Philosophy of Mathematics
Handout #2

The ZFC Axioms

Russell's Paradox

- By the 1920s, it was emerging that mathematics could be "reduced" to set theory in that all familiar mathematical objects, from natural numbers to differentiable functions, could be defined as sets of certain sorts, and theorems about them could be viewed as theorems about sets. However, it remained to say what sets were, and what principles they obeyed.
  - *Note*: This does *not* mean that mathematics generally could be *epistemically* reduced to set theory -- i.e., *justified* on the basis of set-theoretic principles.
- The Naive (or Logical) Conception of set begins with the intuition that any predicate has an extension. If a condition is contradictory, no matter. The extension is empty. But it should always be possible to collect the things that satisfy it (and the complement of this).
- One can try to make this argument airtight, by appealing to the Law of the Excluded Middle. "By the law of the excluded middle, any...predicate...applies to a given object or ...not. So...to any predicate there correspond two sorts of thing: the sort of thing to which the predicate applies...and the sort...to which it does not apply" (Boolos 1971, 216).
- However, even if a given predicate either applies or not to any thing, it does not follow that *among* the things are extensions -- i.e., sets of things to which a predicate applies.
- An axiomatization of (the first-order fragment of) Naive Set Theory includes the axiom:
  - *Comprehension Schema*: $(\exists y)(x)(x \in y \longleftrightarrow \Phi))$ [where y is not free in Φ]
    - *Note*: A schema is an infinite array of axioms, one for each formula Φ.
- Comprehension Schema has innumerable unproblematic instances. But one instance is:
  - $(\exists y)(x)(x \in y \longleftrightarrow x \notin x))$
- Unfortunately, $\sim(\exists y)(x)(x \in y \longleftrightarrow x \notin x))$ is a *logical truth*, since otherwise $y \in y \longleftrightarrow y \notin y$.
- Hence, Comprehension Schema, and, thus, the Naive Conception of set, must be false.

Zermelo-Fraenkel Set Theory with Choice (ZFC)

- By the mid-20th century, set theorists had substantially converged on alternative axioms. They are written in a first-order language with one non-logical binary predicate, $\in$. Their quantifiers are assumed to range over sets alone (including an "empty set"). They are:
  - *Extensionality*: Sets are identical if they have the same members.
    - $(x)(y)(z)(z \in x \longleftrightarrow z \in y) \rightarrow x=y)$
    - *Note*: The converse is a logical truth in first-order logic with identity.
  - *Pairing*: For any sets, z and w, there is a set containing exactly z and w.
    - $(z)(w)(\exists y)(x)(x \in y) \longleftrightarrow (x=z \vee x=w))$

