

The Incompleteness Theorems

- Here are some fundamental philosophical questions with mathematical answers:
 - (1) Is there a (recursive) algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is true?
 - (2) Is there an algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is a theorem of Peano or Robinson Arithmetic?
 - (3) Is there an algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is a theorem of pure (first-order) logic?
 - (4) Is there a complete (even if not recursive) recursively axiomatizable theory in the language of first-order arithmetic?
 - (5) Is there a recursively axiomatizable sub-theory of Peano Arithmetic that proves the consistency of Peano Arithmetic (even if it leaves other questions undecided)?
 - (6) Is there a formula of arithmetic that defines arithmetic truth in the standard model, $\mathbb{N}$ (even if it does not represent it)?
 - (7) Is the (non-recursively enumerable) set of truths in the language of first-order arithmetic categorical? If not, is it ω -categorical (i.e., categorical in models of cardinality ω)?
- Questions (1) -- (7) turn out to be linked. Their philosophical interest depends partly on the following philosophical thesis, of which we will make frequent, but inessential, use.
 - *Church-Turing Thesis*: A function is (intuitively) computable if/f it is recursive.
 - *Church*: “[T]he notion of an effectively calculable function of positive integers should be identified with that of recursive function (quoted in Epstein & Carnielli, 223).”
 - *Note*: Since a function is recursive if/f it is Turing computable, the Church-Turing Thesis also implies that a function is computable if/f it is Turing computable.

Robinson Arithmetic (Q)

- Q is the following finitely-axiomatized first-order theory. It is of interest because it is among the weakest theories for which *recursiveness is equivalent to representability*. A function is recursive just in case it is representable (in a sense to be explained) in Q.

- (Q1) $(\forall x)(\forall y)[(x' = y') \rightarrow (x=y)]$
 - If the successors of two numbers are equal, then the numbers are equal.
- (Q2) $(\forall x)\sim[0 = x']$
 - Zero is not the successor of any number.
- (Q3) $(\forall x)[(\sim[x = 0]) \rightarrow ((\exists y)(x=y'))]$
 - Every number other than zero is the successor of a number.
 - *Note:* (Q1) and (Q2) already imply that Q has *only infinite models*.
- (Q4) $(\forall x)[(x + 0) = x]$
 - Any number plus zero is equal to that number.
- (Q5) $(\forall x)(\forall y)[(x + y') = (x + y)']$
 - The sum of a number and the successor of a number is equal to the successor of the sum of the two numbers.
- (Q6) $(\forall x)[(x * 0) = 0]$
 - Any number times zero is zero.
- (Q7) $(\forall x)(\forall y)(x * y') = [(x * y) + x]$
 - The product of some number and the successor of a number is equal to the product of the two numbers plus the first number.
- *Note:* We will write **n** to abbreviate the formal numeral $0'''\dots'$ (n primes)

Basic Arithmetic in Q

- Q is great with particulars. It is Σ_1 -complete (i.e., Q can prove all true Σ_1 sentences).
- *Recall:* A Σ_1 formula is of the form $(\exists 1x)(\exists 2x)\dots(\exists nx)\Delta_0$, where Δ_0 is a formula with only *bounded* quantifiers. Similarly, a Π_1 formula is of the form $(\forall 1x)(\forall 2x)\dots(\forall nx)\Delta_0$. It follows that the negation of a Σ_1 formula is Π_1 , while the negation of a Π_1 formula is Σ_1 .

