Spatial price discrimination with heterogeneous firms*

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Abstract

In this paper we aim to explain intuitively heterogeneous firms' optimal location decisions in a simple spatial market. To do so, we present and solve a four-stage game of entry, location, pricing, and consumption in a spatial price discrimination framework with arbitrarily many heterogeneous firms. We provide a unique equilibrium outcome without imposing restrictions on the distribution of marginal costs across firms.

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1 Introduction

In this paper we aim to explain intuitively heterogeneous firms' optimal location decisions in a simple spatial market. To do so, we present and solve a four-stage game of entry, location, pricing, and consumption in a spatial price discrimination framework with arbitrarily many heterogeneous firms. We provide a unique equilibrium outcome without imposing restrictions on the distribution of marginal costs across firms.

Spatial price discrimination represents the ability of a firm to charge different prices to consumers at different locations in space. Spatial price discrimination is possible in markets in which firms are geographically differentiated, such as ready-mixed concrete. In such a market, a producer observes the location of each of its customers and can condition its customer-specific price on this location. Spatial price discrimination is also possible in markets in which producers sell goods tailored to the desired specifications of their customers, such as differentiated intermediate input producers. In such a market, a producer customizes its output to match the requirements of each of its customers and can condition its price on these requirements.

The present paper makes contributions along three dimensions. First, it reproduces recent results obtained in Vogel (2008)—e.g. that more productive firms are more isolated (all else equal)—that were obtained in a framework that applies to very different types of industries than the present paper. In particular, spatial price discrimination was not allowed in Vogel (2008). Second, it generalizes these predictions in two important respects: (*i*) it does not impose restrictions on the distribution of marginal costs across firms,[1] and (*ii*) it includes an entry stage in which, in equilibrium, less productive firms do not enter. Finally, by greatly simplifying the game—precisely by allowing for spatial price discrimination—the present paper highlights in a clearer way the economic intuition behind these results.

Section 2 introduces our theoretical framework. The market is represented by the unit circumference, which is populated with uniformly distributed consumers. There is a potentially large set of potential entrants with different constant marginal costs of production. Firms and consumers play a four-stage game of complete information. In the first stage, potential entrants simultaneously choose whether to enter, where entrants incur a fixed cost. In the second stage, the entrants (firms) simultaneously choose their locations in the market. In the third stage, firms simultaneously set their prices, where each firm can price discriminate, potentially choosing a different price for each location in the market. In the final

[1] Most spatial competition models that incorporate firm heterogeneity impose a restriction on the extent of permissible asymmetry between firms; see e.g. Aghion and Schankerman (2004), Syverson (2004), Alderighi and Piga (2008), and Vogel (2008). Wiseman (2010) is a very interesting and notable exception.

stage, consumers choose from which firm, if any, to purchase. In Sections 3-5 we solve for an equilibrium using backwards induction. In equilibrium, more productive firms are more isolated (all else equal), supply more consumers, and earn higher profits.

It is worth emphasizing that the assumption of spatial price discrimination is key to the present paper's ability to relax the restriction on the distribution of marginal costs across firms. Spatial price discrimination greatly simplifies the game. As is well known, under the assumption of mill pricing—in which a firm charges a single price regardless of customer location—solving for an equilibrium to a location-and-pricing game is complicated because of issues that arise in the price stage; see e.g. d'Aspremont, Gabszewicz, and Thisse (1979). The model with spatial price discrimination is significantly more tractable: in the third stage, in which firms choose prices, firms engage in Bertrand competition with undifferentiated goods at each location.

This is not the first paper to consider price discrimination in a spatial competition model; see e.g. Hoover (1937), Lederer and Hurter (1986), Hamilton, Thisse, and Weskamp (1989), Hamilton, MacLeod, and Thisse (1991), and MacLeod, Norman, and Thisse (1992).[2] Building on these papers, the primary focus of which was existence of equilibria, we emphasize the determinants of isolation for arbitrarily many heterogeneous firms. This paper also contributes to a growing spatial competition literature concerned with heterogeneous firms; see e.g. Aghion and Schankerman (2004), Syverson (2004), Alderighi and Piga (2008), and Vogel (2008). Unlike Aghion and Schankerman (2004), Syverson (2004), and Alderighi and Piga (2008), the present paper considers not only endogenous prices, but also endogenous locations.

2 Setup

The market is represented by the unit circumference, the points of which are indexed in a clockwise direction by $z \in [0, 1]$, and is populated by a uniformly distributed unit mass of consumers. Each consumer purchases one unit of a homogeneous good—buying from the lowest price source—if and only if the lowest price at which she can purchase the good is no greater than her reservation value, $v > 0$.

There is a set N containing $|N| \geq 2$ potential entrants, each of which has a unique marginal cost (of production) $c_i \in [0, v - t/2)$.[3] A firm i that is located at point η_i and sells

[2] Spatial discrimination in the Cournot setting with homogeneous firms has also been studied; see e.g. Anderson and Neven (1991) and Chapter 9 of Combes, Mayer, and Thisse (2008). Bernard, Eaton, Jensen, and Kortum (2003) consider a spatial price discrimination model of international trade.

[3] The assumption of an upper bound on firm costs is to insure that at least one firm enters the market.

to a consumer at point z incurs a *delivered marginal cost* of $k_i(\eta_i, z) \equiv c_i + t\|\eta_i - z\|$, where $\|\eta_i - z\|$ is the shortest arc-length separating the firm from the consumer, and $t \in (0, 2v)$ is the cost of transportation.

