Cohen Forcing ~CH[1]

- We have claimed that ~CH is consistent with ZF, if ZF is consistent. It might be thought that we could prove this in the way that we proved that $Con(ZF) \rightarrow Con(ZF + V{=}L)$.
    - However, this is not the case, due to a result of Shepherdson. Suppose there is a model M which is provably a transitive class satisfying $ZFC + V{\neq}L$. By the properties of L (proved by Shepherdson), $ZFC + V = L \vdash M = L$. So, $ZFC + V= L$ proves $M \neq L$ and $M = L$, contradicting Gödel's relative consistency result.

- A more promising method is to begin with the countable transitive set – not class! – model, M, and adjoin to it a so-called 'generic' set, G. G is constructed from a partial order, P, which is an element of M. P is chosen so as to consist of pieces of the object that want to supplement M with. (G will be a '<u>M-generic filter</u>' for P, and $\cup G$ will be a union of pieces. We will assume that a least element $\emptyset_P$ is always a member of P, and, consequently, G.) While $G \subseteq P$ and $P \in$ M, we will have $G \notin$ M. P is called a <u>notion of forcing</u>, and the result of the construction is known as a <u>generic extension</u>, $M[G]$. $M[G]$ is the <u>smallest</u> superset of M containing G and all else such that $M[G] \models ZF(C) +$ ~CH.
    - *Example*: If P consists of all the finite partial functions from $\aleph_2 \times \omega \rightarrow \{0,1\}$ then G codes a set of subsets of ω indexed by $\aleph_2$. This adds at least $\aleph_2$ reals to $M[G]$.

- *Note*: The method of forcing cannot be used to prove sentences of arithmetic independent (as currently understood, at least). Forcing works only for statements about infinite sets.

<u>Overview</u>

- We want to add subsets, a_η, of ω to our initial model, M, called the <u>ground model</u>. (Of course, once we add a_η we need to add $<a_\eta, a_\eta>$, $a_\eta \times a_\eta$, and much more.) Since $P(\omega) > \omega$, we have many sets to choose from (living in V). This is one reason why a countable transitive model (*ctm*) recommends itself in the case of ~CH. However, it should not be thought that one can only force over countable sets. There is also (proper) <u>class forcing</u>.
    - One subtlety is that we want to <u>maintain M's ordinals</u>. Maybe M does not contain a_η only because M does not contain $M[G]$'s ordinals. If so, a_η may be constructible *in* $M[G]$. Another is that we do not want to <u>collapse any cardinals</u> in $M[G]$ by adding new bijections. If we collapsed $\aleph_1$ into $\aleph_0$, then $\aleph_2$ would become $\aleph_1$ in $M[G]$! It is the fact that G is 'generic' that will let us avoid these problems.

- If we begin with a (standard) *ctm*, $M = <$M, $\in>$, then M's ordinals really are ordinals (in V). They are all the ordinals up to some α. If we assume that $M \models ZF(C) + CH$ (which we know is consistent if ZF is), then M's powerset of ω will be minimal. Cohen's idea was to add κ distinct subsets, a_η, $\eta < \kappa$, of ω to M, where κ is <u>any cardinal</u> less than α.

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[1] Thanks to Juliette Kennedy for helpful discussion.

- *Note*: This will ultimately let us conclude more than just that *Con(ZFC)* → *Con(ZFC + ~CH)*. We will have: *Con(ZFC)* → *Con(ZFC* + $2^{\aleph_0} = \kappa = \aleph_k$), k ∈ ω.
  - *Note*: *M*[*G*] satisfies *ZF*, and *AC*, if *M* does, thanks to the fact that the 'forcing relation' (introduced below) is definable in *M* and *M* |= *ZF(C)*.