    - *Note*: If z = w, then y = {z}, the *singleton* of z, which is unique by Extensionality.
- *Union*: For any set, z, there is a set, Uz, containing exactly the members of members of z.
    - $(z)(\exists y)(x)(x \in y \longleftrightarrow (\exists w)(w \in z \,\&\, x \in w))$
- *Powerset*: For any set, z, there is a set containing exactly the subsets of z, P(z).
    - $(z)(\exists y)(x)(x \in y \longleftrightarrow (w)(w \in x \rightarrow w \in z))$
    - *Note*: This tells us nothing about *how many* subsets of z there are!
- *Subsets (Restricted Comprehension) Schema*: For any set, z, and any condition Φ, there is a set that contains exactly those members of z which satisfy Φ.
    - $(z)(\exists y)(x)(x \in y \longleftrightarrow (x \in z \,\&\, \Phi))$ [where y is not free in Φ]
    - *Note*: Since it is a logical truth that $(\exists z)(z = z)$, Subsets Schema implies $(\exists y)(x)(x \in y \longleftrightarrow (x \in z \,\&\, x \neq x)) = \varnothing$, which is unique by Extensionality.
- *Infinity*: There is a set containing ∅, and containing the *successor* of x (i.e., x u {x}) whenever it contains x.
    - $(\exists y)(((\exists x)(x \in y \& \ (z)(z \notin x) \,\&\, (x)(x \in y \rightarrow (\exists z)(z \in y \,\&\, (w)(w \in z \longleftrightarrow (w \in x \vee w{=}x)))))$
    - *Note*: Without Infinity, one can prove the existence of *infinitely-many things* (e.g., ∅,{∅}, {{∅}}, etc.), but not the existence of a *set* of them.
- *Foundation (Regularity) Schema*: For any condition Φ, if there is something that satisfies Φ, then there is a *minimal* x that does -- i.e., an x such that Φ and no y∈x such that Φ.
    - $(\exists x)\Phi \rightarrow (\exists x)[\Phi \,\&\, (y)(y \in x \rightarrow {\sim}\Phi^*)]$ [where Φ does not contain y, and Φ* is just Φ but contains y wherever Φ contains free occurrences of x]
        - Contraposing, we get a *Principle of Set-Theoretic Induction*.
            - If Φ is a condition such that, whenever all members of a set, x, satisfy Φ, then x does too, then every set satisfies Φ.
- *Replacement Schema*: For any set, z, and any formula Φ such that, for every t∈z, there is exactly one x with Φ(t, x), there is a set which contains just those things, x, for which Φ(t, x) holds for some t∈z.
    - $(a)[(u)(v)(w)(u \in a \,\&\, \Phi(u, v) \,\&\, \Phi(u, w) \rightarrow u{=}w) \rightarrow (\exists y)(x)(x \in y \longleftrightarrow (\exists t)(t \in a \,\&\, \Phi(t, x)))]$ [where u, v, w, and y are not free in Φ(t, x)]
    - *Note*: Replacement is equivalent (in ZF-Replacement) to a *Reflection Principle*, which states that if a formula is true of the set-theoretic universe, V, then it is already true in an *initial segment* of the universe.
- *Choice*: If t is a disjointed set not containing ∅, then there is a subset of Ut whose intersection with each member of t is a singleton.
    - $(t)[(x)[x \in t \rightarrow \exists z(z \in x) \,\&\, (y)(y \in t \,\&\, y \neq x \rightarrow {\sim}\exists z(z \in x \,\&\, z \in y))] \rightarrow \exists u(x)(x \in t \rightarrow \exists w(v)[v{=}w \longleftrightarrow (v \in u \,\&\, v \in x)])]$

- *Note*: When t is *finite* (not bijective with any of its proper subsets), Choice is actually redundant. But it is not redundant for infinite sets of finite sets.

Observations about ZFC

- ZFC does not allow the above derivation of Russell's Paradox. If x is a set, the Subsets Schema lets us form $y = \{z \in x : x \notin x\}$. Moreover, if $y \in y$, then indeed $y \notin y$. However, there is no evident contradiction in the assumption that $y \notin y$. There would be a contradiction if we added that $y \in x$. But we may conclude from this that $y \notin x$.
- Moreover, ZFC is strong enough to interpret standard mathematics (including, of course, syntax), and the set-theoretic theories of transfinite ordinal and cardinal arithmetic.
    - *Illustration*: One can identify the natural numbers with the *von Neumann ordinals* so that $0 = \varnothing$, and $S(x) = x$ u $\{x\}$. In general, the class of *ordinal numbers*, On, is the class of those transitive sets (i.e., sets, x, such that whenever $y \in x$, $y \subseteq x$) which are well-ordered by $\in$, while the class of *cardinal numbers* is the class of those ordinal numbers which are not bijective with any prior ordinal number. One may go on to distinguish *successor* from *limit* ordinals and cardinals in a natural way.
- ZFC is *impredicative*. In particular, whenever the condition Φ refers to the powerset of z in the Subsets Schema, it defines a subset z_Φ in terms of a set, P(z), to which z_Φ belongs.
- By the Reflection Principle, and Godel's Second Incompleteness Theorem, ZFC is not *finitely-axiomatizable*, if it is consistent (though, of course, its axioms are recursive).
- ZFC justifies definition by *transfinite recursion*. If $G: V \rightarrow V$ is a class function, then there is a class function, $F: On \rightarrow V$ such that (a) $F(0) = 0$, (b) $F(o + 1) = G(F(o))$, and (c) $F(u) = U\{F(o) \mid o < u\}$, for limit, u. This allows us to define the *cumulative hierarchy*:
    - $V_0 = 0$
    - $V_{o+1} = P(V_0)$
    - $V_u = UV_0 \mid o < u$, for limit, u.

