The Consistency of the Continuum Hypothesis

- Gödel’s Theorems supply concrete examples of statements that are not provable from standard axioms, namely their Gödel and consistency sentences. Moreover, by the MRDP Theorem, if \( G \) is some \( \Pi^0_1 \) Gödel sentence of \( PA \), for example, then the claim that a corresponding diophantine equation has no solutions in the integers is undecidable in \( PA \).

- However, some deny that these count as ‘ordinary’ mathematical claims of ‘intrinsic mathematical interest’. Even if that were so (which is doubtful in the MRDP case), we think we know – or can find out – their truth-values. The fact that \( ZF \) neither proves \( Con(ZF) \) nor \( \neg Con(ZF) \) [given that \( Con(ZF) \)] just shows that \( ZF \) – and any consistent recursively axiomatizable extension of it – is weak. We still know that \( ZF \) is consistent!

- There are other claims that are independent of standard mathematics, \( ZFC \), whose intrinsic interest is universally conceded but whose truth-values are not known, and not readily knowable, even by everyday, non-philosophical, standards. The Continuum Hypothesis (\( CH \)) is the most celebrated of these. \( CH \) says that, for any uncountable subset of the real numbers, there is a bijection between it and the set of real numbers.

  - Note: \(|\mathbb{R}| = 2^\aleph_0\). Hence, if every set has a well-order, \( CH \) says that \( 2^\aleph_0 = \aleph_1 \). The so-called Generalized Continuum Hypothesis (\( GCH \)) then says that \( 2^\aleph_\alpha = \aleph_{\alpha+1} \).

  - The conjecture that every set has a well-order is actually equivalent to the Axiom of Choice (\( AC \)). Recall that \( AC \) says that if \( t \) is a disjointed set not containing the empty set, \( \emptyset \), then there is a subset of \( \bigcup t \) whose intersection with each member of \( t \) is a singleton.

  - Although \( AC \) has become a mainstay of mathematics, this should not be confused with consensus among those (few!) who concern themselves with foundational questions. It would, thus, be desirable if \( AC \) could be proved relatively consistent with \( ZF \), i.e., if \( Con(ZF) \rightarrow Con(ZFC) \). Even better, we would like to prove that \( ZF \models \neg AC \). But while we can prove \( Con(ZF) \rightarrow Con(ZFC) \), we can also prove that \( Con(ZF) \rightarrow Con(ZF + \neg AC) \). So, \( AC \) is independent of \( ZF \) [if \( Con(ZF) \)] as \( CH \) is independent of \( ZFC \) [if \( Con(ZF) \)].

  - Note: One upshot of the this is that belief in \( ZFC + CH \) and \( ZFC + \neg CH \) (or, indeed, \( ZF + \neg AC + CH \) and \( ZF + \neg AC + CH \)) is no more ‘risky’ than belief in \( ZF \) as far as the danger of believing a contradiction goes. This contrasts with belief in \( ZF(C) + \text{Large Cardinals} \).

- These varied facts have related proofs. So, we focus on the proof that \( Con(ZFC) \rightarrow Con(ZFC + CH) \) and that \( Con(ZFC) \rightarrow Con(ZF + \neg CH) \). Let us begin with the first.

The Constructible Universe

- The Soundness Theorem says that if \( \neg Con(T) \), then there is no model, \( M \), such that \( M \models \varphi \), for every \( \varphi \in T \), or, equivalently, that if \( \exists M : M \models \varphi \), for each \( \varphi \in T \), then \( Con(T) \).

1 Thanks to Tim Button for helpful comments.
• **Problem:** Gödel’s Second Incompleteness Theorem guarantees that ZFC $\vdash \text{Con}(\text{ZFC})$ only if $\neg\text{Con}(\text{ZFC})$. Thus, hopefully (!), ZFC $\vdash \exists M : M \models \phi$, for each $\phi \in \text{ZFC} + \text{CH}$.  

• What Gödel’s theorem allows is that, for any finite subset of axioms of ZFC + CH, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, ZFC $\vdash \exists M : M \models \alpha_1, \alpha_2, \ldots, \alpha_n$. (Recall the Reflection Principle.) But this must not be confused with the claim that ZFC $\vdash \exists M : M \models \phi$, which contravenes Gödel’s theorem by Compactness.

  - **Theorem 1:** Suppose that ZFC $\subseteq P$ and ZFC $\subseteq Q$ and that for any finite $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq P, Q \vdash \exists M : M \models \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\text{Con}(Q) \rightarrow \text{Con}(P)$.

