A New Preference Model That Allows for Narrow Framing

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Abstract

We show that the preference model proposed by Barberis and Huang (2009) to allow for narrow framing may fail to define the utility process of the agent and may result in multiple solutions when applied to portfolio selection problems. We then show that a new preference model with narrow framing that is proposed by Guo and He (2016) defines a unique utility process and the corresponding dynamic programming equation in portfolio selection admits a unique solution as well. Finally, we show that the latter model is more tractable than the former when applied to explain individuals’ attitudes toward timeless gambles, non-participation of households in the stock market, and the equity premium puzzle.

Key words: narrow framing, recursive utility, existence and uniqueness, dynamic programming, risk attitudes, portfolio selection, asset pricing

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1 Introduction

Barberis and Huang (2009) propose a preference model that features narrow framing, referred to as the BH model, in which individuals evaluate risks in isolation, separately from

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other risks, such as consumption and investment risks, they are already facing. This model is formulated by generalizing the classical recursive utility model (Epstein and Zin, 1989, Kreps and Porteus, 1978) in that (i) the risks that are evaluated in isolation by the individuals at the end of each period are assessed according to prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992) so that the utility of gains and losses experienced by the individuals in these risks is calculated, (ii) the utility of gains and losses and the certainty equivalent of the individuals’ total utility from next period are added linearly with the weight for former to be a positive constant in the linear addition, and (iii) the sum of the utility of gains and losses and the certainty equivalent of the total utility from next period is aggregated with the individuals’ consumption in the current period via an aggregation function, resulting in the individuals’ total utility at the beginning of the current period. The certainty equivalent and aggregation function are chosen so that the so-called relative risk aversion degree (RRAD) and elasticity of intertemporal substitution (EIS) are constant, and the time horizon is usually set to be infinite.

The BH model is used by Barberis et al. (2006) to explain individuals’ attitude towards some monetary gambles that cannot be easily explained by many models of preferences and attitude towards large gambles such as non-participation into the stock market. Moreover, this model and its variants have been extensively employed in the literature of portfolio selection and asset pricing; see for instance Barberis and Huang (2009), De Giorgi and Legg (2012), He and Zhou (2014), and Easley and Yang (2015).

Guo and He (2016) show that the BH model is not well behaved in that the total utility process in this model may not uniquely exist. Indeed, Guo and He (2016) show that when (i) an agent consumes a constant fraction of her wealth and invests another constant fraction of her wealth into some assets whose returns are independent and identically distributed (i.i.d.) over time, (ii) the EIS is less than or equal to one, and (iii) the utility of gains and losses experienced by the agent is negative, it is either the case in which the total utility process of the agent in the BH model does not exist or the case in which there exists two solutions to the recursive equation that defines the total utility process. Recall that in many applications of the BH model, EIS indeed takes a value that is less than one and the utility of gains and losses is negative; see for instance Barberis and Huang (2009) and De Giorgi and Legg (2012). Moreover, in those applications, the agents also consume constant fractions of their wealth and the asset returns are i.i.d. Thus, the ill-behaved property for the BH model that is observed by Guo and He (2016) poses a challenge for the applicability of this model. To solve this issue, Guo and He (2016) then propose a new
model that features narrow framing, referred to as the GH model, in which the total utility process uniquely exists. In this new model, the utility of gains and losses and the certainty equivalent of the individuals’ total utility from next period are also added linearly, but instead of a constant, the weight for the utility of gains and losses is scaled in a sense that it is proportional to the certainty equivalent of the total utility from next period per unit wealth.

The present paper continues the work of Guo and He (2016) to further study the GH model and compare it with the BH model. We first show that even in the finite-horizon setting, the BH model may fail to define the total utility process. Intuitively, the failure of the BH model arises from the restriction that the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period must be nonnegative. Imagine that in the BH model, because the weight for the utility of gains and losses is a constant, the magnitude of a negative utility of gains and losses can dominate the certainty equivalent of the total utility from next period, and thus the sum of these two can be negative. In the GH model, the weight for the utility of gains and losses is proportional to the the certainty equivalent of the total utility from next period per unit wealth, so in the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period, the latter always dominates the former, ensuring that the aggregation is nonnegative. Thus, the GH model defines a unique total utility process whether the time horizon is finite or infinite. Moreover, the GH model in the finite-horizon setting with any given utility at the terminal time converges to that in the infinite-horizon setting as the number of periods in the former goes to infinity. This results also implies that by starting from any positive value and applying the recursive equation in the GH model repeatedly, we can obtain, as a limit, the total utility process in the infinite-horizon setting.

When asset returns are i.i.d. and portfolio and consumption strategies are constant, the total utility per unit wealth in the GH model becomes a constant and thus the weight for the utility of gains and losses in the GH model is also a constant. The total utility process in the GH model, however, cannot be computed from the BH model using a recursive algorithm no matter how one chooses the weight for the utility of gains and losses in the BH model.

Guo and He (2016) show that when the GH model is applied to portfolio selection, the resulting dynamic programming equation admits a unique solution. Moreover, this solution can be computed by applying the equation repeatedly with any starting point. For the BH model, one can also derive a dynamic programming equation heuristically.
We show that this equation can have multiple solutions. When applying the dynamic programming equation repeatedly, different starting points can lead to different solutions. Even worse, different solutions to the equation lead to different portfolios solved from the dynamic programming equation.

Finally, we apply the GH model to study individuals’ attitudes toward risk and portfolio selection and asset pricing implications. We first show that similar to the BH model, the GH model can explain an aversion to a small, independent, actuarially favorable gamble and acceptance of a large, independent gamble over a reasonable range of wealth levels. We then apply to the GH model to portfolio selection and find that it can explain why many households do not participate in the stock market. We also apply the GH model to asset pricing in a production-consumption economy that is studied in Barberis and Huang (2009) and find that this model can generate large equity premia. In all these applications, the GH model is more tractable than the BH model.

The remainder of the paper is organized as follows: In Section 2 we review the BH and GH models and show in various aspects why the GH model is better behaved than the BH model. In Section 3, we apply the GH model to study individuals’ attitudes toward timeless gambles. In Section 4, we show that the GH model can explain why many households do not participate in the stock market. In Section 5, we study the asset pricing implication of the GH model. Section 6 concludes. In Appendix A, we review the existence and uniqueness of the total utility process in the GH model and of the solution to the dynamic programming equation when this model is applied to portfolio selection in a general Markovian setting. All proofs are placed in Appendix B.
2 A New Preference Model that Allows for Narrow Framing

2.1 BH Model, the Existing Model That Allows for Narrow Framing

Consider an agent who consumes $C_t$ at time $t$. Denote $U_t$ as the total utility of the agent from time $t$. In the BH model, the agent’s total utility is defined recursively by

$$U_t = H \left( C_t, M_t(U_{t+1}) + \sum_{i=1}^{n} \bar{b}_i G_{i,t} \right), \quad (2.1)$$

where

$$H(c,z) := \begin{cases} (1 - \beta)c^{1-\rho} + \beta z^{1-\rho} \frac{1}{1-\rho}, & 0 < \rho \neq 1, \\ e^{(1-\beta)\ln c + \beta \ln z}, & \rho = 1 \end{cases} \quad (2.2)$$

is an aggregation function with $1/\rho$ and $\beta \in (0, 1)$ standing for the EIS and discount rate of the agent, respectively,

$$M_t(X) := u^{-1}(E_t[u(X)]), \quad u(x) := \begin{cases} x^{1-\gamma}/(1 - \gamma), & 0 < \gamma \neq 1, \\ \ln(x), & \gamma = 1 \end{cases} \quad (2.3)$$

stands for the certainty equivalent of $X$ with $\gamma$ representing the RRAD of the agent, $G_{i,t}$ stands for the agent’s utility for risk $i$ that is evaluated in isolation from other risks, such as consumption and investment risk, and $\bar{b}_i$ is a nonnegative constant.

Here and hereafter, $E_t$ and $M_t$ stand for the expectation and certainty equivalent, respectively, that are computed based on the information at time $t$. If $X$ is independent of the information at time $t$, we simply drop the subscript $t$ when calculating its expectation and certainty equivalent. In the following, when $\rho \geq 1$, we set $H(c,0) = \lim_{z\downarrow 0} H(c,z) = 0$ and $H(0,z) = \lim_{c\downarrow 0} H(c,z) = 0$. As a result, $H(c,z)$ is well defined, takes real values, and continuous in $(c,z) \in [0,\infty)^2$. Similarly, when $\gamma \geq 1$, we define $u(0) := -\infty$ and $u^{-1}(-\infty) := 0$. As a result, $M_t(X)$ is well defined for any nonnegative random variable $X$. Moreover, when $\gamma \geq 1$ and $X = 0$ with a positive probability, $M_t(X) = 0$.

Examples of risks that are evaluated in isolation include a monetary gamble that is
offered to the agent and the gain and loss incurred by holding a stock. Barberis and Huang (2009) assume that the agent employs prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992) to evaluate these risks. More precisely, suppose the gain and loss experienced in one of these risks is $X$, then the utility for taking this risk is $\mathbb{E}_t[\nu(X)]$, where

$$\nu(x) := x \mathbb{1}_{\{x \geq 0\}} + \lambda x \mathbb{1}_{\{x < 0\}}$$

(2.4)

for some $\lambda \geq 1$ that represents the loss aversion degree (LAD) of the agent.

Guo and He (2016) show that the BH model is not well behaved in that the total utility process in this model may not uniquely exist. Suppose the agent consumes a constant fraction $c$ of her wealth, so the remaining $(1 - c)$ fraction of her wealth is used for investment. Suppose $\theta$ fraction of her investment is in a stock and the remaining is in a risk-free asset. Suppose the agent frames the investment in the stock separately and uses the risk-free return as a reference point to calculate the gain and loss she experiences from holding the stock. Denote $W_t$ as the agent’s wealth at time $t$. Denote the gross return rate of the stock in period $t$ to $t + 1$ as $R_{S,t+1}$ and assume $R_{S,t+1}$’s to be i.i.d. Assume the gross return rate of the risk-free asset is constant over time and denote it as $R_f$. Then, the agent’s gain and loss for holding the stock in period $t$ to $t + 1$ is $(1 - c)W_t \theta (R_{t+1} - R_f)$ and the resulting utility is

$$G_t = \mathbb{E}_t \left[ \nu((1 - c)W_t \theta (R_{S,t+1} - R_f)) \right] = (1 - c)W_t \mathbb{E}_t \left[ \nu(\theta (R_{S,t+1} - R_f)) \right].$$

In consequence, the agent’s total utility process $\{U_t\}$ in the BH model for this consumption and investment strategy is defined recursively by

$$U_t = H(cW_t, \mathcal{M}_t (U_{t+1}) + \bar{b}(1 - c)W_t \mathbb{E}_t \left[ \nu(\theta (R_{S,t+1} - R_f)) \right])$$

(2.5)

for some constant $\bar{b} > 0$. Dividing both sides of the above equation by $W_t$ and recalling that $W_{t+1} = (1 - c)W_t (R_f + \theta (R_{S,t+1} - R_f))$, we obtain

$$\frac{U_t}{W_t} = H\left( c, (1 - c)\mathcal{M}_t \left( \frac{U_{t+1}}{W_{t+1}} R_{p,t+1} \right) + \bar{b}(1 - c)\mathbb{E}_t \left[ \nu(\theta (R_{S,t+1} - R_f)) \right] \right),$$

(2.6)

where $R_{p,t+1} := R_f + \theta (R_{S,t+1} - R_f)$ stands for the gross return of the agent’s portfolio in period $t$ to $t + 1$. 
Because the fraction of wealth for consumption and the fraction of wealth for investment in
the stock are constant over time, the risk-free return rate is constant over time, and the
stock return rates are i.i.d. over time, it is expected that $U_t/W_t$ is a constant over time
and we denote this constant as $\Psi$. Denote

$$\delta := (1 - c)\mathcal{M}_t(R_{p,t+1}), \quad \zeta := (1 - c)\mathbb{E}\left[\nu(\theta(R_{S,t+1} - R_f))\right]/c. \quad (2.7)$$

Then, we conclude from (2.6) that

$$\Psi = H(c, \delta \Psi + \bar{c}\bar{b}\zeta). \quad (2.8)$$

Guo and He (2016, Theorem 1) show that when $\beta \delta^{1-\rho} < 1$ and $\zeta \geq 0$, (2.8) has a unique
solution in $(0, +\infty)$. Moreover, starting from any positive number for $\Psi$ and applying the
recursive equation (2.8) repeatedly, the resulting sequence converges to the solution to this
equation. When $\zeta < 0$, however, Guo and He (2016, Example 1) show that the solution
to (2.8) can be nonunique or nonexistent even when $\beta \delta^{1-\rho} < 1$. The following proposition
expands the observation therein.

**Proposition 1** Suppose $\beta \delta^{1-\rho} < 1$ and $\zeta < 0$. Then, the right-hand side of (2.8), denoted
as $\mathbb{V}(\Psi)$, is well defined only for $\Psi \geq -\bar{c}\bar{b}\zeta/\delta$.

(i) If $\rho < 1$ and $-\bar{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)}$, then there exists a unique solution, denoted as $\Psi^*$, to (2.8) in $[-\bar{c}\bar{b}\zeta/\delta, +\infty)$.

(ii) If $\rho < 1$ and $(1 - \beta)^{1/(1-\rho)} < -\bar{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)} \left[1 - (\beta \delta^{1-\rho})^{1/\rho}\right]^{-\rho/(1-\rho)}$, then there exists two solutions, denoted as $\Psi^*_1 < \Psi^*_2$, to (2.8) in $(-\bar{c}\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-\bar{c}\bar{b}\zeta/\delta, \Psi^*_1)$, there exists positive integer $n_0$ such that $\mathbb{V}^{n_0}(\Psi) < -\bar{c}\bar{b}\zeta/\delta$; for any $\Psi > \Psi^*_1$, $\{\mathbb{V}^n(\Psi)\}_{n \geq 1}$ converges to $\Psi^*_2$.

(iii) If $\rho < 1$ and $-\bar{b}\zeta/\delta > (1 - \beta)^{1/(1-\rho)} \left[1 - (\beta \delta^{1-\rho})^{1/\rho}\right]^{-\rho/(1-\rho)}$, then there does not exist any solution to (2.8) in $[-\bar{c}\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-\bar{c}\bar{b}\zeta/\delta, +\infty)$, there exists positive integer $n_0$ such that $\mathbb{V}^{n_0}(\Psi) < -\bar{c}\bar{b}\zeta/\delta$.