- *Note:* A theory can be Σ_1 -complete but not Σ_1 -sound! (That is, it could prove all true Σ_1 sentences *as well as* some false Σ_1 sentence.) We will construct just such a theory when we supplement Peano Arithmetic with the negation of its own consistency sentence.
- *Theorem 1:* For all natural numbers, i and j :
 - (a) If $i = j$, then $Q \vdash i = j$
 - (b) If $i \neq j$, then $Q \vdash i \neq j$
- *Theorem 2:* For all natural numbers, i, j , and k , such that $i + j = k$:
 - (a) $Q \vdash (i + j) = k$
 - (b) $Q \vdash (\forall x) [(i + j) = x \rightarrow (x = k)]$
- *Theorem 2.1:* For any natural numbers, i and j :
 - $Q \vdash (i + j) = (j + i)$
- *Theorem 3:* For all natural numbers i, j , and k , such that $i \times j = k$:
 - (a) $Q \vdash (i \times j) = k$
 - (b) $Q \vdash (\forall x) [(i \times j) = x \rightarrow x = k]$
- *Theorem 3.1:* For all natural numbers i and j :
 - $Q \vdash (i \times j) = (j \times i)$
 - *Caveat:* Just because Q proves that addition and multiplication are commutative for its *numerals* (by Theorems 2.1 and 3.1), it does *not* follow that Q proves that these operations are commutative *for all natural numbers* (i.e., universal generalizations). Indeed, we will see that Q proves neither of these general laws.
- *Proof:* Part (b) of *Theorem 2 & 3* follow from logic (of identity). All other claims are of the form $\alpha = \beta$ or $\alpha \neq \beta$, where α and β are (perhaps complex) *numerical terms*. If such a term, θ , refers to n in the standard interpretation, N , then $Q \vdash \theta = n$. So, if $i = j$, and $\alpha = \beta$ says so on N , then $Q \vdash \alpha = \beta \leftrightarrow i = j$, and $Q \vdash i = j$, as i is numeral j . Likewise, if $i \neq j$, and $\alpha \neq \beta$ says so on N , then, by the logic of identity, $Q \vdash i \neq j$, and again $Q \vdash \alpha \neq \beta$.
- *Note:* These (meta)theorems suffice to significantly constrain models of Q . That part of any model which interprets Q 's numerals must look *just like the standard model, N* .

- *Upshot*: Any non-standard model of Q must look just like the standard model, N , but with some additional elements “tacked onto the end” (and additional values for $'$, $+$, and $*$).

Representability

- We have really been showing that addition and multiplication are *representable* in Q .
 - *Caveat*: The proof of this uses Induction, which Q itself does not have!
- *Definition 1*: A total function $f: N^n \rightarrow N$ is *representable* in theory, T , if/f there is a formula, Φ , in which $x_1, x_2, \dots, x_n, x_{n+1}$ occur free, such that when $q_1, q_2, \dots, q_n, k$ are natural numbers with $f(q_1, q_2, \dots, q_n) = k$, we have:
 - (a) $T \vdash \Phi(q_1, q_2, \dots, q_n, k)$
 - (b) $T \vdash (\forall x)[\Phi(q_1, q_2, \dots, q_n, x) \rightarrow (x = k)]$
- In this case, the formula, Φ , is said to *represent* function, f , in theory, T .
 - *Note*: Part (b) reflects the fact that f is a function, giving a unique output.
 - *Example*: By *Theorem 2*, $(x_1 + x_2) = x_3$ represents the plus function in Q , while, by *Theorem 3*, $(x_1 * x_2) = x_3$ represents times.
 - *Example*: The zero, successor, and projections functions --i.e., the basic *primitive recursive* functions -- are representable in Q .
- *Theorem 4 (Representability Theorem)*: Every total recursive function is representable in any theory, T , which extends Q (where T *extends* Q when $Q \subseteq T$).
 - *Note*: Q *explicitly* represents $'$, $+$ and $*$, but implicitly represents far more.
 - In fact, despite being so weak, Q is *optimal*, in the following sense:
- *Theorem 5*: If T is any consistent theory with a recursive set of axioms, then every total function representable in T is itself recursive.
 - *Proof*: Suppose that $f: N^n \rightarrow N$ is a total function represented in T by Φ . Then $\forall q_1 q_2 \dots q_n, k \in N, T \vdash \Phi(q_1 q_2 \dots q_n, k)$ if/f $f(q_1 q_2 \dots q_n) = k$, and the theorems of T are algorithmically enumerable. This gives an algorithm for computing f . Given $q_1 q_2 \dots q_n$, wait for a theorem of the form $\Phi(q_1 q_2 \dots q_n, k)$. (Since T is consistent, there will be exactly one of this form.) By the Church-Turing Thesis, f is therefore recursive.
- *Theorem 6 (Equivalence)*: A total function is representable in Q if/f it is recursive.

- *Proof:* From *Theorems 4* and *5* (assuming that Q is consistent).

