"Zeno's arguments, in some form, have afforded grounds for almost all theories of space and time and infinity which have been constructed from his time to our own" -- Bertrand Russell

Zeno's Paradoxes of Motion

(1) Achilles and the Tortoise

- If Achilles, a stealth runner, races a dawdling tortoise, he could never win so long as the tortoise were granted some finite head start. In the finite time it takes for Achilles to reach the tortoise's starting point, the tortoise will have moved some finite distance, d_1; in the finite time it takes Achilles to reach the new point, d_1, the tortoise will have moved another finite distance, d_2; and so on forever. Achilles can never catch up to the tortoise.

(2) The Dichotomy (Progressive Form)

- Achilles cannot even make it to the tortoise's departure point, d_0. To do so, he would first need to make it to $(\frac{1}{2})d_0$, and then to $(\frac{1}{2} + \frac{1}{4})d_0$, and then to $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8})d_0$, and, in general, (1/(2^n)) as n runs through all N (this is really just (1) without the Tortoise).
- *Note*: The *Regressive* form, to which we will return, is in some ways more worrisome. To make it to d_0, Achilles would *first* need to make it to $(\frac{1}{2})d_0$. And to make it *there*, Achilles would first need to make it to $(\frac{1}{4})d_0$. And so on forever. But this sequence does not even have a *first member*! So, it would appear, Achilles could never even *get started*.

(3) The Arrow

- An arrow shot from a bow never actually moves. For, at any given instant, it is stationary, occupying an arrow-sized region of space. Nothing can move *at* an instant. Since the arrow's flight is given by a sequence of instants, the arrow never moves.

19th Century Calculus

- 19th century calculus seems *prima facie* to resolve the paradoxes. (The modifier "19th century" is essential. Prior to the work of Cauchy and others calculus lacked a coherent foundation, as witnessed by Bishop Berkeley's 1734 critique of the subject.)

- Paradoxes (1) and (2) are supposed to be premised on the following intuitively compelling assumption: *an infinite sum of finite quantities cannot itself be finite*.
- *Note*: It does not help much to complain, as Aristotle did, that the Achilles traverses the infinite sum in a correspondingly infinite sum of finite temporal quantities. That just multiplies the mysteries. How could an infinite sum of *those* add up to less than infinity?
- It was not until the 19th century that the notion of an infinite sum was even rigorously *defined*. The basic idea is to say that the infinite *series*, $s_1 + s_2 + s_3 \ldots = L$ when it can be approximated arbitrarily closely to L by adding a sufficient (but finite) number of terms.
    - *Details*: An infinite *sequence* of numbers $\{S_k\}$ *converges* to, or has a *limit* of, real number, L, when, for any real $\varepsilon > 0$, there is a natural number, δ, such that for any term S_n, where $n > \delta$, $|S_n - L| < \varepsilon$. (That is, we can make the difference between L and terms following S_δ as small as we like by choosing a large δ.)
        - *Examples*: {1/(2^n)} converges to 0, while {1 - 1/(2^n)} converges to 1. However, {n} fails to converge, since "infinity" is not a real number.
    - An infinite *series*, $\Sigma(S_n)_{n\to\infty} = s_1 + s_2 + s_3 \ldots$ can now be said to *equal* L when the infinite *sequence* of partial sums, $\{S_k\}$, where $S_k = s_1 + s_2 + s_3 \ldots s_k$, converges to L.
        - *Example*: Σ(1/(2^n)) = 1.
- *Note*: Infinite sums are not just like finite ones! The order of a finite sum is irrelevant. But an infinite series can have different limits depending on the order in which it is given.
- With this definition of infinite sums in hand, it might be thought that we can see how (1) and (2) can be resolved. We just saw that Σ(1/(2^n)) can be defined to be 1. But we also saw that the problem was precisely how Achille could traverse (½ + ¼ + ⅛ +...) = Σ(1/(2^n)). The answer is that he could because the series Σ(1/(2^n)) converges.
- Calculus also lets us see how to make sense of the notion of *instantaneous velocity*, contra paradox (3). We can hold that the arrow has velocity, v, at time, t, when *the limit of the average velocities (dx/dt) is v* (i.e., we take the limit of the fraction *dx/dt* as $dt \to 0$).
    - *Note*: On this view, motion *just is* a functional relationship between position and time coordinates. To say that the arrow *moved* from point x_1 to point x_2 during time $(t_2 - t_1)$ is to say that *at* time, t_1, the arrow *is* at x_1, and *at* time t_2, the arrow *is* at x_2. If we ask *how it got* from x_1 to x_2, the answer is just that it (tenselessly) *occupies* intervening position coordinates at intervening time coordinates!