- *Clarification*: Although we assume a *ctm*, this is avoidable. One can carry out the entire proof in Peano Arithmetic (PA) [Weaver (2014)]. This is because the forcing relation is definable in *M* without appeal to *G*, is preserved under provability, and precludes the possibility that a formula and its negation are both forced (by the same 'condition').
  - *Recall*: If *Con*(*ZFC*) → *Con*(*ZFC* + ~*CH*), then ~*Con*(*ZFC* + ~*CH*) → ~*Con*(*ZFC*). As proofs are finite, Γ ∪ ~*CH* |- ⌜**0** = **1**⌝ for some finite Γ.

The Forcing Language

- In order to talk about the envisioned model, *M*[*G*], from the standpoint of the ground model, *M*, let us use an expanded version of the predicate calculus, $\mathfrak{L}$(*M*) (representable *via* Gödel 'numbering'), depending on *M*, containing labels (perhaps many) for *all* elements of our target, *M*[*G*]. (Remember that *M*, and thus *M*[*G*], is really countable.)
  - One constant – called in the literature a name – $\mathbf{a}_\eta$ for each η < κ corresponding to a *new* subset of ω, a_η, and one constant **m** corresponding to each element of *M,* m
  - The logical symbols, **~**, **&**, and ∃
  - The predicates, $\in_{\mathbf{M[G]}}$ and ≡, corresponding to membership and equality in *M*[*G*]
  - The symbols, $\exists_\alpha$, and $\{...:...\}_\alpha$, for each α ∈ On^M

- Terms in $\mathfrak{L}$(*M*) are defined so as to be *stratified into levels of M's ordinals*. However, the objects in the generic extension, *G*[*M*], are not themselves correspondingly stratified. Forcing is defined by recursion on the rank of *names*, not the rank of their *referents*.

- Every $\mathbf{a}_\eta$ is of level 1; each **m** is of the level of m's rank in *M*. Finally, $\{\mathbf{x} : \psi(\mathbf{x}\ldots)\}_\alpha$ is a term of level α when **ψ(x…)** does *not* contain ∃**,** and only contains $\exists_\beta$, and $\{...:...\}_\beta$ β < α.

Conditions

- Definition 1: A **condition**, *p*, is a finite set of ordered triples <n ∈ ω, η ∈ κ, i ∈ {0,1}>; *p* is consistent in that for no n and η is it the case that both <n, η, 0> ∈ *p* and <n, η, 1> ∈ *p*.

- *Intuition*: Each p encodes a bit of information about $M[G]$. So, if $\langle n, \eta, 1\rangle \in p$, then n will turn out to be an element of a_η; not if $\langle n, \eta, 0\rangle \in p$.

  - *Note*: The fact that each p is finite will let those 'living in M' determine what is true in $M[G]$ without knowing about G!

- Definition 2: A condition, q, **extends** condition, p, when $p \subseteq q$. (This is the so-called 'Jerusalem-convention'. The American convention flips the order of inclusion.)

  - *Intuition*: If one conceives of conditions as worlds in a Kripke model, then $p \subseteq q$ will correspond to the requirement that q be *accessible* from p.

- Those 'living in M' can talk about $M[G]$ in $\mathcal{L}(M)$ by talking about what sentences of $\mathcal{L}(M)$ conditions 'force'. Conversely, anything true in $M[G]$ will be 'forced' by some (finite) p. p **forces** (the Gödel number of) φ, $p \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$, when p 'says' $M[G] \models \varphi$. The purpose of the stratification of terms, mentioned previously, is to ensure that whether a formula is forced depends only on sub-formulas of *lower* levels ultimately down to 'prime' ones.

  - Atomic Cases*:* $p \Vdash \ulcorner \mathbf{n} \in_{\mathbf{M[G]}} \mathbf{a_\eta} \urcorner$ if/f $\langle n, \eta, 1\rangle \in p$. These are prime formulas. G will determine them. $p \Vdash \ulcorner \mathbf{l} \in_{\mathbf{M[G]}} \mathbf{m} \urcorner$ if/f $l \in m$ and $m \in M$. $p \Vdash \ulcorner \mathbf{l} \equiv \mathbf{m} \urcorner$ if/f $l = m$ and $l, m \in M$. (There are more complicated atomic cases too, irrelevant for our purposes, defined by transfinite recursion. Forcing is all about recursion.)