Four-stage game: Consumers and firms play a four-stage game of complete information. In the first stage, *the entry stage*, potential entrants simultaneously choose whether or not to enter, where entrants must incur a fixed cost $f > 0$. Entrants become firms and move to the second stage, *the location stage*, in which firms simultaneously choose their locations in the market. In the third stage, *the price stage*, firms simultaneously choose price schedules. Each firm i can price discriminate, choosing a price $p_i(z)$ for each location z on the circle. In the final stage, the *consumption stage*, consumers choose which offer, if any, to accept.

Equilibrium concept and equilibrium outcome: Throughout the paper we focus on pure strategy subgame perfect Nash equilibria in strategies that are limit points of undominated strategies. We refer to these simply as *equilibria*.

We define an *equilibrium outcome* as the set of firms that enter the market ($K \subseteq N$), the market shares of these entrants ($x_i \geq 0$ for all $i \in K$) such that if $|K| \geq 1$ then $\sum_{i \in K} x_i = 1$, and the variable profits of the entrants. We say that there is a *unique equilibrium outcome* if the set of firms that enter and each firm's market share and profit are the same across all equilibria.

In the following sections we solve for the unique equilibrium outcome in the case in which the fixed cost of entry $f > 0$ is small.

3 Price and consumption stages

Suppose that the set of firms in the market is $K \subseteq N$, and that the location of each firm is fixed at η_i. In what follows, suppose that $|K| \geq 2$.[4]

With perfect price discrimination, at each point in the market we have a homogeneous good Bertrand oligopoly with asymmetric marginal costs. It is well known that the only equilibrium that satisfies our refinement has an equilibrium outcome in which, at each location z, the firm with the lowest delivered marginal cost makes the sale at a price equal to the second lowest delivered marginal cost.[5]

The assumption that no two firms have the same marginal cost of production is for exposition only. Both assumptions could be dispensed with easily.

[4] If $|K| = 1$, the monopoly charges each location the reservation price v.

[5] Note that all our results would be unaffected if we allowed for a stage of consumer arbitrage in which consumers faced a transport cost equal to (or greater than) that incurred by firms, since at these prices consumers would have no incentive to arbitrage.

Given these prices, market shares are determined as follows. Suppose that firm $i-1$ is firm i's closest neighbor (that supplies a positive mass of consumers) in the counterclockwise direction and that firm $i+1$ is firm i's closest neighbor (that supplies a positive mass of consumers) in the clockwise direction. Let $d_{i-1,i}$ and $d_{i,i+1}$ denote the distance between firm $i-1$ and firm i and between firm i and firm $i+1$ in the clockwise direction, respectively. By standard arguments in a Hotelling-style model, firm i's market share is given by

$$x_i = \max\left\{\frac{1}{2}\left(d_{i-1,i} + d_{i,i+1}\right) + \frac{1}{t}\left[\frac{c_{i+1} + c_{i-1}}{2} - c_i\right], 0\right\}. \tag{1}$$

If firm i has a positive market share, then firm i supplies half the consumers between its two neighbors in addition to a (potentially negative) mass of consumers that depends on i's marginal cost relative to the average marginal cost of its neighbors. On the other hand, firm i's market share is zero if its delivered marginal cost is not the lowest over any interval in the market. Finally, note that x_i in equation (1) denotes firm i's market share whether or not there are additional firms between i and $i-1$ or between i and $i+1$ that have a market share of zero.

4 Location stage

In this section, we obtain a set of preliminary results. First, we show how firm i chooses its optimal location between firm $i-1$ and firm $i+1$ for given locations of these two firms, under the restriction that each of these three firms supplies a positive mass of consumers. This will build intuition for the results that follow. In the remainder of the section we provide five lemmas that lead, in Section 5, to our existence and uniqueness result.

4.1 Optimal location in a special case

In this section we construct market shares and profits, and solve for firm i's optimal location between firms $i-1$ and $i+1$ in the special case in which all firms supply a positive mass of consumers. This will build intuition for the more general results that follow.

Normalize firm $i-1$'s location as point zero and define all other points by their distance from $i-1$ in the clockwise direction. Denote by $X_{i,i-1}$ (and $X_{i,i+1}$) the point at which the delivered marginal costs of firms $i-1$ and i (firms i and $i+1$) are equal. We say that the consumers located at $X_{i,i-1}$ and $X_{i,i+1}$ are firm i's *boundary consumers*; they are the consumers at the boundary between the sets of consumers supplied by i and its neighbors. Finally, firm i's price at point $z \in [X_{i,i-1}, X_{i,i+1}]$ equals firm $i-1$'s delivered marginal cost

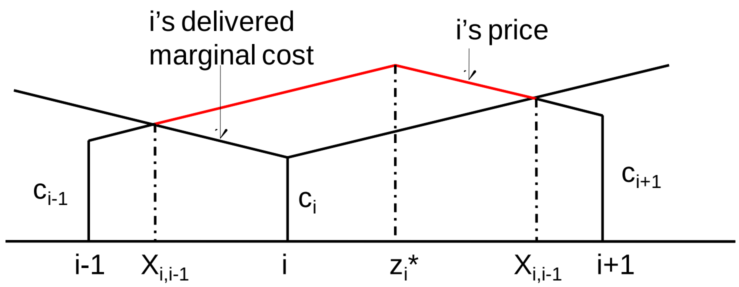

Figure 1: Firm i's price, delivered marginal cost, and boundary consumers for given locations.

