NOTES ON "RELATIVITY, GRAVITATION & COSMOLOGY"

LAMBOURNE (ed.)

SPECIAL RELATIVITY (SR)

- AN EVENT IS AN INSTANTANEOUS OCCURRENCE AT A PARTICULAR PLACE IN SPACE (A POINT IN SPACE-TIME).

- A FRAME OF REFERENCE IS A SYSTEM FOR ASSIGNING COORDINATES – e.g. (t, x, y, z) OR (t, r, θ, ϕ) – TO AN EVENT.

- AN INERTIAL FRAME OF REFERENCE IS ONE IN WHICH NEWTON'S FIRST LAW OF MOTION IS TRUE.

- NOTE: IF S IS INERTIAL, AND S' MOVES AT A CONSTANT VELOCITY RELATIVE TO S, THEN S' IS INERTIAL TOO. BUT IF S' ACCELERATES (OR ROTATES) RELATIVE TO S, THEN S' IS NOT INERTIAL.

- AN OBSERVER IS ONE WHO USES A PARTICULAR FRAME OF REFERENCE TO MEASURE (NOT "DIRECTLY OBSERVE") EVENTS.

- NOTE: SR IS MAINLY CONCERNED WITH MEASUREMENTS MADE IN INERTIAL FRAMES OF REFERENCE.

• IT IS CONJECTURED THAT THE SAME LAWS OF PHYSICS HOLD TRUE AT ALL PLACES AND TIMES. WE KNOW THAT THEY DO NOT HOLD FOR ALL STATES OF MOTION (SINCE NEWTON'S FIRST LAW HOLDS IN AN INERTIAL FRAME, BUT NOT IN AN ACCELERATING ONE). BUT MIGHT THE SAME LAWS HOLD IN ALL INERTIAL FRAMES?

• GALILEAN RELATIVITY SAYS THAT AT LEAST THE SAME LAWS OF MECHANICS HOLD FOR ALL INERTIAL OBSERVERS. EINSTEIN WENT FURTHER.

SR

• FIRST POSTULATE OF SR: THE LAWS OF PHYSICS – NOT JUST OF MECHANICS – TAKE THE SAME FORM IN ALL INERTIAL FRAMES

• SECOND POSTULATE OF SR: THE SPEED OF LIGHT, C, IN A VACUUM IS $3 \times 10^8 \frac{m}{s}$ IN ALL INERTIAL FRAMES.

• NOTE: GIVEN THE SECOND POSTULATE, WE CAN USE LIGHT SIGNALS TO SYCHRONIZE OUR SYSTEM OF CLOCKS.

• SR CONCERNS THE RELATIONSHIP BETWEEN MEASUREMENTS MADE BY INERTIAL OBSERVERS IN RELATIVE MOTION – ESPECIALLY THEIR MEASUREMENTS OF THE TIME AND SPACE COORDINATES OF EVENTS.

- WE SAY THAT THE <u>FRAMES OF REFERENCE</u>, S AND S', OF OBSERVERS, O AND O', ARE IN <u>STANDARD CONFIGURATION</u> WHEN:

  (1) THE ORIGIN OF S' MOVES AT A <u>CONSTANT VELOCITY</u>, $\vec{V}$, IN THE DIRECTION OF INCREASING VALUES ALONG S'S X-AXIS, AS MEASURED IN S.

  (2) THE X, Y, AND Z AXES OF S REMAIN PARALLEL TO THE X', Y', AND Z' OF S'.

  (3) THE EVENT AT WHICH THE ORIGINS OF S AND S' COINCIDE OCCUR AT TIME $t=0$ IN S' AND AT $t'=0$ IN S.

  - <u>NOTE</u>: <u>ANY</u> INERTIAL FRAMES IN UNIFORM RELATIVE MOTION CAN BE PLACED IN <u>STANDARD CONFIGURATION</u>.

- <u>COORDINATE TRANSFORMATIONS</u> RELATE THE COORDINATES ASSIGNED TO ANY EVENT BY O TO THE COORDINATES ASSIGNED TO THE SAME EVENT IN O'.

- THE "OBVIOUS" COORDINATE TRANSFORMATIONS ARE THE <u>GALILEAN TRANSFORMATIONS</u>. THESE ARE:

ASSUMES THAT TIME INTERVAL BETWEEN EVENTS IS THE SAME FOR O AND O'

$$t' = t \qquad (1.1)$$

$$x' = x - Vt \qquad (1.2)$$

$$y' = y \qquad (1.3)$$

$$z' = z \qquad (1.4)$$

WHERE THE <u>RELATIVE SPEED</u> OF S' WITH RESPECT TO S IS $|\vec{V}|$

• HOWEVER, SR ENTAILS THE <u>LORENTZ TRANSFORMATIONS</u>:

"MIXES UP" TIME & SPACE

$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

t' IS A MIX OF O's t <u>AND</u> x

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

x' IS A MIX OF O's x <u>AND</u> t

$$y' = y$$

$$z' = z$$

<u>LORENTZ FACTOR</u>

$$\gamma(V) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

DEPENDS ON V

• REWRITING USING THE <u>LORENTZ FACTOR</u>:

$$t' = \gamma(V)\left(t - \frac{Vx}{c^2}\right) \qquad (1.5)$$

$$x' = \gamma(V)(x - Vt) \qquad (1.6)$$

$$y' = y \qquad (1.7)$$

$$z' = z \qquad (1.8)$$

WHERE

$$\boxed{\gamma(V) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}}$$ ← LORENTZ FACTOR ≈ 1 WHEN $V \ll C$.

· NOTE: $\gamma(V) > 1$, FOR ALL (POSITIVE) V

· EXAMPLE: IF V IS 10% THE SPEED OF LIGHT, THEN $\gamma(V) = \frac{1}{\sqrt{1 - \frac{(.1c)^2}{c^2}}} = 1.01$

BUT IF V WERE 90% THE SPEED OF LIGHT, THEN $\gamma(V) = 2.29$!

• IT IS SOMETIMES CONVENIENT TO WRITE THE THE LORENTZ TRANSFORMATION IN MATRIX FORM:

UNITS OF DISTANCE

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(V) & -\gamma(V)V/c & 0 & 0 \\ -\gamma(V)V/c & \gamma(V) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$\boxed{S' \text{ FRAME}}$ $\boxed{S \text{ FRAME}}$

• $ct' = \gamma(V)ct - \frac{\gamma(V)Vx}{c}$

$= \gamma(V)\left(ct - \frac{Vx}{c}\right)$

So, $t' = \gamma(V)\left(t - \frac{Vx}{c^2}\right)$

• $x' = -\frac{\gamma(V)Vct}{c} + \gamma(V)x$

$= \gamma(V)(x - Vt)$

• $y' = y$

• $z' = z$

• WE CAN REPRESENT THIS MATRIX EQUATION MORE SIMPLY AS:

ROW

$$[x'^{\mu}] = [\Lambda^{\mu}{}_{\nu}][x^{\nu}]$$

COLUMN

LORENTZ TRANSFORMATION MATRIX

$$\begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}$$

• NOTE: $[x^{\mu}]$ IS THE FOUR-POSITION (WHERE $x^0 = ct$)

• FINALLY, WE CAN WRITE THE LORENTZ TRANSFORMATIONS THUS

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} \quad (\mu = 0, 1, 2, 3)$$

"FREE INDEX"

"DUMMY INDEX"

SO, THIS IS REALLY FOUR EQUATIONS

• THE INVERSE LORENTZ TRANSFORMATIONS (LT) ARE GIVEN B[Y]

WE JUST SWITCHED PRIMED AND UNPRIMED QUANTITIES, AND V FOR −V

$$\begin{cases} t = \gamma(V)(t' + Vx'/c^2) & (1.14) \\ x = \gamma(V)(x' + Vt') & (1.15) \\ y = y' & (1.16) \\ z = z' & (1.17) \end{cases}$$

• SIMILARLY, THE TRANSFORMATION RULES FOR INTERVALS ARE:

INTERVALS WHERE $\Delta t = (t_2 - t_1)$, $\Delta x = (x_2 - x_1)$, etc.

$$\begin{cases} \Delta t' = \gamma(V)(\Delta t - V\Delta x/c^2) & (1.32) \\ \Delta x' = \gamma(V)(\Delta x - V\Delta t) & (1.33) \\ \Delta y' = \Delta y & (1.34) \\ \Delta z' = \Delta z & (1.35) \end{cases}$$

• THE INVERSE TRANSFORMATION SIMPLY SUBSTITUTES $-V$ FOR V IN THE ABOVE.

• DERIVATION OF LT

• WITHOUT LOSS OF GENERALITY, WE CAN CONSIDER TRANSFORMATIONS FROM (x, t) TO (x', t') IGNORING y AND z. IN GENERAL, SUCH A TRANSFORMATION TAKES THE FORM:

S S'

$$t' = a_0 + a_1 t + a_2 x + a_3 t^2 + a_4 x^2 \ldots$$

$$x' = b_0 + b_1 x + b_2 t + b_3 x^2 + b_4 x^2 t \ldots$$

SINCE S AND S' ARE IN STANDARD CONFIGURATION $a_0 = b_0 = 0$. MOREOVER, SINCE S AND S' ARE INERTIAL, THE TRANSFORMATION MUST BE LINEAR

$$\left.\begin{array}{ll} (1.20) & t' = a_1 t + a_2 x \\ (1.21) & x' = b_1 x + b_2 t \end{array}\right\} \text{SO WE NEED TO SOLVE FOR } a_1, a_2, b_1 \text{ AND } b_2$$

• THE ORIGIN OF S' IS $x = Vt$ IN S ($x' = 0$ IN S').
SO, BY (1.21), WE KNOW:

$$0 = b_1 Vt + b_2 t$$

SO, $b_2 = -b_1 V$ (1.22)

• DIVIDING (1.21) BY (1.20) GIVES:

$$\frac{x'}{t'} = \frac{b_1 x - b_1 Vt}{a_1 t + a_2 x} \qquad (1.23)$$

• SIMILARLY,

$$\frac{-Vt'}{t'} = \frac{-b_1 Vt}{a_1 t}$$

SO, $b_1 = a_1$

• SUBSTITUTING INTO (1.23)

$$\frac{x'}{t'} = \frac{b_1 \left(\frac{x}{t}\right) - Vb_1}{b_1 + a_2 \left(\frac{x}{t}\right)}$$

• MOREOVER, SINCE $c = \frac{x'}{t'} = \frac{x}{t}$, WE HAVE:

$$c = \frac{b_1 c - Vb_1}{b_1 + a_2 c}, \quad \text{SO: } a_2 = \frac{-Vb_1}{c^2} = \frac{-V_1 a_1}{c^2}$$

• SO, WE CAN NOW WRITE:

$$t' = a_1\left(t - \frac{Vx}{c^2}\right)$$

$$x' = a_1\left(x - Vt\right)$$

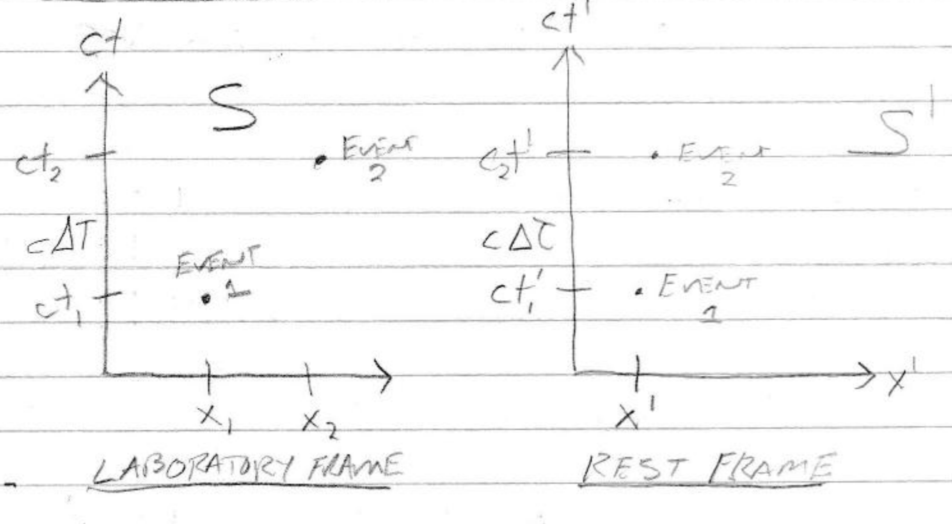
• CALCULATION THEN GIVES:

$$a_1 = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \gamma(V)$$

• TIME DILATION

$(\Delta T, ?)$

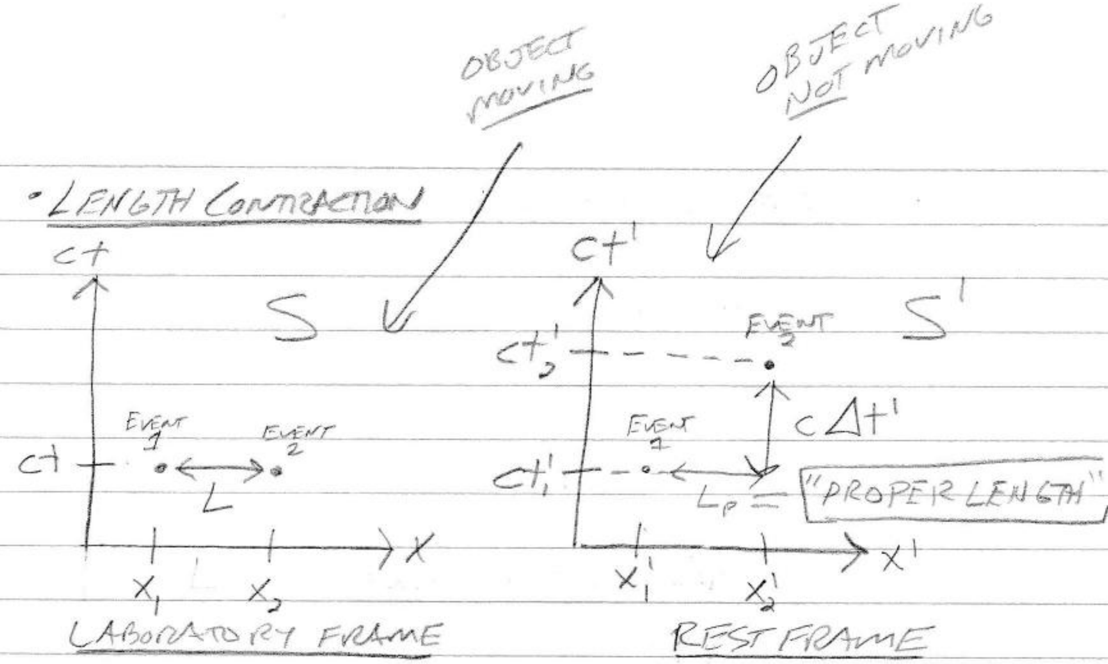
$(\Delta \tau, 0)$



• USING THE INVERSE OF (1.32) WE GET:

$$\boxed{\Delta T = \underbrace{\gamma(V)}_{>1}\Delta\tau} \qquad (1.40)$$

- **LENGTH CONTRACTION**



- THE LENGTH OF AN OBJECT IN A FRAME IS THE DISTANCE BETWEEN ITS ENDPOINTS AT A SINGLE TIME AS MEASURED IN THE FRAME.

- USING (1.33), WE GET $L_p = \gamma(V)(L-0)$. HENCE,

"LORENTZ–FITZGERALD CONTRACTION" $\boxed{L = \dfrac{L_p}{\underbrace{\gamma(V)}_{>1}}}$ ← CONTRACTION IS ALONG THE DIRECTION OF MOTION

- **RELATIVITY OF SIMULTANEITY**

- IN THE FIGURE ABOVE, EVENTS 1 AND 2 OCCUR AT THE SAME TIME IN S BUT NOT IN S'. USING (1.32), IF $\Delta t = 0$ IN S, THEN:

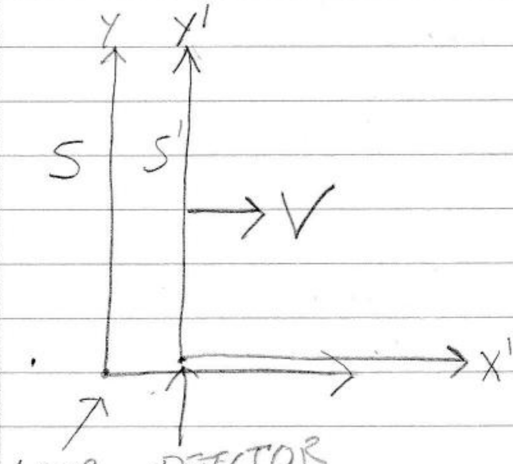
$$\boxed{\Delta t' = -\frac{\gamma(V) \overbrace{L}^{\Delta x}}{c^2}} \quad (1.41)$$

- THE DOPPLER EFFECT

  - THE DOPPLER EFFECT, KNOWLEDGE OF WHICH PREDATES RELATIVITY, ACCOUNTS FOR THE DIFFERENCE BETWEEN EMITTED AND RECEIVED FREQUENCIES (AND WAVELENGTHS) OF RADIATION. IT ARISES FROM THE RELATIVE MOTION OF THE EMITTER AND RECEIVER.

  - ASTRONOMERS USE THE DOPPLER EFFECT TO DETERMINE THE SPEED OF APPROACH OR RECESSION OF DISTANT STARS.

  - SR IMPLIES THAT THE TRADITIONAL FORMULA DESCRIBING THE EFFECT WAS INCORRECT.



- f_{em} = FREQUENCY OF LAMP'S WAVES AS MEASURED IN S (PROPER FREQUEN

- f_{rec} = FREQUENCY OF LAMP'S WAVES AS MEASURED IN S'

• THE NODE, OR POINT OF ZERO DISTURBANCE, OF THE EMITTED WAVES ARE SEPARATED BY PROPER WAVELENGTH:

$$\lambda_{em} = \frac{c}{f_{em}} \quad \text{(AS MEASURED IN S)}$$

• SO (IN S) THE TIME INTERVAL BETWEEN EMISSION OF NODES, Δt, IS THE PROPER PERIOD OF THE WAVE, T_{em}. THAT IS:

$$\Delta t = T_{em} = \frac{1}{f_{em}}$$

• NOW, O' IN S' WILL FIND THAT THE INTERVAL IS DIALATED ACCORDING TO (BY 1.40):

$$\Delta t' = \gamma(V)\Delta t$$

• A COMPUTATION THEN REVEALS:

FOR APPROACHING FRAME, REPLACE -V TO GET BLUESHIFT.

$$\boxed{f_{rec} = f_{em}\underbrace{\sqrt{\frac{c-V}{c+V}}}_{<1}} \qquad (1.42)$$

"REDSHIFT" { • UPSHOT: THE RECEIVED WAVELENGTH, $\lambda_{rec} = \frac{c}{f_{rec}}$ WILL BE GREATER THAN THE PROPER WAVELENGTH, λ_{em}.

· VELOCITY TRANSFORMATION

· DIVIDING (1.32) BY (1.33) AND REDUCING GIVES:

$$\frac{\Delta x'}{\Delta t'} = \frac{\left(\frac{\Delta x}{\Delta t} - V\right)}{\left(1 - \left(\frac{\Delta x}{\Delta t}\right)\frac{V}{c^2}\right)}$$

· LETTING $\frac{\Delta x}{\Delta t} \to 0$, AND SIMILARLY FOR Δy AND Δz

REDUCES TO GALILEAN TRANSFORM

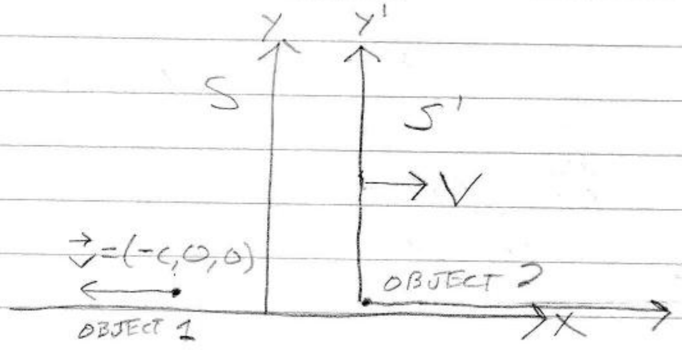
$$\boxed{v_x' = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}} \quad (1.43)$$

NOTE: WHEN $v_x V \ll c$, $\frac{v_x V}{c}$ IS SMALL.

$$\boxed{v_y' = \frac{v_y}{\gamma(V)\left(1 - \frac{v_x V}{c^2}\right)}} \quad (1.44)$$

$$\boxed{v_z' = \frac{v_z}{\gamma(V)\left(1 - \frac{v_x V}{c^2}\right)}} \quad (1.45)$$

• SPEED LIMIT OF C



$$\bullet\ v_x' = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

$$= \frac{-c - V}{1 - \frac{(-c)V}{c^2}}$$

$$= -c$$

• UPSHOT: S AND S AGREE ABOUT THE SPEED OF LIGHT EVEN THOUGH S' IS MOVING AWAY FROM OBJECT 1 AT V.

• MINKOWSKI SPACETIME CONTAINS ALL EVENTS.

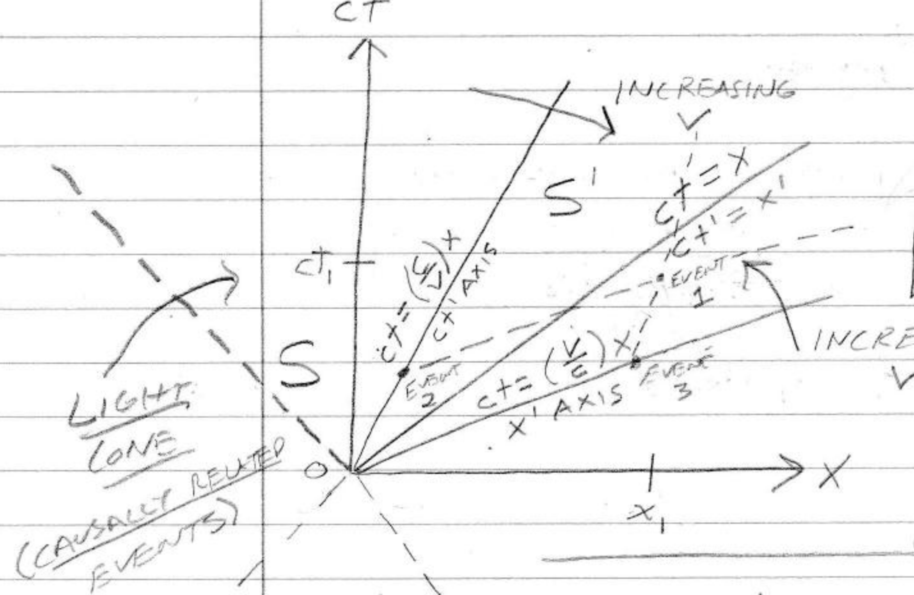
• GIVEN FRAMES S AND S', WE CAN PLOT THE ct' AND x' AXES OF S' ON THE SPACETIME DIAGRAM FOR S. THE COORDINATES OF x' AND c' ARE RELATED BY THESE LORENTZ TRANSFORMATIONS:

MULTIPLY (1.5) BY C

$$ct' = \gamma(V)\left(ct - \frac{Vx}{c}\right)$$

$$x' = \gamma(V)\left(x - Vt\right)$$

• SETTING ct' AND x' = 0 LETS US PLOT THE x' AND ct' AXES IN S.



- EVENTS 2 AND 1 ARE SIMULTANEOUS IN S', BUT NOT IN S.