  - Conjunction: $p \Vdash \ulcorner \boldsymbol{\theta} \ \&\ \boldsymbol{\varphi} \urcorner$ if/f $p \Vdash \ulcorner \boldsymbol{\theta} \urcorner$ and $p \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$.

  - Quantification: $p \Vdash \ulcorner \exists(\mathbf{x\ldots})\boldsymbol{\varphi}(\mathbf{x\ldots}) \urcorner$ if/f $p \Vdash \ulcorner \boldsymbol{\varphi}(\mathbf{t\ldots}) \urcorner$ for some term(s) of $\mathcal{L}(M)$, $\mathbf{t\ldots}$ (keeping in mind that all elements of $M[G]$ have names).

    - *Note*: We want forcing to be a *semantic* idea. So, while an arbitrary theory may *prove* that $\exists(x\ldots)\varphi(x\ldots)$ without proving that $\varphi(t\ldots)$, we do not want this to be the case for forcing. Term(s) $t\ldots$ correspond to *objects* in $M[G]$).

  - Limited Quantification: $p \Vdash \ulcorner \exists_{\alpha}(\mathbf{x\ldots})\boldsymbol{\varphi}(\mathbf{x\ldots}) \urcorner$ if/f $p \Vdash \ulcorner \boldsymbol{\varphi}(\mathbf{t\ldots}) \urcorner$ for some term(s) of $\mathcal{L}(M)$, $\mathbf{t\ldots}$, (whose Gödel number is) of level $< \alpha$.

  - Negation: $p \Vdash \ulcorner {\sim}\boldsymbol{\varphi} \urcorner$ if/f for all q extending p, $q \nVdash \ulcorner \boldsymbol{\varphi} \urcorner$.

- *Illustration*: $p \Vdash \ulcorner {\sim}\mathbf{n} \in_{\mathbf{M[G]}} \mathbf{a_\eta} \urcorner$ if/f $\langle n, \eta, 0\rangle \in p$. If $\langle n, \eta, 0\rangle \in p$ then for no q extending p is it the case that $\langle n, \eta, 1\rangle \in q$. So, for no q is it the case that $q \Vdash \ulcorner \mathbf{n} \in_{\mathbf{M[G]}} \mathbf{a_\eta} \urcorner$. Thus, $p \Vdash \ulcorner {\sim}\mathbf{n} \in_{\mathbf{M[G]}} \mathbf{a_\eta} \urcorner$. Conversely, let $\langle n, \eta, 0\rangle \notin p$. Then $q = p \cup \{\langle n, \eta, 1\rangle\}$ is consistent and finite, and, hence, a condition. Since $\langle n, \eta, 1\rangle \in q$, $q \Vdash \ulcorner \mathbf{n} \in_{\mathbf{M[G]}} \mathbf{a_\eta} \urcorner$.

Forcing & Truth

- Definition 3: Let P be a partial order and element of *ctm*, M, of *ZFC* with $G \subseteq P$. Then G is generic for M when:
    - For each sentence, φ, of $\mathfrak{L}(M)$, there is a $p \in G$, such that $p \Vdash \ulcorner\boldsymbol{\varphi}\urcorner$ or $p \Vdash \ulcorner\boldsymbol{\sim\varphi}\urcorner$.
    - It is not the case that there are conditions $p, q \in G$, and a sentence, φ, of $\mathfrak{L}(M)$, such that $p \Vdash \ulcorner\boldsymbol{\varphi}\urcorner$ and $q \Vdash \ulcorner\boldsymbol{\sim\varphi}\urcorner$.
    - *Note*: The second condition is guaranteed by the negation clause above.
- Theorem 1 (Generic Existence Theorem): If M is a *ctm* of *ZFC*, and $P \in$ M is a partial order, then there exists a $G \subseteq P$ that is *generic for M*. G is known as a generic filter (for reasons that will become apparent shortly).
    - *Proof Outline*: Enumerate the sentences in the language of forcing, $<\varphi_k \mid k \in \omega>$. At stage $k+1$, given $p_1, p_{2'}, p_3, \ldots p_k$, let $p_{k+1} \Vdash \alpha_{k+1}$ or $p_{k+1} \Vdash \sim\alpha_{k+1}$. Finally, build G and a model, $M[G]$, from $<\alpha_k \mid k \in \omega>$ of the sentences forced by some $p \in G$.
    - *Note*: This is the only place in which we use the assumption that M is countable.
- Definition 4: If $P \in$ M is a partial order and $D \subseteq P$, then D is dense in P if/f for all $p \in P$, there is a $d \in D$ such that d extends p.
    - *Intuition*: D is dense in P when, however you construct G at some stage, there is a subsequent stage at which you may incorporate some of set D.
- Theorem 2 (Equivalence): If $P \in$ M is a partial order, then the following are equivalent.