Gödel Numbering

- A formula in the language of arithmetic is a finite string of symbols. So, we can pair those symbols, and strings of them, with natural numbers, called their *Gödel numbers*.
- There are many ways to do this. But the pairing must satisfy the following conditions.
 - (a) Different strings of symbols are assigned to different Gödel numbers
 - (b) There is a (recursive) algorithm such that, given a natural number, it decides whether the number codes a string of basic symbols, and, if it does, what string it codes.
 - (c) There is a (recursive) algorithm such that, given a string of basic symbols, it decides what the Gödel number of that string is.
- Here is one pairing, β , that will do the trick:

<i>Symbol:</i>	(	)	&	$\forall$	$\rightarrow$	$\leftrightarrow$	$\sim$	∇	$\exists$	0	'	+	*	=	x_i
<i>Number:</i>	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$i+15$

- This generates, corresponding to each string of *symbols*, $\alpha_1\alpha_2\alpha_3\alpha_4\dots$, a string of Gödel numbers, given by the above pairing, $\beta(\alpha_1)\beta(\alpha_2)\beta(\alpha_3)\dots$
- We can now assign the overall string $\alpha_1\alpha_2\alpha_3\alpha_4\dots$ the Gödel number:
 - $(p_1^{\beta(\alpha_1)})(p_2^{\beta(\alpha_2)})(p_3^{\beta(\alpha_3)})\dots$, where $p_1p_2p_3\dots p_k$ are the first k primes.
 - *Note:* This will generally be a very big number! But, given it, we can check after finitely-many steps that it is the Gödel number of the string.
 - *Note:* We use here the *Fundamental Theorem of Arithmetic*, that every natural number has a unique decomposition into powers of primes.
- If Ψ is a string of symbols, then we will write $\gamma(\Psi)$ for the Gödel number of Ψ .
- It remains to code arbitrary formal proofs, F . If the following is such a proof,
 - Φ_1
 - Φ_2
 - Φ_3
 - ...

- Φ_k

then we can code it with the function, $\Gamma(F)=(2^{\gamma(\Phi_1)})(3^{\gamma(\Phi_2)})...(p_k^{\gamma(\Phi_k)})$.

- *Upshot:* We can now describe facts about our theory, Q , in terms of Godel numbers. For instance, the second of our initial questions, (2), can now be phrased: is there an algorithm for deciding whether a given number is the Godel number of a theorem of Q ?

The Diagonal Lemma

- *Theorem 6:* If Φ is a formula in which at most x and y occur free, then for all natural numbers n and k :

- $\Phi(n, k), \forall y[\Phi(n, y) \rightarrow (y = k)] \mid \dashv \vdash \forall y[\Phi(n, y) \leftrightarrow (y = k)]$

■ *Note:* This is a theorem of first-order logic.

- *Clarification:* Variables like x and y are doing double-duty, as meta-variables in our meta-language, and as variables in Q .

- *Theorem 7:* If $f: N \rightarrow N$ is a total function represented in theory, T , by formula, Φ , then, for all natural numbers n and k , if $f(n) = k$,

- $T \mid \dashv \vdash \forall y[\Phi(n, y) \leftrightarrow y = k]$

- *Proof:* Suppose that $f(n) = k$. Then, by the *Definition 1*, we have that T proves both of the antecedents of *Theorem 6*.

- *Notation:* If Φ is a formula with Godel number, n , then $\ulcorner \Phi \urcorner$ is the term, n .

- *Note:* $\ulcorner \Phi \urcorner$ is not $\gamma(\Phi)$. $\gamma(\Phi)$ is a *natural number*, while $\ulcorner \Phi \urcorner$ is the *numeral* in the language of arithmetic which, in the standard interpretation, N , refers to $\gamma(\Phi)$.

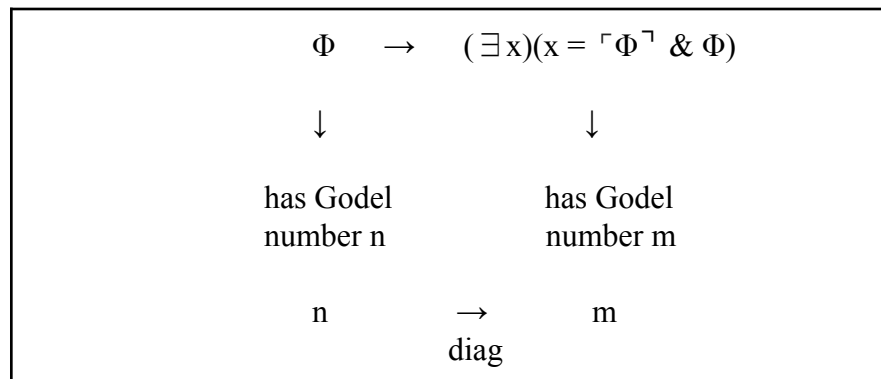
- *Example:* If Φ is the formula $(\exists x)[y = 0'' * x]$, then, given our numbering, $\gamma(\Phi) = (2^9)(3^{16})(5^{15})(7^{14})(11^1)(13^{10})(17^{11})(19^{11})(23^{13})(29^{16})(31^2)$. On the other hand, $\ulcorner \Phi \urcorner$ is the term **0** followed by $\gamma(\Phi)$ occurrences of the symbol $\prime$.