- ALSO, EVEN 1 AND 3 OCCU AT THE SAME PLAC IN S', BUT NOT IN

- ALTHOUGH O AND O' DISAGREE ON THE COORDINATES OF EVENTS, THEY DO AGREE ABOUT WHICH EVENTS ARE ON, INSIDE, AND OUTSIDE OF THE LIGHTCONE.

- UPSHOT: ALL OBSERVERS AGREE ABOUT CAUSATION, EVEN THOUGH THEY DISAGREE ABOUT THE ORDER OF EVENTS.

SO LONG AS NO CAUSAL SIGNALS TRAVEL FASTER THAN C!

- NOTE: EVENTS O AND 2 CANNOT SWITCH ORDER. INCREASING V JUST COLLAPSES THE ct' AXIS ONTO THE LIGHT RAY $ct = x$. HOWEVER, EVENTS O AND 3 CAN.

- EVERY EVENT HAS AN OBJECTIVE LIGHTCONE

• <u>MINKOWSKI METRIC</u>

• IN 3-DIMENSIONAL EUCLIDEAN SPACE THE SEPARATION BETWEEN POINTS (x_1, y_1, z_1) AND (x_2, y_2, z_2) IS DESCRIBED BY:

$$(\Delta \ell)^2 = \underbrace{(\Delta x)^2}_{x_2 - x_1} + \underbrace{(\Delta y)^2}_{y_2 - y_1} + \underbrace{(\Delta z)^2}_{z_2 - z_1} \qquad (1.46)$$

• <u>NOTE:</u> $(\Delta \ell)^2$ IS <u>INVARIANT UNDER ROTATIONS</u> OF THE COORDINATE SYSTEM DESCRIBING THE POINTS.

• IN <u>4-DIMENSIONAL</u> <u>MINKOWSKI</u> <u>SPACETIME</u>, THE SEPARATION BETWEEN <u>EVENTS</u> IS GIVEN BY:

SIGNS ARE JUST A CONVENTION

• <u>SPACETIME SEPARATION:</u>

$$\boxed{(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} \qquad (1.49)$$

• <u>KEY POINT:</u> $(\Delta s)^2$ IS <u>INVARIANT UNDER LORENTZ TRANSFORMATIONS.</u>

$$(\Delta s)^2 = (c\Delta t)^2 - \underbrace{(\Delta \ell)^2} = (c\Delta t')^2 - \underbrace{(\Delta \ell')^2}_{\text{EUCLIDEAN SEPARATION}} = (\Delta s')^2$$

$$(\Delta s)^2 = (\Delta s')^2$$

• PROOF: IGNORING $\Delta y'$ AND $\Delta z'$, WE HAVE:

• $(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2$

• BY THE LORENTZ TRANSFORMATION,

$$(\Delta x')^2 = \gamma\left(c^2(\Delta t)^2 - 2V\Delta x\Delta t + \frac{V^2(\Delta x)^2}{c^2}\right)$$

$$= \gamma^2 c^2(\Delta t)^2\left(1-\frac{V^2}{c^2}\right) - \gamma^2(\Delta x)^2\left(1-\frac{V^2}{c^2}\right)$$

• Now, $\gamma^2 = \left[1-\frac{V^2}{c^2}\right]^{-1}$, SO WE GET:

$$(\Delta s')^2 = c^2(\Delta t)^2 - (\Delta x)^2 = (\Delta s)^2,$$ AS DESIRED.

• WE CAN REWRITE THE SPACETIME SEPARATION MORE EFFICIENTLY AS:

SEPARATION $\rightarrow$ $$\boxed{(\Delta s)^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \quad (1.51)}$$

WHERE Δx^0, Δx^1, Δx^2, AND Δx^3 ARE THE COMPONENTS OF

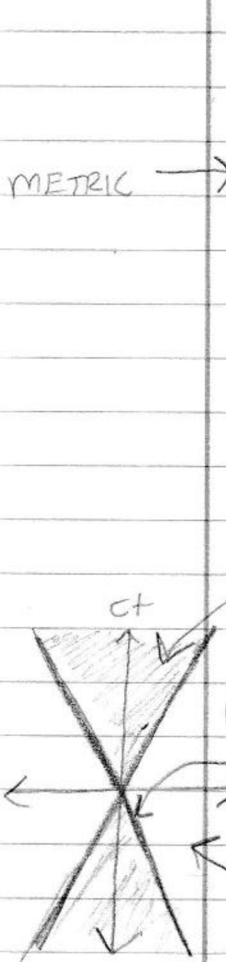
$$[\Delta x^\mu] = (c\Delta t, \Delta x, \Delta y, \Delta z)$$

AND $\eta_{\mu\nu}$ IS THE FOLLOWING MINKOWSKI METRIC:

16 COMPONENTS

$$\text{METRIC} \rightarrow [\eta_{\mu\nu}] \equiv \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.52)$$

- THE SPACETIME SEPARATION IS NOT ONLY INVARIANT (LORENTZ), WE CAN IDENTIFY THREE CLASSES OF CAUSAL RELATIONSHIPS:



- $(\Delta s)^2 > 0$: TIMELIKE = CAUSALLY RELATED
  → THERE IS A FRAME IN WHICH THE TWO EVENTS HAPPEN AT THE SAME PLACE BUT DIFFERENT TIMES.

- $(\Delta s)^2 = 0$: LIGHT-LIKE = CAUSALLY RELATED BY LIGHT SIGNALS
  → ALL FRAMES AGREE

- $(\Delta s)^2 < 0$: SPACE-LIKE = CAUSALLY UNRELATED
  → THERE IS A FRAME IN WHICH THE TWO EVENTS HAPPEN AT THE SAME TIME BUT DIFFERENT PLACES.

- THE SPACETIME SEPARATION ALSO GIVES THE LORENTZ INVARIANT PROPER TIME, τ:

WHERE $(\Delta s)^2 > 0$ (EVENTS ARE TIMELIKE SEPARATED)

$$\boxed{(\Delta\tau)^2 = \frac{(\Delta s)^2}{c^2}} \quad (1.53)$$

- **NOTE:** GIVEN TWO TIMELIKE SEPARATED EVENTS, THE PROPER TIME BETWEEN THEM IS THE LEAST AMOUNT OF TIME MEASURED BY ANY INERTIAL OBSERVER.

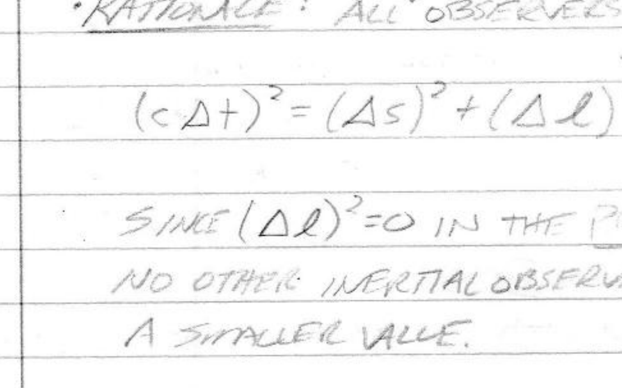
- **RATIONALE:** ALL INERTIAL OBSERVERS AGREE THAT:

$$(c\Delta t)^2 = (\Delta s)^2 + (\Delta \ell)^2 \text{ WHERE } (\Delta \ell)^2 \geq 0$$

SINCE $(\Delta \ell)^2 = 0$ IN THE PROPER FRAME, NO OTHER INERTIAL OBSERVER COULD MEASURE A SMALLER VALUE.

- THE TWIN EFFECT

PROPER TIME IS T $\longrightarrow$



- NOTE: WHILE SR SAYS THAT VELOCITY IS RELATIVE (LAWS ARE THE SAME IN ALL INERTIAL REFERENCE FRAMES), IT DOES NOT SAY THAT ACCELERATION I

- NO SINGLE INERTIAL FRAME CAN REPRESENT ASTRA'S VIEW.

· SR AND PHYSICAL LAWS

· NEWTON'S LAWS ARE NOT LORENTZ INVARIANT, SO, BY THE FIRST POSTULATE OF SR, THEY ARE FALSE.

· IN SR, WE SEEK LORENTZ INVARIANT QUANTITIES.

· EXAMPLES: SPEED OF LIGHT IN A VACUUM

SPACETIME SEPARATION, $(\Delta s)^2$

FOR TIMELIKE SEPARATED EVENTS $\longrightarrow$ PROPER TIME $(\Delta \tau)^2 = \dfrac{(\Delta s)^2}{c^2}$

· WE CAN DEFINE PROPER TIME, τ, BETWEEN TWO EVENTS AS THE TIME BETWEEN THEM AS MEASURED IN AN INERTIAL FRAME IN WHICH THE EVENTS OCCUR AT THE SAME PLACE. WE WILL SIMILARLY DEFINE THE ELECTRIC CHARGE OF A PARTICLE AS THE CHARGE IT IS MEASURED TO HAVE IN AN INERTIAL FRAME IN WHICH IT IS AT REST, SIMILARLY FOR ITS MASS.

AKA "REST MASS" (AS OPPOSED TO RELATIVISTIC MASS)

· IN SUMMARY, LORENTZ INVARIANT QUANTITIES INCLUDE:

- SPEED OF LIGHT IN VACUUM, c
- SPACETIME SEPARATION $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z$
- PROPER TIME $(\Delta \tau)^2 = (\Delta s)/c^2$ (FOR TIMELIKE SEPARATED EVENTS)
- THE CHARGE OF A PARTICLE, q
- THE MASS OF A PARTICLE, m

- JUST AS WE SEEK THE INVARIANCE OF QUANTITIES, WE CAN SEEK LAWS THAT ARE FORM-INVARIANT.

IGNORING ELECTROMAGNETIC FORCES, ETC

- EXAMPLE: NEWTON'S LAWS OF MOTION ARE FORM-INVARIANT UNDER GALILEAN TRANSFORMATIONS.

APPLIED FORCE

MASS OF BODY

ITS ACCELERATION

- $\vec{F} = m\vec{a} = m\left(\frac{d^2x}{dt}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$ FORMULATED FRAME S

IN STANDARD COORDINATION

- THE COORDINATES OF ANOTHER FRAME, S' ARE RELATED TO THOSE OF S BY THE GALILEAN TRANSFORMATIONS:

$$t' = t$$
$$x' = x - Vt$$
$$y' = y$$
$$z' = z$$

- SINCE MASS IS GALILEAN INVARIANT, AND $\frac{\partial x^i}{\partial t} = \frac{\partial x^i}{\partial t'}$ FOR $i = x, y, z$, WE HAVE:

$$m\left(\frac{d^2x'}{dt'^2}, \frac{d^2y'}{dt'^2}, \frac{d^2z'}{dt'^2}\right) = ma'$$

- IT REMAINS ONLY TO SHOW THAT F IS GALILEAN INVARIANT ITSELF. THIS IS MORE INVOLVED.

• AN EQUATION THAT IS <u>FORM-INVARIANT</u> UNDER A GIVEN TRANSFORMATION IS CALLED "<u>COVARIANT</u>" UNDER THAT TRANSFORMATION.

• <u>NOTE</u>: A LAW CAN BE <u>COVARIANT</u> UNDER A TRANSFORMATION EVEN THOUGH THE QUANTITIES IT GOVERNS ARE <u>NOT INVARIANT</u> UNDER IT. (THOUGH WE HAVE <u>BOTH</u> KINDS OF INVARIANCE IN THE CASE OF $\vec{F} = m\vec{a}$.)

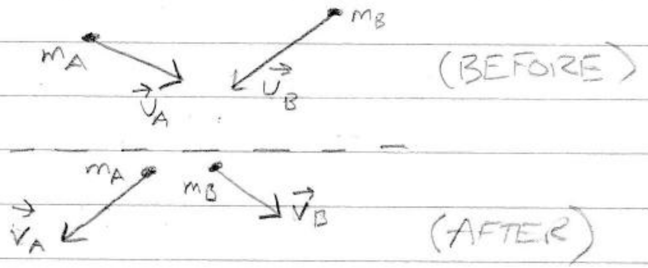
• <u>RELATIVISTIC MOMENTUM</u>

• IN <u>NEWTONIAN MECHANICS</u> (NM), <u>THE MOMENTUM OF A PARTICLE OF MASS</u>, m, TRAVELING WITH <u>VELOCITY</u>, v, IS:

$$\vec{P}_{NEWTON} = m\vec{v}$$

MOREOVER, <u>IN NM THE TOTAL MOMENTUM OF A SYSTEM IS CONSERVED</u> (i.e., REMAINS <u>CONSTANT</u>), PROVIDED THAT NO <u>EXTERNAL FORCES</u> ACT ON THE SYSTEM.

$$\boxed{m_A\vec{U}_A + m_B\vec{U}_B = m_A\vec{V}_A + m_B\vec{V}_B}$$



• BY CONTRAST, IN SR, EVEN IF MOMENTUM IS CONSERVED IN ONE INERTIAL FRAME, IT IS NOT CONSERVED IN ALL OTHERS. ACCORDINGLY, WE SEEK A NEW DEFINITION OF MOMENTUM, RELATIVISTIC MOMENTUM THAT SIMPLY TRANSFORMS UNDER LORENTZ TRANSFORMATION, AND THAT PROVIDES A LAW THAT IS LORENTZ-COVARIANT.

$\left(\frac{\Delta x}{\Delta \tau}\right)$ TRANSFORMS IN THE SAME WAY AS DISPLACEMENT

• NEWTONIAN MOMENTUM TRANSFORMS IN A COMPLICATED WAY BECAUSE BOTH Δx AND Δt (LORENTZ) TRANSFORM IN MODERATELY COMPLICATED WAYS. BUT PROPER TIME, $\Delta \tau$, IS INVARIANT. THIS SUGGESTS DEFINING:

$$(\text{RELATIVISTIC}) \text{ MOMENTUM} = \vec{p} = \underset{\uparrow}{m}\left(\frac{\Delta x}{\Delta \tau}, \frac{\Delta y}{\Delta \tau}, \frac{\Delta z}{\Delta \tau}\right)$$

INVARIANT

$$= m\gamma(v)\left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)$$

$$(\text{BECAUSE } \Delta t = \gamma(v)\Delta \tau)$$

$$= m\gamma(v)\vec{v}$$

• THIS TURNS OUT TO WORK. SO, WE HAVE:

$$\boxed{\text{RELATIVISTIC MOMENTUM: } \vec{p} = \gamma(v)\vec{v} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

· RELATIVISTIC KINETIC ENERGY:

· HERE TOO WE MUST MODIFY THE NEWTONIAN DEFINITION IN ORDER TO ACHIEVE A LORENTZ-COVARIANT FORMULATION OF MECHANICS.

· IN NEWTONIAN MECHANICS, WE HAVE:

$$E_K = W = \int_{u=0}^{u=v} f \, dx$$

WORK DONE ACCELERATING PARTICLE TO ITS VELOCITY v

MAGNITUDE OF FORCE CAUSING ACCELERATION

$$= \int_{u=0}^{u=v} \frac{dp}{dt} \, dx$$

IN NEWTONIAN MECHANICS!

MAGNITUDE OF MOMENTUM

$$\text{Now, } \int \frac{dp}{dx} dx = \int \frac{dp}{dx}\frac{dx}{dp} dp = \int \frac{dx}{dt} dp = \int u \, dp$$

· THIS SUGGESTS DEFINING:

ENSURES THAT WHAT IS AN ELASTIC COLLISION IN ONE FRAME IS IN ANOTHER LORENTZ-TRANSFORMED FRAME TOO

RELATIVISTIC KINETIC ENERGY:

$$E_K = \int_0^v u \, d\left(\frac{mu}{\sqrt{1-\frac{mu}{\sqrt{1-\left(\frac{u}{c}\right)^2}}}}\right) = \left(\frac{mc^2}{\sqrt{1-\left(\frac{v^2}{c^2}\right)^2}}\right) - mc^2$$

$$(2.22) \qquad = \left(\gamma(v) - 1\right) mc^2$$

· NOTE: THIS REDUCES TO NEWTONIAN KINETIC ENERG
FOR LOW VELOCITIES, V, BECAUSE:

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \underbrace{1 + \frac{v^2}{2c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \dots}_{\text{TAYLOR EXPANSION}}$$

SO, FOR V<<C,

$$E_k = (\gamma(v) - 1)mc^2 = \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 - 1\right]$$

NEWTONIAN APPROXIMATION FOR V<<C

$$\boxed{\approx \frac{1}{2}mv^2 + \text{TERMS OF ORDER}}$$

A.K.A "REST ENERGY"

· RELATIVISTIC AND MASS ENERGY

· EQUATION (2.22) CAN BE REWRITTEN:

"MASS ENERGY" (E_0)

$$\gamma(v)mc^2 = E_k + mc^2 \qquad (2.23)$$

$E = E_k + E_0$

IN AN INERTIAL FRAME, S, WHERE A PARTICLE OF MASS, M, HAS SPEED, V, THE PARTICLE HAS TOTAL RELATIVISTIC ENERGY

$E = \gamma(v)mc^2$ = KINETIC ENERGY + MASS ENERGY

· EINSTEIN'S REVOLUTIONARY ADDITION TO THIS REARRANGEMENT OF TERMS WAS TO

PROPOSE THAT, ABSENT EXTERNAL FORCES, TOTAL RELATIVISTIC ENERGY IS CONSERVED EVEN THOUGH NEITHER KINETIC NOR MASS ENERGY GENERALLY IS INDIVIDUALLY.

"EQUIVALENCE OF MASS AND ENE

- UPSHOT: PARTICLES WITH MASS MAY BE CREATED AT EXPENSE OF KINETIC ENERGY, AND ENERGY MAY BE CREATED AT THE EXPENSE OF MASS.

- TOTAL RELATIVISTIC & MASS ENERGY

AND VELOCITY v

TOTAL ENERGY OF A PARTICLE

$$\boxed{E = \gamma(v)\, mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (2.24)}$$

W/ MASS m

MASS/REST ENERGY OF PARTICLE

$$\boxed{E_0 = mc^2 \qquad (2.25)}$$

- FOUR-MOMENTUM

- RECALL THAT THE FOUR-POSITION OF AN OBJECT IS:

UNITS OF DISTANCE

$$[x^\mu] = (x^0, x^1, x^2, x^3) = (ct, \underbrace{x, y, z}_{\vec{r}})$$

$$= (ct, \vec{r})$$

EACH COORDINATE, ct, x, y, z, CAN BE THOUGHT OF AS A FUNCTION OF τ

• WE CAN DEFINE A NEW QUANTITY BY DIFFERENTIATING WITH RESPECT TO PROPER TIME — i.e. BY TAKING THE "PROPER DERIVATIVE" — TO DEFINE:

• FOUR-VELOCITY: $[U^\mu] = \left[\frac{dx^\mu}{d\tau}\right] = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$ (2

→ SINCE $\Delta t = \gamma(v)\Delta\tau$, WE HAVE:

BEHAVES LIKE FOUR-POSITION UNDER LORENTZ TRANSFORMATION

$$\boxed{[U^\mu] = (U^0, U^1, U^2, U^3) = (c\gamma(v), \gamma(v)\vec{v})} \quad (2.28)$$

• RECALL THE INVARIANT DISTANCE:

$$(\Delta s)^2 = \sum_{\mu,\nu=0}^{3} \underbrace{\eta_{\mu\nu}}_{\text{MINKOWSKI METRIC}} \Delta x^\mu \Delta x^\nu$$

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^\mu U^\nu = \gamma(v)^2 c^2 - \gamma(v)^2\left[(V_x)^2 + (V_y)^2 + (V_z)^2\right]$$

$$= \gamma(v)^2\left[c^2 - v^2\right]$$

Now, $\gamma(v)^2 = \frac{c^2}{c^2 - v^2}$

So,

(2.31) $$\boxed{\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^\mu U^\nu = c^2}$$ WHICH IS ALSO INVARIANT.

· MULTIPLYING FOUR-VELOCITY $[U^\mu]$ BY MASS, m, WHICH IS INVARIANT, GIVES ANOTHER INVARIANT QUANTITY:

· FOUR-MOMENTUM

$$[P^\mu] = m\underset{\substack{\uparrow \\ \text{FOUR-VELOCITY}}}{[U^\mu]} = (\overbrace{\underbrace{\gamma(v)mc}_{\substack{\text{TOTAL RELATIVISTIC} \\ \text{ENERGY DIVIDED BY } c}}}^{E/c}, \overbrace{\underbrace{\gamma(v)mv_x, \gamma(v)mv_y, \gamma(v)\ldots}_{\text{RELATIVISTIC MOMENTUM COMPONENTS}}}^{\vec{p}}$$

FOUR-MOMENTUM → $\boxed{= [P^0, P^1, P^2, P^3] = (E/c, \vec{p})}$ (2.33)

TRANSFORMS AGAIN JUST LIKE FOUR-POSITION (b/c τ AND m ARE INVARIANT)

CONTAINS ALL INFORMATION ABOUT RELATIVISTIC ENERGY & RELATIVISTIC MOMENTUM OF ANY PARTICLE

· UPSHOT: IF INERTIAL FRAMES S AND S' ARE IN STANDARD CONFIGURATION, THEN A PARTICLE WITH MASS, m, VELOCITY, $\vec{v}$, RELATIVISTIC ENERGY, E, AND RELATIVISTIC MOMENTUM, $\vec{p}$, IN FRAME, S, THEN THAT PARTICLE WILL HAVE ENERGY AND MOMENTUM IN S':

$v' = \frac{(v-\ldots}{(1-\ldots}$

$$E' = \gamma(V)(E - Vp_x) \quad (2.34)$$

FRAME SPEED, NOT PARTICLE SPEED, v

TRANSFORMATION OF FOUR-MOMENTUM

$$p_x' = \gamma(V)\left(p_x - \frac{VE}{c^2}\right) \qquad (2.35)$$

$$p_y' = p_y \qquad (2.36)$$

$$p_z' = p_z \qquad (2.37)$$

• AS WITH FOUR-POSITION, WE CAN REWRITE THIS AS:

(2.38)
$$\begin{pmatrix} \frac{E'(v')}{c} \\ p_x'(v') \\ p_y'(v') \\ p_z'(v') \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma(V) & -\frac{\gamma(V)V}{c} & 0 & 0 \\ -\frac{\gamma(V)V}{c} & \gamma(V) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{V = \text{FRAME SPEED}} \begin{pmatrix} \frac{E(v)}{c} \\ p_x(v) \\ p_y(v) \\ p_z(v) \end{pmatrix}$$

v' = PARTICLE SPEED

v = PARTICLE SPEED

OR, MORE SIMPLY:

$$[p'^{\mu}] = [\Lambda^{\mu}{}_{\nu}][p^{\nu}] \qquad (2.40)$$

WHICH MEANS:

$$p'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} p^{\nu} \quad (\mu = 0,1,2,3) \qquad (2.40)$$

"MOMENERGY"

• KEY POINT: UNDER LORENTZ TRANSFORMATION, THE ENERGY & MOMENTUM COMPONENTS INTERTWINE, JUST AS SPACE & TIME COMPONENTS DO. WE NOW HAVE SPACETIME AND "MOMENERGY".

• ENERGY-MOMENTUM RELATION

• RECALL FROM (2.31) THAT:

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^\mu U^\nu = c^2$$

NOW $[P^\mu]$ IS JUST $m[U^\mu]$, SO:

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} P^\mu P^\nu = m^2c^2 \qquad (2.41)$$

USING THE MINKOWSKI METRIC, WE KNOW:

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} P^\mu P^\nu = \frac{E^2}{c^2} - (P_x)^2 - (P_y)^2 - (P_z)^2$$

$$= \frac{E^2}{c^2} - p^2 \qquad (2.42)$$

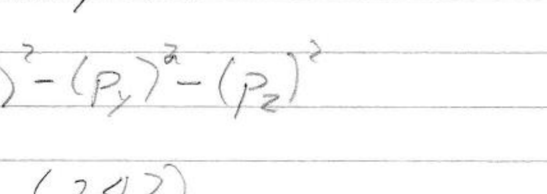
AS MOMENTUM MAGNITUDE INCREASES, TOTAL ENERGY BECOMES MORE AND MORE NEARLY PROPORTIONAL TO MOMENTUM MAGNITUDE.