- *Upshot*: Being in motion and being at rest *makes no sense as applied to a single point*.

Infinite Processes

- Calculus provides a consistent (we think!) account of motion. *But it is up to its ears in infinity* (more on this next time). Is the notion of an *infinite process* really coherent?
- *Note*: So far, we have not (on its face) "discovered" anything about infinite sums! We have simply *stipulated* an apparently coherent definition for (countably) infinite sums.
- *Thompson's Lamp*: At least in Zeno's cases, the series converge. Consider a case where even this is not so. Consider a lamp which obeys the following instruction. At t_0 the lamp is off; at $t_1 = 1$ switch the lamp on; at $t_2 = 1 + ½$ switch it off; at $t_3 = 1 + ¾$ switch it on, and so on. These seem to be coherent instructions. But at t = 2 is the lamp on or off? (For every moment it was on, there was a later moment when it was off, and vice versa.)
- *Problem*: If switching the light involved a *fixed* finite movement, then we would require that an infinite distance be traversed in a finite time, whereas in the case of (1) and (2) we merely require that a finite distance -- which can be divided into an infinite number of pieces -- can be. But if we modified the switch ('in principle') so that this was not so, the location of the switch, and hence the state of the light, would seem to approach a limit!
- *Note*: Even allowing for infinite processes that fail to approach limits, it is not right to say that there is a *contradiction* in the assumption that at t = 2, the light is on, say. The function is not defined at t = 2, so we are free -- logically -- to stipulate a value as we please. There is something intuitively, and physically, *unnatural* about the state of the light floating free from its state at times arbitrarily close in the past. But that is different.

What is the Question?

- We have been speaking as though the question of whether an infinite process 'makes sense' was clear. But there are actually multiple questions that we could be trying to ask.
    a. Are infinite processes (first-order) consistent (logically possible)?
        - *Ray*: "If we set aside the question of whether or not it is physically possible for an infinite number of such tasks to be performed in a finite time, we may still ask...whether or not it is logically possible (14)."
    b. Are infinite processes conceptually consistent (conceptually possible)?

c. Are infinite processes metaphysically possible?
d. Are infinite processes physically possible?
   - *Salmon*: "[N]o definition...can provide the answer to a *physical* question ….The inescapable consequence of [Zeno's Paradoxes] would seem to be that mathematical physics needs a radically different mathematical foundation if it is to deal adequately with physical reality (45)."
e. Are infinite processes physically *natural*?

- If ordinary mathematics is (first-order) consistent, then the answer to (a) is clearly 'yes'. Our thought experiments most directly seem to address (b). But whether (b) makes sense depends on whether the notion of conceptual truth (analyticity) does, which is contested.
- Moreover, who cares what is conceptually possible, unless conceptual possibility is at least a defeasible guide to metaphysical possibility (another controversial matter)?
- Actually, who cares unless conceptual consistency is a guide to what is allowed given *the actual physical laws*? Even if conceptual consistency is a guide to metaphysical possibility, we only get from the latter to *physical possibility* if (1) the scenarios in question are metaphysically possible, given the actual physical laws and (2) the way to understand the claim that P is physically possible, <N>P, is as the claim that <M>(T & P), where <M> is metaphysical possibility and T is a (perhaps infinite) conjunction of statements of the physical laws (contra, e.g., Fine, "The Varieties of Necessity").
- *Salmon*: "The speed of switching demanded is, of course, beyond human capability, but we are concerned with logical possibilities, not 'medical' limitations" (44)."
- *Actually*, who cares unless conceptual consistency (given the laws of physics) is a guide to *natural physical scenarios*? Malament-Hogarth spacetimes are models of General Relativity (so physically possible in at least one ordinary sense, assuming the physical possibility of General Relativity itself), but generally thought to be 'unreasonable'.

- *Upshot*: A deflationary spin on our question would be that it is trivial, if not just verbal. There is *some* sense of 'possible' such that infinite processes -- 'supertasks' -- are possible, and *some* sense such that they are not. What is left to dispute, if not just how select academics use the word 'possible'?