Incompleteness Undecidability & Undefinability

- Here are some philosophical questions with mathematical answers:
  - (1) Is there a (recursive) algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is true?
  - (2) Is there an algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is a theorem of Peano or Robinson Arithmetic?
  - (3) Is there an algorithm for deciding whether an arbitrary sentence in the language of first-order arithmetic is a theorem of pure (first-order) logic?
  - (4) Is there a complete (even if not recursive) recursively axiomatizable theory in the language of first-order arithmetic?
  - (5) Is there a recursively axiomatizable sub-theory of Peano Arithmetic that proves the consistency of Peano Arithmetic (even if it leaves other questions undecided)?
  - (6) Is there a formula of arithmetic that defines arithmetic truth in the standard model, N (even if it does not represent it)?
  - (7) Is the (non-recursively enumerable) set of truths in the language of first-order arithmetic categorical? If not, is it ω-categorical (i.e., categorical in models of cardinality ω)?
- Questions (1) -- (7) turn out to be linked. Their philosophical interest depends partly on the following philosophical thesis, of which we will make frequent, but inessential, use.
  - *Church-Turing Thesis:* A function is (intuitively) computable if/f it is recursive.
    - *Church*: "[T]he notion of an effectively calculable function of positive integers should be identified with that of recursive function (quoted in Epstein & Carnielli, 223)."
  - *Note*: Since a function is recursive if/f it is Turing computable, the Church-Turing Thesis also implies that a function is computable if/f it is Turing computable.

Robinson Arithmetic (Q)

- Q is the following finitely-axiomatized first-order theory. It is of interest because it is among the weakest theories for which *recursiveness is equivalent to representability*. A function is recursive just in case it is representable (in a sense to be explained) in Q.

- *(Q1)* $(\forall x)(\forall y)[(x' = y') \rightarrow (x=y)]$
  - If the successors of two numbers are equal, then the numbers are equal.
- *(Q2)* $(\forall x){\sim}[\mathbf{0} = x']$
  - Zero is not the successor of any number.
- *(Q3)* $(\forall x)[({\sim}[x = \mathbf{0}]) \rightarrow ((\exists y)(x=y')]$
  - Every number other than zero is the successor of a number.
  - *Note*: (Q1) and (Q2) already imply that Q has *only infinite models.*
- *(Q4)* $(\forall x)[(x + \mathbf{0}) = x]$
  - Any number plus zero is equal to that number.
- *(Q5)* $(\forall x)(\forall y)[(x + y') = (x + y)']$
  - The sum of a number and the successor of a number is equal to the successor of the sum of the two numbers.
- *(Q6)* $(\forall x)[(x * \mathbf{0}) = \mathbf{0}]$
  - Any number times zero is zero.
- *(Q7)* $(\forall x)(\forall y)(x * y') = [(x * y) + x]$
  - The product of some number and the successor of a number is equal to the product of the two numbers plus the first number.

- *Note*: We will write **n** to abbreviate the formal numeral $\mathbf{0}'''...'$ (n primes)

<u>Basic Arithmetic in Q</u>

- Q is great with particulars. It is *Σ_1-complete* (i.e., Q can prove all true Σ_1 sentences).
- *Recall*: A Σ_1 formula is of the form $(\exists_1 x)(\exists_2 x)....(\exists_n x)\Delta_0$, where Δ_0 is a formula with only *bounded* quantifiers. Similarly, a Π_1 formula is of the form $(\forall_1 x)(\forall_2 x)....(\forall_n x)\Delta_0$. Hence, the negation of a Σ_1 formula is Π_1, while the negation of a Π_1 formula is Σ_1.
- *Note*: A theory can be Σ_1-complete but not *Σ_1-sound!* (That is, it could prove all true Σ_1 sentences *as well as* some false Σ_1 sentence.) We will construct just such a theory when we supplement Peano Arithmetic with the negation of its own consistency sentence.