■ We can think of $\ulcorner \Phi \urcorner$ as the *name* of Φ in the language of arithmetic, since, given a Godel numbering, it is *uniquely determined* by Φ .

- *Definition 2 (Diagonalization):* If Φ is a formula in which at most some variable, x , occurs free, then the *diagonalization* of Φ , written D_Φ , is the (closed) sentence:

- $(\exists x)(x = \ulcorner \Phi \urcorner \ \& \ \Phi)$

- *Example:* If Φ is $(\exists y)[x = (2 * y)] \ \& \ \sim(\exists y)[x = (4 * y)]$, and m is $\gamma(\Phi)$, then $D\Phi$ is $(\exists x)(x = m \ \& \ (\exists y)[x = (2 * y)] \ \& \ \sim(\exists y)[x = (4 * y)])$.
 - *Note:* If x is free in Φ , then the diagonalization of Φ “says” that $\gamma(\Phi)$ has the property, Φ , and $(\exists x)(x = \ulcorner \Phi \urcorner \ \& \ \Phi) \dashv\vdash \Phi(\ulcorner \Phi \urcorner)$.
 - *Note:* The diagonalization of a formula need not be true (under the standard interpretation), though it might be (and is in *Example* above).
- *Theorem 8:* If Φ is a formula in which some variable, x , occurs free, and n is a natural number:
 - $\vdash\vdash [(\exists x)(x = n \ \& \ \Phi) \longleftrightarrow \Phi(n)]$
 - *Note:* This is a logical truth.
- *Theorem 9:* If Φ is a formula in which at most some variable, x , occurs free:
 - $\vdash\vdash [D\Phi \longleftrightarrow \Phi(\ulcorner \Phi \urcorner)]$
 - *Proof:* Immediate from *Theorem 8*.
- *Theorem 10:* There is a (primitive) recursive function, $*$, such that, for all strings α and β of basic symbols, if $m = \gamma(\alpha)$ and $n = \gamma(\beta)$, then $m*n = \gamma(\alpha\beta)$.
- *Theorem 11:* There is a (primitive) recursive function, $\text{diag}(x)$, with the following property. If n is the Godel number of a formula Φ in which at most some variable, x , occurs free, then $\text{diag}(n)$ is the Godel number of the *diagonalization* of Φ , $D\Phi$.



- *Note:* It is intuitively clear that the function, $\text{diag}(x)$ is recursive since all we have to do to calculate it is check if n is the Godel number of a formula, and, if it is, add the code for “ $(\exists x)(x = n \ \& \$ ” in front and add the code for “ $)$ ” in the back.

- *Theorem 12 (Diagonal Lemma)*: If diag is representable in a theory, T , then, for each formula, Φ , in which at most some variable, y , occurs free, there is a sentence, G_Φ , such that:
 - $T \vdash [G_\Phi \longleftrightarrow \Phi(\ulcorner G_\Phi \urcorner)]$
 - *Proof Sketch*: Let Θ represent the function, diag , in theory T , with at most x and y free. Let ψ be a formula with at most y free, and take Ω to be the formula $(\exists y)(\Theta \ \& \ \psi)$, where at most x may occur free. Finally, identify G_Θ with the diagonalization of Ω , D_Ω , to get $T \vdash [G_\Phi \longleftrightarrow \Phi(\ulcorner G_\Phi \urcorner)]$.
 - *Example*: If Φ is the formal analog of “is even”, then G_Φ is a sentence which is true just in case the Godel number of G_Φ is even.
 - *Example*: If Φ is the formal analog of “codes a sentence which is provable”, then G_Φ is a sentence which is true just in case the Godel number of G_Φ codes a sentence which is, indeed, provable.
 - *Note*: The *Diagonal Lemma* means that *every predicate in T has a fixed point* -- roughly, a point where you get the same thing out that you put in.
- *Upshot*: Every theory extending Q has the capacity for self-reference.