(iv) If $\rho \geq 1$ and $-\bar{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)} \left[1 - (\beta \delta^{1-\rho})^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1 - \beta)(\beta \delta)^{\beta/(1-\beta)}$ when $\rho = 1$, then there exists two solutions, denoted as $\Psi^*_1 < \Psi^*_2$, to (2.8) in $(-\bar{c}\bar{b}\zeta/\delta, +\infty)$. Moreover, for any $\Psi \in (-\bar{c}\bar{b}\zeta/\delta, \Psi^*_1)$, there
exists positive integer \( n_0 \) such that \( V^{n_0}(\Psi) < -c\bar{b}\zeta/\delta \); for any \( \Psi > \Psi_1^\ast \), \( \{V^n(\Psi)\}_{n \geq 1} \) converges to \( \Psi_2^\ast \).

(v) If \( \rho \geq 1 \) and \(-\bar{b}\zeta/\delta > (1 - \beta)^{1/(1 - \rho)} \left[ 1 - (\beta\delta^{1-\rho})^{1/\rho} \right]^{-\rho/(1-\rho)}\), where the right-hand side is defined to be \((1 - \beta)(\beta\delta)^{\beta/(1-\beta)}\) when \( \rho = 1 \), then there does not exist any solution to (2.8) in \([-c\bar{b}\zeta/\delta, +\infty)\). Moreover, for any \( \Psi \in (-c\bar{b}\zeta/\delta, +\infty) \), there exists positive integer \( n_0 \) such that \( V^{n_0}(\Psi) < -c\bar{b}\zeta/\delta \).

Figure 2.1: Plots of \( V(\Psi) = H(c, \delta\Psi + c\bar{b}\zeta) \) when \( \beta\delta^{1-\rho} < 1 \) and \( \zeta < 0 \). The top three panels correspond to cases (i)–(iii) in Proposition 1, respectively, and the bottom two panels correspond to cases (iv) and (v) in Proposition 1, respectively. In each panel, the solid line stands for \( V(\Psi) \), the dashed diagonal line represents the identity function, and the dash-dotted vertical line indicates that the domain of \( V \) is \([-c\bar{b}\zeta/\delta, +\infty)\).

Proposition 1 is illustrated by Figure 2.1, with the five panels from top to bottom and from left to right representing cases (i)–(v) in the proposition, respectively. Proposition 1 confirms the finding in Guo and He (2016, Example 1) that the solution to (2.8) can be nonunique or nonexistent when \( \zeta < 0 \). In particular, when \( \rho \geq 1 \), for any negative value of \( \zeta \) and any positive value of \( \bar{b} \), it is either the case in which the solution to (2.8) does not
exist or the case in which the solution is not unique.\(^1\) Thus, the BH model fails to define the total utility process of an agent who derives a negative utility, i.e., a disutility, of gains and losses and whose EIS is less than or equal to one, no matter how small the disutility is and no matter how small the weight of the utility of gains and losses is. This failure is essential for the BH model because many applications of the BH model in the literature, such as those in Barberis et al. (2006) and Barberis and Huang (2009), assume EIS to be less than one and a disutility of the gain and loss incurred by holding a stock.

One may argue that when an agent derives a sufficiently negative utility of gains and losses for a strategy, the nonexistence of her total utility process means that the strategy is not favorable. When the disutility of gains and losses is sufficiently small, Proposition 1-(iv) shows that with \(\rho \geq 1\), the total utility process exists but is not unique. In this case, one possible remedy is to simply choose one of the utility processes, such as the maximal. Such choice, however, seems to be arbitrary. Furthermore, Proposition 1-(iv) shows that applying the recursive equation (2.8) repeatedly does not necessarily lead to its solution. Even worse, Proposition 1 shows that for any given negative utility of gains and losses, even in a finite-horizon setting, the agent’s total utility may not exist. Indeed, consider the total utility process defined by (2.5) in \(n + 1\) periods. Recall \(\zeta, \delta, \nabla, \) and \(\Psi_i^*, i = 1, 2\) in Proposition 1-(iv). Suppose that at the terminal time, \(U_T = \Psi W_T\) for some \(\Psi \in (-c\bar{b}\zeta/\delta, \Psi_1^*)\). Then, the agent’s total utility at time 0 is \(W_0 \nabla^{n+1}(\Psi)\). Proposition 1-(iv) shows that \(\nabla^{n_0}(\Psi) < -c\bar{b}\zeta/\delta\), so \(\nabla^{n_0+1}(\Psi)\) does not exist. In consequence, when \(n \geq n_0\), the agent’s total utility at time 0 does not exist.

Note that the BH model has been used in the literature even though the total utility process in this model does not necessarily exist. For instance, in Barberis et al. (2006) and Barberis and Huang (2009), \(\rho\) is set to be larger than one and the utility of gains and losses is negative. Typically, the utility per unit capital is solved from (2.8) by starting from some \(\Psi > 0\) and applying the right-hand side of the equation repeatedly. The successful application of the BH model in Barberis et al. (2006) and Barberis and Huang (2009) implies that the authors set \(\bar{b}\) sufficiently small so that case-(iv) of Proposition 1 is in effect. Proposition 1-(iv) then implies that for the recursive algorithm to work, the authors must choose starting point \(\Phi > \Phi_1^*\) and the resulting solution to (2.8) must be \(\Phi_2^*\).

\(^1\)Here, to simplify exposition, we exclude a marginal case in which \(\bar{b}\zeta = -\delta(1 - \beta)^{1/(1-\rho)} \left[1 - (\beta(1-\rho)^{1/\rho})^{1/(1-\rho)}\right]^{\rho/(1-\rho)}\), where the right-hand side is defined to be \((1 - \beta)(\beta\delta)^{\beta/(1-\beta)}\) when \(\rho = 1\). In this case, (2.8) has a unique solution, \(\Psi^*\), in \((-c\bar{b}\zeta/\delta, +\infty)\). Moreover, for any \(\Psi \in (-c\bar{b}\zeta/\delta, \Psi^*)\), there exists positive integer \(n_0\) such that \(\nabla^{n_0}(\Psi) < -c\bar{b}\zeta/\delta\); for any \(\Psi \geq \Psi^*\), \(\nabla^n(\Psi)\) for \(n \geq 1\) converges to \(\Psi^*\).
2.2 GH Model, A New Preference Model That Allows for Narrow Framing

Intuitively, the failure of the BH model arises from the restriction that the aggregation of the utility of gains and losses and the certainty equivalent of the total utility from next period must be nonnegative. In the BH model, the weight for the utility of gains and losses is constant over time. If the utility of gains and losses derived by the agent is constant over periods and its magnitude dominates the utility of immediate consumption in every period, then the agent’s total utility is reduced by a constant amount in every period. In consequence, the sum of the utility of gains and losses and the certainty equivalent of the total utility from next period can eventually become negative, and thus the total utility fails to exist.

The GH model that is proposed by Guo and He (2016) resolves the issue of nonexistence or nonuniqueness of the total utility process in the BH model by modeling the weight for the utility of gains and losses to be proportional to the certainty equivalent of the total utility per unit wealth from next period. More precisely, the agent’s total utility process \( \{U_t\} \) in the GH model is defined recursively by

\[
U_t = H \left( C_t, M_t(U_{t+1}) + \sum_{i=1}^{n} b_i \frac{M_t(U_{t+1})}{M_t(W_{t+1})} G_{i,t} \right),
\]

(2.9)

where \( b_i \geq 0 \) is a constant, \( i = 1, \ldots, n \). Alternatively, (2.9) can be written as

\[
U_t = H \left( C_t, \left( 1 + \sum_{i=1}^{n} \frac{b_i}{M_t(W_{t+1})} G_{i,t} \right) M_t(U_{t+1}) \right).
\]

(2.10)

This reformulation shows that in the BH model experiencing a loss (a gain) in risks that are evaluated separately reduces (increases) the certainty equivalent of the total utility proportionally.

In Appendix A, we review some results obtained by Guo and He (2016): In a finite-state Markovian setting in which randomness is driven by a finite-state Markov process \( \{X_t\} \) representing the market state and by an independent time series \( \{Y_t\} \) representing random shocks, the total utility process \( \{U_t\} \) in the GH model uniquely exists provided
that a growth condition holds and that
\[ M_t(W_{t+1}) + \sum_{i=1}^{n} b_i G_{i,t} > 0. \]

See Theorem 1 in Appendix A. The latter condition stipulates that the agent’s disutility of the losses experienced in risks in certain period that are evaluated separately cannot exceed the certainty equivalent of the agent’s wealth at the end of the same period. In consequence, we can observe from (2.10) that if the agent’s total utility at time \( t + 1 \) is positive, her total utility at time \( t \) remains positive.

Theorem 1 also shows that to compute the total utility per unit wealth \( U_t/W_t \), which is a deterministic function of the market state \( f(X_t) \), one can start from any positive function of the market state and apply the right-hand side of (2.9) repeatedly, and this recursive algorithm leads to the function \( f \). This result not only provides an easy algorithm to compute the total utility process in the GH model but also implies that the total utility in the finite-horizon setting converges to that in the infinite-horizon setting when the number of periods in the former setting goes to infinity. In particular, in contrast to the BH model, the total utility process in the GH model is also well defined in the finite-horizon setting.

Barberis and Huang (2001), Barberis et al. (2001), and Li and Yang (2013) consider the following preference model for the representative agent in their equilibrium asset pricing studies:

\[ E_t \left[ \sum_{s=t}^{\infty} \left( \beta^s u(C_s) + \beta^{s+1} u'(\bar{C}_s) \sum_{i=1}^{n} \hat{b}_i G_{i,s} \right) \right], \quad (2.11) \]

where \( \hat{b}_i \)'s are constants and \( \{\bar{C}_t\} \) is the aggregate consumption process in the whole economy. The idea of scaling the weight for the utility of gains and losses in GH model is similar to using the ad-hoc factor \( u'(\bar{C}_s) \) in (2.11). Indeed, in the asset pricing models studied by Barberis and Huang (2001), Barberis et al. (2001), and Li and Yang (2013), the aggregate consumption is equal to the consumption of the representative agent. On the other hand, as argued by Barberis and Huang (2008, p.210), the model (2.11) does not admit an explicit value function and thus is difficult to use to study individuals’ attitudes towards timeless gambles and is intractable when applied to portfolio selection. The GH model, however, is tractable when applied to studying individuals’ attitudes towards timeless gambles and to portfolio selection and asset pricing; see Sections 3–5.
2.3 Connection between the GH and BH Models

Consider an agent who consumes a constant fraction $c$ of her wealth. For the remaining wealth, the agent invests $\theta$ fraction in a stock and the rest in a risk-free asset. Suppose that the agent frames the investment in the stock separately and uses the risk-free return as a reference point to calculate the gain and loss she experiences from holding the stock. Denote the gross return rate of the stock in period $t$ to $t+1$ as $R_{S,t+1}$ and assume $R_{S,t+1}$’s to be i.i.d. Assume the gross return rate of the risk-free asset is constant over time and denote it as $R_f$. Recall $\delta$ and $\zeta$ in (2.7) and assume $\beta\delta^{1-\rho} < 1$. Suppose $\rho \geq 1$ and $\zeta < 0$.

Then, Proposition 1 shows that the total utility process of the agent in the BH model is either nonexistent or nonunique if the agent frames the investment in the stock separately (i.e., if $\bar{b} > 0$).

Now, suppose the agent’s preferences are represented by the GH model. Assume $\delta + cb\zeta > 0$ and $\beta(\delta + cb\zeta)^{1-\rho} < 1$, which actually imply $\beta\delta^{1-\rho} < 1$ because $\zeta < 0$ and $\rho \geq 1$. Then, according to Theorem 1, the total utility process in the GH model uniquely exists. Moreover, the total utility per unit wealth in the GH model, denoted as $\Psi^*$, is the unique solution to

$$\Psi = H\left(c, \delta \Psi + cb\zeta \Psi\right). \quad (2.12)$$

Now, define $\bar{b} := b\Psi^*$. By comparing (2.8) and (2.12), we conclude that $\Psi^*$ is also a solution to (2.8). Then, according to Proposition 1-(iv) and -(v), one can expect that $-\bar{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)} \left[1 - (\beta\delta^{1-\rho})^{1/\rho}\right]^{-\rho/(1-\rho)}$, where the right-hand side is defined to be $(1 - \beta)(\beta\delta)^{(1-\rho)/(1-\beta)}$ when $\rho = 1$, in which case (2.8) has two solutions $\Psi_1^* < \Psi_2^*$. We want to see whether $\Psi^*$ is equal to $\Psi_1^*$ or $\Psi_2^*$.

**Proposition 2** Suppose $\rho \geq 1$, $\zeta < 0$, $\delta + cb\zeta > 0$, and $\beta(\delta + cb\zeta)^{1-\rho} < 1$. Let $\Psi^*$ be the unique solution to (2.12). Define $\bar{b} := b\Psi^*$ and recall (2.8) that defines the total utility process in the BH model.

(i) If $cb\zeta < (\delta\beta)^{1/\rho} - \delta$, then $\Psi^* = \Psi_1^*$, where $\Psi_1^* < \Psi_2^*$ are the two solutions to (2.8) in Proposition 1-(iv).

(ii) If $cb\zeta > (\delta\beta)^{1/\rho} - \delta$, then $\Psi^* = \Psi_2^*$, where $\Psi_1^* < \Psi_2^*$ are the two solutions to (2.8) in Proposition 1-(iv).
Because \( \zeta < 0 \) and \( \rho \geq 1 \), \( \beta(\delta + cb\zeta)^{1-\rho} < 1 \) implies \( \beta\delta^{1-\rho} < 1 \), i.e., implies \( \beta^{1/(\rho-1)} < (\delta\beta)^{1/\rho} \). Moreover, \( \beta(\delta + cb\zeta)^{1-\rho} < 1 \) if and only if \( cb\zeta > \beta^{1/(\rho-1)} - \delta \). Thus, both cases (i) and (ii) of Proposition 2 are non-redundant. In other words, the solution to (2.12) in the GH model can be the smaller one or the larger one of solutions to (2.8), depending on model parameters. The left and right panels of Figure 2.2 plot cases (i) and (ii), respectively, of Proposition 2. Recall that the smaller solution to (2.8) is not computable in a sense that it cannot be obtained by applying the right-hand side of (2.8) repeatedly with any starting points (unless the starting point happens to be this fixed point). Thus, the total utility per unit wealth in the GH model may not correspond to any computable total utility per unit wealth in the BH model even if this quantity is constant.