- *Theorem 1*: For all natural numbers, i and j:
  - (a) If $i = j$, then Q |-- $\mathbf{i} = \mathbf{j}$
  - (b) If $i \neq j$, then Q |-- $\mathbf{i} \neq \mathbf{j}$
- *Theorem 2:* For all natural numbers, i, j, and k, such that $i + j = k$:
  - (a) Q |-- $(\mathbf{i} + \mathbf{j}) = \mathbf{k}$
  - (b) Q |-- $(\forall x)\ [((\mathbf{i} + \mathbf{j}) = x) \rightarrow (x = \mathbf{k})]$
- *Theorem 2.1*: For any natural numbers, i and j:
  - Q |-- $(\mathbf{i} + \mathbf{j}) = (\mathbf{j} + \mathbf{i})$
- *Theorem 3*: For all natural numbers i, j, and k, such that i x j = k:
  - (a) Q |-- $(\mathbf{i} * \mathbf{j}) = \mathbf{k}$
  - (b) Q |-- $(\forall x)[((\mathbf{i} * \mathbf{j}) = x) \rightarrow x = \mathbf{k}]$
- *Theorem 3.1*: For all natural numbers i and j:
  - Q |-- $(\mathbf{i} * \mathbf{j}) = (\mathbf{j} * \mathbf{i})$
  - *Caveat*: Just because Q proves that addition and multiplication are commutative for its *numerals* (by Theorems 2.1 and 3.1), it does *not* follow that Q proves that these operations are commutative *for all natural numbers* (i.e., universal generalizations). Indeed, we will see that Q proves neither of these general laws.

- *Proof*: Part (b) of *Theorem 2 & 3* follow from logic (of identity), given the other theorems. The other theorems are each of the form $\alpha = \beta$ or $\alpha \neq \beta$, where α and β are (perhaps complex) *numerical terms*. If such a term, θ, refers to n in the standard interpretation, N, then Q |-- $\theta = \mathbf{n}$. Consequently, if $i = j$, and $\alpha = \beta$ says so on N, then Q |-- $\alpha = \beta \longleftrightarrow \mathbf{i} = \mathbf{j}$, and Q |-- $\mathbf{i} = \mathbf{j}$, as **i** is the same numeral as **j**. Likewise, if $i \neq j$, and $\alpha \neq \beta$ says so on N, then, by the logic of identity, Q |-- $\mathbf{i} \neq \mathbf{j}$, and again Q |-- $\alpha \neq \beta$.

- *Note*: These (meta)theorems suffice to significantly constrain models of Q. That part of any model which interprets Q's numerals must look *just like the standard model, N.*

- *Upshot*: Any non-standard model of Q must look just like the standard model, N, but with some additional elements "tacked onto the end" (and additional values for ', +, and *).

Representability

- We have really been showing that addition and multiplication are *representable* in Q.
    - *Caveat*: The proof of this uses Induction, which Q itself does not have!
  - *Definition 1*: A total function f: N^n → N is *representable* in theory, T, if/f there is a formula, Φ, in which $x_1, x_2, \ldots x_n, x_{n+1}$ occur free, such that when $q_1, q_2, \ldots q_n$, k are natural numbers with $f(q_1, q_2, \ldots q_n) = k$, we have:
    - (a) T |-- Φ(**q_1, q_2,...q_n, k**)
    - (b) T |-- (∀x)[Φ((**q_1, q_2,...q_n,** x) → (x = **k**)]
  - In this case, the formula, Φ, is said to *represent* function, f, in theory, T.
    - *Note*: Part (b) reflects the fact that f is a function, giving a unique output.
      - *Example*: By *Theorem 2*, $(x_1 + x_2) = x_3$ represents the plus function in Q, while, by *Theorem 3*, $(x_1 * x_2) = x_3$ represents times.
      - *Example*: The zero, successor, and projections functions --i.e., the basic *primitive recursive* functions -- are representable in Q.
  - *Theorem 4 (Representability Theorem)*: Every total recursive function is representable in any theory, T, which extends Q (where T *extends* Q when Q⊆T).
    - *Note*: Q *explicitly* represents ', + and *, but implicitly represents far more.
    - In fact, despite being so weak, Q is *optimal*, in the following sense:
  - *Theorem 5*: If T is any consistent theory with a recursive set of axioms, then every total function representable in T is itself recursive.
    - *Proof*: Suppose that f: N^n → N is a total function represented in T by Φ. Then $\forall q_1 q_2 \ldots q_n$, k ∈ N, T |-- Φ(**$q_1 q_2 \ldots q_n$, k**) if/f $f(q_1 q_2 \ldots q_n) = k$, and the theorems of T are algorithmically enumerable. This gives an algorithm for computing f. Given $q_1 q_2 \ldots q_n$,, wait for a theorem of the form Φ(**$q_1 q_2 \ldots q_n$, k**). (Since T is consistent, there will be exactly one of this form.) By the Church-Turing Thesis, f is therefore recursive.
  - *Theorem 6 (Equivalence)*: A total function is representable in Q if/f it is recursive.
    - *Proof*: From *Theorems 4* and *5* (assuming that Q is consistent).