Undecidable Theories

- *Definition 3*: A set, A , is representable in theory, T , if/f its *characteristic function* is (where the characteristic function, f_A , of set, A , is $f_A(n) = 1$ when $n \in A$ and 0 otherwise.)
- *Theorem 13*: Every *recursive set* is representable in any theory, T , which extends Q .
 - *Proof*: Immediate from the *Representability Theorem*.
- *Theorem 14*: If T is a theory extending Q , and A is a set of natural numbers which is representable in T , then there is a formula, δ_A , in which at most x occurs free, such that, for each natural number, n :
 - (a) If $n \in A$, then $T \vdash \delta_A(n)$
 - (b) If $n \notin A$, then $T \vdash \sim \delta_A(n)$
 - *Proof*: Let Φ_A represent the characteristic function, f_A , of A in T . Then $\forall n, k \in \mathbb{N}$, if $f_A(n) = k$, $T \vdash \Phi_A(n, k)$, and $T \vdash \forall y[\Phi_A(n, y) \rightarrow y=k]$. Define $\delta_A(x) = \Phi_A(x, 1)$. If $n \in A$, then $f_A(n) = 1$, so $T \vdash \Phi_A(n, 1)$ and, hence, $T \vdash \delta_A(n)$. Conversely, if $n \notin A$, then $f_A(n) = 0$, and, thus, $T \vdash \forall y[\Phi_A(n, y) \rightarrow y=0]$. But since T extends Q , $T \models 1 \neq 0$. Hence, by logic in T , $T \vdash \sim \Phi_A(n, 1)$, and, therefore, $T \vdash \sim \delta_A(n)$.

- In fact, the other direction holds as well.
- *Theorem 15:* If T extends Q , then the following conditions on a set, A , of natural numbers are equivalent:
 - (a) The set, A , is representable in T .
 - (b) There is a formula, δ_A , in which at most some variable, x , occurs free, such that, for each natural number, n , if $n \in A$, then $T \vdash \delta_A(n)$, and, if $n \notin A$, then $T \vdash \sim \delta_A(n)$.
- *Theorem 16 (Undecidability of Extensions of Q):* If T is a consistent theory extending Q , then the set of Godel numbers of the theorems of T , $GN(T)$, is *not* representable in T .
 - *Proof:* Assume for reductio that $GN(T)$ is representable in T . Then by *Theorem 15*, there is a formula, $\delta(x)$, whose only free variable is x , such that, for any $n \in \mathbb{N}$,
 - If $n \in GN(T)$, then $T \vdash \delta(n)$
 - If $n \notin GN(T)$, then $T \vdash \sim \delta(n)$
 - Moreover, since $GN(T)$ is the set of Godel numbers of theorems of T , for each Φ ,
 - If $T \vdash \Phi$, then $\gamma(\Phi) \in GN(T)$
 - If $\sim(T \vdash \Phi)$, then $\gamma(\Phi) \notin GN(T)$
 - Hence, if $T \vdash \Phi$, then $T \vdash \delta(\ulcorner \Phi \urcorner)$, and if $\sim(T \vdash \Phi)$, then $T \vdash \sim \delta(\ulcorner \Phi \urcorner)$. So, applying the *Diagonal Lemma* to $\sim \delta$, we are ensured a sentence, $G_{\sim \delta}$ such that:
 - $T \vdash (G_{\sim \delta} \longleftrightarrow \sim \delta(\ulcorner G_{\sim \delta} \urcorner))$.
 - *Note:* The sentence $\sim \delta(\ulcorner G_{\sim \delta} \urcorner)$ “says” that $G_{\sim \delta}$ is not a theorem of T .
 - However, both the assumption that $T \vdash G_{\sim \delta}$ and $\sim(T \vdash G_{\sim \delta})$ lead to contradiction.
 - *Note:* *Theorem 16* allows that the set of theorems of T is representable in extensions of T . Moreover, the assumption that T is consistent is essential.
- *Definition 4: True Arithmetic, TA ,* is the set of all sentences, S , such that $\mathbb{N} \models S$.
- *Theorem 17:* The set of Godel numbers of sentences of True Arithmetic, $GN(TA)$, is not a recursive set.

- *Proof:* Immediate from *Theorem 16*.
- *Theorem 18 (Church-Turing Theorem):* The set of theorems of first-order logic is not recursive.
 - *Proof:* If it were, then the set of theorems of Q would be too, contrary to *Theorem 16*. For an arbitrary Φ , check whether $AxQ \rightarrow \Phi$ is a theorem of first-order logic.