  - **Proof:** Q proves the Soundness Theorem for first-order logic. So, for any finite $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq P, Q \vdash \text{Con}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ since $Q \vdash \exists M : M \models \alpha_1, \alpha_2, \ldots, \alpha_n$. So, suppose that $\neg\text{Con}(P)$. By Compactness, there is a finite $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq P$ such that $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash (\phi \land \neg\phi)$. Moreover, as an extension of ZFC, Q is certainly $\Sigma_i$-complete. Thus, Q proves the $\Sigma_i$ sentence $\exists M : M \models (\phi \land \neg\phi)$. But, again, as Q proves Soundness, $Q \vdash \forall M [M \models \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \rightarrow M \models (\phi \land \neg\phi)]$. Because $Q \vdash \exists M : M \models (\phi \land \neg\phi)$, Q $\vdash \exists M : M \models (\phi \land \neg\phi)$. This contradicts the ZFC-provable fact that no model satisfies $\phi \land \neg\phi$.

• What kind of model would make CH true? A model that is thin. We want to add the minimum number of subsets at each stage of the cumulative construction. But our model will contain all ordinals (see below). So, it will not really be a model, because it will not be a set. We will define a class model, L, in analogy with the cumulative hierarchy, $V$.

• A way to execute this idea is to add only subsets that are definable at each stage.

• **Problem:** Since L is not a (set) model, we cannot define satisfaction in L. So, what do we mean in claiming that a sentence is ‘true in L’?

• **Answer:** We mean that the sentence with its quantifiers restricted to L is provable in ZFC.

• **Definition 1:** If $\mathcal{L}$ is a language and $M$ a model for it, then $S \subseteq M$ (the domain of $M$) is definable over $M$ just in case there is a formula $\phi(x_1, x_2, \ldots, x_n)$ of $\mathcal{L}$ and $m_1, m_2, \ldots, m_n \in M$ such that: $S = \{m \in M : M \models \phi(m, m_1, \ldots, m_n)\}$.

  - **Example:** If $a_1, a_2 \in M$, then $\{a_1, a_2\}$ is definable in $M$ by ‘$z = x \lor z = y$’.  

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• **Definition 2:** If \( \mathcal{L} \) is a language and \( M \) a model for \( \mathcal{L} \), then the definable powerset over \( M \), written \( \text{Def}(M) \), is \( \{ X \subseteq M : X \text{ is definable over } M \} \).

  ○ **Intuition:** These are the sets that someone ‘living in \( M \)’ could name.

• We can now characterize the class model, \( L \), Gödel’s **Constructible Universe**, as follows.

  ○ \( L_0 = \emptyset \)
  ○ \( L_{\alpha+1} = \text{Def}(L_\alpha) \)
  ○ \( L_\alpha = \bigcup L_\beta : \alpha < \beta \), for limit, \( \beta \).

  ○ **Note:** This is a class function from \( \text{On} \to V \) with the feature that \( \alpha < \beta \rightarrow L_\alpha \subset L_\beta \). Gödel also gave a second definition of \( L \) that avoids talk of definability (which requires Gödel numbering to formulate in the language of \( \in \)) in favor of eight ‘fundamental operations’ under which any (standard – ‘\( \in \)’ means \( \in \)) transitive model is closed. Part of the inspiration for \( L \) was **Russell’s ramified type theory**.

  ○ **Notation:** We will write \( L(x) \) to mean that \( \exists \alpha (\alpha \in \text{On} & x \in L_\alpha) \).

  ● **Transitivity:** Each \( L_\alpha \) is a transitive set and \( L \) is a transitive class. Recall that \( M \) is transitive just in case, when \( x \in M \) and \( y \in x \), then \( y \in M \) (i.e., \( x \subseteq M \)) . If \( M \) is not transitive, then it might contain, say, the set of real numbers, without containing any real numbers! From the perspective of such a model \( M \), the set of real numbers would just be the empty set, \( \emptyset \).

  ● Every subset of a finite set is definable. So, \( L_n = V_n \) for all \( n \in \omega \). But \( L_{\omega+1} \neq V_{\omega+1} \) since \( L_\omega \) is countable so \( L_{\omega+1} \) is too. (Indeed, \( L_\alpha \) is countable whenever \( \alpha \) is.) But \( V_{\omega+1} \) is not. However, it might be that \( V = L \). The \( L \) function adds more subsets to \( L_\omega \) at later stages.