at all locations for which firm $i-1$'s delivered marginal cost is less than firm $i+1$'s delivered marginal cost. Firm i's price equals firm $i-1$'s delivered marginal cost if $z \leq z_i^*$, where

$$z_i^* = \frac{1}{2t}\left(c_{i+1} - c_{i-1}\right) + \frac{1}{2}\left(d_{i-1,i} + d_{i,i+1}\right). \tag{2}$$

Figure 1 clarifies this notation. In addition to depicting the location of the consumer at which firm $i-1$ and firm $i+1$ have the same delivered marginal cost, it depicts firm i's price and delivered marginal cost for each consumer that it supplies, and its boundary consumers. Figure 1 also shows firm i's variable profit π_i, which is the area under firm i's price and above its delivered marginal cost, integrated between firm i's boundary consumers:

$$\pi_i = \int_{X_{i,i-1}}^{z_i^*} \left[c_{i-1} + tz - k_i\left(z\right)\right] dz + \int_{z_i^*}^{X_{i,i+1}} \left[c_{i+1} + t\left(d_{i-1,i} + d_{i,i+1} - z\right) - k_i\left(z\right)\right] dz. \tag{3}$$

Now consider firm i's optimal *local* choice of location (local in the sense that i's location satisfies the restrictions that i is between firms $i-1$ and $i+1$ and that firms $i-1$, i, and $i+1$ all supply a positive mass of consumers). Firm i's choice of location has no impact either on its market share x_i, equation (1), on the location z_i^* at which its price changes from being determined by firm $i-1$'s delivered cost to firm $i+1$'s delivered cost, equation (2), or on the price it can charge any location z. Hence, all the effects on firm i's variable profit of a change in its location arise from the impact of this change on the location of the consumers that it supplies, since this location affects the total transportation costs it incurs and the total revenue it receives. To minimize its transportation costs or to maximize its revenue, firm i would choose to locate at z_i^*, where it is equidistant from its boundary consumers in the clockwise and counterclockwise directions. Hence, firm i will choose to locate at z_i^*.

This observation has two important implications. First, if one of firm i's neighbor's costs were to fall, then firm i would optimally move away from this neighbor. For instance, if c_{i-1}

falls, then z_i^* moves to the right, and firm i will optimally move to the right as well. Second, if firm i's cost changes, this has no direct effect on its optimal location between $i-1$ and $i+1$.

4.2 Equilibrium

Let $K \subseteq N$ denote the set of firms in the market, where $|K| \geq 2$.[6] Denote by $\overline{c}(K') \equiv \frac{1}{|K'|}\sum_{n\in K'} c_n$ the average marginal cost (of production) of the firms in $K' \subseteq K$.[7] We first characterize equilibrium in a special case in which all firms supply a positive mass of consumers.[8]

Lemma 1 *Suppose $|K| \geq 2$. In any equilibrium in the location subgame in which all firms supply a positive mass of consumers, firm i's market share and variable profit, for all $i \in K$, are given by*

$$x_i(K) = \frac{1}{|K|} + \frac{2}{t}\left[\overline{c}(K) - c_i\right] \tag{4}$$

$$\pi_i(K) = \frac{t}{2} x_i(K)^2, \tag{5}$$

and the distance between two neighboring firms i and $i+1$ is given by

$$d_{i,i+1}(K) = \frac{1}{|K|} + \frac{2}{t}\left[\overline{c}(K) - \frac{c_i + c_{i+1}}{2}\right]. \tag{6}$$

Lemma 1 implies that firm i's market share and variable profit are common across all equilibria in which all firms supply a positive mass of consumers, and depend on another producer's marginal cost only through its impact on the average marginal cost $\overline{c}(K)$. This result may seem surprising. Given arbitrary locations, a firm's profit clearly depends more on the marginal costs of its neighbors than on the costs of more distant firms. However, as discussed in Section 4.1, locations are not arbitrary in equilibrium. Instead, in any equilibrium in which all firms supply a positive mass of consumers, each firm is equidistant from its two boundary consumers.

Because each firm is equidistant from its two boundary consumers, the delivered marginal cost of supplying each boundary consumer between any two firms must equal $\frac{t}{2|K|} + \overline{c}(K)$. This directly implies that firm i's market share and profit depend on its competitors' marginal costs only through their impact on $\overline{c}(K)$. Figure 2 depicts locations of firms and boundary

[6] If $|K| = 1$, then the monopoly firm is indifferent between all locations.
[7] We set $\overline{c}(\emptyset) = 0$.
[8] All Proofs are relegated to the Appendix.

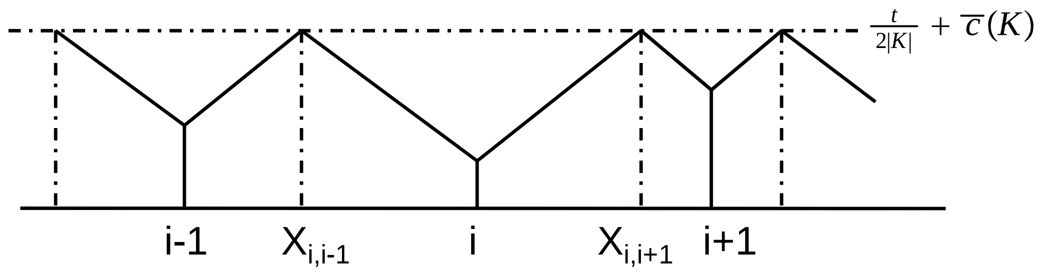

Figure 2: Equilibrium firm and boundary consumer locations if all firms supply a positive mass of consumers.

consumers and shows the price charged to each boundary consumer (in any equilibrium in which all firms supply a positive mass of consumers).