The Iterative Conception

- ZFC contains no known contradiction. But does it enjoy any *intrinsic plausibility*?
- Quine thought not: "[C]ommon sense is bankrupt, for it wound up in contradiction…. [T]he [set theorist] has to resort to mythmaking. The myth will be best that engenders a...logic most convenient for mathematics and the sciences (quoted in Woods 2003, 332, fn. 4)."
- The view of most set theorists is different. They take ZFC to stem from the *iterative conception of set*. Contra the naive conception, iterative sets "depend on their members".
    - *Boolos*: "A set is any collection that is formed at some stage of the following process: Begin with [non-sets]...At stage zero...form all possible collections of individuals…..[I]f there are n individuals, 2^n sets are formed….At stage one,

form all possible collections of individuals and sets formed at stage zero…. Immediately after all of stages zero, one, two, three,...there is a stage… omega. At stage omega, form all possible collections of individuals formed at stages zero, one, two,...,and omega….Keep on going this way (Boolos 1971, 221)."

- *Note*: "According to this description, sets are formed *over and over again* (Boolos, *Ibid*.)
- Since sets according to the iterative conception are supposed to depend on their members, the Comprehension Schema would be false of it, even if -- counterpossibly -- it were consistent. For one apparently consistent instance of Comprehension is the following:
    - $(\exists y)((x)(x \in y \longleftrightarrow x=x))$
- Since such a y = y, like everything else, we have $y \in y$, contrary to the iterative conception.
- Does "stage theory", in fact, imply ZFC? In order to check, it must be regimented in a first-order language. Boolos has done this as follows (where x, y, z, w,… range over sets and r, s, t,.... range over stages, 'E' means *is earlier than*, and 'F' means *is formed at*).
    - (I) No state is earlier than itself.
        - $(x){\sim}sEs$
    - (II) Earlier than is transitive.
        - $(r)(s)(t)((rEs \ \& \ sEt) \rightarrow rEt)$
    - (III) Earlier than is connected.
        - $(s)(t)(sEt \vee s=t \vee tEs)$
    - (IV) There is an earliest stage.
        - $(\exists s)(t)(t \neq s \rightarrow sEt)$
    - (V) Immediately after any stage is another.
        - $(s)(\exists t)(sEt \ \& \ (r)(rEt \rightarrow (rEs \vee r=s)))$
    - (VI) There is a limit stage.
        - $(\exists s)((\exists t)tEs \ \& \ (t)(tEs \rightarrow (\exists r)(tEr \ \& \ rEs)))$
    - (VII) Every set is formed at a unique stage.
        - $(x)(\exists s)(xFs \ \& \ (t)(xFt \rightarrow t=s)$
    - (VIII) If a set is formed at a stage, then at or after any earlier stage, at least one of its members is formed.
        - $(x)(y)(t)((y \in x \ \& \ tEs \rightarrow (\exists y)(\exists r)(y \in x \ \& \ yFr \ \& \ (t=r \vee tEr)))$
- In addition, the following two schemas are added:
    - *Specification*: For any stage, there is a set of exactly the sets to which Φ applies which are formed before that stage.
        - $(s)(\exists y)(x)(x \in y \longleftrightarrow (\Phi \ \& \ (\exists t)(tEs \ \& \ xFt)))$ [where y is not free in Φ]
    - *Induction*: If each stage is such that Φ applies to all sets formed at it whenever Φ applies to all sets formed at each earlier stage, then, for every stage, Φ applies to all sets formed at it.