Godel Numbering

- A formula in the language of arithmetic is a finite string of symbols. So, we can pair those symbols, and strings of them, with natural numbers, called their *Godel numbers*.

- There are many ways to do this. But the pairing must satisfy the following conditions.

    - (a) Different strings of symbols are assigned to different Godel numbers

    - (b) There is a (recursive) algorithm such that, given a natural number, it decides whether the number codes a string of basic symbols, and, if it does, what string it codes.

    - (c) There is a (recursive) algorithm such that, given a string of basic symbols, it decides what the Godel number of that string is.

- Here is one pairing, β, that will do the trick:

Symbol:	(	)	&	v	→	←→	~	∀	∃	**0**	‘	+	*	=	x_i
Number:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	i+15

    - This generates, corresponding to each string of *symbols*, $\alpha_1\alpha_2\alpha_3\alpha_4$…, a string of Godel numbers, given by the above pairing, $\beta(\alpha_1)\beta(\alpha_2)\beta(\alpha_3)$...

- We can now assign the overall string $\alpha_1\alpha_2\alpha_3\alpha_4$… the Godel number:

    - $(p_1\^\beta(\alpha_1))(p_2\^\beta(\alpha_2))(p_3\^\beta(\alpha_3))$..., where $p_1p_2p_3...p_k$ are the first k primes.

        - *Note*: This will generally be a very big number! But, given it, we can check after finitely-many steps that it is the Godel number of the string.

        - *Note*: We use here the *Fundamental Theorem of Arithmetic*, that every natural number has a unique decomposition into powers of primes.

- If Ψ is a string of symbols, then we will write γ(Ψ) for the Godel number of Ψ.

- It remains to code arbitrary formal proofs, F. If the following is such a proof,

    - Φ_1
    - Φ_2
    - Φ_3
    - …
    - Φ_k

then we can code it with the function, $\Gamma(F)=(2^{\wedge}(\gamma(\Phi_1)))(3^{\wedge}(\gamma(\Phi_2)))...(p_k{}^{\wedge}(\gamma(\Phi_k)))$.

- *Upshot*: We can now describe facts about our theory, Q, in terms of Godel numbers. For instance, the second of our initial questions, (2), can now be phrased: is there an algorithm for deciding whether a given number is the Godel number of a theorem of Q?

<u>The Diagonal Lemma</u>

- *Theorem 6*: If Φ is a formula in which at most x and y occur free, then for all natural numbers n and k:
  - Φ(**n**, **k**), ∀y[Φ(**n**, y) → (y = **k**)] | -- ∀y[Φ(**n**, y) ←→ (y = **k**)]
    - *Note*: This is a theorem of first-order logic.
      - *Clarification*: Variables like x and y are doing double-duty, as meta-variables in our meta-language, and as variables in Q.
- *Theorem 7*: If f: N → N is a total function represented in theory, T, by formula, Φ, then, for all natural numbers n and k, if f(n) = k,
  - T |-- ∀y[Φ(**n**, y) ←→ y = **k**]
  - *Proof*: Suppose that f(n) = k. Then, by the *Definition 1*, we have that T proves both of the antecedents of *Theorem 6*.
- *Notation*: If Φ is a formula with Godel number, n, then ⌜Φ⌝ is the term, **n.**
  - *Note*: ⌜Φ⌝ is not γ(Φ). γ(Φ) is a *natural number*, while ⌜Φ⌝ is the *numeral* in the language of arithmetic which, in the standard interpretation, N, refers to γ(Φ).
  - *Example*: If Φ is the formula (∃x)[y = **0''** * x], then, given our numbering, γ(Φ) = (2^9)(3^16)(5^15)(7^14)(11^1)(13^10)(17^11)(19^11)(23^13)(29^16)(31^2). On the other hand, ⌜Φ⌝ is the term **0** followed by γ(Φ) occurrences of the symbol ’.
    - We can think of ⌜Φ⌝ as the *name* of Φ in the language of arithmetic, since, given a Godel numbering, it is *uniquely determined* by Φ.
- *Definition 2 (Diagonalization)*: If Φ is a formula in which at most some variable, x, occurs free, then the *diagonalization* of Φ, written D_Φ, is the (closed) sentence:
  - (∃x)(x = ⌜Φ⌝ & Φ)
    - *Example*: If Φ is (∃y)[x = (**2** * y)] & ~(∃y)[(x = (**4** * y)], and m is γ(Φ), then D_Φ is (∃x)(x = **m** & (∃y)[x = (**2** * y)] & ~(∃y)[(x = (**4** * y)]).

