \rightarrow V$ with the feature that $\alpha < \beta \rightarrow L_\alpha \subset L_\beta$. Gödel also gave a second definition of L that avoids talk of definability (which requires Gödel numbering to formulate in the language of $\in$) in favor of eight 'fundamental operations' under which any (standard – '$\in$' means $\in$) transitive model is closed. Part of the inspiration for L was Russell's ramified type theory.
  - *Notation*: We will write $L(x)$ to mean that $\exists\alpha(\alpha \in On \;\&\; x \in L_\alpha)$.
    - *Transitivity*: Each L_α is a transitive set and L is a transitive class. Recall that M is transitive just in case, when $x \in M$ and $y \in x$, then $y \in M$ (i.e., $x \subseteq M$). If M is not transitive, then it might contain, say, the set of real numbers, without containing any real numbers! From the perspective of such a model M, the set of real numbers would just be the empty set, Ø.
- Every subset of a finite set is definable. So, $L_n = V_n$ for all $n \in \omega$. But $L_{\omega+1} \neq V_{\omega+1}$ since L_ω is countable so $L_{\omega+1}$ is too. (Indeed, L_α is countable whenever α is.) But $V_{\omega+1}$ is not. However, it might be that $V = L$. The L function adds more subsets to L_ω at later stages.

Consistency of *CH*

- Definition 3: If M and N are models in a language, $\mathcal{L}$, such that $M \subseteq N$, then M is an elementary submodel of N, written $M \preccurlyeq N$, just in case, for all $m, m_1,\dots m_n \in M$, $\varphi(x_1, x_2,\dots x_n)$ from $\mathcal{L}$, $M \models \varphi(m, m_1,\dots m_n) \longleftrightarrow N \models \varphi(m, m_1,\dots m_n)$.
    - *Note:* M and N are elementary equivalent if they are the same as regards what one can say in $\mathcal{L}$. But this does *not* mean that M and N are isomorphic. They may fail to be, by the *Löwenheim–Skolem* theorems.
- Definition 4: If ψ is a formula in the language of *ZFC*, and M is a transitive model, then the relativization of ψ to a M, written ψ^M, is defined recursively as follows.
  - If φ is $x \in y$, then φ^M is φ
  - If ψ is $\sim\varphi$, then ψ^M is $\sim(\varphi^M)$.

- If ψ is $(\varphi \ \& \ \chi)$, then ψ^M is $(\varphi^M \ \& \ \chi^M)$
- If ψ is $\forall x\varphi(x)$, then ψ^M is $\forall x[x \in M) \rightarrow \varphi(x)^M]$

  - *Truth and Relativization*: we will write $M \models \varphi$ just in case φ^M.

  - *ZFC Relativized to L*: For every axiom, φ, of *ZFC*, *ZFC* $\vdash \varphi^L$.

  - *Example*: *Pairing* is: $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y) \longleftrightarrow (x=z \vee x=w))$. So, *Pairing*^{L} is: $(\forall z \in L)(\forall w \in L)(\exists y \in L)(\forall x \in L)(x \in y) \longleftrightarrow (x=z \vee x=w))$. Since L is transitive, this means: $(\forall z \in L)(\forall w \in L)(\{z, w\} \in L)$. But $z \in L$ and $w \in L$ just in case $\exists \alpha \in On$, and $\exists \beta \in On$ such that $z \in L_\alpha$ and $w \in L_\beta$. So, without loss of generality, suppose that $\beta \geq \alpha$. Then $\{z, w\} \in L_{\beta+1}$.

    - The most difficult case is *Replacement*. *Choice* is true in L because the well-order of *On* induces a well-order on the L_αs, and each member of a L_α maps to a formula, and these are well-orderable. Moreover, this well-order exists in L.

- Definition 5: A formula $\varphi(x_1, x_2, \ldots x_n, x)$ is absolute for *M* (where *M* is a transitive set or class model) if/f $(\forall x_1, x_2, \ldots x_n) \in M\ [\varphi(x_1, x_2, \ldots x_n, x) \longleftrightarrow \varphi(x_1, x_2, \ldots x_n, x)^M]$.