COMBINING (2.41) AND (2.42) THEN GIVES:

$$E^2 - c^2p^2 = m^2c^4$$

OR:

$$\boxed{E = \sqrt{m^2c^4 + c^2p^2}}$$



• ENERGY-MOMENTUM RELATION:

$$\boxed{E^2 = p^2c^2 + m^2c^2 \qquad (2.43)}$$

→ UPSHOT: IN ANY INERTIAL FRAME THE DIFFERENCE BETWEEN THE SQUARED TOTAL ENERGY

HAS NO ANALOG IN NEWTONIAN MECHANICS

AND THE SQUARED MOMENTUM MAGNITUDE TIMES C^2 IS PROPORTIONAL TO THE SQUARED (INVARIANT) MASS.

- THE ENERGY-MOMENTUM RELATION IMPLIES A SYMMETRY PRINCIPLE KNOWN AS "GAUGE INVARIANCE". IT IMPLIES THAT A PHOTON IS MASSLESS.

SO, FOR A PHOTON: $p = \frac{E}{c}$

i.e. A PHOTON HAS MOMENTUM WITHOUT MASS - IN CONTRAST TO THE NEWTONIAN CONCEPT.

"SOLAR SAILS"

- THE CONSERVATION OF ENERGY & MOMENTUM

- WE CAN NOW SEE THAT THE LAWS OF RELATIVISTIC ENERGY AND MOMENTUM ARE LORENTZ COVARIANT. INDEED, THEY ARE MANIFESTLY COVARIANT.

SOME OF WHICH MAY HAVE 0 MASS

- EXAMPLE: IN A COLLISION OF N PARTICLES FROM WHICH $\bar{N}$ PARTICLES EMERGE, OUR CONSERVATION LAW SAYS:

FREE INDEX $(\nu = 0,1,2,3)$

IN FRAME S $\longrightarrow$ $P^{\nu}_{(1)} + P^{\nu}_{(2)} + \dots P^{\nu}_{(N)} = \bar{P}^{\nu}_{(1)} + \bar{P}^{\nu}_{(2)} + \dots \bar{P}^{\nu}_{(\bar{N})}$ (2.46)

IN FRAME S' $\longrightarrow$ $\sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} \left(P^{\nu}_{(1)} + P^{\nu}_{(2)} + \dots P^{\nu}_{(N)}\right) = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} \left(\bar{P}^{\mu}_{(1)} + \bar{P}^{\mu}_{(2)} + \dots \bar{P}^{\nu}_{(\bar{N})}\right)$ (

LORENTZ TRANSFORM

SINCE $[P'^{\mu}] = [\Lambda^{\mu}_{\nu}][P^{\nu}]$, WE GET:

$$P'^{\mu}_{(1)} + P'^{\mu}_{(2)} + \dots P'^{\mu}_{(N)} = \bar{P}'^{\mu}_{(1)} + \bar{P}'^{\mu}_{(2)} + \dots \bar{P}'^{\mu}_{(\bar{N})}$$

WHICH HAS MANIFESTLY THE SAME FORM.

• FOUR-FORCE

• IN NEWTONIAN MECHANICS, FORCE IS THE RATE OF CHANGE OF MOMENTUM, $\vec{F} = m\left(\frac{d\vec{v}}{dt}\right) = d\vec{p}$.

• CORRESPONDINGLY, IN RELATIVISTIC MECHANICS, WE DEFINE THE FOUR-FORCE AS THE MANIFESTLY COVARIANT RELATION:

$$[F^{\mu}] = \left[\frac{dP^{\mu}}{d\tau}\right] = \left[\frac{1}{c}\frac{dE}{d\tau}, \underbrace{\frac{dp_x}{d\tau}, \frac{dp_y}{d\tau}, \frac{dp_z}{d\tau}}_{\left[\gamma(v)\vec{f}\right]}\right] \qquad (2.49$$

INVARIANT PROPER TIME

$$\frac{1}{c}\frac{dE}{d\tau} = \left[\frac{\gamma(v)}{c}\frac{dE}{dt}\right]$$

$\vec{f} = (f_x, f_y, f_z)$

(CONVENTIONAL FORCE)

$\frac{dE}{dt}$ = RATE OF CHANGE OF TOTAL ENERGY $= \vec{f} \cdot \vec{v}$

SO:

$$[F^{\mu}] = \left[\frac{\partial p^{\mu}}{\partial \tau}\right] = \left(\frac{\gamma}{c}\vec{f}\cdot\vec{v}, \gamma f_x, \gamma f_y, \gamma f_z\right)$$

$$= \left(\frac{\gamma}{c}\vec{f}\cdot\vec{v}, \gamma\vec{f}\right) \qquad (2.51)$$

- NOTE: THE "CONVENTIONAL" FORCE, $\vec{f}$, IS NOT EXACTLY THE NEWTONIAN FORCE, SINCE THE FORMER, BUT NOT THE LATTER, IS INVARIANT UNDER LORENTZ TRANSFORMATION.

- UPSHOT: WE CANNOT TREAT THE NEWTONIAN GRAVITATIONAL FORCE AS PART OF THE FOUR-FORCE (BUT WE WILL SEE THAT WE CAN TREAT THE ELECTROMAGNETIC LORENTZ FORCE THAT WAY).

  → HENCE THE NEED FOR A NEW THEORY OF GRAVITY THAT GOES BEYOND SPECIAL RELATIVITY.

FOR S AND S':

$F'^0 = \gamma(V)\left(F_0 - \frac{VF^1}{c}\right)$

$F'^1 = \gamma(V)\left(F_1 - \frac{VF^0}{c}\right)$

$F'^2 = F^2$

$F'^3 = F^3$

- FOUR-VECTORS

  - WE HAVE NOW INTRODUCED A SERIES OF IMPORTANT CONTRAVARIANT FOUR-VECTORS:

SPECIAL CASE OF FOUR-DISPLACEMENT WHERE COORDINATE INTERVALS ARE MEASURED FROM ORIGIN

- FOUR-DISPLACEMENT: $[\Delta x^\mu] = (c\Delta t, \Delta\vec{r})$
- FOUR-POSITION: $[x^\mu] = (ct, \vec{r})$
- FOUR-VELOCITY: $[U^\mu] = (\gamma c, \gamma\vec{v})$
- FOUR-MOMENTUM: $[p^\mu] = \left(\frac{E}{c}, \vec{p}\right)$
- FOUR-FORCE: $\left(\gamma\vec{f}\cdot\frac{\vec{v}}{c}, \gamma\vec{f}\right)$

- CONTRAVARIANT FOUR-VECTORS ARE IMPORTANT BECAUSE OF HOW THEY BEHAVE UNDER LORENTZ TRANSFORMATION.

  - GIVEN FRAMES S AND S' IN STANDARD CONFIGURATION, THE COMPONENTS, A^μ, OF A CONTRAVARIANT FOUR-VECTOR, $[A^\mu] = (A^0, A^1, A^2, A^3)$ TRANSFORM THUS:

$$A'^0 = \gamma(V)\left(A^0 - \frac{VA^1}{c}\right) \qquad (2.56)$$

$$A'^1 = \gamma(V)\left(A^1 - \frac{VA^0}{c}\right) \qquad (2.57)$$

$$A'^2 = A^2 \qquad (2.58)$$

$$A'^3 = A^3 \qquad (2.59)$$

• OR, MORE SIMPLY:

$$[A'^{\mu}] = [\Lambda^{\mu}{}_{\nu}][A^{\nu}] \qquad (2.60)$$

$$A'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} A^{\nu} \; (\mu = 0,1,2,3) \qquad (2.6$$

• <u>NOTE</u>: NOT ALL FOUR-VECTORS ARE <u>CONTRAVARIANT</u>, AND NOT ALL <u>FOUR-COMPONENT OBJECTS</u> ARE <u>FOUR-VECTORS</u>.

<u>CONTRAVARIANT</u> VS. <u>COVARIANT</u> FOUR-VECTORS

• WHILE <u>CONTRAVARIANT</u> <u>FOUR-VECTORS</u> — WRITTEN WITH <u>UPPER INDEX</u>, $[A^{\mu}]$ — TRANSFORM LIKE <u>FOUR-POSITIONS</u>, SO-CALLED <u>COVARIANT FOUR-VECTORS</u> — WRITTEN WITH A <u>LOWER INDEX</u>, $[B_{\mu}]$ — TRANSFORM LIKE <u>DERIVATIVES OF SCALAR FUNCTIONS</u>.

• LET ϕ BE A FUNCTION OF x^0, x^1, x^2, x^3 THAT IS <u>INVARIANT</u> UNDER <u>LORENTZ TRANSFORMATION</u> SO, $\phi'(x'^0, x'^1, x'^2, x'^3) = \phi(x^0, x^1, x^2, x^3)$.

• HOW DOES THE DERIVATIVE $\frac{\partial \phi}{\partial x^0}$, WRITTEN B_0, TRANSFORM? THE <u>CHAIN RULE</u> TELLS US:

(2.62) $$B'_0 = \frac{\partial \phi'}{\partial x'^0} = \frac{\partial \phi}{\partial x^0}\frac{\partial x^0}{\partial x'^0} + \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial x'^0}$$

$$+ \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial x'^0} + \frac{\partial \phi}{\partial x^3}\frac{\partial x^3}{\partial x'^0}$$

• NOW, $\frac{\partial x^0}{\partial x'^0}$, $\frac{\partial x^1}{\partial x'^0}$, $\frac{\partial x^2}{\partial x'^0}$ AND $\frac{\partial x^3}{\partial x'^0}$

CAN BE DETERMINED FROM THE <u>INVERSE LORENTZ TRANSFORMATIONS</u>:

$$\left\{\begin{array}{ll} t = \gamma(V)\left(t' - \frac{Vx'}{c^2}\right) & (1.14) \\ x = \gamma(V)(x' + Vt') & (1.15) \\ y = y' & (1.16) \\ z = z' & (1.17) \end{array}\right.$$

THESE GIVE:

$$\frac{\partial x^0}{\partial x'^0} = \gamma(V) \qquad \frac{\partial x^1}{\partial x'^0} = \gamma(V)\frac{V}{c}$$

$$\frac{\partial x^2}{\partial x'^0} = 0 \qquad \frac{\partial x^3}{\partial x'^0} = 0$$

SUBSTITUTING INTO (2:62) AND WRITING B_μ FOR $\frac{\partial \phi}{\partial x^\mu}$, WE GET:

$$B'_0 = \gamma(V)\left(B_0 + \frac{VB_1}{c}\right) \qquad (2.63)$$

• SIMILAR CALCULATIONS GIVE:

$$B'_2 = \gamma(V)\left(B_1 + \frac{VB_0}{c}\right) \qquad (2.64)$$

$$B'_2 = B_2 \qquad (2.65)$$

$$B'_3 = B_3 \qquad (2.66)$$

• USING $\left[(\Lambda^{-1})_\mu{}^\nu\right]$ TO REPRESENT THE INVERSE LORENTZ TRANSFORMATION:

$$\left[(\Lambda^{-1})_\mu{}^\nu\right] = \begin{pmatrix} (\Lambda^{-1})_0{}^0 & (\Lambda^{-1})_0{}^1 & (\Lambda^{-1})_0{}^2 & (\Lambda^{-1})_0{}^3 \\ (\Lambda^{-1})_1{}^0 & (\Lambda^{-1})_1{}^1 & (\Lambda^{-1})_1{}^2 & (\Lambda^{-1})_1{}^3 \\ (\Lambda^{-1})_2{}^0 & (\Lambda^{-1})_2{}^1 & (\Lambda^{-1})_2{}^2 & (\Lambda^{-1})_2{}^3 \\ (\Lambda^{-1})_3{}^0 & (\Lambda^{-1})_3{}^1 & (\Lambda^{-1})_3{}^2 & (\Lambda^{-1})_3{}^3 \end{pmatrix}$$

WE CAN WRITE THE TRANSFORMATION RULE FOR $[B_\mu]$ AS FOLLOWS:

ANY FOUR-COMPONENT ENTITY THAT SO TRANSFORMS IS A COVARIANT FOUR-VECTOR

$$\rightarrow [B'_\mu] = [(\Lambda^{-1})_\mu{}^\nu][B_\nu] \qquad (2.68)$$

$$B'_\mu = \sum_{\nu=0}^{3} (\Lambda^{-1})_\mu{}^\nu B_\nu \quad (\mu = 0,1,2,3) \qquad (2.69)$$

COVARIANT (LOWER INDEX)

- CONTRAVARIANT AND COVARIANT FOUR-VECTORS

(1) RAISING & LOWERING FOUR-VECTOR INDICES

- FOR EVERY CONTRAVARIANT FOUR-VECTOR, THERE IS A CORRESPONDING COVARIANT FOUR-VECTOR, AND VICE VERSA. THEY ARE RELATED VIA THE MINKOWSKI METRIC:

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (1.52)$$

- IF A^0, A^1, A^2, AND A^3 TRANSFORM AS CONTRAVARIANT FOUR-VECTORS, THEN WE GET FOUR QUANTITIES THAT TRANSFORM AS COVARIANT VECTORS, A_0, A_1, A_2, AND A_3 AS FOLLOWS:

COVARIANT (LOWER INDEX)

CONTRAVARIANT (RAISED INDEX)

(2.70) $$A_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} A^\nu \quad (\mu = 0,1,2,3)$$

→ UPSHOT: THE MINKOWSKI METRIC CAN BE USED TO LOWER THE INDICES OF FOUR-VECTORS.

• IN THIS CASE, THE COMPUTATION IS EASY TO CARRY OUT:

IF $[A^\mu] = (a, b, c, d)$, THEN $[A_\mu] = (a, -b, -c, -d)$

• THIS GIVES:

• COVARIANT FOUR-DISPLACEMENT: $[\Delta x_\mu] = (ct, -\Delta$

• COVARIANT FOUR-VELOCITY: $[U_\mu] = (\gamma c, -\gamma \vec{v})$

• COVARIANT FOUR-MOMENTUM: $[P_\mu] = \left(\frac{E}{c}, -\gamma \vec{v}\right)$

• COVARIANT FOUR-FORCE: $[F_\mu] = \left(\gamma \vec{f} \cdot \frac{\vec{v}}{c}, -\gamma \vec{f}\right.$

• TO CONVERT FROM COVARIANT VECTORS TO CONTRAVARIANT VECTORS, WE DEFINE:

$$[\eta^{\mu\nu}] = \begin{pmatrix} \eta^{00} & \eta^{01} & \eta^{02} & \eta^{03} \\ \eta^{10} & \eta^{11} & \eta^{12} & \eta^{13} \\ \eta^{20} & \eta^{21} & \eta^{22} & \eta^{23} \\ \eta^{30} & \eta^{31} & \eta^{32} & \eta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (2.71)$$

WHICH GIVES:

COVARIANT (LOWER INDEX)

$$A^{\mu} = \sum_{\nu=0}^{3} \eta^{\mu\nu} A_{\nu} \quad (\mu = 0, 1, 2, 3) \qquad (2.72)$$

CONTRAVARIANT (UPPER INDEX)

• NOTE: ALTHOUGH $[\eta_{\mu\nu}]$ AND $[\eta^{\mu\nu}]$ HAVE IDENTICAL COMPONENTS, THEY ARE INVERSELY RELATED, THAT IS:

$$\sum_{\nu} \eta^{\alpha\nu} \eta_{\nu\beta} = \delta^{\alpha}_{\beta}$$

WHERE $[\delta^{\alpha}_{\beta}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) INVARIANTS BY CONTRACTION

- RECALL THAT WE OBTAIN INVARIANTS BY SUMMING COMPONENTS, AS IN:

$$\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^{\mu} U^{\nu} = c^2 \qquad (2.31)$$

BUT THIS INVOLVES A SUM OF THE COMPONENTS OF A CONTRAVARIANT FOUR-VECTOR AND ITS COVARIANT COUNTERPART.

Used

$$\sum_{\nu=0}^{3} U_{\nu} U^{\nu} = U_0 U^0 + U_1 U^1 + U_2 U^2 + U_3 U^3 \qquad (2.74$$

- SINCE THE CONTRAVARIANT AND COVARIANT COMPONENTS TRANSFORM INVERSELY UNDER LORENTZ TRANSFORMATIONS, SUCH SUMS ARE INVARIANT.

→ OTHER EXAMPLES:

$$\sum_{\nu=0}^{3} P_{\nu} P^{\nu} = m^2 c^2$$

$$\sum_{\nu=0}^{3} \Delta x_{\nu} \Delta x^{\nu} = (\Delta s)^2$$

- THE PROCESS OF SUMMING OVER A RAISED INDEX AND LOWERED INDEX IS CALLED CONTRACTION.

- QUANTITIES THAT ARE INVARIANT UNDER LORENTZ TRANSFORMATION ARE CALLED "LORENTZ SCALARS".

(3) LORENTZ TRANSFORMATION GENERALLY

- ALTHOUGH WE HAVE FOCUSED ON THE CASE WHERE S AND S' ARE IN STANDARD CONFIGURATION, THERE IS NO LOSS OF GENERALITY IN DOING THIS. IN GENERAL, A CONTRAVARIANT FOUR-VECTOR, $[A^\mu]$, TRANSFORMS LIKE FOUR-DISPLACEMENT:

WHERE: $x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$

$$A'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu$$

- UNDER THE SAME LORENTZ TRANSFORMATION, A COVARIANT FOUR-VECTOR, $[B_\mu]$, TRANSFORMS:

TRANSFORMS LIKE A SET OF DERIVATIVES

$$B'_\mu = \sum_{\nu=0}^{3} \underbrace{(\Lambda^{-1})^\nu{}_\mu}_{\text{MATRIX INVERSE OF } [\Lambda^\mu{}_\nu]} B_\nu$$

- INDICES ON FOUR-VECTORS CAN BE RAISED OR LOWERED USING THE MINKOWSKI METRIC, $\eta_{\mu\nu}$, OR $\eta^{\mu\nu}$ (WHERE $\sum_\gamma \eta^{\alpha\gamma}\eta_{\gamma\beta} = \delta^\alpha_\beta$) AS FOLLOWS:

$$\cdot A_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} A^\nu \qquad (2.70)$$

$$\cdot A^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} A_\nu \qquad (2.72)$$

• FINALLY, WE GET <u>LORENTZ</u> <u>INVARIANTS</u> BY CONTRACTING – i.e., <u>SUMMING</u> OVER ONE RAISED AND ONE LOWERED INDEX AS FOLLOWS:

$$\sum_{\nu=0}^{3} A^\nu B_\nu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \qquad (2.75)$$

EXAMPLES:

$$\boxed{\sum_{\nu=0}^{3} U_\nu U^\nu = c^2}$$

$$\boxed{\sum_{\nu=0}^{3} P_\nu P^\nu = m^2 c^2}$$

• <u>KEY POINT</u>: <u>FOUR VECTORS</u> CAN BE USED TO FORMULATE THE <u>LAWS</u> OF <u>MECHANICS</u> IN A <u>MANIFESTLY</u> <u>LORENTZ COVARIANT WAY</u>, AS IN:

$$F^\mu = \frac{dP^\mu}{d\tau}$$

HOWEVER, THE FORCE IN NEWTON'S <u>UNIVERSAL LAW OF GRAVITATION</u> DOES <u>NOT</u> TRANSFORM IN THE NECESSARY WAY. SO, WE WILL NEED TO REVISIT LAWS OF GRAVITY.

· LAWS OF ELECTROMAGNETISM

· THE LAWS OF ELECTROMAGNETISM (UNLIKE THOSE OF NEWTONIAN MECHANICS) ARE ALREADY CONSISTENT WITH SR. WE NEED MERELY TO REWRITE THEM.

· A BASIC LAW OF ELECTROMAGNETISM IS THE CONSERVATION OF CHARGE (CHARGE CAN NEITHER BE CREATED OR DESTROYED).

· EQUATION OF CONTINUITY:

J = ELECTRIC CURRENT DENSITY

ρ = DENSITY OF ELECTRIC CHARGE

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

· THE CHARGE DENSITY AND CURRENT DENSITY CAN BE EXPRESSED AS COMPONENTS OF A FOUR-VECTOR, $[J^\mu] = (c\rho, J_x, J_y, J_z)$, KNOWN AS THE ELECTRIC FOUR-CURRENT.

· THE EQUATION OF CONTINUITY NOW BECOMES:

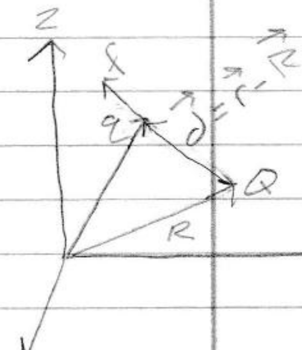
$$\boxed{\sum_{\nu=0}^{3} \frac{\partial J^\nu}{\partial x^\nu} = 0} \leftarrow \boxed{\text{COVARIANT}} \quad (2.77)$$

INVARIANT FORMED BY CONTRACTION

"LOWER" INDEX

• LORENTZ FORCE LAW

• THE ELECTROSTATIC FORCE ON A PARTICLE OF CHARGE q AT POSITION $\vec{r}$ DUE TO ANOTHER PARTICLE OF CHARGE Q AT POSITION $\vec{R}$ IS GIVEN BY:



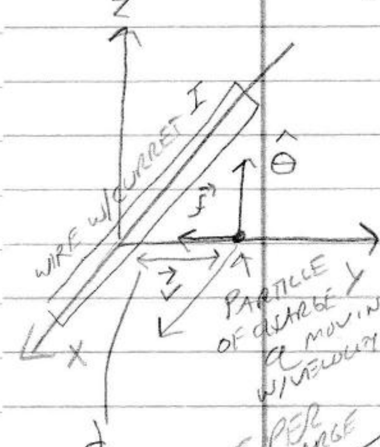
• COULOMB'S LAW:

$$\boxed{\vec{f} = \frac{Qq}{4\pi\varepsilon_0 \mathcal{r}^2}\,\hat{\mathcal{r}}} \qquad (2.78)$$

UNIT VECTOR IN DIRECTION OF $\vec{\mathcal{r}}$

CONSTANT

$\vec{\mathcal{r}} = \vec{r} - \vec{R}$



• THE ELECTRIC FIELD, $\mathcal{E}(\vec{r})$, IS A VECTOR FIELD (i.e., A FUNCTION OF POSITION THAT ASSIGNS A VECTOR, $\vec{\mathcal{E}}$, TO EACH POINT, $\vec{r}$, THROUGHOUT A GIVEN REGION). AT ANY GIVEN POINT, $\vec{r}$:

$$\boxed{\vec{\mathcal{E}} = \frac{\vec{f}}{q}} \rightarrow \boxed{\vec{f} = q\,\vec{\mathcal{E}}(\vec{r})} \qquad (2.79)$$

FORCE PER UNIT CHARGE THAT WOULD ACT ON A TEST CHARGE q PLACED AT $\vec{r}$

• LIKEWISE, THE MAGNETIC FIELD, $\vec{B}(\vec{r})$, IS GIVEN BY:

VECTOR PRODUCT

$$\boxed{\vec{f} = q\,\vec{v} \times \vec{B}(\vec{r})} \qquad (2.81)$$

VELOCITY OF CHARGE

VECTOR PRODUCT

• WE CAN COMBINE THESE DESCRIPTIONS OF ELECTRIC AND MAGNETIC FORCES.

$$\boxed{\text{• LORENTZ FORCE LAW: } \vec{f} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{B}\right)} \qquad (2.82)$$

• THIS HAS COMPONENTS:

$$f_x = q\left(\mathcal{E}_x + v_y B_z - v_z B_y\right)$$

$$f_y = q\left(\mathcal{E}_y + v_z B_x - v_x B_z\right)$$

$$f_z = q\left(\mathcal{E}_z + v_x B_y - v_y B_x\right)$$

OR, MORE SIMPLY:

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = q \begin{pmatrix} \mathcal{E}_x/c & 0 & B_z & -B_y \\ \mathcal{E}_y/c & -B_z & 0 & B_x \\ \mathcal{E}_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} \qquad (2.83)$$

• IN ORDER TO WRITE THE LORENTZ FORCE LAW IN A MANIFESTLY LORENTZ COVARIANT WAY, WE INTRODUCE THE ELECTROMAGNETIC FOUR-TENSOR (a.k.a. THE FIELD TENSOR), $[F^{\mu\nu}]$:

NOT TO BE CONFUSED W/ THE FOUR-FORCE!