In the rest of the paper we use the following definition:

$$K^*(K) = \left\{ i \in K \,\middle|\, c_i - \bar{c}\left[K^*(K)\right] < \frac{t}{2\left|K^*(K)\right|} \right\}.$$

This subset $K^*(K)$ contains all the firms in K with a marginal cost below a certain cutoff, where the value of this cutoff depends on the set K: to be in the set $K^*(K)$, a firm's marginal cost cannot be too high relative to the average marginal cost of the firms in $K^*(K)$. The set $K^*(K)$ has an important property: if each firm $i \in K^*(K)$ were located at a distance $d_{i,i+1}[K^*(K)]$, as defined in equation (6), from its clockwise neighbor in the set $K^*(K)$—ignoring all firms not in the set $K^*(K)$—then any firm not in the set $K^*(K)$ located at any point in the market would be unable (i) to obtain a positive market share or (ii) to affect the location of boundary consumers.

Figure 3 helps clarify this key property, and also provides an additional important insight. The figure depicts the locations of four firms, where $i-1, i, i+1 \in K^*(K)$ and $j \notin K^*(K)$. Firms $i-1$ and i are separated by a distance $d_{i-1,i}[K^*(K)]$, and firms i and $i+1$ are separated by a distance $d_{i,i+1}[K^*(K)]$. Given these distances, firm j's marginal cost is too high to affect any firm's market share or the location of any boundary consumers, such as $X_{i,i-1}$ and $X_{i,i+1}$, as discussed above. This is true for any location firm j chooses. However, at the location depicted for firm j, firm i's profit is affected by firm j, since j's delivered marginal cost is the second lowest over a range of consumers.

The following lemma states two additional properties of $K^*(K)$.

Lemma 2 $K^*(K)$ *is unique and non-empty.*

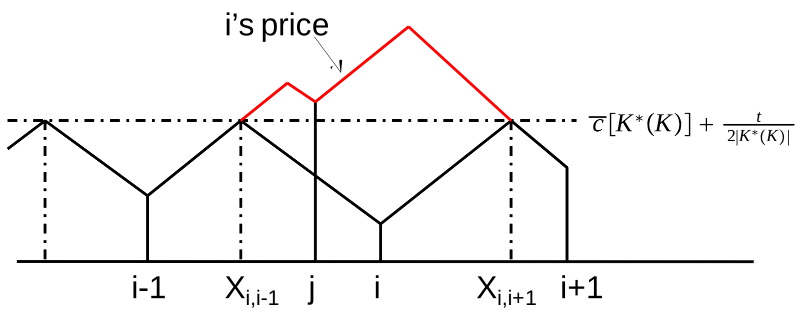

Figure 3: The important properties of the set $K^*(K)$, where $i-1, i, i+1 \in K^*(K)$ and $j \notin K^*(K)$.

We first use this definition to show that certain firms must obtain positive market shares and, therefore, positive variable profits in any equilibrium.

Lemma 3 *In any equilibrium in the location subgame, $x_i > 0$ for any $i \in K^*(K)$.*

At a broad level, the intuition behind Lemma 3 is simple. If firm $i \in K^*(K)$ does not have a strictly positive market share, then neither can any firm $j \notin K^*(K)$, since i has a strictly lower cost of production than j. If i does not have a strictly positive market share, then the entire market must be supplied by some subset of firms $M \subset K^*(K)$. But in this case, the firms in M are separated by a sufficiently large distance so that a boundary consumer faces a price strictly greater than i's marginal cost. Hence, firm i could locate at the same point as this boundary consumer and earn strictly positive variable profits.

While Lemma 3 shows that each $i \in K^*(K)$ must have a positive market share in any equilibrium, the next result provides a necessary and sufficient condition under which all firms have a positive market share in any equilibrium.

Lemma 4 *In any equilibrium in the location subgame, $x_i > 0$ for all $i \in K$ if and only if $K^*(K) = K$.*

The proof of Lemma 4 formally proceeds in two steps. If $K = K^*(K)$, then all firms in the market must have a positive market share in any equilibrium, as shown in Lemma 3. The proof in the other direction is also simple. Any equilibrium in which each firm has a positive market share must be characterized by Lemma 1. But if $K^*(K) \subset K$, then the market share of any firm $j \notin K^*(K)$ would be bounded above by zero, according to equation (4).

Lemma 4 implies that firms must be sufficiently productive relative to the average firm in the market in order for each to have a strictly positive market share in any equilibrium.

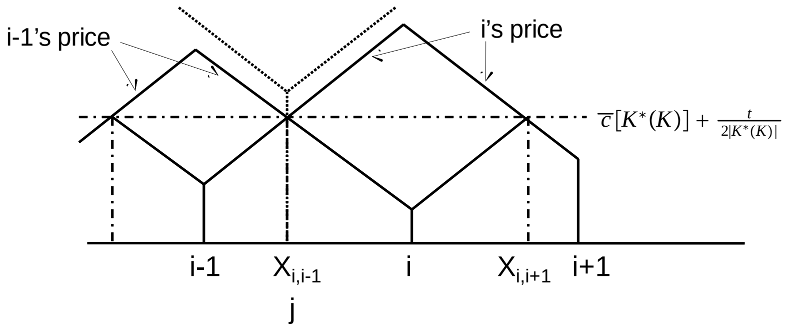

Figure 4: Equilibrium locations, prices, and market shares when $K^*(K) \neq K$.