- *Note*: If x is free in Φ, then the diagonalization of Φ "says" that *$\gamma(\Phi)$ has the property, Φ*, and $(\exists x)(x = \ulcorner\Phi\urcorner \;\&\; \Phi)$ --||-- $\Phi(\ulcorner\Phi\urcorner)$.

    - *Note*: The diagonalization of a formula need not be true (under the standard interpretation), though it might be (and is in *Example* above).

- *Theorem 8*: If Φ is a formula in which some variable, x, occurs free, and n is a natural number:

    - |-- $[(\exists x)(x = \mathbf{n} \;\&\; \Phi) \longleftrightarrow \Phi(\mathbf{n})]$

    - *Note*: This is a logical truth.

- *Theorem 9*: If Φ is a formula in which at most some variable, x, occurs free:

    - |-- $[D_\Phi \longleftrightarrow \Phi(\ulcorner\Phi\urcorner)]$

    - *Proof*: Immediate from *Theorem 8*.

- *Theorem 10*: There is a (primitive) recursive function, *, such that, for all strings α and β of basic symbols, if $m = \gamma(\alpha)$ and $n = \gamma(\beta)$, then $m*n = \gamma(\alpha\beta)$.

- *Theorem 11*: There is a (primitive) recursive function, diag(x), with the following property. If n is the Godel number of a formula Φ in which at most some variable, x, occurs free, then diag(n) is the Godel number of the *diagonalization* of Φ, D_Φ.

Φ	$\rightarrow$	$(\exists x)(x = \ulcorner\Phi\urcorner \;\&\; \Phi)$
$\downarrow$		$\downarrow$
has Godel number n		has Godel number m
n	$\rightarrow$ diag	m

- *Note*: It is intuitively clear that the function, diag(x) is recursive since all we have to do to calculate it is check if n is the Godel number of a formula, and, if it is, add the code for "$(\exists x)(x = \mathbf{n}$ &" in front and add the code for ")" in the back.

- *Theorem 12 (Diagonal Lemma)*: If diag is representable in a theory, T, then, for each formula, Φ, in which at most some variable, y, occurs free, there is a sentence, G_Φ, such that:

- T |-- [G_Φ ←→ Φ(⌜G_Φ⌝)]

  - *Proof Sketch*: Let Θ represent the function, diag, in theory T, with at most x and y free. Let ψ be a formula with at most y free, and take Ω to be the formula (∃y)(Θ & ψ), where at most x may occur free. Finally, identify G_Θ with the diagonalization of Ω, D_Ω, to get T |-- [G_Φ ←→ Φ(⌜G_Φ⌝)].

    - *Example*: If Φ is the formal analog of "is even", then G_Φ is a sentence which is true just in case the Godel number of G_Φ is even.

    - *Example*: If Φ is the formal analog of "codes a sentence which is provable", then G_Φ is a sentence which is true just in case the Godel number of G_Φ codes a sentence which is, indeed, provable.

  - *Note*: The *Diagonal Lemma* means that *every predicate in T has a fixed point* -- roughly, a point where you get the same thing out that you put in.

- *Upshot*: Every theory extending Q has the capacity for self-reference.

<u>Undecidable Theories</u>

- *Definition 3*: A *set*, A, is representable in theory, T, if/f its *characteristic function* is (where the characteristic function, f_A, of set, A, is $f_A(n) = 1$ when n∈A and 0 otherwise.)