- *Absolute Formulas*: Important formulas are absolute for *all* transitive models. These include those expressing that x is $\varnothing$, $x \subseteq y$, x is transitive, $x \in On$, x is a limit ordinal, x is a finite ordinal, and x is ω, x is a relation, x is a function, x is the Cartesian product (union, intersection, etc.) of y and z. This is because *these are expressible by Δ_0 formulas*, and Δ_0 *formulas* are absolute. This is provable by induction on complexity.

  - *Note*: The restriction to transitive models is essential. Consider a model containing only Ø and {{Ø}}. Then 'Ø = {{Ø}}' is true in that model!

  - *Note*: Agreement that x is an ordinal is not agreement on *which* ordinal x is!

- *Relative Formulas*. Other key formulas are not absolute (thankfully, if we want to have hope of proving independence results!). These include those expressing that x is a well-order, x is a cardinal, x is countable, y is the powerset of x, and that x is bijective with y.

- Theorem 2 (Mostowski Collapse): If *M* is a model and *the relativization of Extensionality to M, Extensionality^M*, holds, then there is a unique transitive model, *N*, the collapse, and:

  - There is a an *isomorphism*, f: $N \cong M$

  - $f(x) = x$ for any *transitive* set, $x \in M$

- *Rationale*: We can remove whatever gaps there were in *M*. If $x, y \in M$, and $y = \{x, z\}$ but $z \notin M$, then replace y by $\{x\}$ to get a transitive set (which *M* already thinks of as y).

- Robustness of L: If M is a *transitive* model, then for x ∈ M, $(x \in L)^M$ just in case (x ∈ L). Although the formula expressing that x ∈ L_o is not absolute for all transitive models, it is absolute for L. So, $(V{=}L)^L$ is: $(\forall x \in L)((\exists \alpha \in L)[\alpha \in On)^L \ \& \ (x \in L_o)^L)$, which is trivial. L is the *smallest* inner model, i.e., transitive model containing *On* satisfying *ZFC*.

- Theorem 3 (Condensation): If $M \preccurlyeq L_\alpha$, for some limit α, and N is the collapse of M, then $N \cong L_\beta$, for some $\beta \leq \alpha$.

- *Rationale*: Since all transitive models agree on what is in L, anything true of elements of an elementary submodel, M, of some L_o must be true of L_o as well. Moreover, M can always be collapsed to intro a transitive, N, by Mostowski Collapse, and this will believe it is some initial segment of L_o, L. Thus, by the Robustness of L, it must actually be one.

- Theorem 4: *CH* is true in L (i.e., *ZFC* |- $(CH)^L$).

- *Proof*: Since $|L_{\aleph_1}| = \aleph_1$, it suffices to show that $P(\omega) \subseteq L_{\aleph_1}$, i.e., that if $x \subseteq \omega$, then $x \in L_\alpha$, for some $\alpha < \aleph_1$. So, let $x \in M \preccurlyeq L_\beta$, where M may have cardinality $\aleph_0$, by the (downward) *Löwenheim–Skolem theorem*. Let N be the transitive collapse of M. Then x, being transitive, collapses to itself, and, by *Condensation*, $N = L_\alpha$, for $\alpha < \aleph_1$, as $|L_\alpha| = |\alpha|$.

  - *Note*: The levels at which new subsets of ω emerge turns out to be bounded by $\aleph_1$.

- *Upshot*: *Con(ZFC) → Con(ZFC + CH)*. Since '*V=L*' can be expressed in the language of set theory, we can even show: *Con(ZFC) → Con(ZFC + V = L)*. So, the hypothesis that the universe of sets *actually is* Gödel's Constructible Universe is consistent if *ZFC* is.