![Figure 2.2: Plots of \( H(c, \delta \Psi + cb\zeta \Psi) \) in the GH model and \( H(c, \delta \Psi + c\bar{b}\zeta) \) in the BH model with \( \bar{b} := b\Psi^* \), where \( \Psi^* \) is the unique fixed point of the former. In each of the two panels, the solid line stands for \( H(c, \delta \Psi + cb\zeta) \), the dash-dotted line stands for \( H(c, \delta \Psi + c\bar{b}\zeta \Psi) \), the dashed diagonal line stands for the identical function. The left panel plots the case in which the fixed point of \( H(c, \delta \Psi + cb\zeta \Psi) \) is the same as the smaller fixed point of \( H(c, \delta \Psi + c\bar{b}\zeta) \) and the right panel plots the case in which the fixed point of \( H(c, \delta \Psi + cb\zeta \Psi) \) is the same as the larger fixed point of \( H(c, \delta \Psi + c\bar{b}\zeta) \).](image)

2.4 Dynamic Programming

2.4.1 Dynamic Programming in the GH Model

An important application of a model of preferences is portfolio selection. Assuming a finite-state Markovian setting in which randomness is driven by a finite-state Markov process \( \{X_t\} \)
representing the market state and by an independent time series \( \{ Y_t \} \) representing random shocks, Guo and He (2016) prove that portfolio selection problems in the GH model can be solved by dynamic programming; see Theorem 2 in Appendix A.2. More precisely, the optimal utility per unit wealth is the unique solution to the dynamic programming equation (A.13), and the optimal portfolio and consumption can be obtained by solving a maximization problem; see equations (A.14)–(A.15). Moreover, the solution to the dynamic programming equation can be obtained by a recursive algorithm that starts from any positive value and then applies the equation repeatedly. This algorithm not only provides us with a method to compute the solution but also shows that the optimal portfolio and consumption in a finite-horizon portfolio selection model converge to those in an infinite-horizon model when the number of periods in the former goes to infinity.

The maximization problem we need to solve to obtain the optimal portfolio is easy to deal with when there is only one market state or when the agent’s RRAD is one; see the discussion following Theorem 2 and in particular equation (A.16). The first special case is commonly assumed in the literature. Indeed, in all their applications of the BH model in Barberis et al. (2006) and Barberis and Huang (2009), the authors assume asset returns are i.i.d. over time, which simply means that there is only one market state. In the second special case, we need to assume that the agent’s RRAD to be one. In Guo and He (2017), the authors propose a measure, referred to as implied RRAD, to compute the overall risk aversion degree of an agent whose preferences are represented by the GH model. It turns out that the implied RRAD is insensitive to the RRAD \( \gamma \) for reasonable values of \( b \), i.e., for \( b \geq 1 \); see Table 3 therein. Thus, in the GH model, the RRAD \( \gamma \) is not very critical in determining the agent’s overall risk aversion degree, so assuming it to be one should have little impact on portfolio selection and asset pricing results in this model.

2.4.2 Dynamic Programming in the BH Model

We have already showed the total utility process in the BH model can be nonexistent or nonunique. This model, however, has been applied to portfolio selection in the literature without being aware of the issue of nonexistence or nonuniqueness of the total utility process. The approach taken in the literature to portfolio selection with the BH model is to derive the dynamic programming equation heuristically and solve it numerically. In the following, we study whether this approach works.

We consider a simple economy in which an agent can invest in a risk-free asset with constant risk-free gross return \( R_f \) and in a stock with i.i.d. gross returns \( R_{S,t+1} \)’s. Thus,
it is reasonable to assume that the agent’s strategy is to consume a constant fraction \( c \) of her wealth \( W_t \) at time \( t \), invest a constant fraction \( \theta \) of the remaining wealth \((1-c)W_t\) in the stock and the remaining in the risk-free asset. Suppose the agent can choose \( c \in I \) for some compact subset \( I \) of \((0, 1)\) and \( \theta \in J \) for some compact subset \( J \) of \([0, +\infty)\) such that the portfolio return \( R_f + \theta(R_{S,t+1} - R_f) > 0 \) for any \( \theta \in J \), so short selling is not allowed. Assume the agent derives a negative utility of the gain and loss experienced by each dollar invested in the stock, in which the reference point is set to be the risk-free return; i.e., \( \mathbb{E}[\nu(R_{S,t+1} - R_f)] < 0 \). Suppose the agent’s preferences are represented by the BH model.

If we take the issue of nonexistence and nonuniqueness of the total utility process in the BH model aside, we can derive the dynamic programming equation heuristically:

\[
\Phi = \max_{c \in I, \theta \in J} \left\{ \Phi H(c, (1-c) \left( \Phi M(R_f + \theta(R_{S,t+1} - R_f)) + \bar{b}\theta \mathbb{E}[\nu(R_{S,t+1} - R_f)] \right)) \right\}, \tag{2.13}
\]

where \( \Phi \) stands for the optimal total utility per unit capital of the agent. Note that the right-hand side of \( (2.13) \) is not well defined for all \( \Phi \geq 0 \) if \( \mathbb{E}[\nu(R_{S,t+1} - R_f)] < 0 \). Noting that the maximization in \( c \) and in \( \theta \) can be separated, however, we can rewrite \( (2.13) \) as

\[
\Phi = \max_{c \in I} \left( c, (1-c) \max_{\theta \in J} \left( \Phi M(R_f + \theta(R_{S,t+1} - R_f)) + \bar{b}\theta \mathbb{E}[\nu(R_{S,t+1} - R_f)] \right) \right). \tag{2.14}
\]

Now, the right-hand side of \( (2.14) \) is well defined for any \( \Phi \geq 0 \) if \( 0 \in J \), i.e., if investing only in the risk-free asset is feasible. Thus, \( (2.14) \) appears to be a better formulation than \( (2.13) \) as the dynamic programming equation in the BH model. Moreover, the following proposition shows that \( (2.14) \) admits a solution provided that a growth condition holds.

**Proposition 3** Suppose \( 0 \in J \) and

\[
\beta \max_{c \in I, \theta \in J} \left[ (1-c)M(R_f + \theta(R_{S,t+1} - R_f)) \right]^{1-\rho} < 1. \tag{2.15}
\]

Then, the solution to \( (2.14) \) exists.

Although the dynamic programming equation \( (2.14) \) admits a solution as shown in Proposition 3, the solution is not unique in general. This is not surprising because the total utility process in the BH model for each consumption-investment strategy can be non-unique. The non-uniqueness of the solution to \( (2.14) \) can even result in non-uniqueness of the optimal portfolio solved from this equation because the maximization in \( \theta \) in this equation depends on \( \Phi \).
We provide an example to illustrate the issue of non-uniqueness of the solution to (2.14). Set $\beta = 0.5$, $\rho = 1.5$, $\gamma = 0.25$, $\lambda = 5$, $b = 10$, $R_f = 1$, and $R_{S,t+1} = \begin{cases} 10.3911, & \text{with probability 0.1,} \\ 0.78983, & \text{with probability 0.9.} \end{cases}$

Set $I = \{0.1, 0.5\}$ and $J = \{0, 3\}$. One can see that for any $\theta \in J$, the portfolio return $R_f + \theta(R_{S,t+1} - R_f) > 0$. Moreover, $\mathcal{M}(R_f + \theta(R_{S,t+1} - R_f))$ is 1 when $\theta = 0$ and is 2 when $\theta = 3$. In consequence, straightforward calculation yields that condition (2.15) holds.

Figure 2.3 plots in the solid line the right-hand side of the dynamic programming equation (2.14) as a function of $\Phi$. The intersections of this line with the dashed diagonal line, which represents the identity function, in the region $(0, +\infty)$ are the solutions to (2.14). We can see that there are three solutions: $\Phi^*_1 = 0.17157$, $\Phi^*_2 = 0.2447$, and $\Phi^*_3 = 0.30236$. Moreover, we solved that the optimal consumption-investment strategy corresponding to $\Phi^*_1$ is $c^* = 0.5, \theta^* = 0$, but that corresponding to $\Phi^*_2$ and $\Phi^*_3$ is $c^* = 0.5, \theta^* = 3$. Furthermore, from Figure 2.3, we can observe that with any starting point in $(0, \Phi^*_2)$, e.g., 0.1, applying the dynamic programming equation (2.14) repeatedly leads to $\Phi^*_1$ and thus the corresponding optimal consumption-investment strategy $c^* = 0.5, \theta^* = 0$; with any starting point in $(\Phi^*_2, +\infty)$, e.g., 0.35, however, this algorithm leads to $\Phi^*_3$ and thus the corresponding optimal consumption-investment strategy $c^* = 0.5, \theta^* = 3$. Therefore, this example shows that the dynamic programming equation in the BH model can have multiple solutions, corresponding to different portfolios. Moreover, when solving the equation using a recursive algorithm, the resulting solution depends heavily on the choice of the starting point.

2.5 Discussions on Alternative Modeling of Preferences with Narrow Framing

In this section, we discuss two alternatives to weight the utility of gains and losses, leading to two variants of the GH model. We show that each of the variants has its own issues.
Figure 2.3: Plot of $\mathbb{W}(\Phi)$, the right-hand side of the dynamic programming equation (2.14) in the BH model as a function of $\Phi$. The solid line represents $\mathbb{W}(\Phi)$ and the dashed diagonal line represents the identity function, so their intersections in $(0, +\infty)$ represent the solutions to (2.14). The parameter values are as follows: $\beta = 0.5$, $\rho = 1.5$, $\gamma = 0.25$, $\lambda = 5$, $b = 10$, $R_f = 1$, $R_{S,t+1} = 10.3911$ with probability 0.1 and $R_{t+1} = 0.78983$ with probability 0.9, $I = \{0.1, 0.5\}$, and $J = \{0, 3\}$. There are three intersection points: $\Phi_1^* = 0.17157$, $\Phi_2^* = 0.2447$, and $\Phi_3^* = 0.30236$, corresponding to optimal consumption-investment strategies $(c^* = 0.5, \theta^* = 0)$, $(c^* = 0.5, \theta^* = 3)$, and $(c^* = 0.5, \theta^* = 3)$, respectively.

2.5.1 Alternative I

The first variant of the GH model, referred to as GH-I model, is as follows

$$U_t = H \left( C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^n b_i \mathcal{M}_t \left( \frac{U_{t+1}}{W_{t+1}} \right) G_{i,t} \right).$$

(2.16)

Note that when $U_t/W_t$ is a constant in $t$ or when $\gamma = 1$, we have

$$\mathcal{M}_t \left( \frac{U_{t+1}}{W_{t+1}} \right) = \frac{\mathcal{M}_t(U_{t+1})}{\mathcal{M}_t(W_{t+1})},$$

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showing that the GH-I model is the same as the GH model in this case. In general, however, these two models can be different.

Appendix A.3 shows that if we formulate the dynamic programming equation for the GH-I model heuristically, the resulting equation is the same as that in the GH model when there is only one market state or when the agent’s RRAD is one. In general, the dynamic programming equations in the GH and GH-I models can be different, and the latter appear to be simpler because the maximization problem therein for deriving the optimal portfolio is easier to solve. However, in contrast to the GH model, we do not know whether the solution to the dynamic programming equation in the GH-I model uniquely exists. Even worse, we do not know whether the total utility process in the GH-I model uniquely exists.

2.5.2 Alternative II

The second variant of the GH model, referred to as GH-II model, is as follows

\[ U_t = H \left( C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i \frac{\mathcal{M}_t(U_{t+1})}{W_t - C_t} G_{i,t} \right). \]  

Compared to the GH model, we replace \( \mathcal{M}_t(U_{t+1})/\mathcal{M}_t(W_{t+1}) \), the scaling factor in the weight for the utility of gains and losses, with \( \mathcal{M}_t(U_{t+1})/(W_t - C_t) \).

We can prove that the total utility process in the GH-II model uniquely exists and the solution to the dynamic programming equation in a portfolio selection problem with the GH-II model uniquely exists. Furthermore, a standard recursive algorithm can be used to find the total utility process and the solution to the dynamic programming equation with any positive starting point. However, when there is only one market state or when the agent’s RRAD is one, the resulting dynamic programming equation in the GH-II model is more complicated than in the GH model. In general, the dynamic programming equation in the GH-II model seems to be simpler; see a full discussion in Appendix A.4.

3 Attitudes towards Timeless Gambles

One of the important implications for the BH model is to explain individuals’ attitudes toward timeless gambles that are not easily explained by other models of preferences under risk. In this section, we show that the GH model is also able to explain these attitudes. Moreover, the GH model turns out to be more tractable than the BH model in explaining
these attitudes.

3.1 Model

Following Barberis et al. (2006) and Barberis and Huang (2009), we suppose the current time is $t$ and the agent faces a timeless gamble that is a 50:50 bet to gain $x$ and lose $y$. Denote the payoff of this gamble as $\xi$ and assume it to be independent of other risks. Denote the agent’s wealth at time $t$ as $W_t$.

Following Barberis et al. (2006) and Barberis and Huang (2009), we assume that the agent is offered the gamble before she decides how much to consume and invest at time $t$ and consider three possible approaches for the agent to evaluate the gamble.

In the first approach, the outcome of the timeless gamble is revealed in an infinitesimal time period $\Delta t$, and other risks have not been revealed in this period. The agent then applies a recursive equation over this time step to calculate her utility at $t$. In this case, the agent must prefer to decide the consumption and investment strategy after the gamble outcome is revealed, so the agent makes the first consumption immediately after $t + \Delta t$.