Undefinability of Truth

- *Definition 5:* A relation $R(x_1, x_2, \dots, x_k)$ of natural numbers is *definable* in the standard model, N , if/f there is a formula $\phi(x_1, x_2, \dots, x_k)$ in the language of arithmetic such that for every $n_1, n_2, \dots, n_k$, $R(n_1, n_2, \dots, n_k)$ if and only if $N \models \phi(n_1, \dots, n_k)$.
- *Theorem 19 (Godel-Tarski Theorem).* The set of Godel numbers of sentences, S , true in N (i.e., $\{S: N \models S\}$) is not definable in arithmetic.
 - *Proof.* A set is definable in N if/f it is representable in TA . But TA consistently extends Q . Hence, again, this set is not representable in TA , by *Theorem 16*.
- *Upshot:* Extensions of Q can *define* typical syntactic properties, but not semantic ones.

Axiomatizability, Enumerability, and Completeness

- *Definition 6:* A theory, T , is (recursively) *axiomatizable* if/f there is a set, S , of axioms of T (i.e., a set $S \subseteq T$ such that, for every $\Phi \in T$, $S \vdash \Phi$) where the set of Godel numbers of S is recursive.
 - *Note:* There is a big difference between a theory's having a recursive set of axioms and its having a recursive set of theorems. Q has a recursive -- indeed, finite -- set of axioms, but we have seen that its set of theorems is not recursive.
- *Definition 7:* A set, A , of natural numbers is recursively enumerable if/f either $A = \emptyset$ or $A = \{f(0), f(1), f(2), \dots\}$ where f is a total recursive function.
 - *Note:* If a set is recursive, then it is recursively enumerable.
- *Theorem 20:* If T is a recursively axiomatized theory, then the set of Godel numbers of theorems of T , $GN(T)$, is *recursively enumerable*.
 - *Proof:* Since the set of theorems of T is not empty, call the Godel number of $(\forall y)(y = y)$, n_0 . Now, define a computable function, f , as follows. Given $n \in N$, first check whether n is the Godel number of a proof in T . If not, let $f(n) = n_0$. If so, we can compute from n what assumptions the proof relied on. Moreover, since the set of axioms of T is recursive, we can check whether each of these assumptions is a member of that set. If it is, let $f(n)$ be the Godel number of the

last line of the proof, and if not, let $f(n) = n_0$. Conversely, if Φ is a theorem, then some number, $\Gamma(F)$, codes its proof, F , in T , meaning $f(\Gamma(F)) = \gamma(\Phi)$, as desired.

- *Upshot*: Though being recursive implies being recursively enumerable, the converse implication fails. But there is one circumstance when it does not.
- *Theorem 21*: If T is an axiomatizable theory which is *complete* (i.e., $T \vdash \Phi$ or $T \vdash \sim\Phi$, for all sentences), then the set of Godel numbers of theorems of T is recursive.
 - *Proof*: If T is inconsistent, it is trivially recursive. So, assume that T is consistent. By *Theorem 20*, there is a recursive function which enumerates its theorems. For any sentence, Φ , simply wait for Φ or $\sim\Phi$ to appear after a finite number of steps.
- *Theorem 22 (Godel's First Incompleteness Theorem)*: If T is a axiomatizable extension of Q , then either T is incomplete or inconsistent.
 - *Proof*: By *Theorem 21*, if T were complete, then it would have a recursive set of theorems, contradicting *Theorem 16*.
- *Theorem 23*: True Arithmetic is not axiomatizable.
 - *Proof*: Immediate from *Theorem 22*.
- *Theorem 24*: True Arithmetic is not recursively enumerable.
 - *Proof*: It is a complete and consistent extension of Q , so, if it were so enumerable, then, by *Theorem 21*, its theorems would be recursive, contrary to *Theorem 16*.

Hilbert's Program and Consistency

- *Definition 7*: Peano Arithmetic (PA) is Q conjoined with all instances of:
 - *Induction Schema*: $((\Phi(0)) \ \& \ \forall x[(\Phi(x) \rightarrow \Phi(x'))]) \rightarrow \forall \Phi(x)$
 - *Note*: Φ may also contain parameters.
- As an axiomatizable (albeit not finitely axiomatizable) extension of Q , *Theorem 20* implies that PA is incomplete, if consistent. But PA is much stronger than Q .
 - Although Q is good with particulars, it is bad with generalizations. Simple model-theoretic arguments show that it does not prove any of the following.
 - $\forall x(x \neq x')$
 - $\forall x([(x + (y + z)) = ((x + y) + z)])$
 - $\forall x \forall y[(x + y) = (y + x)]$
 - $\forall x[(0 + x) = x]$

- $\forall y \forall z \sim [\exists x((x' + y) = z) \ \& \ \exists x((x' + z) = y)]$
- $\forall x \forall y \forall z[(x * (y * z)) = ((x * y) * z)]$
- $\forall x \forall y[(x * y) = (y * x)]$
- $\forall x \forall y \forall z[(x * (y + z)) = ((x * y) + (x * z))]$

- The problem is that Q has no way of saying that the things picked out by its numerical terms are the only numbers. This idea is regimented by Induction. And, indeed, in PA, all of the above are provable using the Induction Schema.