**Consistency of CH**

• **Definition 3:** If \( M \) and \( N \) are models in a language, \( \mathcal{L} \), such that \( M \subseteq N \), then \( M \) is an **elementary submodel** of \( N \), written \( M \subseteq N \), just in case, for all \( m, m_1, \ldots, m_n \in M \), \( \varphi(x_1, x_2, \ldots, x_n) \) from \( \mathcal{L} \), \( M \models \varphi(m, m_1, \ldots, m_n) \iff N \models \varphi(m, m_1, \ldots, m_n) \).

  ■ **Note:** \( M \) and \( N \) are **elementary equivalent** if they are the same as regards what one can say in \( \mathcal{L} \). But this does not mean that \( M \) and \( N \) are isomorphic. They may fail to be, by the **Löwenheim–Skolem** theorems.

• **Definition 4:** If \( \psi \) is a formula in the language of \( ZFC \), and \( M \) is a transitive model, then the relativization of \( \psi \) to a \( M \), written \( \psi^M \), is defined recursively as follows.

  ○ If \( \varphi \) is \( x \in y \), then \( \varphi^M \) is \( \varphi \)
  ○ If \( \psi \) is \( \neg \varphi \), then \( \psi^M \) is \( \neg (\varphi^M) \).
If $\psi$ is ($\phi & \chi$), then $\psi^M$ is ($\phi^M & \chi^M$)

If $\psi$ is $\forall x\phi(x)$, then $\psi^M$ is $\forall x[x \in M] \rightarrow \phi(x)^M$

- **Truth and Relativization**: we will write $M \models \phi$ just in case $\phi^M$.

- **ZFC Relativized to L**: For every axiom, $\phi$, of ZFC, $\text{ZFC} \vdash \phi^L$.

- **Example**: Pairing is: $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y) \iff (x=z \lor x=w))$.

  So, Pairing$^L$ is: $(\forall z \in L)(\forall w \in L)(\exists y \in L)(\forall x \in L)(x \in y) \iff (x=z \lor x=w))$. Since $L$ is transitive, this means: $(\forall z \in L)(\forall w \in L)(\exists \alpha \in On)$ and $(\exists \beta \in On)$ such that $z \in L^\alpha$ and $w \in L^\beta$. So, without loss of generality, suppose that $\beta \geq \alpha$. Then $\{z, w\} \in L^{\beta+1}$.

- The most difficult case is Replacement. Choice is true in $L$ because the well-order of $On$ induces a well-order on the $L^\alpha$s, and each member of a $L^\alpha$ maps to a formula, and these are well-orderable. Moreover, this well-order exists in $L$.

- **Definition 5**: A formula $\phi(x_1, x_2, \ldots, x_n, x)$ is absolute for $M$ (where $M$ is a transitive set or class model) if $\forall f(\forall x_1, x_2, \ldots, x_n) \in M [\phi(x_1, x_2, \ldots, x_n, x) \iff \phi^M(x_1, x_2, \ldots, x_n, x)]$.

- **Absolute Formulas**: Important formulas are absolute for all transitive models. These include those expressing that $x$ is $\emptyset$, $x \subseteq y$, $x$ is transitive, $x \in On$, $x$ is a limit ordinal, $x$ is a finite ordinal, $x$ is a relation, $x$ is a function, $x$ is the Cartesian product (union, intersection, etc.) of $y$ and $z$. This is because these are expressible by $\Delta_0$ formulas, and $\Delta_0$ formulas are absolute. This is provable by induction on complexity.

  - Note: The restriction to transitive models is essential. Consider a model containing only $\emptyset$ and $\{\{\emptyset\}\}$. Then ‘$\emptyset = \{\{\emptyset\}\}$’ is true in that model!

  - Note: Agreement that $x$ is an ordinal is not agreement on which ordinal $x$ is!

- **Relative Formulas**. Other key formulas are not absolute (thankfully, if we want to have hope of proving independence results!). These include those expressing that $x$ is a well-order, $x$ is a cardinal, $x$ is countable, $y$ is the powerset of $x$, and that $x$ is bijective with $y$.

- **Theorem 2 (Mostowski Collapse)**: If $M$ is a model and the relativization of Extensionality to $M$, $\text{Extensionality}^M$, holds, then there is a unique transitive model, $N$, the collapse, and:

  - There is an isomorphism, $f$: $N \cong M$

  - $f(x) = x$ for any transitive set, $x \in M$

- **Rationale**: We can remove whatever gaps there were in $M$. If $x, y \in M$, and $y = \{x, z\}$ but $z \notin M$, then replace $y$ by $\{x\}$ to get a transitive set (which $M$ already thinks of as $y$).
- **Robustness of $L$:** If $M$ is a transitive model, then for $x \in M$, $(x \in L)^M$ just in case $(x \in L)$. Although the formula expressing that $x \in L_o$ is not absolute for all transitive models, it is absolute for $L$. So, $(V=L)^L$ is: $(\forall x \in L)((\exists \alpha \in L)[\alpha \in \text{On}^L] \& (x \in L_o)^L)$, which is trivial. $L$ is the smallest inner model, i.e., transitive model containing $\text{On}$ satisfying $ZFC$.