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & -\mathcal{E}_x/c & -\mathcal{E}_y/c & -\mathcal{E}_z/c \\ \mathcal{E}_x/c & 0 & -B_z & B_y \\ \mathcal{E}_y/c & B_z & 0 & -B_x \\ \mathcal{E}_z/c & -B_y & B_x & 0 \end{pmatrix} \qquad (2.84)$$

IN STANDARD CONFIGURATION

• $[F^{\mu\nu}]$ TRANSFORMS BETWEEN S AND S' AS FOLL[OWS]:

CONTRAVARIANT

$$\boxed{F'^{\mu\nu} = \sum_{\alpha,\beta=0}^{3} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}} \qquad (2.85)$$

• NOW USING THE MINKOWSKI METRIC, WE [...] TO LOWER ONE OF THE INDICES:

• FIELD TENSOR ("MIXED" VERSION):

$$F^{\mu}{}_{\beta} = \sum_{\nu=0}^{3} \eta_{\beta\nu} F^{\mu\nu} \qquad (2.86)$$

• LOWERING THE OTHER INDEX, WE GET:

• FIELD TENSOR (COVARIANT FORM):

$$F_{\alpha\beta} = \sum_{\mu=0}^{3} \eta_{\alpha\mu} F^{\mu}{}_{\beta} \qquad (2.87)$$

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & \mathcal{E}_x/c & \mathcal{E}_y/c & \mathcal{E}_z/c \\ -\mathcal{E}_x/c & 0 & -B_z & B_y \\ -\mathcal{E}_y/c & B_z & 0 & -B_x \\ -\mathcal{E}_z/c & -B_y & B_x & 0 \end{pmatrix} \qquad ($$

• NOTE: THE COVARIANT FORM TRANSFORMS WIT[H] THE INVERSE LORENTZ TRANSFORMAT[ION] MATRIX, $[(\Lambda^{-1})_{\mu}{}^{\nu}]$. THAT IS:

$$\boxed{F'_{\alpha\beta} = \sum_{\mu,\nu=0}^{3} (\Lambda^{-1})_{\alpha}{}^{\mu} (\Lambda^{-1})_{\beta}{}^{\nu} F_{\mu\nu}} \qquad (2.8$$

• WE CAN NOW WRITE THE SECOND LAW OF LORENTZ-COVARIANT ELECTROMAGNETISM:

TOP ROW

$\vec{f}\cdot\vec{v} = q$

ONLY ELECTRIC FIELD DOES WORK ON PARTICLE

$$\boxed{\text{COVARIANT LORENTZ FORCE LAW:} \quad F^{\mu} = q\sum_{\nu=0}^{3} F^{\mu\nu} U_{\nu}} \qquad (2.90)$$

• THIS IS CLEARLY LORENTZ COVARIANT SINCE IT MEANS

SPEED OF PARTICLE

$$\begin{pmatrix} (\gamma(v)/c)\,\vec{f}\cdot\vec{v} \\ \gamma(v) f_x \\ \gamma(v) f_y \\ \gamma(v) f_z \end{pmatrix} = q \begin{pmatrix} 0 & -\mathcal{E}_x/c & -\mathcal{E}_y/c & -\mathcal{E}_z/c \\ \mathcal{E}_x/c & 0 & -B_z & B_y \\ \mathcal{E}_y/c & B_z & 0 & -B_x \\ \mathcal{E}_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma(v)c \\ -\gamma(v)v_x \\ -\gamma(v)v_y \\ -\gamma(v)v_z \end{pmatrix} \qquad (2.91)$$

COMPARE TO LORENTZ FORCE LAW (CANCEL THE $\gamma(v)$)

COVARIANT FOUR-VELOCITY

• TRANSFORMATION OF E AND M FIELDS

• THE SIMPLE TRANSFORMATION OF THE ELECTRO-MAGNETIC FOUR-TENSOR SHOWS THAT ELECTRIC AND MAGNETIC FIELDS GET MIXED UP IN RELATIVITY, LIKE ENERGY AND MOMENTUM. WHAT IS JUST AN ELECTRIC FIELD IN FRAME S IS A MIX OF ELECTRIC & MAGNETIC FIELDS IN FRAME S'.

- WE KNOW THAT:

$$F'^{\mu\nu} = \sum_{\alpha,\beta=0}^{3} \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} F^{\alpha\beta} \qquad (2.85)$$

BLENDING OF ELECTRICITY & MAGNETISM

- THE GENERAL LORENTZ TRANSFORMATION IN WHICH S' HAS ARBITRARY VELOCITY $\vec{V}$ IN S/S:

$$\vec{\mathcal{E}}'_{\parallel} = \vec{\mathcal{E}}_{\parallel} \qquad (2.93)$$

PARALL FIELD COMPONENTS

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \qquad (2.94)$$

$$\vec{\mathcal{E}}'_{\perp} = \gamma(V)\left[\mathcal{E}_{\perp} - \vec{V} \times \vec{B}_{\perp}\right] \qquad (2.95)$$

PERPENDICULAR FIELD COMPONENT

$$\vec{\mathcal{E}}'_{\perp} = \gamma(V)\left[\vec{B}_{\perp} - \vec{V} \times \mathcal{E}_{\perp} / c^2\right] \qquad (2.96)$$

- <u>MAXWELL'S EQUATIONS</u>

  - MAXWELL'S EQUATIONS <u>DETERMINE THE ELECTRIC AND MAGNETIC FIELDS</u> IN A REGION BY RELATING THE ELECTRIC AND MAGNETIC FIELDS TO THE <u>CHARGE</u> & <u>CURRENT DENSITIES</u> THAT ARE THEIR <u>SOURCES</u>, & TO <u>EACH OTHER</u>.

  - THE EQUATIONS CAN BE WRITTEN:

VECTOR DERIVATIVE! $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

MAXWELL'S EQUATIONS

- $\nabla \cdot \vec{\mathcal{E}} = \frac{\rho}{\varepsilon_0}$ [ONE-COMPONENT] (2.97)
- $\nabla \cdot \vec{B} = 0$ [ONE-COMPONENT] (2.98)
- $\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t}$ [THREE-COMPONENT] (2.99)
- $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{\mathcal{E}}}{\partial t}$ [THREE-COMPONENT] (2.100

- <u>NOTE</u>: THE CONSTANTS μ_0, ε_0, AND c SATISFY: $\mu_0 \varepsilon_0 c^2 = 1$

- IN ORDER TO <u>COVARIANTLY</u> CONSTRUCT EIGHT COMPONENT EQUATIONS FROM THESE, WE WRITE:

<u>COVARIANT MAXWELL'S EQUATIONS</u>

REPRODUCE 2.97 & 2.100 $\longrightarrow$

$$\boxed{\sum_{\mu=0}^{3} \frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = \mu_0 J^{\nu}} \qquad (2.102)$$

REDUCES T[O] 4 INDEPEN[DENT] EQUATIONS

REPRODUCE 2.98 & 2.99 $\longrightarrow$

$$\boxed{\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} = 0} \qquad (2.103)$$

$\longrightarrow$ <u>UPSHOT</u>: WE HAVE REWRITTEN <u>MAXWELL'S EQUATIONS</u> IN A <u>MANIFESTLY COVARIANT</u> FORM.

LAWS OF VACUUM ELECTROMAGNETISM

• FOUR-TENSORS

• WE HAVE BEEN TALKING ABOUT FOUR-TENSORS. BUT THESE ARE JUST A SPECIAL CLASS OF TENSORS.

$[F^{\mu\nu}]$, $[\eta_{\mu\nu}]$, $[\eta^{\mu\nu}]$ ETC.

• TECHNICALLY, WE HAVE ONLY SEEN A PARTICULAR KIND OF FOUR-TENSOR – NAMELY, FOUR-TENSORS OF RANK 2.

• FOUR-VECTORS ARE ANOTHER KIND OF FOUR-TENSOR – FOUR TENSORS OF RANK 1.

RECALL LORENTZ TRANSFORM!

$$[x'^{\mu}] = \sum_{\nu} [\Lambda^{\mu}{}_{\nu}][x^{\nu}]$$

• WHAT MAKES ALL THESE OBJECTS TENSORS IS THEIR BEHAVIOR UNDER LORENTZ TRANSFORMATION.

• IN GENERAL, A FOUR-TENSOR OF CONTRAVARIANT RANK m CONSISTS OF 4^m COMPONENTS THAT TRANSFORM THUS:

$$\boxed{T'^{\mu_1,\mu_2,\mu_3 \ldots \mu_m} = \sum_{\nu_1 \nu_2 \ldots \nu_m} \Lambda^{\mu_1}{}_{\nu_1} \Lambda^{\mu_2}{}_{\nu_2} \ldots \Lambda^{\mu_m}{}_{\nu_m} T^{\nu_1 \nu_2 \ldots \nu_m}} \qquad (2.104)$$

• SIMILARLY, A FOUR-TENSOR OF COVARIANT RANK n CONSISTS OF 4^n COMPONENTS THAT TRANSFORM:

$$\boxed{T'_{\alpha_1 \alpha_2 \ldots \alpha_n} = \sum_{\beta_1 \beta_2 \ldots \beta_n} (\Lambda^{-1})^{\beta_1}{}_{\alpha_1} (\Lambda^{-1})^{\beta_2}{}_{\alpha_2} \ldots (\Lambda^{-1})^{\beta_n}{}_{\alpha_n} T_{\beta_1 \beta_2 \ldots \beta_n}} \qquad (2.105)$$

• A MIXED FOUR-TENSOR OF CONTRAVARIANT RANK m AND COVARIANT RANK n CONSISTS OF 4^{m+n} COMPONENTS THAT TRANSFORM:

$$\boxed{\begin{aligned} T'^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n} &= \sum_{\nu_1,\nu_2\ldots\nu_m\,\beta_1,\beta_2\ldots\beta_n} \Lambda^{\mu_1}{}_{\nu_1}\Lambda^{\mu_2}{}_{\nu_2}\ldots\Lambda^{\mu_m}{}_{\nu_m} \\ &\times \left(\Lambda^{-1}\right)_{\alpha_1}^{\beta_1}\left(\Lambda^{-1}\right)_{\alpha_2}^{\beta_2}\ldots\left(\Lambda^{-1}\right)_{\alpha_n}^{\beta_n} \\ &\times T^{\nu_1\nu_2\ldots\nu_m}_{\beta_1\beta_2\ldots\beta_n} \qquad (2.107) \end{aligned}}$$

• MEANWHILE, THE ELEMENTS OF THE GENERAL LORENTZ TRANSFORMATION MATRIX ARE:

SAME! NEED NOT BE A SKIPPED CONTRACTION

$$\boxed{\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}} \qquad (2.108)$$

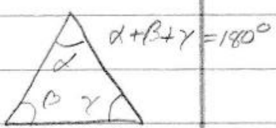
AND THOSE OF THE INVERSE TRANSFORMATION MATRIX ARE:

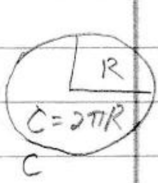
$$\boxed{\left(\Lambda^{-1}\right)_{\mu}^{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}} \qquad (2.109)$$

- GEOMETRY AND CURVED SPACETIME

    - SR CONCERNS THE RELATIONSHIP BETWEEN MEASUREMENTS MADE BY INERTIAL OBSERVERS IN UNIFORM RELATIVE MOTION. IN ORDER TO EXTEND THIS TO INCLUDE A THEORY OF GRAVITY WE NEED RIEMANNIAN GEOMETRY.

    - RECALL THAT IN EUCLIDEAN GEOMETRY,





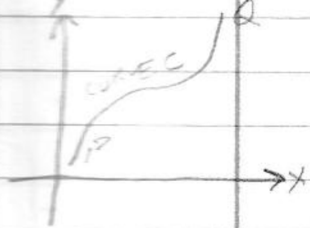
        - THE INTERNAL ANGLES OF TRIANGLES ADD UP TO $\boxed{180°}$.
        - CIRCLES OF RADIUS R HAS CIRCUMFERENCE OF LENGTH $\boxed{C = 2\pi R}$
        - SPHERES OF RADIUS R HAS SURFACE AREA $\boxed{A = 4\pi R^2}$.

    - HOWEVER, NONE OF THESE RESULTS HOLD IN GEOMETRIES GENERALLY.

    - USING THE CONCEPT OF THE METRIC, $g_{\mu\nu}$, WE WILL INTRODUCE METHODS FOR MEASURING THE LENGTH OF A CURVE, DEFINING THE PARALLEL TRANSPORT OF A VECTOR, FINDING GEODESICS, AND QUANTIFYING THE CURVATURE OF NON-EUCLIDEAN SPACES. THEN WE DESCRIBE CURVED SPACETIME.

· LINE ELEMENTS

· IN EUCLIDEAN GEOMETRY, WE COMPUTE THE LENGTH OF A CURVE, C, BY INTEGRATING:



$$L_C(P,Q) = \int_P^Q dl$$

WHERE THE "LINE ELEMENT", dl, IS:

$$dl = \left(dx^2 + dy^2\right)^{\frac{1}{2}} \qquad (3.5)$$

· IN ORDER TO COMPUTE $L_C(P,Q)$ WE NEED TO TAKE ACCOUNT OF THE DIFFERENT DIRECTIONS OF THE LINE ELEMENTS, dl. WE CAN DO THIS BY PARAMETERIZING C.

· WE IDENTIFY EVERY POINT ON C WITH A UNIQUE VALUE OF SOME CONTINUOUSLY VARYING PARAMETER, u. SO, THE X AND Y COORDINATES OF A POINT ON C ARE VALUES OF TWO COORDINATE FUNCTIONS, $X(U)$ AND $y(U)$, WHICH DEFINE THE CURVE.

· EXAMPLE: THE PARABOLA, $y = x^2$, CAN BE PARAMETERIZED USING:

$$X(U) = U$$

$$y(U) = U^2$$

• EXAMPLE: THE CIRCLE, $x^2+y^2=1$, CAN BE PARAMETERIZED USING: $x(u)=\cos(u)$, $y(u)=\sin(u)$.

• GIVEN SUCH A PARAMETERIZATION, ANY TWO POINTS ON C THAT ARE SEPARATED BY COORDINATE INTERVALS Δx AND Δy WILL ALSO BE SEPARATED BY A CORRESPONDING PARAMETER INTERVAL:

$$\left.\begin{aligned}\Delta x &= \frac{\Delta x}{\Delta u}\Delta u \\ \Delta y &= \frac{\Delta y}{\Delta u}\Delta u\end{aligned}\right\}$$

AS $\Delta u, \Delta x, \Delta y \to 0$, $\frac{\Delta x}{\Delta u} \to \frac{dx}{du}$ AND $\frac{\Delta y}{\Delta u} \to \frac{dy}{du}$.

NOTE: $\frac{dx}{du}$ IS JUST THE DERIVATIVE OF $x(u)$ WITH RESPECT TO u (AND SIMILARLY FOR $y(u)$).

SO, $\Delta u \to 0$:

$$dx = \frac{dx}{du}du$$

$$dy = \frac{dy}{du}du$$

HENCE, BY (3.5):

$$dl = \left[\left(\frac{dx}{du}\right)^2 du^2 + \left(\frac{dy}{du}\right)^2 du^2\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2\right]^{\frac{1}{2}} du$$

LENGTH OF C IN A EUCLIDEAN PLANE

- FINALLY, THE <u>LENGTH</u> OF C FROM $P=(x(U_P), y(U_P))$ TO $Q=(x(U_Q), y(U_Q))$ IS:

$$\boxed{L_C(P,Q) = \int_P^Q d\ell = \int_{U_P}^{U_Q} \left[\left(\frac{\partial x}{\partial U}\right)^2 + \left(\frac{\partial y}{\partial U}\right)^2\right]^{\frac{1}{2}} dU} \qquad (3.6)$$

- <u>EXAMPLE</u>: CALCULATE THE <u>LENGTH</u> OF THE CURVE, $y = 2\left(\frac{6}{5}x + 1\right)$ FROM $(0,2)$ TO $(5,14)$ USING <u>PARAMETERIZATION</u>.

  - AS A <u>SINGLE-VALUED</u> FUNCTION, WE CAN WRITE IT AS: $x = u$ AND $y = 2\left(\frac{6}{5}u + 1\right)$

  - DIFFERENTIATING WITH RESPECT TO u,

$$\frac{dx}{\partial u} = 1 \quad \text{AND} \quad \frac{dy}{du} = \frac{12}{5}$$

  - SINCE $x = u$, $x(0,2) = u(0,2) = 0$, AND $x(5,14) = u(5,14) = 5$. SO, BY (3.6):

$$L_C\big((0,2),(5,14)\big) = \int_0^5 \left[1^2 + \left(\frac{12}{5}\right)^2\right]^{\frac{1}{2}} \partial u$$

$$= \left[\frac{13}{5}u\right]_0^5 = 13$$

- EXAMPLE: FIND THE LENGTH OF C, THE CIRCUMFERENCE OF THE CIRCLE GIVEN BY $x^2+y^2=R^2$, IN TERMS OF R, BY PARAMETERIZING C.

- PARAMETERIZE THE CIRCLE BY SETTING:

$$x(u)=R\cos(u)$$

$$y(u)=R\sin(u)$$

- DIFFERENTIATE WITH RESPECT TO u:

$$\frac{dx}{du}=-R\sin(u)$$

$$\frac{dy}{du}=R\cos(u)$$

- LETTING u VARY FROM 0 TO 2π, AND RECALLING THAT $\sin^2(u)+\cos^2(u)=1$, WE GET:

$$C=\int_0^{2\pi}\left[R^2\sin^2(u)+R^2\cos^2(u)\right]^{\frac{1}{2}}du$$

$$=R\left[u\right]_0^{2\pi}=2\pi R$$

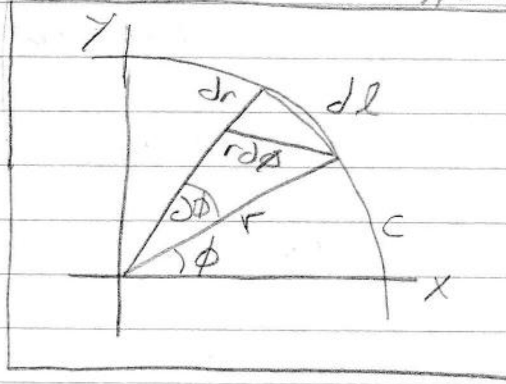
- NOTE: THERE ARE ALWAYS MULTIPLE WAYS TO PARAMETERIZE A CURVE.

- WHEN DEALING WITH A GENERAL CURVE IN A PLANE, IT IS OFTEN MORE CONVENIENT TO USE PLANE POLAR COORDINATES, (r, ϕ), WHICH CAN BE DEFINED IN TERM OF (x, y) BY:

$$x = r\cos\phi$$

$$y = r\sin\phi$$

VARIABLES



- BY THE PRODUCT RULE OF DIFFERENTIATION:

$$\partial x = \cos\phi \, dr - r\sin\phi \, d\phi$$

$$dy = \sin\phi \, dr - r\cos\phi \, d\phi$$

LINE ELEMENT IN POLAR COORDINATES

- SO, SINCE $\partial l^2 = \partial x^2 + dy^2$ (EQUATION 3.4) IN A EUCLIDEAN PLANE, WE CAN WRITE:

$$\boxed{\partial l^2 = \partial r^2 + r^2 \partial \phi^2} \qquad (3.7)$$

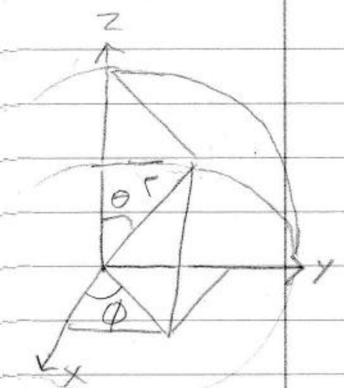
- <u>CURVED SURFACES</u>

LINE ELEMENT FOR 3-DIMENSIONAL EUCLIDEAN SPACE

- THIS DIFFERENTIAL APPROACH TO CURVED SURFACES CAN OF COURSE BE GENERALIZED:

$$dl^2 = dx^2 + dy^2 + dz^2 \qquad (3.8)$$

- IN <u>SPHERICAL</u> COORDINATES, WE SET:



$$x = r\sin\theta\cos\theta$$

$$y = r\sin\theta\sin\theta$$

$$z = r\cos\theta$$

- DIFFERENTIATING (USING THE <u>PRODUCT RULE</u>):

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\theta\, d\theta - r\sin\theta\sin\phi$$

$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

LINE ELEMENT FOR SPHERICAL COORDINATES IN 3-DIMENSIONAL EUCLIDEAN SPACE

WHICH GIVES:

$$\boxed{dl^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2} \qquad (3.9)$$

- <u>NOTE</u>: IF WE FIX r IN (3.9), THEN WE GET THE GEOMETRY OF SPHERE OF RADIUS

CONSTANT
$\downarrow$
$r = R$, AND (3.9) REDUCES TO:

$$d\ell^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2 \qquad (3.10)$$

ONLY VARIABLES

- THIS IS THE GEOMETRY OF A <u>NON-EUCLIDEAN</u> 2-DIMENSIONAL SPACE (EMBEDDED IN A 3-DIMENSIONAL SPACE).

- <u>EXAMPLE</u>: FIND THE LENGTH OF THE CIRCUMFERENCE C OF A CIRCLE OBTAINED BY "SWEEPING AROUND" THE NORTH POLE OF THE SPHERE AT FIXED ANGLE θ (OF RADIUS $R\theta$).

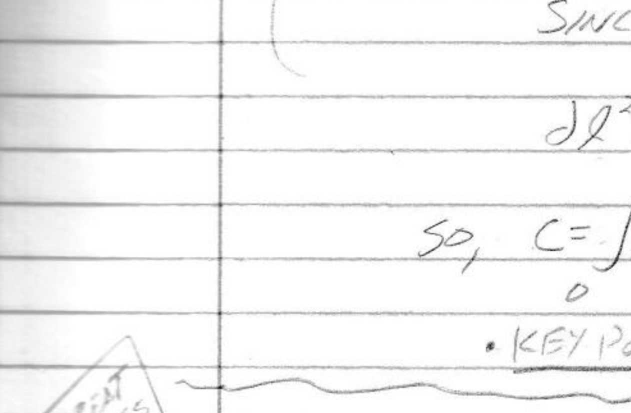
SINCE θ IS CONSTANT, (3.10) GIVES:

$$d\ell^2 = R^2 \sin^2\theta \, d\phi^2$$

TAKING $\sqrt{d\ell^2}$

$$\text{SO, } C = \int_0^{2\pi} R\sin\theta \, d\phi = R\sin\theta \left[\phi\right]_0^{2\pi} = \boxed{2\pi R\sin\theta}$$

- <u>KEY POINT</u>: THIS IS <u>NOT</u> $2\pi \times$ THE RADIUS OF THE CIRCLE, $R\theta$! $2\pi R\sin\theta < 2\pi R\theta$.