Moreover, the permissible cost disadvantage of firm i relative to the average firm becomes smaller as the number of firms in the market rises or the transportation cost falls. Firms move closer to their competitors as the number of firms in the market rises, and at some point high cost firms' market shares shrink to zero. Similarly, the disadvantage of a high cost is magnified as transportation costs fall, since consumers become more willing to substitute to low cost firms that are farther away.

The following lemma provides our final preliminary result.

Lemma 5 *There exists an equilibrium to an arbitrary location subgame.*

In the first step of the proof, we show that if $K = K^*(K)$, then an equilibrium exists, and all such equilibria are characterized by Lemma 1. The proof of this step boils down to showing that no firm has an incentive to make a "large" deviation, since we have already shown in Lemma 1 that each firm is locating optimally for "local" deviations. The remainder of the proof considers the case in which $K \neq K^*(K)$.

The intuition for the proof—if $K \neq K^*(K)$—is constructive, and we sketch the proof in the case in which $|K^*(K)| \geq 2$. Consider locations that satisfy the following two properties. First, each firm $i \in K^*(K)$ is located at a distance $d_{i,i+1}[K^*(K)]$ from its nearest neighbor in the set $K^*(K)$ in the clockwise direction, exactly as in Figure 3. Second, and unlike in Figure 3, each firm $j \in K \backslash K^*(K)$ is located at the same point as a boundary consumer between two firms in $K^*(K)$. These locations are depicted in Figure 4.

As is evident from Figure 4, given the locations of all firms in $K^*(K)$, firm $j \in K \backslash K^*(K)$ cannot supply a positive mass of consumers from any location in the market, since c_j is strictly greater than the lowest delivered marginal cost at almost every point in the market. Hence, firm j has no incentive to deviate from its location. Next, when each firm $j \in K \backslash K^*(K)$ is located at the same point as a boundary consumer between two firms in $K^*(K)$, firm j's delivered marginal cost is always (weakly) greater than the second lowest delivered marginal cost at every point in the market. Therefore, if firm j is located at the same point as a boundary consumer, j has no effect on market shares, boundary consumers, or *prices* for any firm $i \in K^*(K)$. From the perspective of any $i \in K^*(K)$, it is as if only the firms in $K^*(K)$ are in the market. Hence, from the first step of the proof, firm i has no incentive to deviate.

5 Entry stage

A firm chooses to enter if and only if its variable profits are at least large as the fixed entry cost $f > 0$. This implies that the set of entrants K must satisfy $K = K^*(K)$. Otherwise at least one entrant would not earn positive variable profits, by Lemma 4. Hence, if an equilibrium exists, then *given the set of entrants*, the unique equilibrium outcome is as prescribed by Lemma 1. In this section we show that, for a sufficiently small fixed cost of entry, an equilibrium exists and the equilibrium outcome is unique: the set of entrants is $K^*(N)$ and each firm's market share and variable profit are $x[K^*(N)]$ and $\pi[K^*(N)]$, as given in equations (4) and (5).

In what follows we describe the intuition behind why the unique set of entrants K satisfies $K = K^*(N)$. There are 2 cases to consider: (i) $K^*(N) \subset K$, and (ii) there exists a firm $i \in K^*(N)$ where $i \notin K$. Eliminating the first case is straightforward. If $K^*(N) \subset K$, then at least one firm $j \in K \backslash K^*(N)$ must not supply a strictly positive mass of consumers, by Lemma 4. Hence, $K^*(N) \subset K$ cannot hold in equilibrium for any $f > 0$. The key to eliminating the second case is the result that if $i \in K^*(N)$ enters, it can guarantee itself strictly positive variable profits for any set of entrants. This result follows from the same logic used in the proof of Lemma 3. Hence, for a sufficiently small fixed cost of entry $f > 0$, there exists no equilibrium in which $i \in K^*(N)$ does not enter.

We formalize the above discussion and present the central result of the paper in the following proposition.

Proposition 1 *There exists a fixed cost $f^* > 0$ such that for any $0 < f < f^*$:*
1. *an equilibrium exists;*
2. *the equilibrium outcome is unique; and*
3. *if $|K^*(N)| > 1$, then firm i's market share and variable profit are given by $x_i[K^*(N)]$ and $\pi_i[K^*(N)]$; and if $|K^*(N)| = 1$ then firm $i \in K^*(N)$ sets price v at all locations and $x_i = 1$.*

Finally, we are able to discuss the role of our assumption that the fixed cost of entry is sufficiently small. This assumption plays a key role in the uniqueness result. If the fixed cost were sufficiently large, then we could not generically rule out the possibility that a less productive firm enters while a more productive firm does not. While the more productive firm could guarantee itself positive variable profits by entering, these variable profits need not exceed the fixed cost of entry if a less productive firm is in the market (even if they would exceed the fixed cost of entry if the less productive firm were not in the market).[9]

6 Conclusion

In this paper we aimed to explain intuitively heterogeneous firms' optimal location decisions in a simple spatial market. We presented and solved a four-stage game of entry, location, pricing, and consumption in a spatial price discrimination framework with arbitrarily many heterogeneous firms. We obtained a unique equilibrium outcome with a sufficiently small fixed cost of entry and we did not impose restrictions on the distribution of marginal costs across firms. Our main prediction is that more productive firms are more isolated, all else equal.

Our analysis is limited in (at least) three important respects. We have assumed that consumers are uniformly distributed through space, that space is one dimensional, and that the game is static. These are strong and unrealistic assumptions that we made for tractability. Nonetheless, we hope that the paper provides useful insight into the determinants of firm isolation.