- *Theorem 13*: Every *recursive set* is representable in any theory, T, which extends Q.

  - *Proof*: Immediate from the *Representability Theorem*.

- *Theorem 14*: If T is a theory extending Q, and A is a set of natural numbers which is representable in T, then there is a formula, δ_A, in which at most x occurs free, such that, for each natural number, n:

  - (a) If n∈A, then T |-- $\delta_A(\mathbf{n})$

  - (b) If n∉A, then T |-- ~$\delta_A(\mathbf{n})$

  - *Proof*: Let Φ_A represent the characteristic function, f_A, of A in T. Then ∀n,k∈N, if $f_A(n) = k$, T |-- $\Phi_A(\mathbf{n}, \mathbf{k})$, and T |-- ∀y[$\Phi_A(\mathbf{n}, y) \rightarrow y=\mathbf{k}$]. Define $\delta_A(x) = \Phi_A(x, \mathbf{1})$. If n∈A, then $f_A(n) = 1$, so T |-- $\Phi_A(\mathbf{n}, \mathbf{1})$ and, hence, T |-- $\delta_A(\mathbf{n})$. Conversely, if n∉A, then $f_A(n) = 0$, and, thus, T |-- ∀y[$\Phi_A(\mathbf{n}, y) \rightarrow y=\mathbf{0}$]. But since T extends Q, T |= $\mathbf{1} \neq \mathbf{0}$. Hence, by logic in T, T |-- ~$\Phi_A(\mathbf{n}, \mathbf{1})$, and, therefore, T |-- $\delta_A(\mathbf{n})$.

  - In fact, the other direction holds as well.

- *Theorem 15*: If T extends Q, then the following conditions on a set, A, of natural numbers are *equivalent*:
    - (a) The set, A, is representable in T.
    - (b) There is a formula, δ_A, in which at most some variable, x, occurs free, such that, for each natural number, n, if $n \in A$, then T |-- $\delta_A(\mathbf{n})$, and, if $n \notin A$, then T |-- $\sim\delta_A(\mathbf{n})$.
- *Theorem 16 (Undecidability of Extensions of Q)*: If T is a consistent theory extending Q, then the set of Godel numbers of the theorems of T, GN(T), is *not* representable in T.
    - *Proof*: Assume for reductio that GN(T) is representable in T. Then by *Theorem 15*, there is a formula, $\delta(x)$, whose only free variable is x, such that, for any $n \in N$,
        - If $n \in$ GN(T), then T |-- $\delta(\mathbf{n})$
        - If $n \notin$ GN(T), then T|-- $\sim\delta(\mathbf{n})$
    - Moreover, since GN(T) is the set of Godel numbers of theorems of T, for each Φ,
        - If T |-- Φ, then $\gamma(\Phi) \in$ GN(T)
        - If ~(T |-- Φ), then $\gamma(\Phi) \notin$ GN(T)
    - Hence, if T |-- Φ, then T |-- $\delta(\ulcorner\Phi\urcorner)$, and if ~(T |-- Φ), then T |-- $\sim\delta(\ulcorner\Phi\urcorner)$. So, applying the *Diagonal Lemma* to $\sim\delta$, we are ensured a sentence, $G_{\sim\delta}$ such that:
        - T |-- ($G_{\sim\delta} \longleftrightarrow \sim\delta(\ulcorner G_{\sim\delta}\urcorner)$).
        - *Note*: The sentence $\sim\delta(\ulcorner G_{\sim\delta}\urcorner)$ "says" that $G_{\sim\delta}$ is not a theorem of T.
    - However, both the assumption that T |-- $G_{\sim\delta}$ and ~(T |-- $G_{\sim\delta}$) lead to contradiction.
        - *Note*: *Theorem 16* allows that the set of theorems of T is representable in *extensions* of T. Moreover, the assumption that T is consistent is essential.
- *Definition 4*: *True Arithmetic*, TA, is the set of all sentences, S, such that N |= S.
- *Theorem 17*: The set of Godel numbers of sentences of True Arithmetic, GN(TA), is not a recursive set.
    - *Proof*: Immediate from *Theorem 16*.

- *Theorem 18 (Church-Turing Theorem)*: The set of theorems of first-order logic is not recursive.