Denote the agent’s wealth at time $t + \Delta t$ as $W_{t+\Delta t}$. After deciding whether to accept the timeless gamble, the agent needs to decide her consumption propensity $c_s$ (i.e., the fraction of her wealth that is used for consumption) and percentage allocation to $n$ risky assets $\theta_s = (\theta_{1,s}, \ldots, \theta_{n,s})^\top$, $s = t + \Delta t, t + 1, t + 2, \ldots$. Because $\Delta t$ is infinitesimally small and the payoff of the timeless gamble is independent of other risks, the agent’s optimal consumption propensity and percentage allocation to risky assets and optimal utility per unit wealth are independent of whether the agent accepts the gamble; indeed, these variables should be dependent only on market state variables. Denote the optimal utility per unit wealth at time $t + \Delta t$ as $\Phi_{t+\Delta t}$. Then, the agent’s optimal utility at $t + \Delta t$ becomes $W_{t+\Delta t}\Phi_{t+\Delta t}$.

Applying the recursive equation over this time step, we obtain the agent’s utility at $t$, if she accepts the gamble, as follows:

$$\tilde{H} \left(0, M_t(W_{t+\Delta t}\Phi_{t+\Delta t}) + b\frac{M_t(W_{t+\Delta t}\Phi_{t+\Delta t})}{M_t(W_{t+\Delta t})}G(\xi)\right),$$

where $G(\xi) = E_t[\nu(\xi)] = (x - \lambda y)/2$, $b \geq 0$ is a constant, and $\tilde{H}$ is an aggregator. Here, $\tilde{H}$ differs from $H$ because the agent is discounting and aggregating utilities in a smaller time period $\Delta t$. We assume that $\tilde{H}$ is strictly increasing, homogeneous, and $\tilde{H}(0, z) > 0$ for any $z > 0$; the particular form of $\tilde{H}$ does not matter.
Because the only uncertainty that is resolved in the period \( t \) to \( t + \Delta t \) is the outcome of the timeless gamble and the agent’s optimal utility per unit wealth does not depend on this outcome, we conclude that \( \Phi_{t+\Delta t} \) is known given information at \( t \). Then, we conclude, together with the homogeneity of \( \tilde{H} \), that the agent’s total utility at time \( t \) is

\[
\Phi_{t+\Delta t} \tilde{H}(0, \mathcal{M}_t(W_{t+\Delta t}) + bG(\xi)) = \Phi_{t+\Delta t} \tilde{H}(0, \mathcal{M}_t(W_t + \xi) + bG(\xi)).
\]

Similarly, if the agent does not accept the gamble, her total utility at \( t \) is

\[
\Phi_{t+\Delta t} \tilde{H}(0, \mathcal{M}_t(W_t)) = \Phi_{t+\Delta t} \tilde{H}(0, W_t).
\]

In consequence, the agent accepts the gamble if and only if

\[
\mathcal{M}_t \left( 1 + \frac{\xi}{W_t} \right) + bG \left( \frac{\xi}{W_t} \right) > 1. \tag{3.1}
\]

Note that condition (3.1) differs from eq. (28) in Barberis and Huang (2009) in that it is independent of the optimal utility per unit wealth \( \Phi_{t+\Delta t} \) and thus is independent on the consumption and investment opportunities faced by the agent. With such independence, we do not need to assume any model for the agent’s consumption and investment when studying her attitude towards timeless gambles that are evaluated immediately and separately from other risks. In this sense, the GH model is more tractable than the BH model.

In the second approach, the gamble outcome is revealed at time \( t + 1 \) and thus the agent evaluates the gamble over the same time interval she uses to evaluate her other risks, i.e., at time \( t + 1 \). In consequence, if the agent accepts the gamble, she needs to decide how much to consume and invest at time \( t \) before observing the gamble outcome at time \( t + 1 \). Denote \( W_{t+1} \) and \( \Phi_{t+1} \) as the wealth and optimal utility per unit wealth of the agent at time \( t + 1 \), respectively. Again, \( \Phi_{t+1} \) is independent of whether the agent accepts the gamble because the gamble outcome is independent of other risks. As in Barberis and Huang (2009), we assume in this case that the agent takes a fixed portfolio over time that generates return series \( R_{p,s+1} \) in period \( s \) to \( s + 1 \), \( s \geq t \) and she does not derive any utility of gains and losses from the portfolio return. Then, if the agent accepts the gamble and consumes \( c_t W_t \),
at time $t$, her utility becomes
\[
H(c_t W_t, M_t (((1 - c_t) W_t R_{p,t+1} + \xi) \Phi_{t+1})) + b \frac{M_t(W_{t+1} \Phi_{t+1})}{M_t(W_{t+1})} G(\frac{\xi}{W_t}) = W_t H(c_t, M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})),
\]
The agent then maximizes the utility by choosing $c_t$ optimally. Similarly, if the agent does not accept the gamble and consumes $c_t W_t$ at time $t$, her utility becomes
\[
H(c_t W_t, M_t (((1 - c_t) W_t R_{p,t+1} + \xi) \Phi_{t+1})) = W_t H(c_t, M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})),
\]
Therefore, the agent accepts the gamble if and only if
\[
\max_{c_t} H(c_t, M_t (((1 - c_t) W_t R_{p,t+1} + \xi) \Phi_{t+1})) > \max_{c_t} H(c_t, M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})).
\]
If we further assume that $R_{p,s+1}, s \geq t$ are i.i.d., $\Phi_{t+1}$ becomes a constant $\Phi$, so the agent accepts the gamble if and only if
\[
\max_{c_t} H(c_t, \Phi M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})) > \max_{c_t} H(c_t, \Phi M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})).
\]
Moreover, because we assume that $R_{p,s+1}, s \geq t$ are i.i.d., the agent’s optimal total utility per unit wealth should be a constant over time if she does not accept the gamble. In consequence, $\Phi$ can be solved in this case by
\[
\Phi = \max_{c_t} H(c_t, \Phi M_t (((1 - c_t) R_{p,t+1} + \xi) \Phi_{t+1})).
\]
Rigorous establishment of the above dynamic programming equation can be found in Corollary 1.

In the third approach, the gamble outcome is also revealed at time $t + 1$ and thus the agent also evaluates the gamble at time $t + 1$ as in the second case, but the agent makes a portfolio decision and, as a consequence of narrow framing, she derives a utility of gains and losses from some of her investment in risky assets. In consequence, if agent accepts the
gamble, consumes \( c_t W_t \) at time \( t \), and invests \( \theta_{i,t}(1 - c_t)W_t \) in risky asset \( i, i = 1, \ldots, n \), her total utility at time \( t \) is

\[
H \left( c_t W_t, M_t \left( (1 - c_t)W_tR_{p,t+1} + \xi \right) \Phi_{t+1} \right)
\]

\[
+ \frac{M_t(W_{t+1} \Phi_{t+1})}{M_t(W_{t+1})} \left[ bG(\xi) + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1 - c_t)W_tR_{i,t+1}) \right]
\]

\[
= W_t H \left( c_t, M_t \left( (1 - c_t)R_{p,t+1} + \frac{\xi}{W_t} \right) \Phi_{t+1} \right)
\]

\[
+ \frac{M_t(W_{t+1} \Phi_{t+1})}{M_t(W_{t+1})} \left[ bG(\frac{\xi}{W_t}) + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1 - c_t)R_{i,t+1}) \right]
\],

where \( R_{p,t+1} := (1 - \sum_{i=1}^{n} \theta_{i,t}) R_{f,t+1} + \theta_t^\top R_{t+1} \) stands for the gross return of the agent’s portfolio in period \( t \) to \( t+1 \). Similarly, if the agent does not accept the gamble, consumes \( c_t W_t \) at time \( t \), and invests \( \theta_{i,t}(1 - c_t)W_t \) in risky asset \( i, i = 1, \ldots, n \), her utility at time \( t \) is

\[
W_t H \left( c_t, M_t \left( (1 - c_t)R_{p,t+1} \Phi_{t+1} \right) + \frac{M_t(W_{t+1} \Phi_{t+1})}{M_t(W_{t+1})} \left[ \sum_{i=1}^{n} b_i G(\theta_{i,t}(1 - c_t)R_{i,t+1}) \right] \right).
\]

We further assume that \( R_{s+1} := (R_{1,s+1}, \ldots, R_{n,s+1}), s \geq t \) are i.i.d. and \( R_{f,s+1}, s \geq t \) are constant. Then \( \Phi_{s+1}, s \geq t \) becomes a constant \( \Phi \) that can be solved by

\[
\Phi = \max_{c_t, \theta_t} H \left( c_t, \Phi \left[ M_t \left( (1 - c_t)R_{p,s+1} \right) + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1 - c_t)R_{i,t+1}) \right] \right);
\]

see Corollary 1. In consequence, the agent accepts the gamble if and only if

\[
\max_{c_t, \theta_t} H \left( c_t, \Phi \left[ M_t \left( (1 - c_t)R_{p,t+1} + \frac{\xi}{W_t} \right) + bG(\frac{\xi}{W_t}) \right. \right.
\]

\[
\left. + \sum_{i=1}^{n} b_i G(\theta_{i,t}(1 - c_t)R_{i,t+1}) \right] > \Phi. \tag{3.3}
\]
3.2 Examples

3.2.1 Example I

We first consider an example presented in Barberis and Huang (2009): an agent who, at time $t$, has wealth of $500,000, is offered a timeless gamble, a 50:50 bet to gain $200 or lose $100 and the gamble outcome is independent of other risks. Suppose $\beta = 0.98, \gamma = \rho = 1.5$.

As discussed in Section 3.1, there are three possible approaches in which how the agent frames the gamble. In the first approach, the agent accepts the gamble if and only if (3.1) holds. Notes that this condition does not depend on the agent’s consumption and investment strategy. In the second approach, the agent accepts the gamble if and only if (3.2) holds. Following Barberis and Huang (2009), we assume $R_{p,t+1}$ to be i.i.d. over time and $\ln R_{p,t+1}$ to follow normal distribution with mean 4% and standard deviation 3%. In the third approach, the agent accepts the gamble if and only if (3.3) holds. Note that this approach has not been discussed in the literature. We assume that there is only one risky asset to invest (i.e., $n = 1$) and set $b_1 = b$. Assume that $R_{1,t+1}$ is i.i.d. over time and $\ln R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%. Set $R_{f,t+1} = 1.027449$ so that with $b = 5$ and $\lambda = 3$, when the agent does not accept the gamble, her optimal percentage allocation to the stock is 100% and consequently the gross return of her optimal portfolio is the same as that in the second approach.

The left, middle, and right panels of Figure 3.1 show the ranges of the values of $b$ and $\lambda$ for which the agent rejects the gamble using the first, second, and third approaches, respectively. The ranges are highlighted by + signs. Figure 3.1 shows that the specific approach the agent uses to frame the gamble has little effect on whether she rejects the gamble. Moreover, with reasonable parameter values, i.e., with $\lambda \geq 2.5$ and $b \geq 2$, the agent rejects the gamble. Our results are consistent with those in Barberis and Huang (2009) using the BH model. We also compute the ranges of the values of $b$ and $\lambda$ for different values of the standard deviation of $\ln R_{p,t+1}$, such as 17.32%, and the results are almost the same.

3.2.2 Example II

We then consider the example presented in Barberis et al. (2006, p. 1071). In this example, we consider two gambles. The first gamble, denoted as $G_S$, is a 50:50 bet to gain $550 and lose $500. The second gamble, denoted as $G_L$, is a 50:50 bet to gain $20,000,000 and lose $10,000. As argued by Barberis et al. (2006), it is reasonable to posit that individuals
Figure 3.1: Ranges of the values of $b$ and $\lambda$ for which the agent rejects a timeless gamble that is a 50:50 bet to gain $200 or lose $100 and whose outcome is independent of other risks that are faced by the agent. The left, middle, and right panels correspond to three different approaches used by the agent to frame the timeless gamble, in which the agent rejects the gamble if and only if (3.1), (3.2), and (3.3) hold, respectively. We set the agent’s current wealth to be $500,000 and $\beta = 0.98$, $\gamma = \rho = 1.5$. In the second approach, we assume $R_{p,t+1}$ to be i.i.d. over time and $\ln R_{p,t+1}$ to follow normal distribution with mean 4% and variance 3%. In the third approach, we assume that there is only one risky asset to invest (i.e., $n = 1$), set $b_1 = b$, assume that $R_{1,t+1}$ is i.i.d. over time and $\ln R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%, and set $R_{f,t+1} = 1.027449$. The ranges are highlighted by + signs.

Since in Section 3.1, a typical individual may use three approaches to evaluate $G_S$ and $G_L$. We first show theoretically that when using the first approach to evaluate timeless gambles, the GH model can explain the rejection of $G_S$ and the acceptance of $G_L$.

**Proposition 4** Consider an agent who faces a timeless gamble with payoff $\xi$ that is independent of other risks and evaluate this gamble using the GH model. Suppose $E[\xi] > 0$ and the agent uses the first approach to evaluate the gamble. Denote $\xi_+ := \max(\xi, 0)$ and $\xi_- = \max(-\xi, 0)$.

(i) If $\lambda \geq 1 + (1 + 1/b)\left(E[\xi]/E[\xi_-]\right)$, then the agent rejects $\xi$ at any wealth level $W_t$.

(ii) If $\lambda < 1 + (1 + 1/b)\left(E[\xi]/E[\xi_-]\right)$, then the agent accepts $\xi$ if and only if $W_t \geq W^*$, where $W^* \in (0, +\infty)$ is uniquely determined by $M(1 + \xi/W^*) + bE[\nu(\xi/W^*)] = 0$.  

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Proposition 4 shows that for any two gambles \( \xi_i, i = 1, 2 \) such that \( \mathbb{E}[\xi_i] > 0, i = 1, 2 \) and \( \mathbb{E}[^{\xi_i, +}/\mathbb{E}[\xi_i, -] < \mathbb{E}[^{\xi_2, +}/\mathbb{E}[\xi_2, -]] \), where \( \xi_{i, +} \) and \( \xi_{i, -} \) denote the positive and negative parts of \( \xi_i \), respectively, there exists a range of the values of \( \lambda \) for which the agent rejects \( \xi_1 \) at any wealth level and accepts \( \xi_2 \) at sufficiently high wealth levels. This observation indicates that the GH model can possibly explain the rejection of \( G_S \) for wealth levels \( W_t \leq 1,000,000 \) and the acceptance of \( G_L \) at for wealth levels \( W_t \geq 100,000 \).