A Proofs

Proof of Lemma 1. Suppose there exists an equilibrium to the location-stage subgame in which all firms supply a positive mass of consumers. Fix the location of all firms $j \neq i$ and consider the effect of firm i's unilateral ε-deviation towards firm $i+1$ (if $\varepsilon > 0$) or towards firm $i-1$ (if $\varepsilon < 0$). From equations (3) and the definitions of $X_{i,i-1}$ and $X_{i,i+1}$, firm i's first-order condition for a maximum—conditional on all firms supplying a positive mass of consumers—is given by

$$d_{i,i+1}(K) = d_{i-1,i}(K) + \frac{1}{t}(c_{i-1} - c_{i+1}). \tag{7}$$

[9] Of course, we could obtain a uniqueness result without this assumption, if we assumed instead sequential entry of firms (and a positive fixed cost of entry).

Such a location locally maximizes firm i's profits as the second-order condition is satisfied. If an equilibrium exists in which all firms supply a positive mass of consumers, then given an order of firms around the circle: (i) each firm's location must satisfy equation (7) and (ii) the sum of distances between all pairs of firms must sum to 1:

$$d_{n,1}(K) + \Sigma_{i=1}^{n-1} d_{i,i+1}(K) = 1. \tag{8}$$

Solving equation (7) recursively yields

$$d_{i+j,i+j+1}(K) = d_{i-1,i}(K) + \frac{1}{t}(c_{i-1} + c_i - c_{i+j} - c_{i+j+1}).$$

The distance between two arbitrary neighbors as a function of the distance between firms 1 and n, where firm 1 is firm n's clockwise neighbor, is

$$d_{j,j+1}(K) = d_{n,1}(K) + \frac{1}{t}(c_n + c_1 - c_j - c_{j+1}). \tag{9}$$

Substituting equation (9) into equation (8) implies

$$d_{n,1}(K) = \frac{1}{|K|} + \frac{2}{t}\left[\overline{c}(K) - \left(\frac{c_n + c_1}{2}\right)\right].$$

Substituting the solution for $d_{n,1}(K)$ into equation (9) yields equation (6). Given equation (6), it is straightforward to show that market shares and variable profits are given by equations (4) and (5). ■

Proof of Lemma 2. The proof of Lemma 2 requires three preliminary steps.

Step 1. $K^*(K)$ *is non-empty.*

This follows from the fact that $c_i - \overline{c}(i) = 0 < \frac{t}{2}$ for any firm i and the fact that $c_i - \overline{c}(\emptyset) < \infty$ for any firm i.

Step 2. *If* $j \in K'$ *and* $|K'| \geq 2$*, then* $c_j - \overline{c}(K') < \frac{t}{2}\frac{1}{|K'|} \Leftrightarrow c_j - \overline{c}(K' \backslash j) < \frac{t}{2}\frac{1}{|K' \backslash j|}$.

This follows from simple algebra:

$$\begin{aligned} c_j - \overline{c}(K') &< \frac{t}{2}\frac{1}{|K'|} \Leftrightarrow c_j - \frac{1}{|K'|}\sum_{i \in K'} c_i < \frac{t}{2}\frac{1}{|K'|} \\ &\Leftrightarrow c_j - \frac{1}{|K'|}\sum_{i \in K' \backslash j} c_i - \frac{1}{|K'|}c_j < \frac{t}{2}\frac{1}{|K'|} \\ &\Leftrightarrow c_j - \frac{1}{|K'|-1}\sum_{i \in K' \backslash j} c_i < \frac{t}{2}\frac{1}{|K'|-1} \end{aligned}$$

Hence,

$$c_j - \overline{c}(K') < \frac{t}{2}\frac{1}{|K'|} \Leftrightarrow c_j - \overline{c}(K' \backslash j) < \frac{t}{2}\frac{1}{|K' \backslash j|}$$

which concludes the proof of Step 2.

Step 3. *$K^*(K)$ is unique.*

We proceed by contradiction. Suppose that $K', K'' \in K^*(K)$ with $K' \neq K''$. Since $K^*(K)$ contains the most productive firms in K, we have either $K' \subset K''$ or $K'' \subset K'$. Suppose that $K' \subset K''$. This implies that $\overline{c}(K') + \frac{t}{2}\frac{1}{|K'|} < \overline{c}(K'') + \frac{t}{2}\frac{1}{|K''|}$.

Let $K_1 = K''$. Denote by i_1 the highest cost firm in the set $K_1 \backslash K' \neq \emptyset$. Since $i_1 \in K_1$, we have $c_{i_1} - \overline{c}(K_1) < \frac{t}{2}\frac{1}{|K_1|}$. By Step 2, we therefore have $c_{i_1} - \overline{c}(K_1 \backslash i_1) < \frac{t}{2}\frac{1}{|K_1 \backslash i_1|}$. Hence, if $K' = K_1 \backslash i_1$, then we have shown $c_{i_1} - \overline{c}(K') < \frac{t}{2}\frac{1}{|K'|}$, which implies that $i_1 \in K'$, a contradiction. If $K' \neq K_1 \backslash i_1$, then let $K_2 = K_1 \backslash i_1$, and denote by i_2 the highest cost firm in the set $K_2 \backslash K' \neq \emptyset$. Since $c_{i_2} < c_{i_1}$, we have $c_{i_2} - \overline{c}(K_2) < \frac{t}{2}\frac{1}{|K_2|}$. Proceeding by iteration, we obtain $K' = K_n \backslash i_n$, for some finite n, where $c_{i_n} - \overline{c}(K_n) < \frac{t}{2}\frac{1}{|K_n|}$. By Step 2, we therefore have $c_{i_n} - \overline{c}(K') < \frac{t}{2}\frac{1}{|K'|}$, so that $i_n \in K'$, a contradiction.