        - $(s)((t)[tEs \rightarrow (x)(xFt \rightarrow \Phi^*) \rightarrow (x)(xFs \rightarrow \Phi)) \rightarrow (s)(x)(xFs \rightarrow \Phi)$ [where Φ is a formula containing no occurrences of t and Φ^* is just like Φ except for containing free occurrences of t wherever Φ contains free occurrence of s]

Stage Theory as an Epistemic Justification

- *Problem 1*: Stage Theory does not imply Replacement, Choice -- or even Extensionality!
    - *Replacement*: "[T]he reason for adopting the axioms of replacement is [that] they have many desirable consequences and (apparently) no undesirable ones (Boolos 1971, 229)."
    - *Choice*: "[T]he iterative conception is neutral with respect to…[C]hoice...To say this is not to say that the axiom is not both obvious and indispensable (Boolos 1971, 230)."
    - *Extensionality*: "[W]hatever justification for accepting the axiom of extensionality there may be, it is more likely to resemble the justification for accepting most of the classical examples of analytic sentences (Boolos 1971, 230)."
- *Problem 2*: Though Stage Theory is supposed to explain whence sets emerge, it seems to *assume* the ordinal numbers (indexing the stages), which are themselves among the sets.
    - *Note*: It would also seem to assume that we know what we mean by *all* subsets.
- *Problem 3*: The "formation" narrative is obscure to say the least. What kind of priority is in question? Not *temporal* priority. But nor, it would seem, is the sense of prior one that can be explained in terms of the standard notion of *metaphysical possibility*. There is no possible world in which a set exists but its powerset fails to. (Hence, the central idea of collecting all *possible* subsets of a given set is problematic.) A final suggestion would appeal to the hyperintensional notion of *grounding*. But, even supposing that this notion is in good order, whether the grounding relation is well-founded is open.
- *Problem 4*: Even if Stage Theory implied all of ZFC, and Stage Theory was transparently *consistent*, it is hard to see how this would serve show that it, or so ZFC, was *true*. Maybe the iterative concept of set is consistent, but unsatisfied -- like that of the ether.
    - *Bar-Hillel, Fraenkel, & Levy*: "To an extreme Platonist the question of the consistency of a system of set theory is not really a central foundational problem. He will discard a theory at the point when it turns out not to fit the "true facts" about the universe, even if it is consistent. To put it more strongly, an extreme Platonist, like a physicist, will prefer an inconsistent theory some parts of which give a faithful description of the real situation to a demonstrably consistent theory which gives "wrong information" about the universe (1973, 326)."

Alternative Justifications?

- There are at least four other arguments one could try to combine into a defense of ZFC.

- *Argument 1 (Limitation of Size)*: The only sets whose existence we should be able to prove are those that would not be "too big" (e.g., Subsets, Replacement).
- *Argument 2 (Cantorian Finitism)*: Infinite sets resemble finite sets (e.g., Choice).
- *Argument 3 (Regressive Method)*: "The reason for accepting an axiom...is always largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false...(Russell & Whitehead, 59)."
- *Argument 4 (Indispensability)*: "[The Axioms of Set Theory] are to be vindicated...by the indirect systematic contribution which they make to the organizing of empirical data in the natural sciences (quoted in Lakatos 1986, 31)."

Truth and Practice

- *Problem*: Even if ZFC is true, theories corresponding to other conceptions might be true too! So, the *practical* question remains: why treat the iterative conception as canonical?
    - *Shoenfield:* "It is...possible that there is a completely different analysis of the notion of a set, and this might lead to quite a different set of axioms (1985, 324)."
    - *Incurvati*: "[W]e cannot believe in...Foundation on the...grounds that the iterative conception is, as Boolos suggested...'the only natural and (apparently) consistent conception of set we have (GCS, 206)." "[Azcel's] graph conception… emerges as a candidate to be the conception of set embodied by a set theory centred around the idea of a set being depicted by a graph…[I]f sets are what is depicted by an arbitrary graph, then most of the axioms of ZFA are justified (2014, 205)."
    - *Morris*: "[S]et theories such as Quine's New Foundations (NF) cannot...be ruled out on the grounds that they...stray too far from what was originally intended in the notion of set. In fact...the idea of a single intuitive notion of set, especially as the iterative notion, is...a myth of set theory's founding (Morris 2018, 150)."

<u>Works Cited</u>

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