- *Example:* In order to construct a model of $(Q1) \dashv\vdash (Q7) + \sim \forall x(x \neq x')$, add an element, ω , and interpret the successor function as before but with $s(\omega) = \omega$, interpret $+$ as before but with $n + \omega = \omega + n = \omega$, for all numbers, n (including ω), and let $n * \omega = \omega * n = \omega$ when $n \neq 0$, and 0 otherwise.

- Hence, while PA must be incomplete, it might still be hoped that PA, or, even better, a “finitary” fragment of it, proves the consistency of PA. This was Hilbert’s ambition.
- In order to make this precise, we must formalize the sentence “PA is consistent” in the language of PA. Since $PA \vdash \mathbf{0} = \mathbf{1}$ if/f PA is inconsistent, it suffices to formalize “PA does not prove $\mathbf{0} = \mathbf{1}$ ”. The problem is that the set of (Godel numbers of) theorems of PA is not representable in PA, by *Theorem 16*. But there is a formula (actually, different formulas), $Prov(x)$, with only x free, which defines the relation of provability in PA. So:

- $N \models Prov(\ulcorner \Phi \urcorner)$ if and only if $PA \vdash \Phi$

- With this predicate in hand, our question can now be phrased: whether there is an axiomatizable theory, T , of which PA is an extension such that $T \vdash \sim Prov(\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$.

- *Note:* $\sim Prov(\ulcorner \mathbf{0} = \mathbf{1} \urcorner)$ is commonly written $Con(PA)$.

- *Definition 8:* If T extends Q , then the formula, $Prov(x)$, of one free variable is a *provability predicate* for T if/f, for all sentences Ω and Ψ , the following hold:

- (a) If $T \vdash \Omega$, then $T \vdash Prov(\ulcorner \Omega \urcorner)$

- If T proves Ω , then it is a theorem of T that T proves Ω .

- (b) If $T \vdash Prov(\ulcorner \Omega \rightarrow \Psi \urcorner) \rightarrow [(Prov(\ulcorner \Omega \urcorner) \rightarrow Prov(\ulcorner \Psi \urcorner))]$

- It is a theorem of T that if the conditional $\Omega \rightarrow \Psi$, and its antecedent, Ω , are both provable in T , then so is the consequent, Ψ .

- (c) $T \vdash [Prov(\ulcorner \Omega \urcorner) \rightarrow Prov(\ulcorner Prov(\ulcorner \Omega \urcorner) \urcorner)]$