Lemma 2 follows from Step 1 and Step 3. ■

Proof of Lemma 3. The proof of Lemma 3 requires three preliminary steps.

Step 1. *In any equilibrium in the location subgame, if $x_j > 0$ and $c_i < c_j$, then $x_i > 0$.*

We proceed by contradiction. Suppose that there exists an equilibrium in which $x_j > 0$, $x_i = 0$, and $c_i < c_j$. In this equilibrium, $\pi_i = 0$. Suppose that firm i deviates and locates in the same location firm j. Under this deviation, firm i earns counterfactual profits $\pi_i' > x_j(c_j - c_i) > 0$, since firm i sells to all of j's former consumers at an absolute markup of $c_j - c_i$. Finally, $\pi_i' > \pi_i$, a contradiction.

Step 2. *In any equilibrium in the location subgame, if $M = \{i \in K | \, x_i > 0\}$, then the maximum price faced by a boundary consumer is bounded below by $\overline{c}(M) + \frac{t}{2}\frac{1}{|M|}$.*

If $M = \{i \in K | \, x_i > 0\}$, then the maximum price faced by a boundary consumer is minimized if each $i \in M$ is equidistant from both of its boundary consumers. When each firm $i \in M$ is equidistant from each of its boundary consumers, the distance between i and its neighbor $i+1 \in M$ in the clockwise direction (other firms not in M may be located between i and $i+1$) is given by $d_{i,i+1}(M)$, as in equation (6). In this case, boundary consumers face prices $\overline{c}(M) + t/(2|M|)$. Hence, regardless of the locations of all firms $j \in K \backslash M$, the maximum price faced by a boundary consumer is bounded below by $\overline{c}(M) + t/(2|M|)$.

We now prove Lemma 3 by contradiction. Suppose that there exists an equilibrium in which $x_i = 0$ for at least one firm $i \in K^*(K)$. By Step 1, $M \subset K^*(K)$, where $M = \{i \in K | \, x_i > 0\}$, since $c_i < c_j$ for any firm $j \notin K^*(K)$. We consider the case in which $|M| \geq 2$, although the case in which $|M| = 1$ is straightforward. By Step 2, the maximum price faced by a boundary consumer (boundary consumers exist with $|M| \geq 2$) is bounded below by $\overline{c}(M) + \frac{t}{2}\frac{1}{|M|}$.

Let $K_1 = K^*(K)$. Denote by i_1 the highest cost firm in the set $K_1 \backslash M \neq \emptyset$. Since $i_1 \in K_1$, we have $c_{i_1} - \overline{c}(K_1) < \frac{t}{2}\frac{1}{|K_1|}$. By Step 2 in the proof of Lemma 2, we therefore have $c_{i_1} - \overline{c}(K_1 \backslash i_1) <$

$\frac{t}{2}\frac{1}{|K_1\backslash i_1|}$. Hence, if $M = K_1\backslash i_1$, then firm i_1 could obtain a strictly positive market share and variable profit by deviating and locating at the same point as the boundary firm that faces the highest price, a contradiction. If $M \neq K_1\backslash i_1$, then let $K_2 = K_1\backslash i_1$, and denote by i_2 the highest cost firm in the set $K_2\backslash M$. Since $c_{i_2} < c_{i_1}$, we have $c_{i_2} - \overline{c}(K_2) < \frac{t}{2}\frac{1}{|K_2|}$. Proceeding by iteration, we obtain $M = K_n\backslash i_n$, for some finite n, where $c_{i_n} - \overline{c}(K_n) < \frac{t}{2}\frac{1}{|K_n|}$. Firm i_n could deviate by locating on top of the boundary consumer that faces the highest price, and by doing so firm i_n would obtain a strictly positive variable profit, a contradiction. ■

Proof of Lemma 4. The proof of Lemma 4 proceeds in two steps.

Step 1: *In any location subgame in which $K^*(K) \neq K$, there exists no equilibrium in which $x_i > 0$ for all $i \in K$.*

Proof: We proceed by contradiction. Suppose that $K^*(K) \neq K$ and that there exists an equilibrium in which $x_i > 0$ for all $i \in K$. According to Lemma 1, in any such equilibrium each firm's market share must be given by equation (4). However, this equation stipulates a negative market share for at least one firm $n \notin K^*(N)$.

Step 2: *In any location subgame in which $K^*(K) = K$, $x_i > 0$ for all $i \in K$ in any equilibrium.*

Proof: This follows directly from Lemma 3.

The proof of Lemma 4 follow directly from Step 1 and Step 2. ■

Proof of Lemma 5. Suppose that the non-empty set of entrants is K. If $|K| = 1$, then any choice of location is an equilibrium, so in the remainder of the proof we assume that $|K| > 1$. The proof requires 3 steps.

Step 1: *An equilibrium exists to any location subgame in which $K = K^*(K)$.*

Proof: Suppose that all firms $n \in K\backslash i$ locate as prescribed by Lemma 1. Let $k_2^i(z) = \min_{j\neq i} k_j(z)$ denote the minimum delivered marginal cost, taken over all firms but firm i, to a consumer located at point z. Then $k_2^i(z)$ is continuous and $\int_{z\in\vartheta} k_2^i(z)\,dz$ denotes firm i's revenue from selling to a set ϑ of consumers. Let ϑ_i^* denote the set of consumers to whom firm i sells if firm i does not deviate from the location prescribed by Lemma 1. The lowest cost location from which to supply all $z \in \vartheta_i^*$ is the location prescribed by Lemma 1. Step 1 then follows directly from the fact that $k_2^i(z) > k_2^i(z')$ for almost all $z \in \vartheta_i^*$ and $z' \notin \vartheta_i^*$.