    (1) $G \subseteq P$ that is *generic for M*

    (2) $G \subseteq P$ and (i) $G \cap D \neq \emptyset$, for all dense $D \in$ M, (ii) G is a filter – i.e., if $p \in G$ and p extends q, then $q \in G$, and if $p \in G$ and $q \in G$, then some r extends p and q.

    - *Note*: The requirement that G meet every dense set in M is what precludes those living in M from computing the construction of G.
- Theorem 3: If M is a *ctm* of *ZFC* and $P \in$ M is a partial order and $G \subseteq P$ is generic for M, then $G \notin$ M.
- *Proof*: Suppose that $G \in$ M, and consider $D = P \setminus G$. Let $p \in P$ and $p \in G$. Suppose that there are $q_1, q_2 \in P$ such that $q_1, q_2 \supseteq p$ but no r such that $r \supseteq q_1, q_2$ (we only consider partial orders meeting this constraint). Then, $q_1 \in G$ and $q_2 \in$ G, by *Equivalence* there is an r $\in P$ such that $r \supseteq q_1, q_2$. So, one of q_1 and q_2 would not be in G, and D would be dense in P. So, $G \cap D \neq \emptyset$. But, as $d \in D$, if $d \in G \cap D$, $d \in P \setminus G$, and $d \notin G$.

- Construction: Let $P = \{f : f$ is a finite partial function from $\omega \times \aleph_\kappa \rightarrow 2$, where $k \in \omega\}$, ordered by $\supseteq$ (containing the element, $\emptyset_P$). Then let $G \subseteq P$, so $\cup G$ is a function sending pairs of elements of n and η to 0 or 1. This generic set, G, codes a set of subsets of ω indexed by η, and the density argument shows that these sets must be distinct. Thus, we can use this G to specify an interpretation of each term, $\boldsymbol{t}$, of $\mathcal{L}(M)$, I, in $M[G]$ as follows.

  - $I(\mathbf{a}_\eta) = \{n \in \omega : \exists p(p \in G \;\&\; \langle n, \eta, 1\rangle \in p\}$.

    - The term $\mathbf{a}_\eta$ picks out the set of numbers, n, for which some condition, p of G, 'says' that n is a member of a_η. (This *sounds* circular. It is not really. A close examination shows that forcing is not even impredicative!)

    - *Note*: Remember the Skolem 'paradox'. G determines countably-many sets by determining uncountably^M-many. Our countably-many terms correspond to uncountably^M-many new subsets of ω. There is no paradox!

  - $I(\mathbf{m}) = m$, for every $m \in M$.

  - $I(\{\mathbf{x} : \boldsymbol{\psi}(\mathbf{x}\ldots)\}_\alpha) = \{I(\mathbf{t}\ldots) : \mathbf{t}\ldots$ is a term $< \alpha$, and $\exists p(p \in G \;\&\; p \Vdash \ulcorner \boldsymbol{\psi}(\mathbf{t}) \urcorner\}$

- Theorem 4 (The Truth Lemma): $M[G] \models \varphi$ under I just in case $\exists p \in G$ and $p \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$.