Figure 3.2 plots by + signs the ranges of the values of \( b \) and \( \lambda \) for which the agent rejects \( G_S \) for wealth levels \( W_t \leq 1,000,000 \) and accepts \( G_L \) at for wealth levels \( W_t \geq 100,000 \). The left, middle, and right panels correspond to the first, second, and third approaches the agent uses to evaluate the gambles, respectively. The values of other parameters are the same as those in Figure 3.1. We can observe that the specific approach the agent uses has little impact on whether the agent rejects or accepts \( G_S \) and \( G_L \), and with reasonable parameter values, the BH model predicts the rejection of \( G_S \) for wealth levels \( W_t \leq 1,000,000 \) and the acceptance of \( G_L \) at for wealth levels \( W_t \geq 100,000 \). We also compute the ranges of the values of \( b \) and \( \lambda \) for different values of the standard deviation of \( \ln R_{p,t+1} \), such as 17.32%, and the results are almost the same.

4 Non-participation in the Stock Market

Barberis et al. (2006) consider an agent who, at the start of each period, has a fixed fraction \( \bar{\theta}_N \) of her wealth invested in a nonfinancial asset with gross return \( R_{N,s+1}, s \geq t \). The agent decides the fraction \( \theta_{F,s+1} \) of her wealth invested in a stock with gross return \( R_{F,s+1}, s \geq t \) and thus the remaining fraction \( 1 - \bar{\theta}_{N,s+1} - \theta_{F,s+1} \) invested in a risk-free asset with gross return \( R_{f,s+1}, s \geq t \). Assume (log \( R_{N,s+1}, \log R_{F,s+1} \)), \( s \geq t \) are i.i.d. and follow joint normal distribution with mean vector and covariance matrix to be

\[
\begin{pmatrix}
\mu_N \\
\mu_F
\end{pmatrix},
\begin{pmatrix}
\sigma^2_N & \omega \sigma_N \sigma_F \\
\omega \sigma_N \sigma_F & \sigma^2_F
\end{pmatrix}
\]

respectively. Assume \( R_{f,t+1} \) to be a constant. Suppose the agent derives a utility of gains and losses that are experienced by the investment in the stock. The agent needs to decide the consumption propensity and fraction of her wealth invested in the stock at each time. Because the gross returns of the nonfinancial asset and the stock are i.i.d. over time and the risk-free gross return is constant, the agent’s optimal consumption propensity and percentage allocation to the stock must be constant as well. Moreover, the optimal utility
Figure 3.2: Ranges of the values of $b$ and $\lambda$ for which the agent rejects $G_S$ for wealth levels $W_t \leq 1,000,000$ and accepts $G_L$ at for wealth levels $W_t \geq 100,000$. The left, middle, and right panels correspond to three different approaches used by the agent to frame a timeless gamble, in which the agent rejects the gamble if and only if (3.1), (3.2), and (3.3) hold, respectively. We $\beta = 0.98$ and $\gamma = \rho = 1.5$. In the second approach, we assume $R_{p,t+1}$ to be i.i.d. over time and $\ln R_{p,t+1}$ to follow normal distribution with mean 4% and variance 3%. In the third approach, we assume that there is only one risky asset to invest (i.e., $n = 1$), set $b_1 = b$, assume that $R_{1,t+1}$ is i.i.d. over time and $\ln R_{1,t+1}$ follows normal distribution with mean 4% and standard deviation 3%, and set $R_{f,t+1} = 1.027449$. The ranges are highlighted by + signs.

per unit capital of the agent is also constant. Thus, the agent faces the following decision problem

$$
\Phi = \max_{c, \theta_F} H\left(c, (1 - c) \Phi \mathcal{M}_1 \left((R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}) + \theta_F(R_{F,t+1} - R_{f,t+1}))ight) + bG(\theta_F(R_{F,t+1} - R_{f,t+1}))\right),
$$

where $\Phi$ stands for the agent’s optimal utility per unit capital; see Corollary 1.

Following Barberis et al. (2006), we want to know when the agent’s optimal allocation to the stock is non-positive. Because

$$f(\theta_F) := \mathcal{M}_1 \left((R_{f,t+1} + \bar{\theta}_N(R_{N,t+1} - R_{f,t+1}) + \theta_F(R_{F,t+1} - R_{f,t+1})) + bG(\theta_F(R_{F,t+1} - R_{f,t+1}))\right)$$

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is strictly concave in $\theta_F$, this is the case if and only if $f'(0+) \leq 0$, i.e.,

$$\mathbb{E}[u'((R_{f,t+1} + \tilde{\theta}_N(R_{N,t+1} - R_{f,t+1}))(R_{F,t+1} - R_{f,t+1})) - bG(R_{F,t+1} - R_{f,t+1})] \leq 0. \tag{4.1}$$

Note that this inequality does not depend on the discount rate, EIS, or the optimal utility per unit capital $\Phi$ of the agent. Thus, whether the agent’s optimal allocation to the stock is non-positive is easier to check in the GH model than in the BH model.

Note that because $f$ is concave in $\theta_F$, the non-positivity of the optimal $\theta_F$ implies that the agent does not participate in the stock market when short selling is not allowed. When short-selling is allowed, however, condition (4.1) is insufficient to imply the non-participation in the stock market. Indeed, we need one more condition, $f'(0-) \geq 0$, which is equivalent to

$$\mathbb{E}[u'((R_{f,t+1} + \tilde{\theta}_N(R_{N,t+1} - R_{f,t+1}))(R_{F,t+1} - R_{f,t+1})) - bG(R_{F,t+1} - R_{f,t+1})] \geq 0, \tag{4.2}$$

to conclude the nonparticipation in the stock market.

Note that with a higher $\lambda$, both (4.1) and (4.2) are more likely to hold, so it is more likely for the agent to have a non-positive allocation to the stock or not to participate in the stock market. The left panel of Figure 4.1 plots the threshold of $\lambda$, as a function of $\gamma$ and $b$, for which the agent has a non-positive allocation to the stock (i.e., condition (4.1) holds). The right panel of Figure 3 plots a similar threshold for the agent not to participate in the stock market (i.e., conditions (4.1) and (4.2) hold). In both plots, we follow Barberis et al. (2006) to set $\mu_F = 6\%$, $\sigma_F = 20\%$, $\mu_N = 4\%$, $\sigma_N = 3\%$, $\tilde{\theta}_N = 0.75$, $\omega = 0.1$, and $R_{f,t+1} = 1.02$. We can observe that for reasonable parameter values, the agent in the GH model does not participate in the stock market.

5 Asset Pricing

In the following, we consider a consumption-production equilibrium setting in Barberis and Huang (2009) and use the GH model to study the impact of narrow framing on asset prices.

Consider an economy with three assets. The first asset is a risk-free asset with zero net supply. The second asset is a non-financial asset, such as housing wealth or human capital, that has positive net supply. The third asset is a risky stock that has positive net supply. At each time $t$, the agent chooses consumption amount $C_t$ and the remaining is
Figure 4.1: Thresholds of $\lambda$ as functions of $\gamma$ and $b$ for which the agent has a non-positive allocation to the stock (left panel) and for which the agent does not participate in the stock market (right panel). In the plot, we set $\mu_F = 6\%$, $\sigma_F = 20\%$, $\mu_N = 4\%$, $\sigma_N = 3\%$, $\bar{\theta}_N = 0.75$, $\omega = 0.1$, and $R_{f,t+1} = 1.02$.

used for investment, in which $\theta_{N,t}$ fraction is invested in the non-financial asset, $\theta_{S,t}$ fraction is invested in the stock, and the remaining is invested in the risk-free asset.

Following Barberis and Huang (2009), we consider an equilibrium in which (i) the risk-free gross return is a constant $R_f$; (ii) consumption growth $C_{t+1}/C_t$ and stock gross return $R_{S,t+1}$ in period $t$ to $t+1$ are distributed as

$$
\log\left(\frac{C_{t+1}}{C_t}\right) = g_C + \sigma_C \varepsilon_{C,t+1}, \quad \log(R_{S,t+1}) = g_S + \sigma_S \varepsilon_{S,t+1},
$$

where $(\varepsilon_{C,t+1}, \varepsilon_{S,t+1})$’s are i.i.d. over time and follow normal distribution with mean vector $(0, 0)$ and covariance matrix $(1, \rho_{CS}; \rho_{CS}, 1)$; (iii) the consumption-wealth ratio is a constant over time; and (iv) the fraction of total wealth made up by the stock market is a constant $\iota$, i.e.,

$$\frac{S_t}{S_t + N_t} = \iota, \quad \forall t,$$

where $S_t$ and $N_t$ are the value of the stock market and of the non-financial asset, respectively.

Barberis and Huang (2009, Appendix A.3) show that the above equilibrium can be embedded in a general consumption-production economy, so in the following we focus on
solving asset prices in this equilibrium.

**Proposition 5** The equilibrium risk-free gross return $R_f$ and the stock gross return $R_{S,t+1}$ in the GH model are solved together with the consumption-wealth ratio $c^*$ from the following three equations:

\[
\frac{\mathbb{E}\left[u'(C_{t+1}/C_t)R_{S,t+1}\right]}{u'(\mathcal{M}(C_{t+1}/C_t))} - R_f \frac{\mathbb{E}\left[u'(C_{t+1}/C_t)\right]}{u'(\mathcal{M}(C_{t+1}/C_t))} + bG(R_{S,t+1} - R_f) = 0, \tag{5.1}
\]

\[
\frac{1}{1 - c^*}{\mathbb{E}\left[u'(C_{t+1}/C_t)\right]} = (1 - \iota)R_f \mathbb{E}\left[u'(C_{t+1}/C_t)\right] + \iota \mathbb{E}\left[u'(C_{t+1}/C_t)R_{S,t+1}\right], \tag{5.2}
\]

\[
1 - c^* = \left\{ \beta \left[ \frac{1}{1 - c^*}\mathcal{M}\left(\frac{C_{t+1}}{C_t}\right) + \frac{\iota}{1 - \iota} \frac{1}{1 - c^*} \frac{\mathbb{E}\left[u'(C_{t+1}/C_t)R_{S,t+1}\right]}{u'(\mathcal{M}(C_{t+1}/C_t))} \right]^{1 - \rho} \right\}^{1/\rho}, \tag{5.3}
\]

provided that $-(1 - c^*)bG(R_{S,t+1} - R_f) < \mathcal{M}(C_{t+1}/C_t)$ and $\beta[\mathcal{M}(C_{t+1}/C_t) + (1 - c^*)bG(R_{S,t+1} - R_f)]^{1 - \rho} < 1$.

The equilibrium equations (5.1)–(5.3) are easy to solve: we can first solve $1 - c^*$ in terms of $g_S$ from (5.3), then solve $R_f$ in terms of $g_S$ from (5.2), and finally solve $g_S$ from (5.1). Following Barberis and Huang (2009), we set $\rho = 1.5$, $\beta = 0.98$, $\iota = 0.3$, $g_c = 0.0184$, $\sigma_c = 0.0379$, $\sigma_s = .020$, $\rho_{cs} = .10$, and $\lambda$ to be 2 or 3. We set $\gamma$ to be 1.5 or 5, the former being used in Barberis and Huang (2009). Finally, we vary $b$ from 0 to 10. Table 5.1 presents the net return rate of the risk-free asset and the equity premium (i.e., the expected excess return of the stock). We can see that our results are similar to those in Barberis and Huang (2009, Table 4), showing that the GH model can be used to explain a high equity premium due to narrow framing. We also note that for $b \geq 2$, the equity premium is insensitive to the value of $\gamma$.

Finally, when there is only one risky asset in the market and the aggregate consumption is equal to the aggregate dividend paid out by this risky asset, the equilibrium asset pricing equations (5.1)–(5.3) can be simplified. This special case has been studied in Guo and He (2017, Proposition 2).
Table 5.1: Risk-free net return and equity premium in a consumption-production economy. Model parameters are set to be $\rho = 1.5$, $\beta = 9.8$, $\iota = 0.3$ $g_C = 0.0184$, $\sigma_C = 0.0379$, $\sigma_S = 0.20$, and $\rho_{CS} = 0.10$.

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6 Conclusions

The BH model is proposed by Barberis and Huang (2009) to explain individual’s attitudes toward timeless gambles, nonparticipation of households in the stock market, and the equity premium puzzle. We found, however, that the agent’s total utility process in this model may not be well defined when the utility of the gain and loss experienced by the agent is negative. Moreover, we showed that even though one can derive the dynamic programming equation in the BH model heuristically, this equation may have multiple solutions, leading to different optimal consumption-investment strategies.

The issue of non-existence or non-uniqueness of the total utility process and the solution to the dynamic programming equation in the BH model arises because the weight for the utility of gains and losses used by the agent is a constant over time. In consequence, a negative utility of gains and losses in each period can drive the agent’s total utility to be
negative, resulting in the aforementioned ill-behaviors of the BH model.

In the GH model that is proposed by Guo and He (2016), the weight for the utility of gains and losses in each period is scaled by the agent’s total utility per unit wealth from next period. It turns out that in the GH model, the agent’s total utility process uniquely exists and the solution to the dynamic programming equation in this model uniquely exists as well. Moreover, both the total utility process and the solution to the dynamic programming equation can be computed by a recursive algorithm with any starting point. We also showed that even when the agent’s total utility per unit wealth is a constant so that the weight for the utility of gains and losses in the GH model is also a constant, the GH model differs from the BH model in that the total utility process in the former cannot be obtained from the latter.

We applied the GH model to explain individuals’ attitudes toward timeless gambles, nonparticipation of households in the stock market, and the equity premium puzzle. We found that the GH model is as powerful as the BH model in these explanations. More strikingly, the GH model is more tractable than the BH model in these applications.

A Portfolio Selection with the GH Model

In this appendix, we review the general results of the existence and uniqueness of the total utility process in the GH model and portfolio selection results in this model that are obtained in Guo and He (2016).