Step 2: *An equilibrium exists to any location subgame in which $|K^*(K)| \geq 2$ and $K \neq K^*(K)$.*

Here we postulate a set of locations and then prove that no firm has a unilateral incentive to deviate from these. Suppose that each firm $i \in K^*(K)$ is located at a distance $d_{i,i+1}[K^*(K)]$, given by equation (6), from its nearest clockwise neighbor in the set $K^*(K)$. And suppose that each firm $j \in K\backslash K^*(K)$ locates at the same point as a boundary consumer between two firms in $i \in K^*(K)$. In what follows we show that no firm has a unilateral incentive to deviate in the location subgame.

First consider an arbitrary $j \in K \backslash K^* (K)$. Given the locations of all firms $i \in K^* (K)$, the lowest delivered marginal cost—taken over all firms $i \in K^* (K)$ and across all points in the market—is $\bar{c} [K^* (K)] + \frac{t}{2|K^*(K)|} \leq c_j$, where this inequality follows from $j \notin K^* (K)$. Hence, firm j has no incentive to deviate as it cannot earn positive variable profits from any location in the market. Second, consider an arbitrary firm $i \in K^* (K)$. Given the locations of all $j \in K \backslash K^* (K)$ and the fact that $\bar{c} [K^* (K)] + \frac{t}{2|K^*(K)|} \leq c_j$, each firm $j \in K \backslash K^* (K)$ does not impact the price, market share, or variable profits of firm i for any point in the market at which firm i could locate. Then according to Step 1, firm i has no unilateral incentive to deviate.

Step 3: *An equilibrium exists to any location subgame in which* $|K^* (K)| = 1$ *and* $K \neq K^* (K)$.

Suppose that all firms in K locate at the same point. In what follows we show that no firm has a unilateral incentive to deviate in the location subgame.

First consider an arbitrary $j \in K \backslash K^* (K)$. Firm j has no incentive to deviate as it cannot earn positive variable profits from any location in the market. Given that all firms $j \in K \backslash K^* (K)$ locate together, firm $i \in K^* (K)$ earns the same variable profit no matter where it chooses to locate, since it always supplies the full market, incurs the same total delivery costs, and earns the same total revenue. Hence, firm i has no incentive to unilaterally deviate. Thus, no firm has a unilateral incentive to deviate.

Lemma 5 follows directly from Steps 1, 2, and 3. ■

Proof of Proposition 1. We prove the three parts of the proposition in order.

Part 1: Let $f_0 \equiv \min_{j \in K^*(N)} \pi_j [K^* (N)]$, where $f_0 > 0$ follows from Lemma 1 and the definition of $K^* (N)$, and where a monopoly entrant's profit is strictly greater than $\pi_j (j)$. Suppose that $0 < f < f_0$ and consider a potential equilibrium in which the set of entrants is $K^* (N)$ and the equilibrium outcome in the subgame beginning in the location stage is as prescribed by Lemma 1. No firm $i \in K^* (N)$ has an incentive to deviate in the location or price stages, according to Lemma 1. Moreover, $\pi_i [K^* (N)] \geq f_0$ for all $i \in K^* (N)$. Hence, no firm $i \in K^* (N)$ has an incentive to deviate in any stage for any $f < f_0$.

To show that no additional firm has an incentive to enter, we proceed by contradiction. Suppose that $j \notin K^* (N)$ enters and earns positive profits. An equilibrium exists to the location subgame if j also enters, according to Lemma 5. At least one firm has a market share of zero, according to Lemma 4, and this firm must be j, according to Lemma 3. Hence, firm j earns negative profits by entering, a contradiction. Thus, an equilibrium exists in which the set of entrants is $K^* (N)$, and each firm $i \in K^* (N)$ has market share and variable profit given by $x_i [K^* (N)]$ and $\pi_i [K^* (N)]$.

Part 2: We proceed by contradiction. Suppose that the set of entrants is $K \neq K^* (N)$. There are two cases to consider: (i) $K^* (N) \subset K$, and (ii) there exists a firm $i \in K^* (N)$ where $i \notin K$. Case (i) does not correspond to an equilibrium for any $f > 0$. At least one firm has a market share of zero and, therefore, a negative profit, by Lemma 4. Case (ii) also cannot correspond to an equilibrium for a sufficiently small $f > 0$. If $i \notin K$, then $\pi_i = 0$. But if i enters, then either it supplies a strictly positive mass of consumers (in which case it earns positive profits for $f > 0$ sufficiently small) or the entire market is supplied by firms that have lower costs than i. If the entire market is supplied by firms more productive than $i \in K^*(N)$, then we obtain a contradiction using the same argument as in the proof of Lemma 3. Hence, if $f > 0$ is sufficiently small, then $i \in K^*(N)$ must enter in any equilibrium.

Part 3: Part 3 follows directly from Parts 1 and 2. According to Part 2, for $f > 0$ sufficiently close to zero, there exists no equilibrium in which the set of entrants is $K \neq K^*(N)$. According to Part 1, for $f > 0$ sufficiently close to zero, there exists an equilibrium in which the set of entrants is $K^*(N)$. Given that $|K^*(N)| > 1$, in any such equilibrium market shares and variable profits are $x_i[K^*(N)]$ and $\pi_i[K^*(N)]$. Given that $|K^*(N)| = 1$, the monopoly serves the entire market and charges each consumer her reservation value v. ■

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