- SUM OF ANGLES > 180°.

- <u>NOTE</u>: WE NOW HAVE TWO "<u>TESTS</u>" OF CURVATURE: COMPUTE THE CIRCUMFERENCE OF CIRCLES & THE <u>SUMS OF ANGLES OF TRIANGL</u>[ES]

OF CURVATURE

DISTINCTION DUE TO GAUSS

- "TESTS" LIKE THESE REVEAL INTRINSIC CURVATURE, RATHER THAN PROPERTIES OF THE SURFACE THAT ARE ONLY DISCERNABLE IN A HIGHER DIMENSIONAL SPACE. THIS MATTERS BECAUSE NOT ALL SURFACES CAN BE EMBEDDED IN A HIGHER DIMENSIONAL EUCLIDEAN SPACE (e.g., HYPERBOLIC SPACE CANNOT). ALSO, EXTRINSIC CURVATURE IS NOT A RELIABLE GUIDE TO INTRINSIC CURVATURE (e.g. THE GEOMETRY OF A "ROLLED UP" CYLINDER IS INTRINSICALLY "FLAT".)

- METRICS AND CONNECTIONS

  - RIEMANNIAN GEOMETRY IS A BRANCH OF DIFFERENTIAL GEOMETRY USED TO ANALYZE CURVED SPACES.

  - WE HAVE SEEN THAT THE LINE ELEMENT, EXPRESSED AS THE SUM OF SQUARES OF COORDINATE DIFFERENTIALS (e.g., dx, dy, dr, OR $d\theta$), IS CENTRAL TO GEOMETRY.

  - AN n-DIMENSIONAL RIEMANN SPACE IS A SPACE WHERE THE LINE ELEMENT TAKES THE GENERAL FORM:

$$\boxed{dl^2 = \sum_{i,j=1}^{n} g_{ij}\,dx^i dx^j} \qquad (3.11)$$

• THE TERMS $dx^1, dx^2, \ldots dx^n$ ARE THE DIFFERENTIALS OF THE n COORDINATES THAT DEFINE THE GEOMETRY OF THE SPACE.

• THE g_{ij}s ARE FUNCTIONS OF THE COORDINATES CALLED "METRIC COEFFICIENTS".

• NOTE: WE REQUIRE THAT THE g_{ij}s ARE SYMMETRIC, SO THAT $g_{ij} = g_{ji}$.

• EXAMPLE: $\boxed{dl^2 = dr^2 + r^2 d\phi^2}$ (EQUATION 3.7), WHICH IS THE 2-DIMENSIONAL EUCLIDEAN LINE ELEMENT IN POLAR COORDINATES, SETS $n=2$, $x^1 = r$, AND $x^2 = \phi$, AND $g_{11} = 1$, $g_{22} = r^2$, AND $g_{12} = g_{21} = 0$.

• IN n-DIMENSIONAL EUCLIDEAN SPACE DESCRIBED BY n CARTESIAN COORDINATES $(x^1, x^2 \ldots x^n)$, THE LINE ELEMENT IS:

$$dl^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2$$

WHERE THE METRIC COEFFICIENTS CAN BE WRITTEN $g_{ij} = \delta_{ij}$, WITH:

"KRONECKER DELTA"

$$\delta_{ij} = \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j \end{cases}$$

HALF THE OFF-DIAGONAL ELEMENTS + n DIAGONAL ONES

• THE METRIC COEFFICIENTS FORM AN $n \times n$ ARRAY WITH n^2 ELEMENTS. BUT BY SYMMETRY, $n_{ij} = n_{ji}$, ONLY $\frac{n(n+1)}{2}$ ARE INDEPENDENT. THE COMPLETE SET OF METRIC COEFFICIENTS, $[g_{\mu\nu}]$, IS CALLED THE "METRIC TENSOR" (OR "METRIC" FOR SHORT).

TRANSFORMS AS A RANK 2 COVARIANT TENSOR

• EXAMPLE: THE METRIC TENSOR FOR THE 3-DIMENSIONAL EUCLIDEAN DEFINED BY THE LINE ELEMENT $dl^2 = dx^2 + dy^2 + dz^2$ IS:

BRACKETS INDICATE TENSOR

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

• NOTE: IN GENERAL, THE METRIC COEFFICIENTS, g_{ij}s, ARE NOT CONSTANTS. THEY ARE FUNCTIONS OF THE COORDINATES, x^i. THESE RELATE COORDINATE DIFFERENTIALS TO LENGTHS, THUS FIXING THE GEOMETRY.

WE CAN HAVE A FLAT SPACE IN A NON-CARTESIAN COORDINATE SYSTEM

• UPSHOT: THE METRIC DETERMINES THE GEOMETRY.

METRIC VS. GEOMETRY

• HOWEVER, THE GEOMETRY DOES NOT UNIQUELY DETERMINE A METRIC, SINCE THERE ARE ALWAYS MULTIPLE COORDINATE SYSTEMS, AND SO MULTIPLE WAYS TO WRITE THE METRIC.

PRODUCE CONNECTIONS BETWEEN DIFFERENT POINTS ON THE MANIFOLD

• WE HAVE SEEN, IN EFFECT, THAT BOTH FLAT AND CURVED SPACES CAN BE REPRESENTED BY METRICS THAT ARE DIAGONAL ARRAYS (i.e. $g_{ij} = 0$ IF $i \neq j$). IN GENERAL, DIAGONAL METRICS OCCUR WHENEVER THE COORDINATE SYSTEM IS ORTHOGONAL.

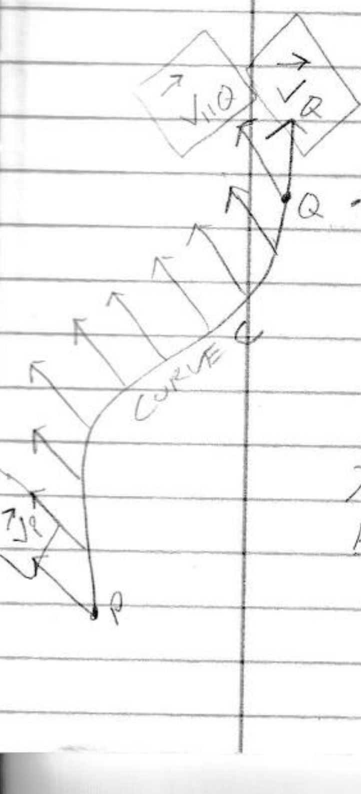
• CONNECTIONS & PARALLEL TRANSPORT

DEPEND ON THE SPACE AND ON THE COORDINATES USED TO DESCRIBE

THEY ARE RELATED TO THE METRIC COEFFICIENTS

• IN AN n-DIMENSIONAL RIEMANNIAN SPACE THERE ARE n^3 "CONNECTION COEFFICIENTS," DENOTED Γ^i_{jk} ($i, j, k = 1, 2, \ldots n$). AS WITH METRICS, THEY ARE NOT ALL INDEPENDENT.

• NOTE: NOTWITHSTANDING THE INDICES, CONNECTION COEFFICIENTS ARE NOT THE COMPONENTS OF A TENSOR, SINCE THEY DO NOT APPROPRIATELY TRANSFORM.



• CONNECTION COEFFICIENTS ARE IMPORTANT FOR A PROCESS CALLED "PARALLEL TRANSPORT." THIS INVOLVES MOVING A VECTOR FROM ONE POINT TO ANOTHER ALONG A PATH SO THAT ITS DIRECTION IS ALWAYS PRESERVED. THIS ALLOWS US TO COMPARE VECTORS AT DIFFERENT POINTS.

• IN EUCLIDEAN SPACE, USING CARTESIAN COORDINATES, THE PROCESS IS TRIVIAL

COMPONENTS IN THE COORDINATE BASIS

EUCLIDEAN SPACE WITH CARTESIAN COORDINATES

- $V_P = V^1 \vec{i} + V^2 \vec{j} + V^3 \vec{k}$, WHERE $\vec{i}, \vec{j}$, AND $\vec{k}$ ARE UNIT VECTORS POINTING ALONG THE DIRECTIONS X^1, X^2 AND X^3 TOWARD INCREASING VALUES. SUCH VECTORS ARE CALLED "COORDINATE BASIS VECTORS".

- A VECTOR CAN BE PARALLEL TRANSPORTED IN THIS CASE MERELY BY KEEPING ITS COMPONENTS CONSTANT (SO THE COMPONENTS OF $\vec{V}_{\parallel Q}$ ARE $V^1_{\parallel Q} = V^1_P$, $V^2_{\parallel Q} = V^2_P$ AND $V^3_{\parallel Q} = V^3_P$).

AN EXAMPLE OF "CURVILINEAR COORDINATES"

BASIS VECTORS: $\hat{r}, \hat{\theta}, \hat{\phi}$

NON-CARTESIAN COORDINATES

- HOWEVER, IF WE USE SPHERICAL COORDINATES INSTEAD, WHERE $X^1 = r$, $X^2 = \theta$, AND $X^3 = \phi$, THE COORDINATE BASIS VECTORS POINT IN DIFFERENT DIRECTIONS AT DIFFERENT POINTS.

- UPSHOT: IN SPHERICAL (OR, MORE GENERALLY, CURVILINEAR) COORDINATES, THE COMPONENTS OF THE PARALLEL TRANSPORTED VECTOR AT Q, $\vec{V}_{\parallel Q}$, WILL BE DIFFERENT FROM THOSE OF THE ORIGINAL VECTOR, $\vec{V}_P$, AT P.

  - HENCE, TO PARALLEL TRANSPORT A VECTOR IN THIS CASE, WE MUST DETERMINE HOW THE COMPONENTS CHANGE DURING EACH INFINITESIMAL DISPLACEMENT ALONG THE CURVE, C.

• THE GENERAL CASE (FOR 3-DIMENSIONS):

• CONSIDER AN ARBITRARY 3-DIMENSIONAL RIEMANNIAN SPACE, WITH COORDINATES x^1, x^2, x^3 AND METRIC $[g_{ij}]$ $(i, j = 1, 2, 3)$.

• WE WANT TO PARALLEL TRANSPORT A VECTOR FROM POINT P ALONG CURVE C TO POINT Q. WE CAN DESCRIBE POSITIONS ALONG C WITH A PARAMETER, U, SO THAT C ITSELF IS DESCRIBED BY THREE COORDINATE FUNCTIONS:

$$x^1(u),\ x^2(u),\ x^3(u)$$

• WE DENOTE THE COORDINATE BASIS VECTORS:

$$\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3$$

SO THAT AT EACH POINT ALONG C, WE CAN EXPRESS THE VALUE OF AN ARBITRARY VECTOR FIELD, $\vec{v}(u)$, AT U IN TERMS OF ITS COMPONENTS IN THE COORDINATE BASIS, AND THE COORDINATE BASIS VECTORS AT U AS FOLLOWS:

$$\boxed{\vec{v}(u) = \sum_j v^j(u)\,\vec{e}_j(u)} \qquad (3.12)$$

• TO DIFFERENTIATE THIS, WE APPLY THE PRODUCT RULE TO GET:

$$\frac{\partial \vec{v}}{\partial u} = \sum_j \left(\frac{\partial v^j}{\partial u} \vec{e}_j + v^j \frac{d\vec{e}_j}{du} \right)$$

BY THE CHAIN RULE

$$= \sum_j \left(\frac{\partial v^j}{\partial u} \vec{e}_j + \sum_k v^j \frac{\partial \vec{e}_j}{\partial x^k} \frac{\partial x^k}{\partial u} \right) \quad (3.$$

EFFECT OF CHANGING COMPONENTS

EFFECT OF CHANGING BASIS VECTORS

• NOTE: THE TERM $\boxed{\frac{\partial \vec{e}_j}{\partial x^k}}$ REPRESENTS A VECTOR QUANTITY. HENCE, IT HAS COMPONENTS IN THE DIRECTION OF EACH OF THE BASIS VECTORS. SO WE HAVE:

THE RATE OF CHANGE OF $\vec{e}_j$ WRT x^k

$$\boxed{\frac{\partial \vec{e}_j}{\partial x^k} = \sum_i \Gamma^i_{jk} \vec{e}_i} \quad (3.14)$$

THE COMPONENT IN THE DIRECTION OF BASIS VECTOR $\vec{e}_i$, OF THE RATE OF CHANGE OF $\vec{e}_j$ WITH RESPECT TO x^k.

• THE n^3 QUANTITIES Γ^i_{jk} DEFINED BY THIS EQUATION ARE THE CONNECTION COEFFICIENTS.

• THESE ARE DETERMINED BY THE METRIC.

SINCE IT INVOLVES ONLY UNIT VECTORS & COORDINATES

ALL INDICES SUMMED OVER (i.e., ALL ARE "DUMMY INDICES")

- NOW, SUBSTITUTING (3.14) INTO (3.13):

(3.15)
$$\boxed{\frac{d\vec{v}}{du} = \sum_j \left(\frac{dv^j}{du}\vec{e}_j + \sum_{i,k} \Gamma^i_{jk}\,\vec{e}_i\, v^j \frac{dx^k}{du}\right)}$$

RELABELING DUMMY INDICES

(3.16)
$$= \sum_i \left(\frac{dv^i}{du} + \sum_{j,k} \Gamma^i_{jk}\, v^j \frac{dx^k}{du}\right) e_i$$

- <u>OBSERVATION</u>: WE CAN THINK OF A PARTICULAR <u>PARALLEL TRANSPORTED VECTOR AS ITSELF BEING THE VECTOR FIELD</u>, $\vec{v}(u)$. BUT, AS SUCH, ITS <u>RATE OF CHANGE MUST BE ZERO</u>. THAT IS, $\vec{V}$ MUST SATISFY

(3.17)
$$\boxed{\sum_i \left(\frac{dv^i}{du} + \sum_{j,k} \Gamma^i_{jk}\, v^j \frac{dx^k}{du}\right) \vec{e}_i = 0}$$

SO, <u>EACH</u> <u>COMPONENT</u> OF $\frac{d\vec{v}}{du}$:

$$\boxed{\frac{dv^i}{du} = -\sum_{j,k} \Gamma^i_{jk}\, v^j \frac{dx^k}{du}} \qquad (3.18)$$

- <u>UPSHOT</u>: GIVEN THE <u>COMPONENTS</u>, $v^i(u)$, OF A VECTOR AT A POINT ON CURVE, C, THE COMPONENTS OF THE <u>PARALLEL TRANSPORTED</u> VECTOR AT A <u>NEIGHBORING POINT</u> ARE:

(3.19) $$\boxed{v^i(u+du) = v^i(u) + \frac{dv^i}{du}du = v^i(u) - \sum_{j,k} \Gamma^i_{jk} v^j \frac{dx^k}{du} du}$$

· QUESTION: WHAT IS THE EXPRESSION FOR THE CONNECTION COEFFICIENT Γ^i_{jk} IN TERMS OF THE METRIC?

· WE BEGIN WITH AN EXPRESSION FOR THE INFINITESIMAL VECTOR SEPARATION BETWEEN TWO NEARBY POINTS:

$$d\vec{l} = \sum_i e_i dx^i$$

$$\text{SO, } dl^2 = d\vec{l} \cdot d\vec{l} = \sum_{i,j} (\vec{e}_i \cdot \vec{e}_j) dx^i dx^j$$

METRIC & BASIS VECTORS

· COMPARING THIS WITH THE GENERAL LINE ELEMENT

$$dl^2 = \sum_{i,j} g_{ij} dx^i dx^j \quad (3.11)$$

WE SEE THAT $\boxed{g_{ij} = (\vec{e}_i \cdot \vec{e}_j)}$ (3.20)

· IN THIS WAY, THE BASIS VECTORS ARE RELATED TO THE METRIC COEFFICIENTS.

· PARTIALLY DIFFERENTIATING (3.20) WRT x^k, WE OBTAIN:

$$\frac{\partial \vec{e}_i}{\partial k^k} \cdot \vec{e}_j + \vec{e}_i \cdot \frac{\partial \vec{e}_j}{\partial x^k} = \boxed{\frac{\partial g_{ik}}{\partial x^k}} \quad (3.21)$$

METRIC

• RECALLING: (3.14) $\boxed{\frac{\partial \vec{e}_j}{\partial x^k} = \sum \Gamma^i_{jk} \vec{e}_i,}$

WE CAN REWRITE (3.21) AS:

(3.22)

$$\sum_\ell \Gamma^\ell_{ik} \vec{e}_\ell \vec{e}_j + \vec{e}_i \cdot \sum_\ell \Gamma^\ell_{jk} \vec{e}_\ell = \frac{\partial g_{ik}}{\partial x^k}$$

FINALLY, THIS YIELDS:

(3.23)

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} \sum_\ell g^{i\ell} \left(\frac{\partial g_{\ell k}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell} \right)}$$

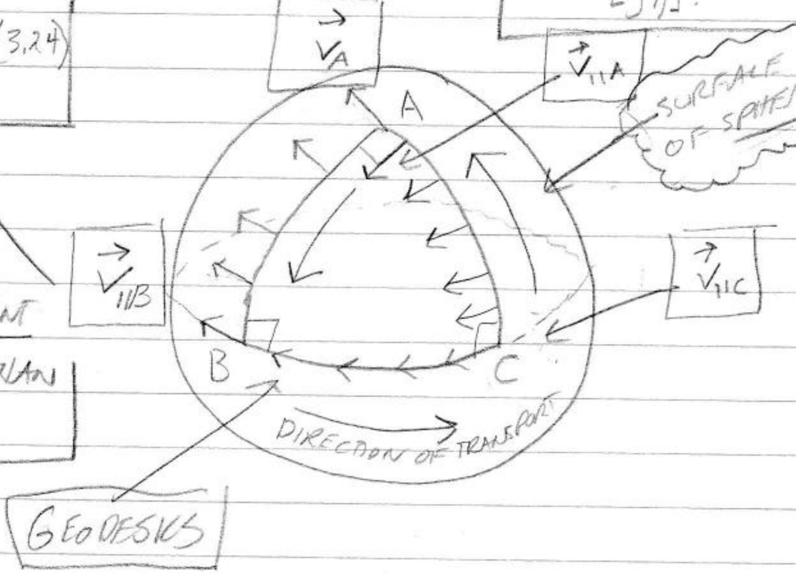
COMPONENT OF CONTRAVARIANT METRIC TENSOR $[g^{ij}]$

CONTAINS ALL INFO ABOUT GEOMETRY OF SPACE JUST LIKE $[g_{ij}]$.

• $[g^{ij}]$ IS CALLED THE "DUAL METRIC" AND IS THE INVERSE OF $[g_{ij}]$:

$$\sum_k g^{ik} g_{kj} = \delta^i_j \quad (3.24)$$

• $\vec{V}_A$ POINTS IN A DIFFERENT DIRECTION THAN $\vec{V}_{\parallel A}$



THE DIFFERENCE BETWEEN THE DIRECTIONS OF $\vec{V}_A$ AND $\vec{V}_{\parallel A}$ GIVES A MEASURE OF THE CURVATURE

- CONNECTION COEFFICIENTS ARE, THUS, INDICATORS OF THE INTRINSIC CURVATURE OF THE SPACE. IN PARTICULAR, PARALLEL TRANSPORT OF A VECTOR AROUND A CLOSED PATH TESTS THE SPACE FOR CURVATURE.

- GEODESICS

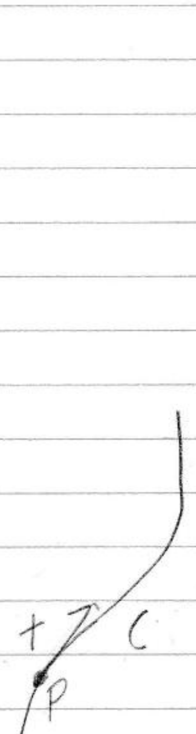
  - IN FLAT SPACE, STRAIGHT LINES PROVIDE THE MOST DIRECT ROUTE BETWEEN POINTS, AND ALSO THE PATH OF SHORTEST DISTANCE.

  - GREAT CIRCLES PLAY THIS ROLE IN THE THE 2-DIMENSIONAL SPACE OF THE SURFACE OF A SPHERE.

  - IN GENERAL, THE "MOST DIRECT PATHS" OF "SHORTEST DISTANCE" IN A RIEMANNIAN SPACE ARE CALLED GEODESICS.

- STRAIGHT LINES

  - A STRAIGHT LINE IN EUCLIDEAN SPACE ALWAYS POINTS "IN THE SAME DIRECTION".



  - FOR A GENERAL CURVE, C, PARAMETERIZED BY U AND DEFINED BY THE COORDINATE FUNCTIONS $x^i(u)$ $(i=1,2\ldots n)$, THE TANGENT VECTOR, $\vec{t}$, AT P IS ONE THAT POINTS ALONG THE CURVE AT P.

· **NOTE:** THE COMPONENTS OF SUCH A VECTOR, $\vec{t}$, WILL BE $\boxed{t^i = \frac{\partial x^i}{\partial u}}$. SO, IF THE CURVE IS ALWAYS GOING IN THE SAME DIRECTION, THE $\vec{t}$ WILL NOT CHANGE ITS DIRECTION AS u VARIES.

IDEA: A GEODESIC IS A PATH THAT PARALLEL TRANSPORTS ITS OWN TANGENT VECTOR.

· **UPSHOT:** IF WE PARALLEL TRANSPORT $\vec{t}$ AT POINT u ALONG C TO $u+du$, WE SHOULD GET A VECTOR THAT IS PROPORTIONAL TO THE TANGENT VECTOR AT $u+du$. THAT IS:

$$? \quad \boxed{\frac{\partial \vec{t}}{\partial u} = f(u)\,\vec{t}} \qquad (3.25)$$

· HENCE, BY THE DEFINITION OF PARALLEL TRANSPORT:

FOR EACH COMPONENT OF $\vec{t}$

$$\longrightarrow \quad \frac{\partial t^i}{\partial u} + \sum_{j,k} \Gamma^i_{jk}\, t^j \frac{\partial x^k}{\partial u} = f(u)\, t^i \qquad (3.26)$$

· AND SINCE $t^i = \frac{\partial x^i}{\partial u}$, WE HAVE:

$$\frac{\partial^2 x^i}{\partial u^2} + \sum_{j,k} \Gamma^i_{jk} \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial u} = f(u)\frac{\partial x^i}{\partial u}$$

THIS AMOUN TO THE CON THAT THE TAN VECTOR PRESE ITS MAGNITUDE ADDITION TO DIREC

AFFINE PARAMETER λ

· BY CHOOSING A PARAMETER, u, SO THAT $f(u) = 0$, WE GET A SO-CALLED "AFFINE PARAMETER", WRITTEN λ.

DIFFERENTIAL EQUATIONS

CONDITION THAT PARAMETERIZED CURVE DEFINED BY COORDINATE FUNCTIONS $x^i(\lambda)$ ALWAYS GO IN "SAME DIRECTION"

• IF WE USE AN AFFINE PARAMETER, λ, THE CONDITION ABOVE REDUCES TO:

$$\boxed{\frac{\partial^2 x^i}{\partial \lambda^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0} \quad (3.27)$$

"GEODESIC EQUATION"

"STRAIGHT LINE" IN CURVED SPACE

• ANY PARAMETERIZED PATH DEFINED BY A SET OF n COORDINATE FUNCTIONS, $x^i(\lambda)$ $(i=1,2,\ldots n)$ THAT SATISFIES THESE DIFFERENTIAL EQUATIONS IS A GEODESIC IN THE n-DIMENSIONAL RIEMANNIAN SPACE WITH METRIC $[g_{ij}]$ AND CONNECTION COEFFICIENTS, Γ^i_{jk}.