    - *Proof*: If it were, then the set of theorems of Q would be too, contrary to *Theorem 16*. For an arbitrary Φ, check whether $Ax_Q \rightarrow \Phi$ is a theorem of first-order logic.

<u>Undefinability of Truth</u>

- *Definition 5*: A relation $R(x_1 x_2,...,x_k)$ of natural numbers is *definable* in the standard model, N, if/f there is a formula $\phi(x_1 x_2,...,x_k)$ in the language of arithmetic such that for every $n_1 n_2,...,n_k$, $R(n_1 n_2,...,n_k)$ if and only if $N \models \Phi(n_1,...,n_k)$.

- *Theorem 19 (Godel-Tarski Theorem)*. The set of Godel numbers of sentences, S, true in N (i.e., {S: N|= S}) is not definable in arithmetic.

    - *Proof*. A set is definable in N if/f it is representable in TA. But TA consistently extends Q. Hence, again, this set is not representable in TA, by *Theorem 16*.

- *Upshot*: Extensions of Q can *define* typical syntactic properties, but not semantic ones.

<u>Axiomatizability, Enumerability, and Completeness</u>

- *Definition 6*: A theory, T, is (recursively) *axiomatizable* if/f there is a set, S, of axioms of T (i.e., a set $S \subseteq T$ such that, for every $\Phi \in T$, $S \vdash \Phi$) where the set of Godel numbers of S is recursive.

    - *Note*: There is a big difference between a theory's having a recursive set of axioms and its having a recursive set of theorems. Q has a recursive -- indeed, finite -- set of axioms, but we have seen that its set of theorems is not recursive.

- *Definition 7*: A set, A, of natural numbers is recursively enumerable if/f either $A=\varnothing$ or A={f(0), f(1), f(2)...} where f is a total recursive function.

    - *Note*: If a set is recursive, then it is recursively enumerable.

- *Theorem 20*: If T is a recursively axiomatized theory, then the set of Godel numbers of theorems of T, GN(T), is *recursively enumerable*.

    - *Proof*: Since the set of theorems of T is not empty, call the Godel number of $(\forall y)(y = y)$, n_0. Now, define a computable function, f, as follows. Given $n \in N$, first check whether n is the Godel number of a proof in T. If not, let $f(n) = n_0$. If so, we can compute from n what assumptions the proof relied on. Moreover, since the set of axioms of T is recursive, we can check whether each of these assumptions is a member of that set. If it is, let f(n) be the Godel number of the last line of the proof, and if not, let $f(n) = n_0$. Conversely, if Φ is a theorem, then some number, Γ(F), codes its proof, F, in T, meaning $f(\Gamma(F)) = \gamma(\Phi)$, as desired.

- *Upshot*: Though being recursive implies being recursively enumerable, the converse implication fails. But there is one circumstance when it does not.

- *Theorem 21*: If T is an axiomatizable theory which is *complete* (i.e., T |-- Φ or T |-- ~Φ, for all sentences), then the set of Godel numbers of theorems of T is recursive.

    - *Proof*: If T is inconsistent, it is trivially recursive. So, assume that T is consistent. By *Theorem 20*, there is a recursive function which enumerates its theorems. For any sentence, Φ, simply wait for Φ or ~Φ to appear after a finite number of steps.

- *Theorem 22 (Godel's First Incompleteness Theorem)*: If T is a axiomatizable extension of Q, then either T is incomplete or inconsistent.

    - *Proof*: By *Theorem 21*, if T were complete, then it would have a recursive set of theorems, contradicting *Theorem 16.*

- *Theorem 23*: True Arithmetic is not axiomatizable.

    - *Proof*: Immediate from *Theorem 22*.

- *Theorem 24*: True Arithmetic is not recursively enumerable.

    - *Proof*: It is a complete and consistent extension of Q, so, if it were so enumerable, then, by *Theorem 21*, its theorems would be recursive, contrary to *Theorem 16*.

<u>Hilbert's Program and Consistency</u>

- *Definition 7*: Peano Arithmetic (PA) is Q conjoined with all instances of:

    - *Induction Schema*: $((\Phi(\mathbf{0})) \ \& \ \forall x[(\Phi(x) \rightarrow \Phi(x'))]) \rightarrow \forall \Phi(x)$

        - *Note*: Φ may also contain parameters.