- It is a theorem of T that if Ω is provable in T , then the fact that Ω is provable in T is itself provable in T .
 - *Note:* This condition can be questioned.
- *Theorem 25 (Lob's Theorem):* If T extends Q , and $\text{Prov}(x)$ is a provability predicate for T , then, for each sentence, Ψ , $T \vdash \Psi$ if and only if $T \vdash \text{Prov}(\ulcorner \Psi \urcorner) \rightarrow \Psi$.
 - *Proof:* Evidently, if $T \vdash \Psi$, then $T \vdash [\text{Prov}(\ulcorner \Psi \urcorner) \rightarrow \Psi]$. So, let us suppose that $T \vdash \text{Prov}(\ulcorner \Psi \urcorner) \rightarrow \Psi$. Since T extends Q , $\text{diag}(x)$ is representable in T , so letting $\delta(x) = [\text{Prov}(\ulcorner \Psi \urcorner) \rightarrow \Psi]$ with only x free, we get that $T \vdash (G_\delta \leftrightarrow [\text{Prov}(\ulcorner G_\delta \urcorner) \rightarrow \Psi])$. A calculation using properties (a) and (b) and logic in T then yields that $T \vdash [\text{Prov}(\ulcorner G_\delta \urcorner) \rightarrow \text{Prov}(\ulcorner \text{Prov}(\ulcorner G_\delta \urcorner) \urcorner)] \rightarrow \text{Prov}(\ulcorner \Psi \urcorner)$, and by property (c) of $\text{Prov}(x)$ we know that the antecedent of this conditional must be true, giving (by logic in T) that $T \vdash (\text{Prov}(\ulcorner G_\delta \urcorner) \rightarrow \text{Prov}(\ulcorner \Psi \urcorner))$. And since we are assuming that $T \vdash \text{Prov}(\ulcorner \Psi \urcorner) \rightarrow \Psi$, we have (by logic in T) that $T \vdash (\text{Prov}(\ulcorner G_\delta \urcorner) \rightarrow \Psi)$. Finally, since $T \vdash [G_\delta \leftrightarrow (\text{Prov}(\ulcorner G_\delta \urcorner) \rightarrow \Psi)]$, we must have that $T \vdash G_\delta$. So, by (a), $T \vdash \text{Prov}(\ulcorner G_\delta \urcorner)$, and, using *modus ponens* in T , $T \vdash \Psi$, as desired.
 - *Upshot:* If T proves “if T proves something, it is true”, then T already proves it!
- *Theorem 26 (Godel's Second Incompleteness Theorem):* If T is a consistent extension of Q , and $\text{Prov}(x)$ is a provability predicate for T , then $\sim(T \vdash \sim \text{Prov}(\ulcorner 0 = 1 \urcorner))$.
 - *Proof:* Let T be a consistent extension of Q and $\text{Prov}(x)$ a provability predicate for it. Suppose $T \vdash \sim \text{Prov}(\ulcorner 0 = 1 \urcorner)$. By logic in T , $T \vdash (\text{Prov}(\ulcorner 0 = 1 \urcorner) \rightarrow 0 = 1)$. So, by *Lob's Theorem*, $T \vdash 0 = 1$, contrary to the assumption that T is consistent.
 - *Note:* One can always choose $\sim \text{Prov}(x)$ to be a Π_1 predicate, making $\sim \text{Prov}(\ulcorner 0 = 1 \urcorner)$ a sentence of *Goldbach Type*. Once chosen we have a *concrete example* of a sentence that is not provable in PA , despite its being much more powerful than Q .

Nonstandard Models

- *Definition 9:* A theory is *categorical* when all its models are isomorphic, and *κ -categorical* when all its models of cardinality κ are isomorphic.
- *Theorem 27:* True Arithmetic is not ω -categorical.
 - *Proof:* Add a name, c , to the language of arithmetic, and define the following set, C , of formulas: $\{x \neq n : n \text{ is any numeral in the (non-augmented) language of arithmetic}\}$. Since every finite subset of the union of True Arithmetic (TA) and C is satisfiable, $TA \cup C$ is too, and in a countable model, by the Compactness and Lowenheim-Skolem Theorems. But no such model can be isomorphic to N .

- *Example:* The theory $PA + \sim \text{Con}(PA)$, which is consistent by Godel's Second Incompleteness Theorem, has ω -models with a non-standard number, greater than the finite numbers, which witnesses a proof of $0 = 1$.
- *Note:* This means that, *even if it were granted that there is a perfectly determinate body of arithmetic truths*, it does not follow that they pick out a determinate model (even up to isomorphism).

Questions Revisited

- We are now in a position to answer all of the questions with which we began.
 - (1) There no algorithm for deciding whether a given sentence in the language of first-order arithmetic is true, by *Theorem 17*.
 - (2) There no algorithm for deciding whether a given sentence in the language of first-order arithmetic is a theorem of Robinson Arithmetic, by the *Undecidability of Extensions of Q*.
 - (3) There is no algorithm for determining whether a given sentence in the language of first-order arithmetic is a theorem of pure (first-order) logic, by the *Church-Turing Theorem*.
 - (4) There is no complete axiomatizable arithmetic theory in the language of first-order arithmetic, by *Godel's First Incompleteness Theorem*.
 - (5) There is no axiomatizable theory of which Peano Arithmetic is an extension that proves the consistency of Peano Arithmetic, by *Godel's Second Incompleteness Theorem*.
 - *Godel:* "For all formal systems for which the existence of undecidable arithmetical propositions was [demonstrated], the assertion of the consistency of the system in question itself belongs to the propositions undecidable in that system.....For a system in which all finitary...forms of proof are formalized, a finitary consistency proof, such as the formalists seek, would thus be...impossible (quoted in Epstein & Carnielli, 214)."
 - (6) There is no formula in the language of first-order arithmetic that defines arithmetic truth in the standard model, by the *Godel-Tarski Theorem*.
 - (7) The (non-recursively enumerable) set of truths in the language of first-order arithmetic, True Arithmetic, is not categorical or ω -categorical, by *Theorem 27*.

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