• SHOREST DISTANCE

• RECALL FROM EQUATION (3.6) THAT IN 2-DIMENSIONAL EUCLIDEAN SPACE THE LENGTH OF A CURVE, C, PARAMETERIZED BY u, AND DEFINED BY FUNCTIONS $x(u)$ AND $y(u)$ BETWEEN POINTS P AND Q IS:

$$L(P,Q) = \int_{u=u_P}^{u=u_Q} d\ell = \int_{u_P}^{u_Q} \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2\right]^{\frac{1}{2}} du$$

• BY EQUATION (3.11) WE CAN EXTEND THE DEFINITIONS OF THE LINE ELEMENT, $d\ell$, TO INCLUDE THE METRIC:

FOR AN n-DIMENSIONAL RIEMANNIAN SPACE

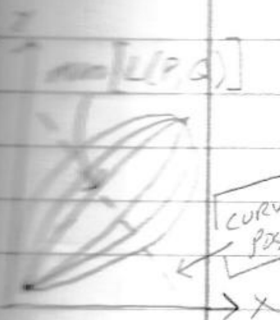
$$d\ell^2 = \sum_{i,j}^{n} g_{i,j}\, dx^i dx^j$$

• SO, WE CAN REWRITE $L(P,Q)$ AS:

$$L(P,Q) = \int_{U_P}^{U_Q} \left[\sum_{ij} g_{ij} \frac{dx^i}{du}\frac{dx^j}{du}\right]^{\frac{1}{2}} du \quad (3.28)$$

ANALOG OF "STRAIGHT LINE"

• WE WANT TO FIND THE PARAMETERIZED CURVE $(x^1(u), x^2(u), \ldots x^n(u))$ BETWEEN P AND Q THAT GIVES THE SMALLEST VALUE (i.e., "SHORTEST DISTANCE").



CURVE POSITION

• WE USE A METHOD FROM THE CALCULUS OF VARIATIONS LIKE THAT OF FINDING THE MINIMUM OF FUNCTION, $f(x)$, BY FINDING $\frac{df}{dx} = 0$.

• CONSIDER ALL POSSIBLE CURVES FROM P TO Q. IN EACH CASE, CONSIDER THE INFINITESIMAL VARIATION OF THE FORM:

ORIGINAL

$$x^i(u) \longrightarrow x^i(u) + \delta x^i(u)$$

THUS CHANGING THE LENGTH OF EACH CURVE BY THE AMOUNT $\boxed{\delta L.}$ HOWEVER, FOR THE CURVE OF MINIMUM LENGTH – i.e., THE GEODESIC – $\delta L = 0$.

• SO WE CAN WRITE:

$$F = \left(\sum_{i,j} g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right)^{\frac{1}{2}} \qquad (3.29)$$

MOREOVER,

$$\delta L = \int_{u_P}^{u_Q} F\, du$$

$$= \int_{u_P}^{u_Q} \sum_m \left[\frac{\partial F}{\partial x^m} \delta x^m + \frac{\partial F}{\partial \left(\frac{\partial x^m}{\partial u}\right)} \delta\left(\frac{dx^m}{du}\right) \right] du$$

• NOTE: $\frac{d}{du}(\delta x^m) = \delta\left(\frac{dx^m}{du}\right)$

SO, INTEGRATING THE SECOND PART (BY PARTS):

$$\delta L = \left[\sum_m \frac{\partial F}{\partial \left(\frac{\partial x^m}{\partial u}\right)} \delta x^m \right]_{u_P}^{u_Q} + \int_{u_P}^{u_Q} \left(\frac{\partial F}{\partial x^m} - \frac{d}{du}\left(\frac{\partial F}{\partial \left(\frac{\partial x^m}{\partial u}\right)} \right) \right) \delta x^m du$$

HOWEVER, $\delta x^m = 0$ AT P AND AT Q FOR ALL m, SO THE FIRST TERM IS ZERO.

• HENCE, FOR $\delta L = 0$,

$$\delta \int_{u_P}^{u_Q} F\, du = \int_{u_P}^{u_Q} \sum_m \left[\frac{\partial F}{\partial x^m} - \frac{d}{dU}\left(\frac{\partial F}{\partial \left(\frac{dx^m}{du}\right)} \right) \right] \delta x^m du = 0$$

AND SINCE THIS HOLDS FOR ARBITRARY VARIATION, δx^m, WE OBTAIN:

$$\boxed{\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial\left(\frac{\partial x^m}{\partial u}\right)}\right)-\frac{\partial F}{\partial x^m}=0} \quad (m=0,1,2,3) \tag{3.30}$$

• THESE ARE THE <u>EULER-LAGRANGE EQUATIONS</u>.

• NOW, SUBSTITUTING (3.29) INTO THE <u>EULER-LAGRANGE EQUATIONS</u>, CHOOSING U SO THAT IT IS AN <u>AFFINE PARAMETER</u> FINALLY GIVES:

$$\boxed{\frac{d^2x^i}{d\lambda^2}+\sum_{j,k}\Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}=0}$$

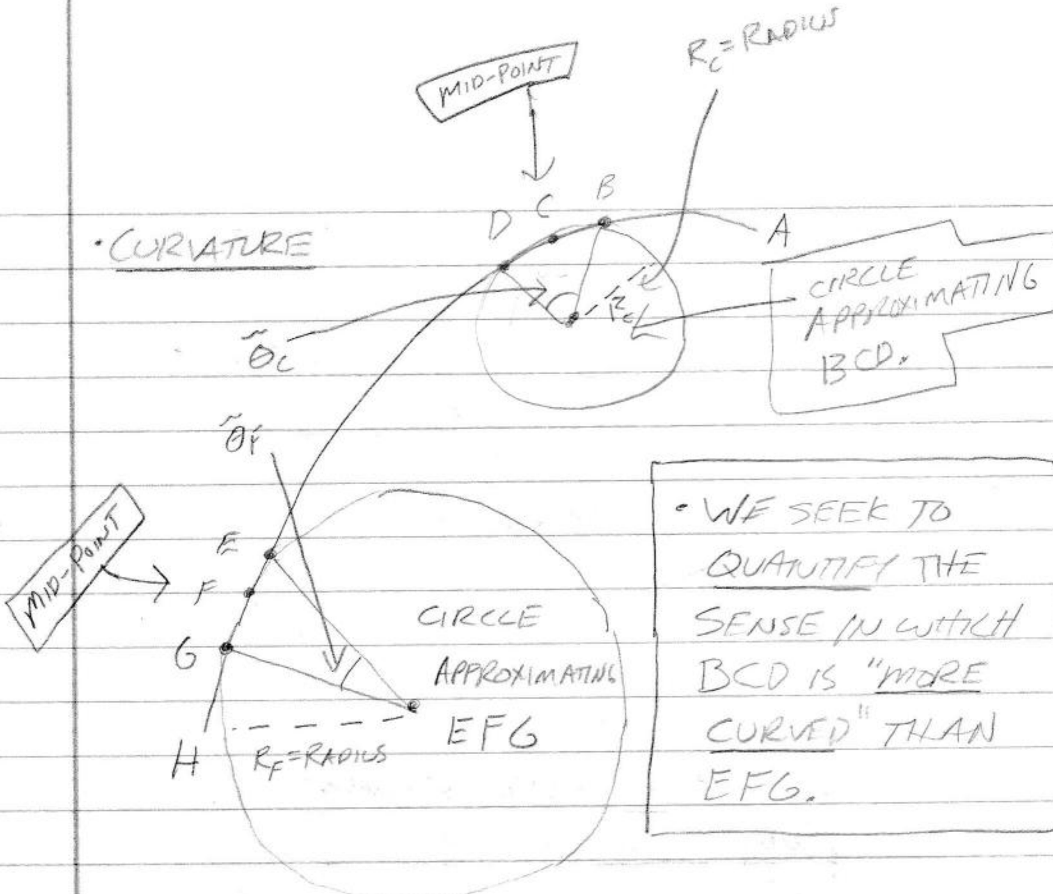
WHICH IS JUST EQUATION (3.27).

i.e. BOTH METHODS OF GENERALIZING THE NOTION OF "STRAIGHT LINE" - IN TERMS OF DIRECTION & DISTANCE - LEAD TO THE SAME DEFINITION

• <u>SUMMARY</u>: IN AN <u>n-DIMENSIONAL RIEMANNIAN SPACE</u>, THE ANALOGS OF <u>STRAIGHT LINES</u> CAN BE REPRESENTED BY <u>CURVES PARAMETERIZED BY AN AFFINE PARAMETER</u>, λ, AND DEFINED BY n COORDINATE <u>FUNCTIONS</u>, $x^i(u)$, THAT SATISFY

~ BOTH IDEAS OF "<u>STRAIGHT LINE</u>" - IN TERMS OF <u>DIRECTION</u> AND <u>DISTANCE</u> - LEAD TO THE SAME <u>NOTION</u> WE CALL "GEODESIC".

- CURVATURE



- WE SEEK TO QUANTIFY THE SENSE IN WHICH BCD IS "MORE CURVED" THAN EFG.

NOTE: SUCH CURVATURE DOES NOT VANISH AS THE REGION AROUND P BECOMES SMALLER! CURVED SURFACES ARE NOT JUST LIKE FLAT ONES LOCALLY.

- AT EACH POINT, P, WE WILL ASSOCIATE A QUANTITY, K_P, WITH THE CURVATURE AT P. IT WILL TURN OUT THAT $K_C > K_P$.

  - A TANGENT SWINGS THROUGH AN ANGLE θ_C AS IT MOVES FROM B TO D, AND AN ANGLE θ_F AS IT MOVES FROM E TO G. THESE CHANGES IN DIRECTION MEASURE THE CURVATURE AT C AND F, RESPECTIVELY. MOREOVER, ANGLES $\tilde{\theta}_C$ AND $\tilde{\theta}_F$ APPROXIMATE THESE ANGLES.

    - IF WE LET THE LENGTH, l, BETWEEN B AND D AND E AND G, RESPECTIVELY, GET ARBITRARILY SMALL, THEN THE FOLLOWING APPROACH EQUALITIES:

$$\ell \approx R_C \tilde{\theta}_C \approx R_C \theta \approx R_F \theta_F$$

- NOW, $\theta_C > \theta_F$, AND $\theta_C \approx \frac{\ell}{R_C}$ WHILE $\theta_F \approx \frac{\ell}{R_F}$. HENCE, $\frac{1}{R_C} > \frac{1}{R_F}$. THIS LETS US IDENTIFY:

$$\boxed{k_X = \frac{1}{R_X}} \qquad (3.31)$$

WITH A MEASURE OF CURVATURE AT ANY POINT, X, OF A CURVE, C, IN THE PLANE (WHERE R_X IS THE RADIUS OF THE CIRCLE THAT BEST APPROXIMATES C IN THE REGION AROUND X).

- FOR MORE COMPLICATED CURVES, WE NEED A BETTER WAY OF FINDING THE RADIUS OF THE APPROXIMATING CIRCLE. IF THE CURVE, C, IS PARAMETERIZED BY COORDINATE FUNCTIONS, $(x(\lambda), y(\lambda))$, THEN ITS CURVATURE, K, IS GIVEN BY:

$$K = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \quad \boxed{\text{1st \& 2nd DERIVATIVES}} \qquad (3.32)$$

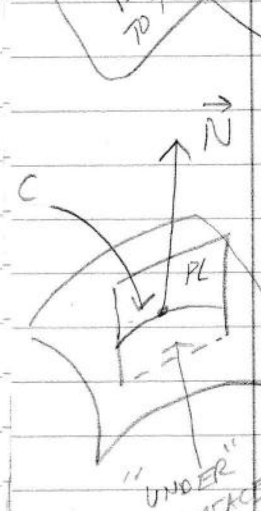
· GAUSSIAN CURVATURE OF 2-D SURFACE

· CONSIDER A 2-DIMENSIONAL SURFACE EMBEDDED IN A 3-DIMENSIONAL SPACE. HOW CAN WE MEASURE THE CURVATURE AT A POINT ON THE SURFACE, A?

i.e. PERPENDICULAR TO THE SURFACE

· IN OUR 3-D EMBEDDING SPACE, WE CAN CONSTRUCT A VECTOR, $\vec{N}$, NORMAL TO THE SURFACE AT A WHICH WILL PARTLY DEFINE A PLANE, PL, WHICH CONTAINS N. THE INTERSECTION OF PL WITH THE SURFACE INDUCES A CURVE, C, WHICH IS A GEODESIC IN THE NEIGHBORHOOD OF A.

OF THE SURFACE



· TO GET THE CURVATURE OF THE 2-D SURFACE — AND NOT MERELY OF C — AT A, USING THE ABOVE METHOD, WE LET PL ROTATE ABOUT N. WE THEN PICK THE LARGEST AND SMALLEST VALUES OF K — DENOTED k_{MAX} AND k_{MIN}.

· THE CURVATURE OF THE 2-DIMENSIONAL SURFACE AT A IS GIVEN BY ITS SO-CALLED "GAUSSIAN CURVATURE" DEFINED BY:

GAUSSIAN CURVATURE

$$\boxed{K = k_{MAX} k_{MIN}} \qquad (3.33)$$

- NOTE: FOR DIFFERENT CURVES AT THE SAME POINT, A, THE APPROXIMATING CIRCLES MAY LIE ON OPPOSITE SIDES OF THE SURFACE.

  → WE SAY THAT CURVATURE, K, IS POSITIVE AT A IF THE CENTER OF THE APPROXIMATING CIRCLE IS ON THE OPPOSITE SIDE OF THE ARROWHEAD OF $\vec{N}$, AND NEGATIVE IF IT IS ON THE SAME SIDE. (BY CONVENTION, WE TAKE NEGATIVE CURVATURES TO BE SMALLER THAN POSITIVE.)

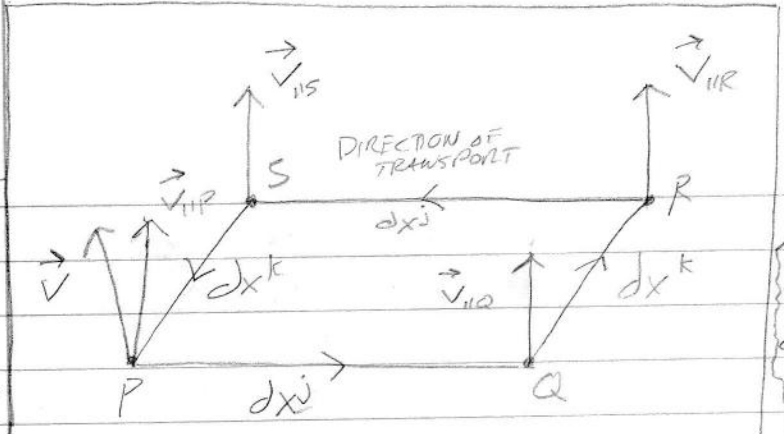
- CURVATURE IN HIGHER DIMENSIONS

  - IT TURNS OUT THAT THE CURVATURE OF A SURFACE CAN BE CALCULATED WITHOUT EMBEDDING IT IN A HIGHER-DIMENSIONAL SPACE. IT CAN BE COMPUTED FROM THE METRIC.

  - CONSIDER AN n-DIMENSIONAL RIEMANNIAN SPACE WITH METRIC $[g_{ij}]$ THAT CAN BE USED TO COMPUTE THE SPACE'S CONNECTION COEFFICIENTS, Γ^{i}_{jk} $(i=1,2,\ldots n)$.

  -

PARALLEL TRANSPORT OF VECTOR, $\vec{V}$, AT P AROUND INFINITESIMAL RECTANGLE w/ SIDES dx^k & dx^j



FOR TRANSPORTED VECTOR, $\vec{V}$, AROUN[D] PARALLELOGRAM w/ S[IDES] dx^k & dx^j GIVEN B[Y] K^ν AND J^μ, WE F[IND]

$\delta V^\rho = (dx^k)(dx^j) K^\nu J^\mu R^\rho$...

CHANGE IN V^ρ

WHICH WORKS BY TENS[OR] CONTRACTION.

• NOTE: THE DIFFERENCE BETWEEN $\vec{V}_{\parallel P}$ AND $\vec{V}_P$ SHOULD HAVE COMPONENTS THAT ARE PROPORTIONAL TO THE INFINITESIMAL DISPLACEMENTS AND TO THE COMPONENTS OF THE ORIGINAL VECTOR. THAT IS:

MEASURE OF CURVATURE

$$\boxed{V_{\parallel P}^{\ell} - V^{\ell} = \sum_{i,j,k} R^{\ell}_{i,j,k} V^i dx^j dx^k} \qquad (3.34)$$

FOR ANY COMPONENT, V^ℓ.

FIXED BY METRIC AN[D] ITS DERIVAT[IVES]

PARALLEL TRANSPORT IS INDEPENDENT OF COORDINATES

• MOREOVER, WHEN WE CARRY OUT THE PARALLEL TRANSPORT, WE FIND THAT:

$$(3.35) \qquad \boxed{R^{\ell}_{ijk} = \frac{\partial \Gamma^{\ell}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{\ell}_{ij}}{\partial x^k} + \sum_m \Gamma^{m}_{ik} \Gamma^{\ell}_{mj} - \sum_m \Gamma^{m}_{ij} \Gamma^{\ell}_{mk}}$$

RIEMANN TENSOR

• KEY POINT: UNDER GENERAL COORDINATE TRANSFORMATION R^{ℓ}_{ijk} TRANSFORMS AS A RANK 4 TENSOR WITH ONE CONTRAVARIANT INDEX AND THREE COVARIANT INDICES.

• WE, THUS, CALL $\boxed{R^{\ell}_{ijk}}$ THE "RIEMANN CURVATURE TENSOR" (OR SIMPLY "RIEMANN TENSOR").

i.e. OF ALL ITS COMPONENTS

MEASURE OF INTRINSIC CURVATURE

· THE IMPORT OF THE RIEMANN TENSOR, $[R^{l}_{ijk}]$, IS THAT THE VANISHING OF IT AT ALL POINTS IS NECESSARY AND SUFFICIENT FOR A SPACE TO BE FLAT. THIS GIVES A MEASURE OF CURVATURE THAT IS RELATED DIRECTLY TO THE METRIC. SO, IT IS AN INTRINSIC QUANTITY AS DESIRED.

ACTUALLY, R^{l}_{ijk} CONTAINS ALL THE INFORMATION ABOUT THE CURVATURE OF THE MANIFOLD.

· NOTE: IN n DIMENSIONS, $[R^{l}_{ijk}]$, HAS n^4 COMPONENTS! BUT (3.23) AND (3.35) ENSURE THAT IT HAS MANY SYMMETRIES LIKE $R^{l}_{ijk} = -R^{l}_{ikj}$. IN ALL, THERE ARE 6 INDEPENDENT COMPONENTS WHEN $n=3$.

· CURVATURE IN SPACETIME

· RIEMANNIAN SPACES ARE TECHNICALLY DEFINED BY A LINE ELEMENT TAKING THE FORM:

$$dl^2 = \sum_{i,j} g_{ij}\, dx^i dx^j \qquad (3.11)$$

WHERE $dl^2 > 0$. HOWEVER, RECALL THAT MINKOWSKI SPACETIME, WHICH WE SEEK TO GENERALIZE, HAS THE LINE ELEMENT:

$$ds^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad (3.37)$$

WHERE:

$$\eta_{\mu\nu} = \begin{cases} 1 \text{ IF } \mu=\nu=0 \\ -1 \text{ IF } \mu=\nu=1,2,3 \\ 0 \text{ OTHERWISE} \end{cases} \quad (3.38)$$

HENCE, THE MINKOWSKI LINE ELEMENT IS:

MINKOWSKI "SPACETIME SEPARATION" (INFINITESIMAL VERSION)

$$ds^2 = c^2dt^2 - d\vec{x}\cdot d\vec{x} = c^2dt - dx^2 - dy^2 - dz^2 \quad (3.3$$

· PROBLEM: ds^2 CAN CLEARLY BE NEGATIVE! SO, MINKOWSKI SPACETIME DOES NOT TECHNICALLY QUALIFY AS A RIEMANNIAN SPACE.

· IN ORDER TO PRESERVE CONSISTENCY, WE DESIGNATE SPACES FOR WHICH THE SQUARED LINE ELEMENT CAN BE POSITIVE, NEGATIVE, OR ZERO (NULL) "PSEUDO-RIEMANNIAN" SPACES.

PHYSICISTS OFTEN IGNORE THIS SUBTLETY

THEY SIMPLY IDENTIFY RIEMANNIAN SPACES W/ THOSE DEFINED BY THEIR METRIC, AS IN (3.11)

· IN ORDER TO GENERALIZE THE FLAT SPACETIME OF MINKOWSKI GEOMETRY TO THE CURVED SPACETIME OF GENERAL RELATIVITY, WE REPLACE THE METRIC COEFFICIENTS, $\eta_{\mu\nu}$, WHICH ARE CONSTANTS $(+1,-1,-1,-1)$ WITH METRIC COEFFICIENTS, $g_{\mu\nu}$, THAT ARE FUNCTIONS OF THE COORDINATES:

$$\boxed{ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu} \quad (3.40)$$

GENERAL (PSEUDO-) RIEMANNIAN METRIC

· **NOTE:** MANY PROPERTIES OF RIEMANNIAN SPACES EXTEND TO PSEUDO-RIEMANNIAN SPACES. IN PARTICULAR, THE VANISHING OF THE RIEMANN TENSOR, $R^{\delta}_{\nu\beta\gamma}$, IS NECESSARY & SUFFICIENT FOR FLATNESS.

TIDAL ACCELERATION

$\frac{\Delta a}{\Delta r} \neq 0$

$\Delta r \to 0$

TEST REGION

NOTE: NOT ALL TIDAL EFFECTS VANISH.

· **KEY POINT:** IF A SPACETIME IS FLAT IT IS ALWAYS POSSIBLE TO CHOOSE A COORDINATE SYSTEM SO THAT THE METRIC IS THE MINKOWSKI METRIC AT EVERY POINT.

"FLAT SPACETIME"

· FOR A CURVED SPACETIME, ONE CAN CHOOSE COORDINATE GIVING THE MINKOWSKI METRIC AT A POINT, BUT NOT EVERYWHERE.

"CURVED SPACETIME"

"REDUCES" TO SR AT A POINT

· **UPSHOT:** IN THE CONTEXT OF GR, THE RESULTS OF SR CONTINUE TO HOLD IN THE NEIGHBORHOOD OF ANY POINT, BUT THEY WILL NOT HOLD GENERALLY

$\rightsquigarrow$ SR APPLIES LOCALLY, AS EUCLIDEAN GEOMETRY APPLIES LOCALLY ON EARTH'S SURFACE.

· **NOTE:** IN A PSEUDO-RIEMANNIAN SPACE, WE CAN HAVE CURVES FOR WHICH ds^2 EQUALS

0 "LENGTH"

ZERO AT ALL POINTS ALONG THE CURVE. SUCH CURVES ARE CALLED "NULL CURVES."