- **Theorem 3 (Condensation):** If $M \preceq L_\alpha$, for some limit $\alpha$, and $N$ is the collapse of $M$, then $N \cong L_\beta$, for some $\beta \leq \alpha$.

- **Rationale:** Since all transitive models agree on what is in $L$, anything true of elements of an elementary submodel, $M$, of some $L_o$ must be true of $L_o$ as well. Moreover, $M$ can always be collapsed to intro a transitive, $N$, by Mostowski Collapse, and this will believe it is some initial segment of $L_0$, $L$. Thus, by the Robustness of $L$, it must actually be one.

- **Theorem 4:** CH is true in $L$ (i.e., $ZFC \vdash (CH)^L$).

  - **Proof:** Since $|L_{\aleph_1}| = \aleph_1$, it suffices to show that $P(\omega) \subseteq L_{\aleph_1}$, i.e., that if $x \subseteq \omega$, then $x \in L_\alpha$, for some $\alpha < \aleph_1$. So, let $x \in M \preceq L_\beta$, where $M$ may have cardinality $\aleph_0$, by the (downward) Löwenheim–Skolem theorem. Let $N$ be the transitive collapse of $M$. Then $x$, being transitive, collapses to itself, and, by Condensation, $N = L_\alpha$, for $\alpha < \aleph_1$, as $|L_\alpha| = |\alpha|$.

  - **Note:** The levels at which new subsets of $\omega$ emerge turns out to be bounded by $\aleph_1$.

- **Upshot:** $\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + CH)$. Since ‘$V=L$’ can be expressed in the language of set theory, we can even show: $\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + V = L)$. So, the hypothesis that the universe of sets actually is Gödel’s Constructible Universe is consistent if $ZFC$ is.

**Assessment of $V=L$**

- The **Axiom of Constructibility** is the statement that $V=L$. By the above argument, it is consistent with $ZF$, if $ZF$ is consistent. Moreover, it turns out to have sweeping consequences for set theory, settling all manner of important, but independent, questions. Indeed, setting aside ‘small’ large cardinal axioms, claims equivalent to those asserting the existence of a model or a transitive model of theories, it is ‘empirically complete’.\(^2\) (Finding additional statements independent of $ZF + V=L$ would require finding a new method of independence proof besides the one to which we turn presently, forcing.)

- **Examples:** $V=L$ implies not only $CH$, but $GCH$, which actually implies AC (we already saw that AC was true in $L$), the **Diamond Principle**, the negation of the **Souslin Hypothesis**, the non-existence of $0^\sharp$, the non-existence of a **Measurable Cardinal**, **Whitehead’s conjecture**, and $V = \text{HOD}$ (or that there is a definable well-ordering on $V$).

  - **Jensen:** “I personally find $[V=L]$ a very attractive axiom.” [1995, 398].

\(^2\) But see Friedman [1981].
Devlin: “[V=L] is...a natural axiom, closely bound up with what we mean by "set"...[and] tends to decide problems in the ‘correct’ direction [1977, 4].”

Eskew: “The axiom V=L...settles ‘nearly all’ mathematical questions...[I]t can be motivated by constructivist views that are still widely held today...[A] wealth of powerful combinatorial principles...follow from...V=L .....[So] why hasn’t there been...a stronger push to adopt it as a[n]...axiom for mathematics [2019]?”

Unfortunately, most set theorists do not find V=L’s consequences to be attractive!

Friedman: “[S]et theorists say that V = L has implausible consequences... [T]hey claim to have a direct intuition which allows them to view these as so implausible that this provides ‘evidence’ against V = L. However, mathematicians [like me] disclaim such direct intuition about complicated sets of reals. Many say they have no direct intuition about all multivariate functions from N into N [2000]!”

Woodin: “Godel’s Axiom of Constructibility, V = L, provides a conception of the Universe of Sets which is perfectly concise modulo only large cardinal axioms which are strong axioms of infinity. However the axiom V = L limits the large cardinal axioms which can hold and so the axiom is false [2010, 1, emphasis in original].”

Any debate over V=L would be moot if it were not just consistent with ZF(C), but a theorem. But, as we show presently, that is not the case either, thanks to Cohen [1966].

References