- As an axiomatizable (albeit not finitely axiomatizable) extension of Q, *Theorem 20* implies that PA is incomplete, if consistent. But PA is much stronger than Q.

    - Although Q is good with particulars, it is bad with generalizations. Simple model-theoretic arguments show that it does not prove any of the following.

        - $\forall x(x \neq x')$
        - $\forall x([(x + (y + z)] = [(x + y) + z])$
        - $\forall x \forall y[(x + y) = (y + x)]$
        - $\forall x[(\mathbf{0} + x) = x]$
        - $\forall y \forall z{\sim}[\exists x((x' + y) = z) \ \& \ \exists x((x' + z) = y]$
        - $\forall x \forall y \forall z[(x * (y * z)) = ((x * y) * z)]$
        - $\forall x \forall y[(x * y) = (y * x)]$

        - $\forall x \forall y \forall z[(x * (y + z)) = ((x * y) + (x * z)]$
    - The problem is that *Q has no way of saying that the things picked out by its numerical terms are the only numbers*. This idea is regimented by Induction. And, indeed, in PA, all of the above are provable using the Induction Schema.
        - *Example*: In order to construct a model of (Q1) -- (Q7) + $\sim\forall x(x \neq x')$, add an element, ω, and interpret the successor function as before but with s(ω) = ω, interpret + as before but with n + ω = ω + n = ω, for all numbers, n (including ω), and let n * ω = ω * n = ω when n ≠ 0, and 0 otherwise.
- Hence, while PA must be incomplete, it might still be hoped that PA, or, even better, a "finitary" fragment of it, proves the consistency of PA. This was Hilbert's ambition.
- In order to make this precise, we must formalize the sentence "PA is consistent" in the language of PA. Since PA |-- **0** = **1** if/f PA is inconsistent, it suffices to formalize "PA does not prove **0** = **1**". The problem is that the set of (Godel numbers of) theorems of PA is not representable in PA, by *Theorem 16*. But there *is* a formula (actually, different formulas), Prov(x), with only x free, which *defines* the relation of provability in PA. So:
        - N |= Prov(⌜Φ⌝) if and only if PA |-- Φ
- With this predicate in hand, our question can now be phrased: whether there is an axiomatizable theory, T, of which PA is an extension such that T |-- ~Prov(⌜**0** = **1**⌝).
    - *Note*: ~Prov(⌜**0** = **1**⌝) is commonly written Con(PA).
- *Definition 8*: If T extends Q, then the formula, Prov(x), of one free variable is a *provability predicate* for T if/f, for all sentences and Ω and Ψ, the following hold:
    - (a) If T |-- Ω , then T |-- Prov(⌜Ω⌝)
        - If T proves Ω , then it is a theorem of T that T proves Ω.
    - (b) If T |-- Prov(⌜Ω → Ψ⌝) → [(Prov(⌜Ω⌝) → Prov(⌜Ψ⌝)]
        - It is a theorem of T that if the conditional Ω → Ψ, and its antecedent, Ω, are both provable in T, then so is the consequent, Ψ.
    - (c) T |-- [Prov(⌜Ω⌝) → Prov(⌜Prov (⌜Ω⌝)⌝)]
        - It is a theorem of T that if Ω is provable in T, then the fact that Ω is provable in T is itself provable in T.
            - *Note*: This condition can be questioned.

- *Theorem 25 (Lob's Theorem)*: If T extends Q, and Prov(x) is a provability predicate for T, then, for each sentence, Ψ, T |-- Ψ if and only if T|-- Prov(⌜Ψ⌝) → Ψ.