· NOTE: ONE EXAMPLE OF A NULL CURVE IS A NULL GEODESIC. BUT THAT MEANS THAT IN PSEUDO-RIEMANNIAN GEOMETRY WE CANNOT DEFINE GEODESICS AS PATHS OF SHORTEST DISTANCE. SO, WE USE THE OTHER DEFINITION - AS A CURVE ALONG WHICH THE TANGENT ALWAYS POINTS IN THE SAME DIRECTION INSTEAD.

NULL GEODESICS REPRESENT POSSIBLE PATHS OF LIGHT RAYS IN CURVED SPACETIME

· GENERAL RELATIVITY & GRAVITATION

· NEWTON'S LAW OF UNIVERSAL GRAVITATION, WHILE VERY ACCURATE, GAVE NO ACCOUNT OF GRAVITY.

· MOREOVER, WE HAVE SEEN THAT UNDER A CHANGE IN INERTIAL FRAME, THE FORCE DESCRIBED BY THE LAW DOES NOT TRANSFORM AS A (THREE-) FORCE SHOULD, ACCORDING TO SR. IT ALSO ACTS INSTANTANEOUSLY CONTRA SR.

· GR IS EINSTEIN'S THEORY OF GRAVITY.

· EINSTEIN'S MOTIVATIONS

· EINSTEIN WAS MOTIVATED BY THREE PRINCIPLES:

(1) THE PRINCIPLE OF EQUIVALENCE

(2) THE PRINCIPLE OF COVARIANCE

(3) THE PRINCIPLE OF CONSISTENCY

· (1) PRINCIPLE OF EQUIVALENCE

· IN NEWTONIAN MECHANICS, THERE ARE TWO NOTIONS MASS - INERTIAL MASS, $m = \frac{|F|}{|a|}$ AND GRAVITATIONAL MASS,

WHICH IS DEFINED BY THE LAW THAT A FORCE, $\vec{F}_{12}$, ON PARTICLE 1 OF GRAVITATIONAL MASS, μ_1, DUE TO PARTICLE 2 OF GRAVITATIONAL MASS, μ_2, IS OF MAGNITUDE:

$$\boxed{|\vec{F}_{12}| = \frac{G\mu_1\mu_2}{|\vec{x}_1 - \vec{x}_2|^2}} \qquad (4.1)$$

- HOWEVER, IT IS AN EXPERIMENTAL FACT THAT THE RATIO OF $\boxed{\mu/m}$ IS THE SAME FOR ALL BODIES. (WE CHOOSE UNITS SO THAT $\mu/m = 1$, AND GENERALLY IGNORE THE DISTINCTION.) WHY IS THIS SO? EINSTEIN SOUGHT AN EXPLANATION.

  - IN NEWTONIAN MECHANICS, $\mu = m$ IMPLIES THAT THE ACCELERATION OF ANY BODY DUE TO THE GRAVITATIONAL FORCE IS INDEPENDENT OF ITS MASS:

    - SINCE $\mu_i = m_i$,

$$|\vec{F}_{12}| = \frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|^2}$$

SO, THE ACCELERATION, $\vec{a}_1$, OF PARTICLE 1 DUE TO THIS FORCE IS: $\vec{F}_{12} = m_1\vec{a}_1$

HENCE, $\cancel{m_1}|a_1| = \dfrac{G\cancel{m_1}m_2}{|\vec{x}_1 - \vec{x}_2|^2}$

AND $|a_1|$ IS <u>INDEPENDENT</u> OF m_1.

- <u>UPSHOT</u>: IN A <u>FREELY FALLING</u> (NON-ROTATING) BOX IN AN AIRLESS SHAFT, AN OBSERVER FINDS HER FRAME TO BE AN <u>INERTIAL</u> ONE.

LAWS OF SR HOLD LOCALLY

- <u>CAVEAT</u>: A <u>FREELY FALLING</u> FRAME IS ONLY <u>LOCALLY</u> INERTIAL IN LIGHT OF <u>TIDAL FORCES</u>. THE <u>GRAVITATIONAL FIELD</u> IN WHICH THE BOX IS LOCATED IS <u>NOT</u> UNIFORM.

- SIMILARLY, IN A <u>ROCKET</u> WITH NO <u>GRAVITATIONAL</u> FIELD <u>ACCELERATED UNIFORMLY</u> AT g, NO <u>SUFFICIENTLY LOCAL</u> EXPERIMENT COULD DISTINGUISH THIS <u>ACCELERATION</u> FROM <u>GRAVITY</u>.

AKA PRINCIPLE OF UNIVERSALITY OF FREE FALL

- EINSTEIN ELEVATED THE ANALOGY BETWEEN <u>ACCELERATION</u> AN <u>GRAVITY</u> TO A <u>PRINCIPLE</u>:

  - <u>WEAK EQUIVALENCE</u>: IN A <u>SUFFICIENTLY LOCAL</u> REGION OF SPACETIME ADJACENT TO <u>MASS</u>, THE MOTION OF BODIES SUBJECT TO GRAVITY <u>ALONE</u> CANNOT <u>BE EXPERIMENTALLY DISTINGUISHED</u> BY ANY <u>EXPERIMENT</u> WITHIN A REGION

FROM THE MOTION OF BODIES

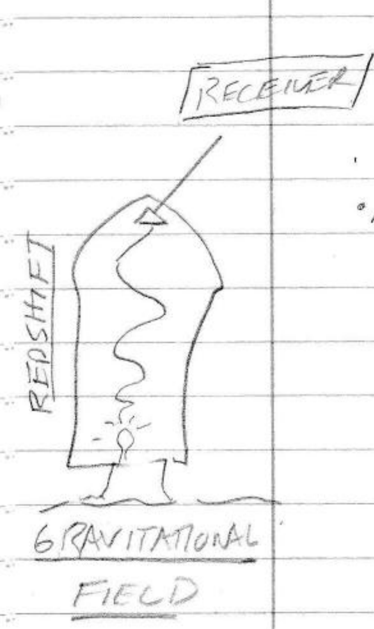
OF APPROPRIATE UNIFORM ACCELERATION.

- NOTE: THIS PRINCIPLE EXCLUDES ELECTRO-MAGNETIC FORCES & NUCLEAR FORCES. ALSO, IT DOES NOT APPLY TO VERY MASSIVE OBJECTS THAT WOULD THEMSELVES SUBSTANTIALLY CHANGE THE NEARBY GRAVITATIONAL FIELD.

AKA, "EQUIVALENCE PRINCIPLE"

- STRONG EQUIVALENCE: IN A SUFFICIENTLY LOCALIZED REGION OF SPACETIME ADJACENT TO MASS, THE PHYSICAL BEHAVIOR OF BODIES CANNOT BE DISTINGUISHED VIA ANY EXPERIMENT FROM THE PHYSICAL BEHAVIOR OF BODIES WITHIN A REGION OF APPROPRIATE UNIFORM ACCELERATION.

  - NOTE: THIS GOES FAR BEYOND THE UNIVERSALITY OF FREE FALL.



- ALTHOUGH THE STRONG EQUIVALENCE PRINCIPLE REQUIRES MORE TESTING, THE WEAK EQUIVALENCE IS HIGHLY CONFIRMED. MOREOVER, THE WEAK PRINCIPLE INSPIRED EINSTEIN TO PREDICT THE GRAVITATIONAL DEFLECTION OF LIGHT AND THE GRAVITATIONAL REDSHIFT OF LIGHT.

- EINSTEIN USED THIS FACT TO ARGUE THAT GRAVITY IS A "FICTIONAL" FORCE (LIKE CENTRIFUGAL FORCE). LOCALLY GRAVITATIONAL FORCES RESULT FROM USING A FRAME THAT IS NOT FREELY FALLING – i.e., LOCALLY INERTIAL.

(2) PRINCIPLE OF GENERAL COVARIANCE

- THIS WAS CONCEIVED BY EINSTEIN AS AN EXTENSION OF THE PRINCIPLE OF RELATIVITY (THAT THE LAWS OF PHYSICS ARE FORM-INVARIANT UNDER LORENTZ TRANSFORMATIONS).

- THE LATTER IS GUARANTEED IF WE CAN WRITE THE LAWS AS FOUR-TENSOR EQUATIONS. IT ALSO PRECLUDES NEWTON'S LAW OF GRAVITATION.

THE PHYSICAL SIGNIFICANCE OF THIS CONSTRAINT IS CONTROVERSIAL

- THE PRINCIPLE OF GENERAL COVARIANCE REQUIRES THAT THE PHYSICAL LAWS ARE FORM-INVARIANT UNDER A MUCH LARGER CLASS OF TRANSFORMATIONS.

"EQUIVALENCE"

- EINSTEIN HIMSELF HOPED TO ESTABLISH THE PHYSICAL EQUIVALENCE OF ALL FRAMES, INERTIAL & NOT. BUT THIS APPEARS UNTENABLE.

• WE HAVE SEEN HOW TO WRITE VARIOUS LAWS AS FORM-INVARIANT UNDER LORENTZ TRANSFORMATION – i.e. AS BALANCED FOUR-TENSOR RELATIONS.

• WRITE THE LAWS OF ELECTROMAGNETISM USING SCALAR INVARIANTS (FOUR-TENSORS OF RANK ZERO), CONTRAVARIANT OR COVARIANT FOUR-VECTORS (FOUR-TENSORS OF RANK 1), AND SOME FOUR-TENSORS OF RANK 2.

CONTRAVARIANT FIELD FOUR-TENSOR $[F^{\mu\nu}]$, MIXED $[F^{\mu}{}_{\nu}]$, AND COVARIANT $[F_{\mu\nu}]$.

• WE NOW EXTEND THE NOTION OF TENSOR BEYOND FOUR-TENSORS, AND TENSOR EQUATIONS BEYOND THOSE WE MET IN SR.

• TENSORS IN GENERAL

• TENSORS ARE MULTI-COMPONENT OBJECTS THAT ARE CATEGORIZED ACCORDING TO HOW THEIR COMPONENTS BEHAVE UNDER GENERAL COORDINATE TRANSFORMATIONS –

$$x'^{\mu} = x'^{\mu}(x^{\nu}), \text{ WITH } \mu, \nu = 0,1,2,3$$

NEW COORDINATES

FUNCTIONS OF THE OLD

• "WELL-BEHAVED" (e.g. DIFFERENTIABLE) BUT NOT NECESSARILY LINEAR AS WITH LORENTZ TRANSFORM

- THE MORE RESTRICTIVE LORENTZ TRANSFORMATIONS WERE LINEAR SO THAT DERIVATIVES, $\partial x'^{\mu}/\partial x^{\nu}$ IS CONSTANT (e.g., c, V, γ, ETC.).

- FOR GENERAL COORDINATE TRANSFORMATIONS, $x'^{\mu} = x'^{\mu}(x^{\nu})$, THE SIXTEEN FUNCTIONS, $\partial x'^{\mu}/\partial x^{\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ AND THE SIXTEEN FUNCTIONS, $\partial x^{\beta}/\partial x'^{\alpha}$ $(\alpha, \beta = 0, 1, 2, 3)$ HAVE NO SUCH RESTRICTIONS.

- IN GENERAL, A TENSOR OF CONTRAVARIANT RANK, m, AND COVARIANT RANK, n, HAS COMPONENTS:

$$(2.110) \quad \boxed{T'^{\mu_1 \mu_2 \ldots \mu_m}_{\alpha_1 \alpha_2 \ldots \alpha_n} = \sum_{\nu_1 \nu_2 \ldots \nu_m, \beta_1 \beta_2 \ldots \beta_n} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \frac{\partial x'^{\mu_2}}{\partial x^{\nu_2}} \ldots \frac{\partial x'^{\mu_m}}{\partial x^{\nu_m}} \times \frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}} \frac{\partial x^{\beta_2}}{\partial x'^{\alpha_2}} \ldots \frac{\partial x^{\beta_n}}{\partial x'^{\alpha_n}} \times T^{\nu_1 \nu_2 \ldots \nu_m}_{\beta_1 \beta_2 \ldots \beta_n}}$$

- EXAMPLES: THE DISPLACEMENT FOUR-VECTOR, $[dx^{\mu}] = (dx^0, dx^1, dx^2, dx^3)$ IS A CONTRAVARIANT TENSOR OR RANK 1. ITS COMPONENTS TRANSFORM ACCORDING TO:

$$\boxed{dx'^{\mu} = \sum_{\alpha=0}^{3} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} dx^{\alpha}} \quad (4.2)$$

METRIC TENSOR

- SIMILARLY, $[g_{\mu\nu}]$, IS A RANK

2 COVARIANT TENSOR, & ITS DUAL IS A RANK 2 CONTRAVARIANT TENSOR WHOSE COMPONENTS TRANSFORM ACCORDING TO:

METRIC TENSOR

CONTRAVARIANT

$$\boxed{g'^{\mu\nu} = \sum_{\alpha=0}^{3} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \sum_{\beta=0}^{3} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}} \qquad (4.3)$$

WHILE ITS COVARIANT FORM TRANSFORMS THUS:

COVARIANT

$$\boxed{g'_{\mu\nu} = \sum_{\alpha=0}^{3} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \sum_{\beta=0}^{3} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}} \qquad (4.4)$$

KRONECKER DELTA

WHERE $\boxed{\sum_{\gamma=0}^{3} g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}_{\beta}}$ (3.24)

- FINALLY, THE RIEMANN CURVATURE TENSOR, $[R^{\alpha}_{\beta\gamma\delta}]$, IS A MIXED TENSOR OF CONTRAVARIANT RANK 1 AND COVARIANT RANK 3. EACH COMPONENT TRANSFORMS ACCORDING TO:

$$\boxed{R'^{\alpha}_{\beta\gamma\delta} = \sum_{\mu=0}^{3} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \sum_{\nu=0}^{3} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \sum_{\rho=0}^{3} \frac{\partial x^{\rho}}{\partial x'^{\gamma}} \sum_{\sigma=0}^{3} \frac{\partial x^{\sigma}}{\partial x'^{\delta}} R^{\mu}_{\nu\rho\sigma}} \qquad (4.5)$$

- RECALL: NOT ALL MULTI-COMPONENT OBJECTS ARE TENSORS. THE 64 CONNECTION COEFFICIENTS, $\Gamma^{\alpha}_{\beta\gamma}$, OF A 4-DIMENSIONAL SPACETIME DO NOT FORM A TENSOR.

• RULES OF TENSOR ALGEBRA

• RECALL THE THE METRIC TENSOR RELATES CONTRAVARIANT AND COVARIANT COMPONENTS VIA:

$$\boxed{A_\mu = \sum_{\alpha=0}^{3} g_{\mu\alpha} A^\alpha} \quad (4.6) \qquad \boxed{\text{LOWERS INDICES}}$$

AND

$$\boxed{A^\mu = \sum_{\alpha=0}^{3} g^{\mu\alpha} A_\alpha} \quad (4.7) \qquad \boxed{\text{RAISES INDICES}}$$

(a) SCALING: A TENSOR $[T^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n}]$ OF CONTRAVARIANT RANK m, AND COVARIANT RANK n, CAN BE MULTIPLIED BY A SCALAR, S, TO PRODUCE A NEW TENSOR, $[U^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n}]$ OF THE SAME RANK.

→ EACH COMPONENT OF $U^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n}$ IS JUST THE RESULT OF MULTIPLYING THE CORRESPONDING COMPONENT OF $[T^{\mu_1\mu_2\ldots\mu_m}_{\alpha_1\alpha_2\ldots\alpha_n}]$ BY S. SO, FOR EACH μ & α:

$$\boxed{S T^\mu_\alpha = U^\mu_\alpha}$$

(b) ADDITION & SUBTRACTION: ADDING & SUBTRACTING TENSORS OF THE SAME TYPE PRODUCE NEW TENSORS. IT IS ALSO CARRIED OUT COMPONENT BY COMPONENT. SO, FOR EACH

VALUE OF μ AND α, $S^{\mu}_{\alpha} + T^{\mu}_{\alpha} = U^{\mu}_{\alpha}$.

THIS FOLLOWS FROM THE TRANSFORMATION PROPERTIES OF CONTRAVARIANT & COVARIANT VECTORS

(c) MULTIPLICATION: TENSORS CAN BE MULTIPLIED TO PRODUCE PRODUCTS OF THEIR COMPONENTS, SO, GIVEN THREE TENSORS $[X^{\mu}]$, $[Y_{\alpha}]$ AND $[Z_{\beta}]$, WE CAN FORM A NEW TENSOR WITH COMPONENTS:

$$A^{\mu}_{\alpha\beta} = X^{\mu} Y_{\alpha} Z_{\beta}$$

→ THE RANK OF THE NEW TENSOR IS THE SUM OF THE RANKS OF THE TENSORS BEING MULTIPLIED (e.g. $A^{\mu}_{\alpha\beta}$ IS OF RANK 3).

(d) CONTRACTION: FOR A SINGLE TENSOR WITH CONTRAVARIANT RANK m AND COVARIANT RANK n, OR FOR A PRODUCT OF TENSORS WITH COMBINED RANK m AND COVARIANT RANK n, WE CAN FORM ANOTHER TENSOR, OF CONTRAVARIANT RANK, $m-1$, AND COVARIANT RANK, $n-1$, BY SUMMING OVER ONE RAISED & ONE LOWERED INDEX

$X^{\epsilon} = T^{\alpha\epsilon}_{\alpha}$ TRANSFORMS:

$$X'^{\epsilon} = T'^{\alpha\epsilon}_{\alpha}$$

$$= \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}} \frac{\partial x^{\eta}}{\partial x'^{\alpha}} T^{\beta\lambda}_{\eta}$$

$$= \frac{\partial x^{\eta}}{\partial x^{\beta}} \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}} T^{\beta\lambda}_{\eta}$$

$$= \delta^{\eta}_{\beta} \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}} T^{\beta\lambda}_{\eta}$$

$$= \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}} T^{\beta\lambda}_{\beta}$$

$$= \frac{\partial x'^{\epsilon}}{\partial x^{\lambda}} X^{\lambda}$$

• EXAMPLE: $B_{\gamma} = \sum_{\sigma=0}^{3} A^{\sigma}_{\sigma\gamma} = \sum_{\sigma=0}^{3} X^{\sigma} Y_{\sigma} Z_{\gamma}$

• NOTE: (a)–(d) MEAN THAT ONLY CERTAIN WELL-DEFINED TENSOR EXPRESSIONS MAKE SENSE.

I.E., EQUATION OF COVARIANT FORM

NEED NOT INVOLVE COVARIANT INDICES!

• KEY POINT: INICES IN A COVARIANT EQUATION FORM A DISTINCT PATTERN SUCH THAT THE EQUATION TAKES THE SAME FORM IN ANY COORDINATE SYSTEM. HERE IS AN ILLUSTRATION:

μ AND ν NOT SUMMED OVER

$$A^{\mu}_{\ \nu} = 5B^{\mu}_{\ \nu} + \sum_{\alpha=0}^{3} C^{\mu}_{\ \alpha} E^{\alpha}_{\ \nu} + \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} X^{\mu\alpha}_{\ \ \ \beta} Y_{\nu\alpha}^{\ \ \ \beta} \quad (4.7)$$

α & β ARE SUMMED OVER

→ μ AND ν ARE "FREE" INDICES, SO EACH IS CONSISTENTLY UP OR DOWN.

→ EACH OF μ AND ν APPEARS ONLY ONCE IN EVERY TERM ON EACH SIDE.

→ α AND β ARE DUMMY INDICES.

→ α APPEAR EXACTLY TWICE IN EACH TERM.

• COVARIANT DIFFERENTIATION

• AS WITH MAXWELL'S EQUATIONS, THE LAWS OF GRAVITATION WILL INVOLVE DIFFERENTIAL EQUATIONS, SO WE NEED TO KNOW HOW TO DIFFERENTIATE GENERAL TENSORS IN A COVARIANT WAY.

"FORM INVARIANT"

• NOTE: SIMPLE PARTIAL DERIVATIVES OF TENSORS ARE NOT GENERALLY COVARIANT, BUT WE WANT SOMETHING THAT REDUCES TO THIS FOR FLAT MINKOWSKI SPACE WITH CARTESIAN COORDINATES.

ALLOWS US TO COMPARE VECTORS AT DIFFERENT POINTS

· RECALL: RECALL THAT PARALLEL TRANSPORTING A RANK 1 TENSOR INVOLVES THE CONNECTION COEFFICIENTS:

(3.23) $$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\sum_{\delta} g^{\alpha\delta}\left(\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right)$$

FIXED BY METRIC

RANK 1 CONTRAVARIANT TENSOR

· TO MAKE PROGRESS, REMEMBER THAT:

COVARIANT DERIVATIVE OF V^{α}

$$\boxed{\nabla_{\beta} V^{\alpha} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + \sum_{\lambda} \Gamma^{\alpha}_{\lambda\beta} V^{\lambda}} \quad (4.11)$$

TRANSFORMS LIKE A COMPONENT OF A RANK 2 TENSOR.

SOME AUTHORS USE A SEMICOLON FOR COVARIANT DERIVATIVES & COMMA FOR PARTIAL DERIVATIVES

→ IDEA: THE "NON-TENSORIAL BEHAVIOR" OF $\partial V^{\alpha}/\partial x^{\beta}$ IS CANCELLED BY THE NON-TENSORIAL BEHAVIOR OF $\sum_{\lambda} \Gamma^{\alpha}_{\lambda\beta} V^{\lambda}$.

*NOTE: WE ASSUME METRIC COMPATIBILITY, i.e., $\nabla_{\beta} g_{\mu\nu} = \nabla_{\beta} g^{\mu\nu} = 0$

· LIKEWISE, WE HAVE:

$$\boxed{\nabla_{\beta} V_{\alpha} = \frac{\partial V_{\alpha}}{\partial x^{\beta}} - \sum_{\lambda} \Gamma^{\lambda}_{\alpha\beta} V_{\lambda}} \quad (4.12)$$

· NOW, THE COVARIANT DERIVATIVE OF HIGHER RANK TENSORS IS JUST A GENERALIZATION OF (4.11) AND (4.12).

· UPSHOT: WHILE THE PROCESS OF PARTIAL DIFFERENTIATION OF A TENSOR DOES NOT NORMALLY PRODUCE ANOTHER TENSOR, COVARIANT DIFFERENTIATION DOES. THIS WORKS FOR TENSORS OF ANY RANK, AND IS EXEMPLIFIED BY:

EXAMPLE OF [illegible] FOR A (RANK 2 TENSOR)

(4.3) $$\boxed{\nabla_\lambda T^\mu{}_\nu = \frac{\partial T^\mu{}_\nu}{\partial x^\lambda} + \sum_\rho \Gamma^\mu{}_{\rho\lambda} T^\rho{}_\nu - \sum_\rho \Gamma^\rho{}_{\nu\lambda} T^\mu{}_\rho}$$

(3) PRINCIPLE OF CONSISTENCY

· A NEW THEORY THAT SUPERCEDES EARLIER ONES SHOULD ACCOUNT FOR THE CORRECT PREDICTIONS OF THE OLD THEORY. SO, GR SHOULD EXPLAIN THE SUCCESS OF SR AND NEWTON'S THEORY OF GRAVITY.

· NOTE: GR EXPLAINS THE SUCCESS OF SR BECAUSE IT USES A SPACETIME THAT IS LOCALLY EQUIVALENT TO MINKOWSKI SPACETIME.

· TO SEE HOW GR IS CONSISTENT WITH NEWTON'S THEORY, WE REFORMULATE THE LATTER AS A "FIELD THEORY."

→ IDEA: POSTULATE A GRAVITATIONAL FIELD THAT OBEYS DIFFERENTIAL EQUATIONS ON ANALOGY WITH THE ELECTRIC AND MAGNETIC FIELDS.