- This remarkable theorem has several upshots. First, any truth about $M[G]$, there is a *finite* bit of information – a 'stage in G's construction' – $p \in G$, such that p forces $\ulcorner \boldsymbol{\varphi} \urcorner$, and $p \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$ is knowable in M. (Indeed, G is *generic* precisely in that everything true of G is forced by some p to be.) Also, if M thinks that $p \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$, then φ is true in $M[G]$.

- Second, the theorem reduces the task of proving the independence of φ to forcing it. To show that a generic extension satisfies a sentence, φ, show that a condition forces it.

- Finally, the *Truth Lemma* tells us that φ is true in *all* generic extensions if/f $\emptyset \Vdash \ulcorner \boldsymbol{\varphi} \urcorner$.

  - *Clarification*: These include all of the axioms of *ZF*, not just logical truths!

- Theorem 5: Let M be the *ctm* above with ordinals up to α satisfying $ZFC + CH$. Then $M[G]$ has the same ordinals, and $M[G] \models ZFC + {\sim} CH$.

- *Proof Method*: This requires showing that $M[G]$ still satisfies *ZFC*, and collapses no cardinals. The novel aspect of the former is showing that $M[G]$ satisfies the *Powerset Axiom*. This proceeds via the *Truth Lemma*, showing that the statement of Powerset is forced. To show that $M[G] \models {\sim}CH$, one confirms that the a_ηs are indeed distinct, and that no cardinals are collapsed when one navigates between M and $M[G]$. Proving that no cardinals are collapsed uses the Δ-System Lemma (which implies the countable antichain condition (c.c.c.) – indeed, no c.c.c.-forcing collapses any cardinals). All in all, this ensures that $M[G]$ acquired $\kappa = \aleph_k$ new subsets, for our choice of k. So, $M[G] \models 2^{\aleph_0} = \kappa$.

- *Note*: Sometimes collapsing cardinals is useful. If one starts with a model satisfying $\sim CH$, and one wants a model of CH, simply collapse $2^{\aleph_0}$ to $\aleph_1$!

- *Note*: The forcing relation and being a condition are absolute for transitive models.

Now What?

- We have seen that $ZFC + CH$ and $ZF(C) + \sim CH$ are both consistent if ZF is. Moreover, **the way that we proved this did nothing to reveal whether *CH* is true**. This contrasts with our proof that, e.g., $PA + Con(PA)$ and $PA + \sim Con(PA)$ are both consistent if PA is.

- There are **five** attitudes one can take toward CH (GCH, the *Diamond Principle*, the *Souslin Hypothesis*, $0^{\#}$, a Measurable Cardinal, Whitehead's conjecture, $V = HOD$, etc.).
    - (1) One can try to argue that CH is true. One way is to argue for $V=L$. Another strategy, due to Woodin, is to argue that $V = Ultimate\ L$, where *Ultimate L* is an *L-like* inner model that is 'close to V', allowing all large cardinals (unlike L).
    - (2) One can try to argue that CH is false. Indeed, an earlier time-slice of Woodin argued that we should want an Ω-complete theory of $H(\omega_2)$, the level at which CH lives. Assuming the so-called 'Strong Ω-conjecture', Woodin then proved that any Omega-complete theory of $H(\omega_2)$ Ω-implies $\sim CH$ (2001a, 2001b). There are also forcing axioms, advocated by Aspero, Magidor, Velickovic, and others that imply that CH is false. For instance,the *Proper Forcing Axiom* (*PFA*) implies this.
    - (3) One can try to argue that the CH has a truth-value, but we may never know it. One way to defend this point of view appeals to the Sorensen-Williamson conception of vagueness, *epistemicism*. According to this view, epistemology is one thing, and metaphysics is another. If there is nothing in our use of '$\in$' revealing the truth-value of CH, then that just shows that we may never know it.
    - (4) One can try to argue that there is no fact of the matter as to whether CH is true. Contra the epistemicist, if nothing about our use of '$\in$' fixes the truth-value of CH, then that truth-value is not determinately fixed. (It is not as if there are causal chains between us and the likes of sets that fix the reference!) The key question for this view is: *what is it about CH that justifies drawing the line there*? After all, every (non-redundant) axiom is undecidable relative to the others, and disagreements over Choice, Foundation, Replacement, Infinity, etc. persist. This tends to undermine the epistemological, as opposed to sociological, import of forcing. Ironically, advocates of (1) and (2) also emphasize analogies between 'the axioms' (ZFC) and extensions of them deciding CH. This cuts both ways!
    - (5) Finally, one can argue that the question of whether CH is true is like that of whether the Parallel Postulate is, understood as a question of pure mathematics.