  - *Proof*: Evidently, if T |-- Ψ, then T |-- [Prov(Ψ) → Ψ]. So, let us suppose that T |-- Prov(⌜Ψ⌝) → Ψ. Since T extends Q, diag(x) is representable in T, so letting δ(x) = [Prov(⌜Ψ⌝) → Ψ] with only x free, we get that T |-- (G_δ ⟷ [Prov(⌜G_δ⌝) → Ψ]). A calculation using properties (a) and (b) and logic in T then yields that T |-- [Prov(⌜G_δ⌝) → Prov(⌜(Prov(⌜G_δ⌝)⌝)] → Prov(⌜Ψ⌝), and by property (c) of Prov(x) we know that the antecedent of this conditional must be true, giving (by logic in T) that T |-- (Prov(⌜G_δ⌝) → Prov(⌜Ψ⌝). And since we are assuming that T |-- Prov(⌜Ψ⌝) → Ψ, we have (by logic in T) that T |-- (Prov(⌜G_δ⌝) → Ψ. Finally, since T |-- [G_δ ⟷ (Prov(⌜G_δ⌝) → Ψ)], we must have that T |-- G_δ. So, by (a), T |-- Prov(⌜G_δ⌝), and, using *modus ponens* in T, T |-- Ψ, as desired.

  - *Upshot*: If T proves "if T proves something, it is true", then T already proves it!

- *Theorem 26 (Godel's Second Incompleteness Theorem)*: If T is a consistent extension of Q, and Prov(x) is a provability predicate for T, then ~(T |-- ~Prov(⌜**0** = **1**⌝)).

  - *Proof*: Let T be a consistent extension of Q and Prov(x) a provability predicate for it. Suppose T |-- ~Prov(⌜**0** = **1**⌝). By logic in T, T |-- (Prov(⌜**0** = **1**⌝) → **0** = **1**). So, by *Lob's Theorem*, T |-- **0** = **1**, contrary to the assumption that T is consistent.

  - *Note*: One can always choose ~Prov(x) to be a Π_1 predicate, making ~Prov(⌜**0** = **1**⌝) a sentence of *Goldbach Type*. Once chosen we have a *concrete example* of a sentence that is not provable in PA, despite its being much more powerful than Q.

<u>Nonstandard Models</u>

- *Definition 9*: A theory is *categorical* when all its models are isomorphic, and *κ-categorical* when all its models of cardinality κ are isomorphic.

- *Theorem 27*: True Arithmetic is not ω-categorical.

  - *Proof*: Add a name, c, to the language of arithmetic, and define the following set, C, of formulas: {x ≠ **n** : n is any numeral in the (non-augmented) language of arithmetic}. Since every finite subset of the union of True Arithmetic (TA) and C is satisfiable, TA u C is too, and in a countable model, by the Compactness and Lowenheim-Skolem Theorems. But no such model can be isomorphic to N.

    - *Example*: The theory PA + ~Con(PA), which is consistent by Godel's Second Incompleteness Theorem, has ω-models with a non-standard number, greater than the finite numbers, which witnesses a proof of **0** = **1**.

- *Note*: This means that, *even if it were granted that there is a perfectly determinate body of arithmetic truths*, it does not follow that they pick out a determinate model (even up to isomorphism).

Questions Revisited

- We are now in a position to answer all of the questions with which we began.
    - (1) There no algorithm for deciding whether a given sentence in the language of first-order arithmetic is true, by *Theorem 17*.
    - (2) There no algorithm for deciding whether a given sentence in the language of first-order arithmetic is a theorem of of Robinson Arithmetic, by the *Undecidability of Extensions of Q*.
    - (3) There is no algorithm for determining whether a given sentence in the language of first-order arithmetic is a theorem of pure (first-order) logic, by the *Church-Turing Theorem*.
    - (4) There is no complete axiomatizable arithmetic theory in the language of first-order arithmetic, by *Godel's First Incompleteness Theorem*.
    - (5) There is no axiomatizable theory of which Peano Arithmetic is an extension that proves the consistency of Peano Arithmetic, by *Godel's Second Incompleteness Theorem*.
        - *Godel*: "For all formal systems for which the existence of undecidable arithmetical propositions was [demonstrated], the assertion of the consistency of the system in question itself belongs to the propositions undecidable in that system…..For a system in which all finitary...forms of proof are formalized, a finitary consistency proof, such as the formalists seek, would thus be...impossible (quoted in Epstein & Carnielli, 214)."
    - (6) There is no formula in the language of first-order arithmetic that defines arithmetic truth in the standard model, by the *Godel-Tarski Theorem*.
    - (7) The (non-recurisvely enumerable) set of truths in the language of first-order arithmetic, True Arithmetic, is not categorical or ω-categorical, by *Theorem 27*.

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