A.1 Existence and Uniqueness of the Total Utility Process

Recall the GH model (2.9). Denote

\[ c_t := C_t / W_t, \quad R_{t+1} := W_{t+1} / (W_t - C_t), \]

which stand for the fraction of the agent’s wealth used for consumption and the gross return of the agent’s investment, respectively. Denote

\[ g_{i,t} := G_{i,t} / (W_t - C_t), \quad A_{t+1} := (1 - c_t) \left( 1 + \sum_{i=1}^{n} b_i M_t(R_{t+1}) g_{i,t} \right) R_{t+1}. \]

We impose the following assumption:
Assumption 1  

(i) \( \{(X_t, Y_t)\} \) is a Markov process and the joint distribution of \( (X_{t+1}, Y_{t+1}) \) conditioned on \( (X_t, Y_t) \) depends only on \( X_t \). Moreover, the state space of \( \{X_t\} \), denoted as \( \mathbb{X} \), is finite.

(ii) \( c_t > 0 \) and \( A_{t+1} > 0 \), \( t = 0, 1, \ldots \)

(iii) \( \log(c_{t+1}) - \log(c_t) + \log A_{t+1} = \kappa(X_t, X_{t+1}, Y_{t+1}), \quad t = 0, 1, \ldots \) for some real-valued function \( \kappa \). Moreover, for any state \( x \), \( \mathbb{E}_t \left[ u\left( e^{\kappa(X_t, X_{t+1}, Y_{t+1})} \right) | X_t = x \right] \) exists.

Assumption 1-(i) posits that given \( \{X_t\}, \{Y_t\} \) is an independent sequence. This assumption also implies that \( \{X_t\} \) is a Markov process. Thus, we can consider \( \{X_t\} \) to be the market state process and \( \{Y_t\} \) to be an innovation process. Assumption 1-(ii) stipulates that \( A_{t+1} \) needs to be positive, which is equivalent to assuming

\[
- \sum_{i=1}^{n} b_i g_{i,t} < \mathcal{M}_t(R_{t+1}). \tag{A.3}
\]

This inequality means that the disutility of gains and losses experienced by the agent as a consequence of narrow framing cannot be too large, i.e., cannot exceed the certainty equivalent of the gross return on the agent’s investment. Assumption-(iii) ensures a Markovian structure for the agent’s consumption and investment.

In view of Assumption 1 and using the homogeneity of \( H \), we can write (2.9) as

\[
\frac{U_t}{C_t} = H \left( 1, \mathcal{M}_t \left( e^\kappa(X_t, X_{t+1}, Y_{t+1}) \frac{U_{t+1}}{C_{t+1}} \right) \right). \tag{A.4}
\]

Because of the Markovian structure imposed in Assumption 1, we can focus on Markovian solutions to (A.4), i.e., \( U_t/C_t = f(X_t) \) for some function \( f \). It is straightforward to see that \( f \) is a fixed point of

\[
\mathbb{T} f(x) := H \left( 1, u^{-1} \left( \mathbb{E}_t \left[ u\left( e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1}) \right) | X_t = x \right] \right) \right), \quad x \in \mathbb{X}. \tag{A.5}
\]

Thus, to find the Markovian solution to (A.4), we only need to find the fixed point of \( \mathbb{T} \). We consider the fixed point of \( \mathbb{T} \) in \( \mathcal{X}_{++} \), the space of positive functions on \( \mathbb{X} \), i.e.,

\[
\mathcal{X}_{++} := \{ f | f(x) > 0, x \in \mathbb{X} \}. \tag{2}
\]

Note that \( \mathbb{T} f \) is well defined for any nonnegative function \( f \). However, as shown by Guo and He (2016), \( \mathbb{T} \) can have multiple fixed points on the space of nonnegative functions. Moreover, because we assume the consumption to be positive (i.e., \( c_t > 0 \)) and the disutility of losses cannot be too large (i.e., \( A_{t+1} > 0 \)), it
The following quantity is crucial for the existence and uniqueness of the total utility process:

\[
\delta = \max_{f \in \mathcal{X}_+} \min_{x \in \mathcal{X}} u^{-1}\left( \mathbb{E}_t \left[ u \left( e^{\kappa(X_t, X_{t+1}, Y_{t+1}) f(X_{t+1})} \right) | X_t = x \right] \right). \tag{A.6}
\]

This quantity is related to the Perron-Frobenius eigenvalue of certain operator; see Proposition 1 in Guo and He (2016). We can see that when the state space \( \mathcal{X} \) is singleton,

\[
\delta = u^{-1}\left( \mathbb{E}_t \left[ u \left( e^{\kappa(X_t, X_{t+1}, Y_{t+1})} \right) \right] \right),
\]

which is nothing but the certainty equivalent of \( \exp[\kappa(X_t, X_{t+1}, Y_{t+1})] \). Thus, in general \( \delta \) can be regarded as a variant of the certainty equivalent of \( \exp[\kappa(X_t, X_{t+1}, Y_{t+1})] \).

**Theorem 1** Suppose Assumption 1 holds and \( \beta \delta^1 - \rho < 1 \). Then, the fixed point of \( T \) in \( \mathcal{X}_+ \) uniquely exists. Moreover, for any \( f \in \mathcal{X}_+ \), \( \{T^nf\}_{n \geq 0} \) converges to the fixed point.

**Proof of Theorem 1.** This theorem is a direct consequence of Theorem 1 in Guo and He (2016). \( \square \)

Theorem 1 proves the existence and uniqueness of the fixed point of \( T \) and thus those of the total utility process in the GH model. Moreover, it provides a simple algorithm to compute the fixed point: one can start from any positive function, e.g., a positive constant function, to do iteration, and one can obtain a sequence that eventually converges to the fixed point.

### A.2 Dynamic Programming

Suppose an agent can invest in a risk-free asset with gross return \( R_{f,t+1} \) in period \( t \) to \( t+1 \) and \( n \) risky assets, indexed by \( i = 1, \ldots, n \), with gross returns \( R_{i,t+1}, \ i = 1, \ldots, n \), respectively, in period \( t \) to \( t+1 \). Suppose at time \( t \), the agent has wealth \( W_t \) prior to any consumption at that time. The agent decides to consume \( c_t W_t \), invests \( \theta_{i,t}(1-c_{i,t}) W_t \) in asset \( i, \ i = 1, \ldots, n \), and thus the remaining amount \( (1-\sum_{i=1}^{n} \theta_{i,t})(1-c_t)W_t \) in the risk-free is reasonable for us to expect a positive total utility process.
asset. For this strategy, we assume the agent’s total utility process is modeled by

\[ U_t = H \left( c_t W_t, M_t(U_{t+1}) + \sum_{i=1}^{n} \frac{M_t(U_{t+1})}{M_t(W_{t+1})} b_i G_{i,t} \right) \]  \( (A.7) \)

where \( b_i \)'s are nonnegative constants and

\[ G_{i,t} := \mathbb{E}_t \left[ \nu \left( \frac{\theta_{i,t}(1 - c_t) W_t R_{i,t+1} - \theta_{i,t}(1 - c_t) W_t R_{f,t+1}}{1} \right) \right] . \]

In other words, the agent may frame the investment in asset \( i \) separately from other risks and evaluate it using the GH model.

With Assumption 1-(i) in force, we assume \( R_{i,t+1} = r_i(X_t, X_{t+1}, Y_{t+1}) \) for some function \( r_i \), \( i = 1, \ldots, n \), and \( R_{f,t+1} = r_0(X_t) \) for some function \( r_0 \). In this Markovian setting, it is reasonable that the agent only considers Markovian strategies, i.e., \( c_t = c(X_t) \) and \( \theta_{i,t} = \theta_i(X_t) \), \( i = 1, \ldots, n \), for some functions \( c \) and \( \theta_i \)'s. Denote \( \theta(x) := (\theta_1(x), \ldots, \theta_n(x))^\top \).

Suppose the feasible set of strategies of the agent is

\[ \mathcal{A} := \{(c, \theta) | c(x) \in I_x, \theta(x) \in J_x, x \in \mathbb{X} \}, \]

where, for each \( x \in \mathbb{X} \), \( I_x \) is a nonempty compact subset of \((0,1)\) and \( J_x \) is a nonempty compact subset of \( \mathbb{R}^n \). To highlight the dependence on the agent’s strategy, we denote the total utility process in \((A.7)\) as \( \{U_t^{c,\theta} \} \). The agent faces the following consumption-investment problem:

\[ \max_{(c, \theta) \in \mathcal{A}} U_t^{c,\theta} . \]  \( (A.8) \)

We need to impose certain assumption on the feasible set \( \mathcal{A} \) so that the total utility process for each feasible strategy uniquely exists. Denote

\[ R_\theta(X_t, X_{t+1}, Y_{t+1}) := r_0(X_t) + \sum_{i=1}^{n} \theta_i(X_t)(r_i(X_t, X_{t+1}, Y_{t+1}) - r_0(X_t)) \]  \( (A.9) \)

as the gross return of the agent’s portfolio and denote

\[ g_i(x, \bar{\theta}_i) := \mathbb{E}_t \left[ \nu \left( \frac{\bar{\theta}_i(r_i(x, X_{t+1}, Y_{t+1}) - r_0(x))}{1} \right) | X_t = x \right] , \quad \bar{\theta}_i \in \mathbb{R}, \quad i = 1, \ldots, n. \]  \( (A.10) \)

In view of Theorem 1, we impose the following:
Assumption 2  

(i) \( \{(X_t, Y_t)\} \) is a Markov process and the joint distribution of \((X_{t+1}, Y_{t+1})\) conditioned on \((X_t, Y_t)\) depends only on \(X_t\). Moreover, the state space of \(\{X_t\}\), denoted as \(\mathcal{X}\), is finite.

(ii) For each \((c, \theta) \in \mathcal{A}\), \(R_\theta(X_t, X_{t+1}, Y_{t+1})) > 0\) and

\[-\sum_{i=1}^n b_i g_i(X_t, \theta_i(X_t)) < \mathcal{M}_t(R_\theta(X_t, X_{t+1}, Y_{t+1})).\]

(iii) For each \((c, \theta) \in \mathcal{A}\) and any state \(x\), \(\mathbb{E}_t\left[u\left(e^{\kappa_{c,\theta}(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1})|X_t = x\right)\right] f(x)\) exists, where

\[
\kappa_{c,\theta}(X_t, X_{t+1}, Y_{t+1}) := \log \left(\frac{c(X_{t+1})}{c(X_t)}\right) - \log \left(\frac{1 - c(X_t)}{c(X_{t+1})}\right) + \log \left(\mathcal{M}_t(R_\theta(X_t, X_{t+1}, Y_{t+1})) + \sum_{i=1}^n b_i g_i(X_t, \theta_i(X_t))\right) + \log \left(R_\theta(X_t, X_{t+1}, Y_{t+1})\right) - \log \left(\mathcal{M}_t(R_\theta(X_t, X_{t+1}, Y_{t+1}))\right).
\]

(A.11)

(iv) For each \((c, \theta) \in \mathcal{A}\), \(\beta (\delta_{c,\theta})^{1-\rho} < 1\), where

\[
\delta_{c,\theta} := \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathcal{X}} u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa_{c,\theta}(X_t, X_{t+1}, Y_{t+1}) f(X_{t+1})|X_t = x\right)\right] f(x)\right).
\]

(A.12)

With Assumption 2, for each \((c, \theta) \in \mathcal{A}\), the total utility process \(\{U_{t}^{c,\theta}\}\) uniquely exists. Denote \(\Phi(x)\) as the agent’s optimal utility per unit capital, i.e.,

\[
\Phi(x) = \max_{(c, \theta) \in \mathcal{A}} \left(U_{t}^{c,\theta}/W_t\right), \text{ given } X_t = x.
\]

Then, we expect the following dynamic programming equation:

\[
\Phi(x) = \mathbb{W} \Phi(x), \quad x \in \mathcal{X},
\]

(A.13)
where
\[
\mathbb{W}\Phi(x) := \max_{\bar{c} \in I_x} H \left( \bar{c}, (1 - \bar{c}) \max_{\theta \in J_x} D_\Phi(x, \theta) \right), \quad x \in X,
\]  
(A.14)
\[
D_\Phi(x, \theta) := u^{-1} \left( \mathbb{E}_t \left[ u \left( r_0(x) + \sum_{i=1}^n \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)) \right) \Phi(X_{t+1}) \right] \right)
\times \left( 1 + \frac{\sum_{i=1}^n b_i g_i(x, \bar{\theta}_i)}{u^{-1} \left( \mathbb{E}_t \left[ u \left( r_0(x) + \sum_{i=1}^n \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)) \right) \right] \right)} \right).
\]  
(A.15)

**Theorem 2** Suppose Assumption 2 holds. Then, the fixed point of \( \mathbb{W} \) in \( X_{++} \) uniquely exists and \( \{\mathbb{W}^n \Phi\}_{n \geq 0} \) converges to this fixed point for any \( \Phi \in X_{++} \). Moreover, if \( (c^*(x), \theta^*(x)) \in I_x \times J_x \) is a maximizer of the maximization in the dynamic programming equation (A.13), which must exist due to the compactness of \( I_x \) and \( J_x \), \( x \in X \), then \( (c^*, \theta^*) \) is an optimal solution to (A.8).

**Proof of Theorem 2.** The proof is exactly the same as for Propositions 2 and 3 in Guo and He (2016) in the case in which \( \varpi_{c, \theta} \) therein is nonnegative for any \( (c, \theta) \). \( \square \)

Theorem 2 shows the existence and uniqueness of the solution to the dynamic programming equation. Moreover, it shows that starting from any \( \Phi \) that is positive and applying the dynamic programming equation repeatedly, one eventually obtains the solution to the equation.

The maximization of \( H(\bar{c}, (1 - \bar{c}) z) \) in \( \bar{c} \) for each fixed \( z > 0 \) is easy, but the maximization of \( D_\Phi(x, \theta) \) in \( \theta \) for each fixed \( \Phi > 0 \) is not straightforward in general. When the state space of \( \{X_t\} \) is singleton or when \( \gamma = 1 \), however, it is easy to solve the maximization of \( D_\Phi(x, \theta) \). Indeed, in these two cases, we have

\[
D_\Phi(x, \theta) = \mathcal{M}_t(\Phi(X_{t+1})) \left[ \sum_{i=1}^n b_i g_i(x, \bar{\theta}_i) \right.
+ u^{-1} \left( \mathbb{E}_t \left[ u \left( r_0(x) + \sum_{i=1}^n \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)) \right) \right] \right].
\]  
(A.16)

Because the certainty equivalent \( \mathcal{M}_t \) is a concave functional and \( g_i(x, \bar{\theta}_i) \) is concave in \( \theta_i \) for each \( i \), \( D_\Phi(x, \theta) \) is concave in \( \theta \). Thus, it is straightforward to find the maximizer of \( D_\Phi(x, \theta) \) in \( \theta \).