• THE NEWTONIAN GRAVITATIONAL FIELD, $\vec{g}(\vec{r})$, IS A FUNCTION OF POSITION, $\vec{r}=(x,y,z)$, THAT GIVES THE NEWTONIAN GRAVITATIONAL FORCE PER UNIT MASS THAT WOULD ACT ON A TEST PARTICLE AT $\vec{r}$. SO, THE NEWTONIAN GRAVITATIONAL

NGF

FORCE (NGF) ON A PARTICLE OF MASS m IS $m\vec{g}(\vec{r})$.

• MOREOVER, RECALL THAT NEWTON'S LAW IMPLIES

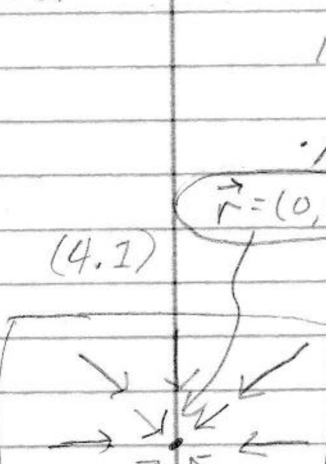
$\vec{r}=(0,0,0)$

UNIFORM SPHERICAL BODY CENTRED AT ORIGIN

(4.1) $$\vec{g}(\vec{r}) = -\frac{GM}{|\vec{r}|^2}\,\vec{e}_r$$

FLIPS DIRECTION

UNIT VECTOR IN RADIAL DIRECTION AWAY FROM ORIGIN



• NOW, IF M IS ENCLOSED IN A LARGER SPHERE OF RADIUS, R, ALSO CENTERED AT ORIGIN, THEN THE FLUX OF THE NGF LEAVING IS GIVEN BY THE SURFACE INTEGRAL:

OUTWARD GRAVITATIONAL FLUX $= \int_S \vec{g}\cdot\hat{n}\,dS$

• NOTE: $\hat{n}$ IS NORMAL TO S EVERYWHERE

OUTWARD POINTING UNIT VECTOR

• NOW, $\vec{e}_r \cdot \hat{n} = -1$, AND $(4\pi R^2) \times (GM/R^2) = 4\pi GM$, SO WE HAVE:

$$\int_S \vec{g} \cdot \hat{n} \, dS = -4\pi GM$$

• THE <u>DIVERGENCE THEOREM</u> SAYS THAT THIS CAN BE REWRITTEN AS THE <u>VOLUME INTEGRAL</u> OF THE <u>DIVERGENCE</u> OF THE FIELD, $\nabla \cdot g$ THROUGHOUT V BOUNDED BY S:

$\nabla \cdot \vec{g} =$

$\dfrac{\partial g_z}{\partial z})$

$$\int_V \nabla \cdot \vec{g} \, dV = -4\pi GM \qquad (4.15)$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

• WRITING THE MASS OF THE SPHERE AS AN INTEGRAL OVER ITS DENSITY, ρ:

$$\int_V \nabla \cdot \vec{g} \, dV = -4\pi G \int_V \rho \, dV \qquad (4.17)$$

• IN FACT, WE HAVE:

$$\boxed{\nabla \cdot \vec{g} = -4\pi G\rho} \qquad (4.18)$$

DERIVATIVE OF NGF

MASS DENSITY (SOURCE OF FIELD)

- NOTE: THE NEWTONIAN GRAVITATIONAL FORCE IS CONSERVATIVE - i.e., THE WORK DONE AGAINST THE GRAVITATIONAL FORCE IN A MOVING BODY IS INDEPENDENT OF THE PATH IT FOLLOWS. THIS ALLOWS US TO ASSOCIATE THE FORCE WITH GRAVITATIONAL POTENTIAL ENERGY.

- NGF LIKEWISE CORRESPONDS TO A GRAVITATIONAL POTENTIAL FIELD, $\Phi(\vec{r})$, WHICH GIVES THE GRAVITATIONAL POTENTIAL ENERGY PER UNIT MASS AT $\vec{r}$.

$$\boxed{\vec{g} = -\nabla\Phi} = -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right) \qquad (4.19)$$

BY (4.18):

BOTH FUNCTIONS OF $\vec{r}$

$$\underbrace{\nabla\cdot\nabla}\Phi = 4\pi G\rho \qquad (4.20)$$

∇^2

(LAPLACIAN OPERATOR)

POISSON'S EQUATION

MASS DENSITY

EQUIVALENT TO NEWTON'S INVERSE SQUARE LAW

$$\Rightarrow \boxed{\nabla^2\Phi = 4\pi G\rho} \qquad (4.21)$$

SOURCE OF FIELD

$$= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho \qquad (4.22)$$

LIKE ELECTRICITY & MAGNETISM

- POISSON'S EQUATION SUMMARIZES NEWTONIAN GRAVITY IN TERMS OF DIFFERENTIAL EQUATIONS.

CONSISTENCY

- WE WILL SEE THAT GR PREDICTS AN EQUATION LIKE POISSON'S, IMPLYING THAT NEWTON'S THEORY PROVIDES AN APPROXIMATION IN APPROPRIATE CIRCUMSTANCES, AS DESIRED.

- INGREDIENTS OF GR

  - EINSTEIN REALIZED THAT IF GRAVITY WAS "BUILT INTO" SPACETIME THEN IT WOULD ACT EQUALLY ON ALL FORMS OF MATTER, THUS EXPLAINING THE UNIVERSALITY OF FREE FALL.

  - THIS LED EINSTEIN TO CONSIDER TWO PARTICULAR TENSORS - ONE DESCRIBING THE PROPERTIES OF MATTER, AND ONE CONCERNED WITH SPACETIME GEOMETRY.

- ENERGY-MOMENTUM TENSOR

  - MASS IS CONSERVED IN NEWTONIAN MECHANICS. IT IS ALSO THE SOURCE OF GRAVITATION.

  - MASS IS NOT CONSERVED IN SPECIAL RELATIVITY. BUT RECALL THAT WE DO HAVE:

$$E = p^2c^2 + m^2c^4 \qquad (2.4)$$

INCLUDING MASS-ENERGY

AND THERE ARE CONSERVATION LAWS FOR ENERGY AND MOMENTUM.

• UPSHOT: THE SOURCE OF GRAVITATION IN GR, IS NOT MASS ALONE – IT ALSO INVOLVES ENERGY AND MOMENTUM.

• THE $\boxed{\text{ENERGY-MOMENTUM TENSOR}}$ DESCRIBES THE DISTRIBUTION AND FLOW OF ENERGY AND MOMENTUM IN A REGION OF SPACETIME.

→ IT IS A RANK 2 TENSOR – i.e., AT ANY EVENT IN SPACETIME IT IS SPECIFIED BY 16 COMPONENTS, $\boxed{T^{\mu\nu}}$

→ IT IS SYMMETRIC – i.e., $T^{\mu\nu} = T^{\nu\mu}$ (SO, ONLY 10 COMPONENTS ARE INDEPENDENT)

→ EACH COMPONENT CAN BE MEASURED IN UNITS OF ENERGY DENSITY (Jm^{-3})

→ EACH COMPONENT IS A FUNCTION OF SPACETIME COORDINATES SUCH THAT:

• T^{00} IS THE LOCAL ENERGY (INCLUDING MASS-ENERGY)

• $T^{0i} = T^{i0}$ IS THE DENSITY OF THE i-COMPONENT OF MOMENTUM TIMES C.

• $T^{ij} = T^{ji}$ IS THE RATE OF FLOW OF THE i-COMPONENT OF MOMENTUM PER UNIT AREA AT RIGHT ANGLES TO THE j-DIRECTION.

• EXAMPLE: A PARADIGM SITUATION IS A REGION OCCUPIED BY A CLOUD OF NON-INTERACTING PARTICLES, ALL OF MASS, m. SUCH MATTER IS CALLED "DUST." THE COMPONENTS OF $T^{\mu\nu}$ FOR DUST AT ANY SPACETIME EVENT CAN BE WRITTEN IN A COVARIANT WAY AS FOLLOWS:

$$\boxed{T^{\mu\nu} = \rho U^{\mu} U^{\nu}} \qquad (4.23)$$

• EXAMPLE: ANOTHER EXAMPLE OF $T^{\mu\nu}$ IS THE ENERGY-MOMENTUM TENSOR OF AN IDEAL FLUID, WHICH ALSO INVOLVES PRESSURE, P. IN THIS CASE WE HAVE:

METRIC

$$\boxed{T^{\mu\nu} = (\rho + p/c^2) U^{\mu} U^{\nu} - p g^{\mu\nu}} \qquad (4.25)$$

• EXAMPLE: FINALLY, CONSIDER A REGION OF SPACE WITH NO MATTER BUT WITH ELECTRIC AND MAGNETIC FIELDS. THEN WE GET:

ELECTROMAGNETIC FIELD TENSOR

(4.28) $$\boxed{T^{\mu\nu} = \frac{1}{\mu_0}\left(\sum_\sigma F^{\mu}{}_{\sigma} F^{\nu\sigma} - \frac{1}{4}\sum_{\rho,\sigma} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)}$$

→ THIS CASE SHOWS THAT IN GR ELECTROMAGNETIC RADIATION ALONE CAN BE THE SOURCE OF GRAVITATION EVEN THOUGH PHOTONS HAVE NO MASS.

• NOTE: IN LIGHT OF THE EQUATION OF CONTINUITY (2.76), IT IS NO SURPRISE THAT WE HAVE CONSERVATION OF RELATIVISTIC ENERGY AND MOMENTUM IN A LOCALLY INERTIAL FRAME (WHERE SR HOLDS):

$$\sum_\mu \frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0 \qquad (4.29)$$

MOREOVER, IN ARBITRARY COORDINATES,

$$\boxed{\sum_\mu \nabla_\mu T^{\mu\nu} = 0} \qquad (4.30)$$

→ i.e., THE COVARIANT DIVERGENCE OF $T^{\mu\nu}$ IS ZERO.

HOWEVER, IN CURVED SPACETIME, (4.30) DOES NOT DESCRIBE CONSERVATION OF ENERGY AND MOMENTUM FOR THE CONTENTS OF SPACETIME. WE NEED TO TAKE ACCOUNT OF GRAVITATIONAL ENERGY TOO.

• THE EINSTEIN TENSOR

• RECALL THAT THE RIEMANN TENSOR, $R^{\delta}{}_{\alpha\beta\gamma}$, INVOLVES CONNECTION COEFFICIENTS, $\Gamma^{\delta}{}_{\alpha\beta}$, WHICH ARE THEMSELVES DEFINED IN TERMS OF THE METRIC AND ITS DERIVATIVES.

• IF WE CONTRACT THE FIRST AND LAST INDICES ON $R^{\delta}{}_{\alpha\beta\gamma}$, THEN WE GET A NEW RANK 2 TENSOR:

INDICES

RICCI TENSOR: $\boxed{R_{\alpha\beta} = \sum_{\gamma} R^{\gamma}{}_{\alpha\beta\gamma}}$ (4.31)

RIEMANN TENSOR

→ BECAUSE OF THE RIEMANN TENSOR'S SYMMETRIES, THE RICCI TENSOR IS SYMMETRIC WITH RESPECT TO INTERCHANGING ITS INDICES, $R_{\alpha\beta} = R_{\beta\alpha}$.

• CONTRACTING THE INDICES ON THE RICCI TENSOR GIVES:

RICCI SCALAR!!

CURVATURE TENSOR: $\boxed{R = \sum_{\alpha,\beta} g^{\alpha\beta} R_{\alpha\beta}}$ ← EXPRESSED IN TERMS OF METRIC TENSOR AND DERIVATIVES.

RICCI TENSOR

• NOW THE BASIC INGREDIENT TO GR IS DEFINED IN TERMS OF THE RICCI TENSOR AND

CURVATURE TENSOR (RICCI SCALAR) WHOSE COMPONENTS ARE GIVEN BY:

METRIC (CONTRAVARIANT)

COVARIANT FORM!
$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

EINSTEIN TENSOR: $\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}$

- 10 INDEPENDENT COMPONENTS

EINSTEIN TENSOR

RICCI (CONTRAVARIANT)

CURVATURE SCALAR

→ $G^{\mu\nu}$ IS SYMMETRIC BECAUSE $R^{\mu\nu}$ AND $g^{\mu\nu}$

- THE COVARIANT DIVERGENCE OF $G^{\mu\nu}$ VANISHES ($\sum_\mu \nabla_\mu G^{\mu\nu} = 0$), LIKE THE ENERGY-MOMENTUM TENSOR

- THE FIELD EQUATIONS

- THE CENTRAL IDEA TO GR IS THAT ENERGY AND MOMENTUM IN A REGION OF SPACETIME DETERMINE THE GEOMETRY OF SPACETIME IN THAT REGION.

- MOREOVER, THE RESULTING SPACETIME GEOMETRY DETERMINES A SPECIAL CLASS OF PATHWAYS THROUGH SPACETIME — THE GEODESICS.

→ UNDER THE INFLUENCE OF GRAVITY ALONE MASSIVE OBJECTS FOLLOW TIME-LIKE

GEODESICS ($ds^2 > 0$), WHILE PHOTONS FOLLOW NULL GEODESICS ($ds = 0$).

→ NON-GRAVITATIONAL EFFECTS, SUCH AS NUCLEAR AND ELECTROMAGNETIC FORCES, CAUSE DEPARTURES FROM GEODESIC MOTION (AND ARE TYPICALLY BEYOND THE SCOPE OF GR).

[illegible]: THE [illegible]SIGNS OF THE METRIC, [illegible], 2 [illegible] TENSORS [illegible] CONVENTIONAL.

· SINCE BOTH THE ENERGY-MOMENTUM TENSOR, $[T^{\mu\nu}]$, AND THE EINSTEIN TENSOR, $[G^{\mu\nu}]$, HAVE ZERO COVARIANT DIVERGENCE, ONE MIGHT THINK THAT THEY ARE PROPORTIONAL. EINSTEIN'S FIELD EQUATIONS SAY THAT THEY ARE:

COVARIANT FORM

EINSTEIN FIELD EQUATIONS:

ANALOG TO POISSON EQUATION

[illegible] EQUATIONS

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} = -\kappa T_{\mu\nu} \qquad (14.34)$$

[illegible] INDEPENDENT, [illegible] SIMULTANEOUS, [illegible]-LINEAR, [illegible] ORDER [illegible]ES.

NON-LINEAR IN $g_{\mu\nu}$

(REPLACING y WITH αy IS NOT THE SAME AS MULTIPLYING BY α.)

EINSTEIN'S CONSTANT → $\boxed{\kappa = \dfrac{8\pi G}{c^4}}$

FOLLOWS FROM CONSISTENCY REQUIREMENT

CONTRAVARIANT FORM:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -\kappa T^{\mu\nu}$$

· SOLVING THE FIELD EQUATIONS REQUIRES FINDING THE METRIC TENSOR, $[g_{\mu\nu}]$, CORRESPONDING TO A GIVEN ENERGY-MOMENTUM TENSOR, $[T_{\mu\nu}]$. THIS IS THE GRAVITATIONAL FIELD.

ENERGY DENSITY

ENERGY FLUX

SHEAR STRESS

$$\begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

PRESSURE

MOMENTUM DENSITY

MOMENTUM FLUX

• IN SOME REGIONS OF SPACETIME, WE MAY HAVE $\boxed{T_{\mu\nu} = 0}$ IN WHICH CASE WE SAY THAT THE REGION IS <u>EMPTY</u>.

• IN <u>EMPTY</u> REGIONS, THE <u>FIELD EQUATIONS</u> REDUCE TO:

SOLUTIONS TO THIS ARE "VACUUM" <u>SOLUTIONS</u>

$$\boxed{R_{\mu\nu} = 0} \qquad (4.36)$$

CAN STILL BE MATTER & RADIATION IN REGION

• NOTE: THIS DOES <u>NOT</u> IMPLY THE <u>FLATNESS</u> OF THE REGION. THE NECESSARY AND SUFFICIENT CONDITION FOR THAT IS THAT $R^{\mu}{}_{\nu\alpha\beta} = 0$, i.e. THE VANISHING OF THE <u>RIEMANN TENSOR</u> AT ALL EVENTS IN THE REGION.

• <u>GEODESIC MOTION</u>

• IN ADDITION TO USING THE <u>FIELD EQUATIONS</u> TO FIND THE <u>SPACETIME METRIC</u>, THE <u>METRIC</u> IS USED TO DETERMINE THE GEODESICS.

• RECALL THAT FOR A <u>FOUR-DIMENSIONAL SPACETIME</u> WITH <u>METRIC TENSOR</u>, $[g_{\mu\nu}]$, THE <u>GEODESIC EQUATIONS</u> ARE:

$$\frac{d^2 x^{\rho}}{d\lambda^2} + \sum_{\alpha,\beta} \Gamma^{\rho}_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0 \qquad (3.27)$$

AFFINE PARAMETER

WHERE:

$$\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\sigma}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\sigma}}\right) \qquad (3.23)$$

• THE FUNCTIONS, $x^{\rho}(\lambda)$, THE SATISFY THE GEODESIC EQUATIONS DESCRIBE THE PARAMETERIZED CURVES THROUGH SPACETIME THAT ARE THE MOST DIRECT ROUTES BETWEEN EVENTS.

"STRAIGHT LINES"

• GIVEN A CURVE SPECIFIED BY $x^{\rho}(\lambda)$, WE KNOW THAT THE TANGENT VECTOR TO THE CURVE AT POINT SPECIFIED BY λ IS GIVEN BY:

$$\boxed{t^{\rho}(\lambda) = \frac{\partial x^{\rho}(\lambda)}{d\lambda}} \qquad (4.37)$$

→ THE NORM OF THE TANGENT VECTOR - INTUITIVELY, ITS LENGTH - IS DEFINED:

$$\boxed{\sum_{\alpha,\beta} g_{\alpha,\beta}\, t^{\alpha} t^{\beta}}$$

• NOTE: THIS WILL BE THE SAME AT ALL POINTS IF THE GEODESIC IS AFFINELY PARAMETERIZED.

• IN THIS CASE, WE GET 3 KINDS OF GEODESIC:

(i) TIME-LIKE GEODESICS (NORM OF TANGENT VECTOR IS ALWAYS POSITIVE)

(ii) NULL GEODESICS (TANGENT VECTOR ALWAYS HAS ZERO NORM).

(iii) SPACE-LIKE GEODESIC (TANGENT VECTOR ALWAYS HAS NEGATIVE NORM)

• UPSHOT: FOR TIME-LIKE AND SPACE-LIKE GEODESICS, THE LINE ELEMENT

"NON-NULL GEODESICS"

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \qquad (3.11)$$

WILL ALWAYS BE NON-ZERO. SO, WE CAN USE THE SQUARE ROOT OF ITS MAGNITUDE, $|ds^2|^{\frac{1}{2}}$ TO DEFINE A DISTANCE ELEMENT WHEN PARAMETERIZING THE GEODESIC. BUT WE CANNOT DO THIS IN THE CASE OF NULL GEODESICS.

• PRINCIPLE OF GEODESIC MOTION: IN GR, TIMELIKE GEODESICS OF A SPACETIME REPRESENT THE POSSIBLE WORLDLINES OF MASSIVE PARTICLES FALLING FREELY (UNDER GRAVITY ALONE), WHILE NULL GEODESICS REPRESENT THE POSSIBLE WORLDLINES OF MASSLESS PARTICLES UNDER INFLUENCE OF GRAVITY ALONE.

• EXAMPLE: THE ORBIT OF A PLANET AROUND A STAR IS, ABSENT OTHER FORCES, A TIMELIKE GEODESIC AROUND IT.

• NOTE: THE GEODESIC PRINCIPLE IS NOT A SEPARATE POSTULATE. IT IS PREDICTED BY THE FIELD EQUATIONS GIVEN:

$$\sum_{\mu} \nabla_{\mu} T^{\mu\nu} = 0 \quad (4.30)$$

• NEWTONIAN LIMIT

• THE "APPROPRIATE CIRCUMSTANCES" IN WHICH THE FIELD EQUATIONS SHOULD APPROXIMATE POISSON'S ($\nabla^2 \phi = 4\pi G\rho$) ARE THAT GRAVITATIONAL EFFECTS ARE WEAK, AND ANY MOTIONS ARE SLOW COMPARED TO LIGHT (NEWTON'S LAW ONLY CONCERNS MATTER).

• WE WANT: $$g_{\mu\nu} \overset{\text{APPROXIMATES}}{\simeq} \eta_{\mu\nu} + \underset{|h_{\mu\nu}| \ll 1}{h_{\mu\nu}} \quad (4.38)$$

AND $g_{\mu\nu}$ IS NOT CHANGING WITH TIME.

• HOWEVER, GR PREDICTS IN THE NEWTONIAN LIMIT THAT $\nabla^2 \phi \simeq \kappa \frac{\rho c^2}{2}$. SO, LETTING $\kappa = 8\pi G/c^4$, WE GET $\nabla^2 \phi \simeq \kappa \frac{\rho c^2}{2}$ AS DESIRED.

- COSMOLOGICAL CONSTANT

    - Λ IS A NEW UNIVERSAL CONSTANT OF NATURE ONCE PROPOSED BY EINSTEIN. WITH IT ADDED, THE FIELD EQUATIONS BECOME:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \underbrace{\Lambda}_{\boxed{\text{COSMOLOGICAL CONSTANT}}} g_{\mu\nu} = -\kappa T_{\mu\nu} \qquad (4.47)$$

    - IN THE NEWTONIAN LIMIT, A POSITIVE VALUE OF Λ PROVIDES A REPULSIVE EFFECT THAT CAN COUNTERBALANCE ORDINARY GRAVITATIONAL ATTRACTION. THIS IS ONE WAY TO EXPLAIN THE ACCELERATING EXPANSION OF THE UNIVERSE (THOUGH EINSTEIN POSTULATED IT IN HOPES OF EXPLAINING HOW THE UNIVERSE COULD BE STATIC!)

    - THE FOLLOWING IS ONE DERIVATION OF "DARK ENERGY":

REARRANGING (4.47) $\longrightarrow$ $$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa \left(T_{\mu\nu} + \frac{\Lambda}{\kappa} g_{\mu\nu} \right)$$

THINKING OF $-\left(\frac{\Lambda}{\kappa}\right) g_{\mu\nu}$ AS ARISING

FROM A NEW PART OF THE ENERGY-MOMENTUM TENSOR, WRITTEN $\bar{T}_{\mu\nu}$:

NEW PART

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\varkappa \left(T_{\mu\nu} + \overline{T}_{\mu\nu} \right) \quad (4.49)$$

AND TREATING $\overline{T}_{\mu\nu}$ AS IF IT COMES FROM AN IDEAL FLUID WITH DENSITY, ρ_Λ, AND PRESSURE, P_Λ; THEN BY (4.25):

$$\overline{T}_{\mu\nu} = \left[(\rho_\Lambda + P_\Lambda)/c^2 \right] U_\mu U_\nu - P_\Lambda g_{\mu\nu} \quad (4.50)$$

WHERE: $\rho_\Lambda c^2$ IS "DARK ENERGY" AND P_Λ IS THE PRESSURE DUE TO IT

FINALLY, ASSUMING:

$$P_\Lambda = -\frac{\Lambda}{\varkappa} \quad \text{AND} \quad \rho_\Lambda = -\frac{P_\Lambda}{c^2} = \frac{\Lambda}{\varkappa c^2} \quad (4.52)$$

WE HAVE THAT:

$$\overline{T}_{\mu\nu} = \frac{\Lambda}{\varkappa} g_{\mu\nu}$$

- THE MODIFIED FIELD EQUATIONS THEN BECOME:

"DARK ENERGY"

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\varkappa \left(T_{\mu\nu} + \rho_\Lambda c^2 g_{\mu\nu} \right)}$$