This need not imply that *CH* is *indeterminate* (in a context). The view is that different (class) models of set theory are as real as different geometries. Balaguer (1995), Shelah (2003), Hamkins (2012) and Clarke-Doane (Forthcoming) develop this view in different ways. It faces the same 'draw the line' problem as the 'no fact of the matter' view, however. The obvious place to draw the line is at (first-order) logical consistency (Balaguer [1995, § 3.5]). But, by *Gödel's Second Incompleteness Theorem*, it is consistent to say false things about consistency, if *PA* is consistent. So, this kind of *mathematical* pluralism engenders a kind of *logical* pluralism. But this kind should not be confused with the view that classical, intuitionistic, paraconsistent, etc. logics are equally legitimate. It is the view that different notions of finite, and so different versions of classical, intuitionistic, paraconsistent, proof are. It is pluralism about *proof-in-logic-L*!

References

Balaguer, Mark. [1995] "A Platonist Epistemology." Synthese. Vol. 103. 303–25
Cohen, Paul. [1966] *Set Theory and the Continuum Hypothesis*. W. A. Benjamin.
Clarke-Doane, Justin. [Forthcoming] *Mathematics and Metaphilosophy*. Cambridge: Cambridge University Press.
Crossley, J.N, Chris Brickhill, Christopher John Ash, John Stillwell, and Neil Hale Williams [1972] What is Mathematical Logic? Oxford University Press. (Reprinted by Dover, 1990.)
Džamonja, Mirna. [2021] Fast Track to Forcing. Cambridge: Cambridge University Press.
Hamkins, Joel David. [2012] "The Set-Theoretic Multiverse." Review of Symbolic Logic. Vol. 5. 416–49.
Honzik, Radek. [2007] "A Quick Guide to Independence Results" Available online at: <https://dl1.cuni.cz/pluginfile.php/888471/mod_resource/content/2/%5BHonzik%5D-Quick_guide.pdf>
Jech, Thomas. [2000] Set Theory (Third Millenium Edition). Springer.
Kunen, Kenneth. [1980] Set Theory: An Introduction to Independence Proofs (Studies in Logic and the Foundations of Mathematics, Volume 102). North Holland.
Levin, Michael & Margarita Levin. [1978] "The Independence Results in Set Theory: An Informal Exposition." Synthese. 1-34.
Meadows, Toby. "Set Theory: Some Basics and a Glimpse of Advanced Techniques." Lecture Notes.
Shelah, Saharon. [2003] "Logical Dreams." Bulletin of the American Mathematical Society. Vol. 40. 203–228.
Sheppard, Barnaby. The Logic of Infinity. Cambridge: Cambridge University Press.
Väänänen, Jouko. [2015] Forcing. University of Amsterdam Lecture Notes.
Weaver, Nick. [2014] Forcing for Mathematicians. World Scientific.
Wolf, Robert. [2005] A Tour through Mathematical Logic. The Mathematical Association of America.
Woodin, Hugh. [2001a] "The Continuum Hypothesis I." Notices of the American Mathematical Society. Vol. 48. 567 –576.
-----. [2001b] "The Continuum Hypothesis II." Notices of the American Mathematical Society. Vol. 48. 681 –690.