Finally, because in many applications of the GH model, the state space is assumed to
be a singleton, for convenience we present the dynamic programming equation in this case. Note that in this case, any function $f$ on $X$ is actually a scalar. Moreover, the gross return of the risk-free asset is a constant $R_f$, the gross returns of the risky assets are i.i.d. over time, and the agent’s consumption strategy $c$ and portfolio strategy $\theta$ are constants.

**Corollary 1** Suppose Assumption 2 holds and the state space $X$ is a singleton. Then, the dynamic programming equation becomes

$$
\Phi = \max_{c \in I} H(c, (1 - c)\Phi\Theta),
$$

(A.17)

$$
\Theta = \max_{\theta \in J} \left( u^{-1}\left( E[u(R_f + \sum_{i=1}^{n} \theta_i (R_{i,t+1} - R_f))]\right) + \sum_{i=1}^{n} b_i g_i(\theta_i) \right),
$$

(A.18)

where $I$ and $J$ are the feasible sets of $c$ and $\theta$, respectively, and $g_i(\theta_i) := E[\nu(\theta_i(R_{t+1} - R_f))]$.

### A.3 Dynamic Programming in the GH-I Model

We consider the GH-I model (2.16). Let us set aside the issue of the existence and uniqueness of the total utility process in this model. Recall the market setting in the portfolio selection problem considered in Section A.2 and suppose the agent’s preferences are represented by the GH-I model. We can write down the dynamic programming equation heuristically:

$$
\Phi(x) = \tilde{\mathcal{V}} \Phi(x), \quad x \in X,
$$

(A.19)

where

$$
\tilde{\mathcal{V}} \Phi(x) := \max_{\tilde{c} \in \tilde{I}_x} H\left( \tilde{c}, (1 - \tilde{c}) \max_{\tilde{\theta} \in \tilde{J}_x} \tilde{D}_\Phi(x, \tilde{\theta}) \right), \quad x \in X,
$$

(A.20)

$$
\tilde{D}_\Phi(x, \tilde{\theta}) := u^{-1}\left( E_t\left[ u\left( r_0(x) + \sum_{i=1}^{n} \tilde{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x)) \Phi(X_{t+1}) | X_t = x \right) \right] \right) + u^{-1}\left( E_t\left[ u(\Phi(X_{t+1}) | X_t = x) \right] \right) \sum_{i=1}^{n} b_i g_i(x, \tilde{\theta}_i).
$$

(A.21)

Clearly, for each $x \in X$ and $\Phi$, $\tilde{D}_\Phi(x, \tilde{\theta})$ is concave in $\tilde{\theta}$, so in general it is easier to maximize $\tilde{D}_\Phi(x, \tilde{\theta})$ than to maximize $D_\Phi(x, \bar{\theta})$ in $\bar{\theta}$. However, neither the existence and
uniqueness of the total utility process in the GH-I model nor those of the solution to the dynamic programming equation (A.19) are known.

A.4 Analysis of the GH-II Model

We consider the GH-II model (2.17). We first establish the existence and uniqueness of the total utility process in this model. To this end, recall \( c_t, R_{t+1}, \) and \( g_{i,t} \) as defined in (A.1) and (A.2), and define

\[
\hat{A}_{t+1} := (1 - c_t) \left( 1 + \sum_{i=1}^{n} b_i g_{i,t} \right) R_{t+1}.
\]  

(A.22)

Suppose Assumption 1 holds with \( A_{t+1} \) replaced by \( \hat{A}_{t+1} \), denote \( \kappa \) therein as \( \hat{\kappa} \), i.e., \( \hat{\kappa}(X_t, X_{t+1}, Y_{t+1}) = \log(c_{t+1}) - \log(c_t) + \log(\hat{A}_{t+1}) \), denote \( \hat{T} \) in (A.5) with \( \kappa \) replaced by \( \hat{\kappa} \) as \( \hat{T} \), and denote \( \delta \) defined in (A.6) with \( \kappa \) replaced by \( \hat{\kappa} \) as \( \hat{\delta} \). Following the same argument as in Section A.1, we can focus on Markovian solutions to (2.17), i.e., \( U_t/C_t = f(X_t) \) for some function \( f \), and \( U_t/C_t \) is a solution to (2.17) if and only if \( f \) is a fixed point of \( \hat{T} \). The existence and uniqueness of the fixed point of \( \hat{T} \) can be established as follows:

**Corollary 2** Suppose Assumption 1 holds with \( A_{t+1} \) replaced by \( \hat{A}_{t+1} \) and \( \beta \hat{\delta}^{1-\rho} < 1 \). Then, the fixed point of \( \hat{T} \) in \( X_{++} \) uniquely exists. Moreover, for any \( f \in X_{++}, \{\hat{T}^n f\}_{n \geq 0} \) converges to the fixed point.

Consider the same portfolio selection setting as in Section A.2. Define

\[
\hat{\kappa}_{c,\theta}(X_t, X_{t+1}, Y_{t+1}) := \log(c(X_{t+1})) - \log(c(X_t)) + \log(1 - c(X_t))
\]

\[
+ \log \left( 1 + \sum_{i=1}^{n} b_i g_i(X_t, \theta_i(X_t)) \right) + \log R_\theta(X_t, X_{t+1}, Y_{t+1}).
\]  

(A.23)

Consider the following dynamic programming equation:

\[
\Phi(x) = \hat{W} \Phi(x), \quad x \in X,
\]  

(A.24)
where
\[
\hat{\Phi}(x) := \max_{\bar{c} \in I_x} H \left( \bar{c}, (1 - \bar{c}) \max_{\bar{\theta} \in J_x} \hat{D}_\Phi(x, \bar{\theta}) \right), \quad x \in \mathbb{X}, \tag{A.25}
\]
\[
\hat{D}_\Phi(x, \bar{\theta}) := u^{-1} \left( \mathbb{E}_t \left[ u \left( (r_0(x) + \sum_{i=1}^n \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x))) \Phi(X_{t+1}) \right) | X_t = x \right] \right)
\times \left( 1 + \sum_{i=1}^n b_i g_i(x, \bar{\theta}_i) \right). \tag{A.26}
\]

**Corollary 3** Consider the same portfolio selection setting as in Section A.2 and assume the agent’s preferences are represented by the GH-II model (2.17). Suppose Assumption 2 holds with \(\kappa_{c, \theta}\) replaced by \(\hat{\kappa}_{c, \theta}\). Then, the fixed point of \(\hat{\Phi}\) in \(X_{++}\) uniquely exists and \(\{\hat{\Phi}^{n}\}_{n \geq 0}\) converges to this fixed point for any \(\Phi \in X_{++}\). Moreover, if \((c^*(x), \theta^*(x)) \in I_x \times J_x\) is a maximizer of the maximization in the dynamic programming equation (A.24), which must exist due to the compactness of \(I_x\) and \(J_x\), \(x \in \mathbb{X}\), then \((c^*, \theta^*)\) is an optimal solution of the agent’s portfolio selection problem.

Note that the maximization of \(\hat{D}_\Phi(x, \bar{\theta})\) in \(\theta\) is more complicated than that of \(D_\Phi(x, \bar{\theta})\) taking a special form in (A.16), showing that the optimal portfolio in the GH-II model is more complicated to solve than in the GH-model when the state space of \(\{X_t\}\) is a singleton or when \(\gamma = 1\). In general, however, \(D_\Phi(x, \bar{\theta})\) takes the general form (A.15), which appears to be more complicated than \(\hat{D}_\Phi(x, \bar{\theta})\), showing that the optimal portfolio in the GH-II model seems easier to compute than in the GH model.

**B Proofs**

*Proof of Proposition 1.* Straightforward calculation yields
\[
\nabla'(\Psi) = \delta \beta \left( H(c/(\delta \Psi + c \delta \zeta), 1) \right)^\rho = \delta \beta \left( \nabla(\Psi)/(\delta \Psi + c \delta \zeta) \right)^\rho, \tag{B.1}
\]
so one can easily see that \( V \) is strictly increasing and concave. Moreover,

\[
\lim_{\Psi \downarrow -\delta \zeta/\delta} V(\Psi) = \begin{cases} 
    c(1 - \beta)^{1/(1-\rho)}, & \rho < 1, \\
    0, & \rho \geq 1,
\end{cases} \quad \lim_{\Psi \uparrow +\infty} V(\Psi) = \begin{cases} 
    +\infty, & \rho \leq 1, \\
    c(1 - \beta)^{1/(1-\rho)}, & \rho > 1,
\end{cases}
\]

\[
\lim_{\Psi \downarrow -\delta \zeta/\delta} V'(\Psi) = \begin{cases} 
    +\infty, & \rho \leq 1, \\
    (\beta \delta^{1-\rho})^{1/(1-\rho)}, & \rho > 1,
\end{cases} \quad \lim_{\Psi \uparrow +\infty} V'(\Psi) = \begin{cases} 
    0, & \rho \geq 1.
\end{cases}
\]

If \( \rho < 1 \) and \(-\overline{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)}\), we have \( \lim_{\Psi \downarrow -\delta \zeta/\delta} V(\Psi) > -c\overline{b}\zeta/\delta \), \( \lim_{\Psi \downarrow -\delta \zeta/\delta} V'(\Psi) = +\infty \), and \( \lim_{\Psi \uparrow +\infty} V'(\Psi) < 1 \), so by the concavity of \( V \), we can see that the solution to (2.8) uniquely exists.

If \( \rho < 1 \) and \(-\overline{b}\zeta/\delta > (1 - \beta)^{1/(1-\rho)}\), we have \( \lim_{\Psi \downarrow -\delta \zeta/\delta} V(\Psi) < -c\overline{b}\zeta/\delta \), \( \lim_{\Psi \downarrow -\delta \zeta/\delta} V'(\Psi) = +\infty \), and \( \lim_{\Psi \uparrow +\infty} V'(\Psi) < 1 \). In this case, the line starting from \((0,0)\) and tangent to \( V(\Psi) \) uniquely exists and the tangent point \( \Psi_0 \) solves \( V(\Psi_0) = V(\Psi_0)'\Psi_0 \). Straightforward yields that

\[
V'(\Psi_0) = \delta(1 - \beta)^{1/(1-\rho)} \left[ (-\overline{b}\zeta)^{-(1-\rho)/\rho} + \left( \frac{\beta}{1 - \beta} \right)^{1/\rho} \right]^{\rho/(1-\rho)}.
\]

Because \( \delta \beta^{1/(1-\rho)} < 1 \), we conclude that \( V'(\Psi_0) > 1 \) if and only if

\[
-\overline{b}\zeta/\delta < (1 - \beta)^{1/(1-\rho)} \left[ 1 - \left( \beta \delta^{1-\rho} \right)^{1/\rho} \right]^{\rho/(1-\rho)}.
\]

In this case, because \( V \) is strictly concave, \( \lim_{\Psi \downarrow -\delta \zeta/\delta} V(\Psi) < -c\overline{b}\zeta/\delta \), and \( \lim_{\Psi \uparrow +\infty} V'(\Psi) < 1 \), we conclude that there exists two fixed points of \( V \) on \([-c\overline{b}\zeta/\delta, +\infty)\), and we denote them as \( \Psi_1^* < \Psi_2^* \). Moreover, \( \Psi_1^* > -c\overline{b}\zeta/\delta \), \( V'(\Psi_1^*) > 1 \), and \( V(\Psi) < \Psi \) for any \( \Psi < \Psi_1^* \). In consequence, for such \( \Psi \), \( \{V^n(\Psi)\} \) is a strictly decreasing sequence. We claim that there exists \( n_0 \) such that \( V^{n_0}(\Psi) < -c\overline{b}\zeta/\delta \). Otherwise, the limit of \( \{V^n(\Psi)\} \) exists in \([-c\overline{b}\zeta/\delta, +\infty)\) and thus must be a fixed point of \( V \) by the continuity of \( V \), and this contradicts with the fact that \( \Psi_1^* \) and \( \Psi_2^* \) are the only two fixed points of \( V \). On the other hand, \( V'(\Psi_2^*) < 1 \), \( V(\Psi) > \Psi \) for \( \Psi \in (\Psi_1^*, \Psi_2^*) \), and \( V(\Psi) < \Psi \) for \( \Psi > \Psi_2^* \). Thus, \( \{V^n(\Psi)\}_{n \geq 1} \) converges to \( \Psi_2^* \) for any \( \Psi > \Psi_1^* \).

Similarly, \( V'(\Psi_0) < 1 \) if and only if

\[
-\overline{b}\zeta/\delta > (1 - \beta)^{1/(1-\rho)} \left[ 1 - \left( \delta \beta^{1/(1-\rho)} \right)^{(1-\rho)/\rho} \right]^{-\rho/(1-\rho)}.
\]
Because \( V \) is strictly concave, \( \lim_{\Psi \rightarrow -c\bar{\zeta}/\delta} V(\Psi) < -c\bar{\zeta}/\delta \), and \( \lim_{\Psi \uparrow +\infty} V'(\Psi) < 1 \), we conclude that there does not exist any fixed point of \( V \) on \([ -c\bar{\zeta}/\delta, +\infty ) \). Using the same argument as in the case in which \( V'(\Psi_0) > 1 \), we can show that for any \( \Psi > -c\bar{\zeta}/\delta \), there exists \( n_0 \) such that \( V^{n_0}(\Psi) < -c\bar{\zeta}/\delta \).