Assessment of *V=L*

- The *Axiom of Constructibility* is the statement that *V=L*. By the above argument, it is consistent with *ZF*, if *ZF* is consistent. Moreover, it turns out to have sweeping consequences for set theory, settling all manner of important, but independent, questions. Indeed, setting aside 'small' large cardinal axioms, claims equivalent to those asserting the existence of a model or a transitive model of theories, it is 'empirically complete'.[2] (Finding additional statements independent of *ZF* + *V=L* would require finding a new method of independence proof besides the one to which we turn presently, *forcing*.)

- *Examples: V=L* implies not only *CH*, but *GCH*, which actually implies *AC* (we already saw that *AC* was true in *L*), the *Diamond Principle*, the negation of the *Souslin Hypothesis*, the non-existence of $0^\#$, the non-existence of a Measurable Cardinal, Whitehead's conjecture, and *V = HOD* (or that there is a definable well-ordering on *V*).

  - *Jensen*: "I personally find [*V=L*] a very attractive axiom." [1995, 398].

[2] But see Friedman [1981].

- *Devlin*: "[*V=L*] is...a natural axiom, closely bound up with what we mean by "set"....[and] tends to decide problems in the 'correct' direction [1977, 4]."

- *Eskew*: "The axiom *V=L*...settles 'nearly all' mathematical questions....[I]t can be motivated by constructivist views that are still widely held today....[A] wealth of powerful combinatorial principles...follow from...*V=L* .....[So] why hasn't there been...a stronger push to adopt it as a[n]...axiom for mathematics [2019]?"

- Unfortunately, most set theorists do not find *V=L*'s consequences to be attractive!

  - *Friedman*: "[S]et theorists say that *V = L* has implausible consequences… [They] claim to have a direct intuition which allows them to view these as so implausible that this provides 'evidence' against *V = L*. However, mathematicians [like me] disclaim such direct intuition about complicated sets of reals. Many say they have no direct intuition about all multivariate functions from N into N [2000]!"

  - *Woodin*: "Godel's Axiom of Constructibility, *V = L*, provides a conception of the Universe of Sets which is perfectly concise modulo only large cardinal axioms which are strong axioms of infinity. However the axiom *V = L* limits the large cardinal axioms which can hold and so the axiom is *false* [2010, 1, emphasis in original]."

- Any debate over *V=L* would be moot if it were not just consistent with *ZF(C)*, but a theorem. But, as we show presently, that is not the case either, thanks to Cohen [1966].

References
Cohen, Paul. [1966] *Set Theory and the Continuum Hypothesis*. W. A. Benjamin.
Devlin, Keith. [1977] The Axiom of Constructability: A Guide for the Mathematician (Lecture Notes on Mathematics: 617). New York: Springer-Verlag.
Eskew, Monroe. [2019] "Re: Why Not Adopt the Constructability Axiom?" May 20.
https://mathoverflow.net/questions/331956/why-not-adopt-the-constructibility-axiom-v-l
Gödel, Kurt. [1938a] "The consistency of the axiom of choice and of the generalized continuum-hypothesis," Proceedings of the U.S. National Academy of Sciences, 24: 556–7.
Gödel, K. [1938b] "Consistency-proof for the generalized continuum-hypothesis," Proceedings of the U.S. National Academy of Sciences, 25: 220–4.
Friedman, Harvey. [1981] "On the Necessary use of Abstract Set Theory." Advances in Mathematics. Vol. 41. 209-280
—--. [2000] *Philosophical Problems in Logic*.
Jensen, Ronald. [1995] "Inner Models and Large Cardinals." Bulletin of Symbolic Logic. Vol. 1. 393 -- 407.
Meadows, Toby. "Set Theory: Some Basics and a Glimpse of Advanced Techniques." Lecture Notes.
Wolf, Robert. [2005] *A Tour through Mathematical Logic*. The Mathematical Association of America.

Woodin, Hugh. [2010] “Strong Axioms of Infinity and the Search for V.” in Bhatia, Rajendra (Ed.), Proceedings of the International Congressof Mathematicians, Hyderabad, August 19-27. Vol. 1. World Scientific.Available online at: http://logic.harvard.edu/EFI_Woodin_StrongAxiomsOfInfinity.pdf