Finally, we consider the case in which \( \rho \geq 1 \). In this case, \( \lim_{\Psi \rightarrow -c\bar{\zeta}/\delta} V(\Psi) = 0 < -c\bar{\zeta}/\delta \). Again, the line starting from \((0,0)\) and tangent to \( V(\Psi) \) uniquely exists and the tangent point \( \Psi_0 \) solves \( V(\Psi_0) = V(\Psi_0)'\Psi_0 \). Straightforward yields that

\[
V'(\Psi_0) = \begin{cases} 
\delta(1-\beta)^{1/(1-\rho)} \left[ (-\bar{b}\zeta)^{(1-\rho)/\rho} + \left( \frac{\beta}{1-\beta} \right)^{1/\rho} \right]^{\rho/(1-\rho)} & \rho > 1, \\
(-\bar{b}\zeta)^{(1-\beta)}(1-\beta)^{1-\beta} & \rho = 1.
\end{cases}
\]

Then, (iv) and (v) follow from the same argument as in the proof of (i)-(iii). \( \square \)

**Proof of Proposition 2.** By definition, \( \Psi^* \) is a fixed point of \( V \). We then examine \( V'(\Psi^*) \). Straightforward calculation yields

\[
V'(\Psi^*) = \delta \beta \left( \frac{V(\Psi^*)}{(\delta \Psi^* + c\bar{\zeta})} \right)^\rho = \delta \beta \left( \frac{\Psi^*}{(\delta \Psi^* + c\bar{\zeta})} \right)^\rho = \frac{\delta}{\delta + c\bar{\zeta}} \beta (\delta + c\bar{\zeta})^{1-\rho}.
\]

Thus, \( V'(\Psi^*) < 1 \) if and only if \( c\bar{\zeta} < (\delta \beta)^{1/\rho} - \delta \). In this case, \( \mathbb{W} \) in Proposition 1 has two fixed points \( \Psi_1^* < \Psi_2^* \) and \( \Psi^* = \Psi_1^* \) because \( V'(\Psi_1^*) < 1 \) and \( V'(\Psi_2^*) > 1 \). Similarly, \( V'(\Psi^*) > 1 \) if and only if \( c\bar{\zeta} > (\delta \beta)^{1/\rho} - \delta \), in which case \( \mathbb{W} \) in Proposition 1 has two fixed points \( \Psi_1^* < \Psi_2^* \) and \( \Psi^* = \Psi_2^* \). \( \square \)

**Proof of Proposition 3.** Denote the right-hand side of (2.14) as \( \mathbb{W}(\Phi) \). First, because \( 0 \in J \), \( \mathbb{W}(\Phi) \) is well defined for any \( \Phi \geq 0 \). Moreover, it is obvious that \( \mathbb{W}(\Phi) \) is continuous in \( \Phi \).

Next, because \( 0 \in J \), we have \( \mathbb{W}(\Phi) \geq H(c_0, (1-c_0)R_f) \) for some fixed \( c_0 \in I \). In consequence, by the homogeneity of \( H \),

\[
\liminf_{\Phi \downarrow 0} \left( \frac{\mathbb{W}(\Phi)}{\Phi} \right) \geq \liminf_{\Phi \downarrow 0} H(c_0/\Phi, (1-c_0)R_f) = \begin{cases} 
+\infty, & \rho \leq 1, \\
\beta^{1/(1-\rho)}(1-c_0)R_f, & \rho > 1.
\end{cases}
\]

In consequence, when \( \rho \leq 1 \), we have \( \liminf_{\Phi \downarrow 0} \left( \frac{\mathbb{W}(\Phi)}{\Phi} \right) = +\infty > 1 \); when \( \rho > 1 \), (2.15) implies that \( \beta^\rho (1-c_0)R_f^{1-\rho} < 1 \), so \( \liminf_{\Phi \downarrow 0} \left( \frac{\mathbb{W}(\Phi)}{\Phi} \right) = \beta^\rho (1-c_0)R_f^{1-\rho} \).
Thus, we conclude that $W(\Phi) > \Phi$ when $\Phi$ is sufficiently small.

Finally, we show $W(\Phi) < \Phi$ when $\Phi$ is sufficiently large so that the fixed point of $W$ exists. Note that because $I \subset (0, 1)$ is compact, there exists $\bar{c} \in (0, 1)$ such that $\bar{c} \geq c, \forall c \in I$. When $\rho \geq 1$, because $I$ and $J$ are compact, there exists $a > 0$ such that

$$(1 - c) \max_{\theta \in J} \{ M(R_f + \theta(R_{s,t+1} - R_f)) + b\theta E[\nu(R_{s,t+1} - R_f)]/\Phi \} < a, \forall c \in I$$

for sufficiently large $\Phi$. In consequence,

$$\limsup_{\Phi \uparrow +\infty} \left( \frac{W(\Phi)}{\Phi} \right) \leq \limsup_{\Phi \uparrow +\infty} \max_{c \in I} H(c/\Phi, a) \leq \limsup_{\Phi \uparrow +\infty} H(\bar{c}/\Phi, a) = 0.$$ 

When $\rho < 1$, because of condition (2.15) and the compactness of $J$, there exists $\alpha < \beta^{-1/(1-\rho)}$ such that

$$(1 - c) \max_{\theta \in J} \{ M(R_f + \theta(R_{s,t+1} - R_f)) + b\theta E[\nu(R_{s,t+1} - R_f)]/\Phi \} < \alpha, \forall c \in I$$

when $\Phi$ is sufficiently large. In consequence,

$$\limsup_{\Phi \uparrow +\infty} \left( \frac{W(\Phi)}{\Phi} \right) \leq \limsup_{\Phi \uparrow +\infty} \max_{c \in I} H(c/\Phi, \alpha) \leq \limsup_{\Phi \uparrow +\infty} H(\bar{c}/\Phi, \alpha) = \beta^{1/(1-\rho)} \alpha < 1.$$ 

Therefore, we conclude that for any value of $\rho$, $W(\Phi) < \Phi$ for sufficiently large $\Phi$. □

**Proof of Proposition 4.** Denote $\delta := 1/W_t$. From (3.1), we only need to investigate when

$$f(\delta) := M(1 + \delta \xi) + bG(\delta \xi), \quad \delta \geq 0$$

is strictly larger than one.

Because $M(\cdot)$ is strictly concave and $G(\delta \xi)$ is linear in $\delta \geq 0$, we conclude that $f$ is strictly concave in $\delta$. Moreover, $f(0) = 1$ and $f'(0) = E[\xi] + bG(\xi)$. In consequence, if $f'(0) \leq 0$, which is the case if and only if $\lambda \geq 1 + (1 + 1/b) (E[\xi]/E[\xi_-])$, we have $f(\delta) < 1$ for any $\delta > 0$, so the agent rejects $\xi$ at any wealth level $W_t$.

If $f'(0) > 0$, which is the case if and only if $\lambda < 1 + (1 + 1/b) (E[\xi]/E[\xi_-])$, there exists unique $\delta^* > 0$ that solves $f(\delta^*) = 1$ such that $f(\delta) \geq 1$ for $\delta \leq \delta^*$. Note that $\delta^* = 1/W^*$, where $W^*$ solves $M(1 + \xi/W^*) + bE[\nu(\xi/W^*)] = 0$, so the proof completes. □

**Proof of Proposition 5.** Denote the constant consumption-wealth ratio as $1/c$, i.e., the
agent consumes a constant fraction $c$ of her wealth at each time. In equilibrium, the net supply of the risk-free asset is zero and the total consumption at time $t$ is $C_t$, so $S_t + N_t = W_t - C_t = (1 - c)W_t$, $\forall t$. On the other hand, because $S_t/(S_t + N_t) = \iota$, we have $N_t/S_t = 1/\iota - 1$ and $S_t/W_t = \iota(1 - c)$.

Denote $C_{N,t}$ and $C_{S,t}$ as the consumption good payout by the non-financial asset and by the stock, respectively, at time $t$. Then,

$$R_{N,t+1} : = \frac{N_{t+1} + C_{N,t+1}}{N_t} = \frac{W_{t+1} - C_{t+1}}{N_t} - \frac{S_{t+1} + C_{S,t+1}}{N_t} = \frac{W_{t+1} - (S_{t+1} + C_{S,t+1})}{N_t}$$

$$= \frac{(W_{t+1}/W_t)(W_t/S_t) - R_{S,t+1}}{N_t/S_t} = \frac{(C_{t+1}/C_t)/((\iota(1 - c)) - R_{S,t+1})}{1/\iota - 1}.$$  

Because $(\log(C_{t+1}/C_t), \log(R_{S,t+1}))$’s are i.i.d. over time, so are $(\log(R_{N,t+1}), \log(R_{S,t+1}))$’s.

Now, consider the optimal consumption and portfolio selection problem faced by the agent. Because the gross returns of the assets are i.i.d. over time, the agent’s optimal utility per unit wealth is a constant, and we denote it by $\Psi$. Then, we have the following dynamic programming equation:

$$\Phi = \max_{c \in I} H \left( c, (1 - c) \max_{\theta_N \in J_N, \theta_S \in J_S} D_\Phi(\theta_N, \theta_S) \right),$$

where $I$ is a subinterval of $(0, 1)$ and $J_N$ and $J_S$ are two subintervals of $\mathbb{R}$ that specify the feasible set of the agent’s strategies, and

$$D_\Phi(\theta_N, \theta_S) = \Phi \left[ \mathcal{M}(R_f + \theta_N (R_{N,t+1} - R_f) + \theta_S (R_{S,t+1} - R_f)) + bG(\theta_S(R_{S,t+1} - R_f)) \right];$$

see Corollary 1. Because the right-hand side of the dynamic programming is strictly concave in $c, \theta_N$ and $\theta_S$, the first-order condition is necessary and sufficient for the optimality of $c^*$, $\theta_N^*$ and $\theta_S^*$. Here, we assume $I, J_N$, and $J_S$ to contain $c^*, \theta_N^*$, and $\theta_S^*$, respectively.

Because we assume the net supplies of the non-financial asset and of the stock are positive, in equilibrium the optimal $\theta_N^*$ and $\theta_S^*$ must be positive as well. Then, taking the first order derivatives of $D_\Phi(\theta_N, \theta_S)$ with respect to $\theta_N$ and $\theta_S$, respectively, we have

$$\frac{\mathbb{E}[u'(R_{p,t+1}) (R_{N,t+1} - R_f)]}{u'(\mathcal{M}(R_{p,t+1}))} = 0,$$  

(B.2)

$$\frac{\mathbb{E}[u'(R_{p,t+1}) (R_{S,t+1} - R_f)]}{u'(\mathcal{M}(R_{p,t+1}))} + bG(R_{S,t+1} - R_f) = 0,$$  

(B.3)
where \( R_{p,t+1} := R_f + \theta^*_N (R_{N,t+1} - R_f) + \theta^*_S (R_{S,t+1} - R_f) \) is the optimal portfolio return. Multiplying \( \theta^*_N \) on both sides of (B.2), and noting that \( \theta^*_N (R_{N,t+1} - R_f) = R_{p,t+1} - R_f - \theta^*_S (R_{S,t+1} - R_f) \), we conclude

\[
\frac{\mathbb{E} [u' (R_{p,t+1}) R_{p,t+1}]}{u' (\mathcal{M}(R_{p,t+1}))} = (1 - \theta^*_S) R_f \frac{\mathbb{E} [u' (R_{p,t+1})]}{u' (\mathcal{M}(R_{p,t+1}))} + \theta^*_S \frac{\mathbb{E} [u' (R_{p,t+1}) R_{S,t+1}]}{u' (\mathcal{M}(R_{p,t+1}))}. \quad (B.4)
\]

Note that in equilibrium, the gross return of the optimal portfolio is the same as the growth rate of the wealth, i.e.,

\[ R_{p,t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{1}{1 - c^*} \frac{C_{t+1}}{C_t}. \]

Moreover, \( \theta^*_N + \theta^*_S = 1 \) and \( \theta^*_S = \ell \). In consequence, we conclude (5.1) and (5.2) from (B.3) and (B.4) immediately. Moreover, we have

\[
D\Phi(\theta^*_N, \theta^*_S) = \Phi [\mathcal{M}(R_{p,t+1}) + b u G(R_{S,t+1} - R_f)]
= \Phi \left[ \frac{1}{1 - c^*} \mathcal{M} \left( \frac{C_{t+1}}{C_t} \right) - \ell \frac{\mathbb{E} [u' (C_{t+1}/C_t) R_{S,t+1}]}{u' (\mathcal{M}(C_{t+1}/C_t))} - \frac{\ell^2}{1 - \ell} \frac{\mathbb{E} [u' (C_{t+1}/C_t) R_{S,t+1}]}{u' (\mathcal{M}(C_{t+1}/C_t))} \right.
+ \frac{\ell}{1 - \ell} \frac{1}{1 - c^*} \left. \frac{\mathbb{E} [u' (C_{t+1}/C_t) (C_{t+1}/C_t)]}{u' (\mathcal{M}(C_{t+1}/C_t))} \right]. \quad (B.5)
\]

Note that

\[
\Phi = \max_{c \in I} H (c, (1 - c) D\Phi(\theta^*_N, \theta^*_S)), \quad c^* = \arg\max_{c \in I} H (c, (1 - c) D\Phi(\theta^*_N, \theta^*_S)). \quad (B.6)
\]

Straightforward calculation yields that for each \( z > 0 \),

\[
\arg\max_{c \in (0,1)} H (c, (1 - c) z)) = 1 / \left[ 1 + (\beta/(1 - \beta))^{1/\rho} z^{(1-\rho)/\rho} \right], \quad (B.7)
\]
\[
\max_{c \in (0,1)} H (c, (1 - c) z)) = (1 - \beta) \left[ 1 + (\beta/(1 - \beta))^{1/\rho} z^{(1-\rho)/\rho} \right]. \quad (B.8)
\]

Recalling \( D\Phi(\theta^*_N, \theta^*_S) \) in (B.5) and setting \( z = D\Phi(\theta^*_N, \theta^*_S) \) in (B.7) and (B.8), we can solve \( \Phi \) and \( c^* \), provided that \( I \) contains \( c^* \). In particular,

\[
1 - c^* = \left[ \beta (D\Phi(\theta^*_N, \theta^*_S))^{1/\rho} \right]^{1/\rho},
\]

which yields (5.3).
Finally, according to Theorem 1, for the total utility process of the agent to exist when she takes the optimal consumption and investment strategy, we need to assume 
\[-b\theta^*_S G(R_{S,t+1} - R_f) < \mathcal{M}(R_{p,t+1}) \text{ and } \beta \delta^{1-\rho} < 1,\]
where \(\delta := (1 - c^*)[\mathcal{M}(R_{p,t+1}) + b\theta^*_S G(R_{S,t+1} - R_f)]\). Recalling \((1 - c^*)\mathcal{M}(R_{p,t+1}) = \mathcal{M}(C_{t+1}/C_t)\) and \(\theta^*_S = \iota\), the proof completes. \(\square\)

References


