Equilibrium Asset Pricing with Epstein-Zin and Loss-Averse Investors *

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Abstract

We study multi-period equilibrium asset pricing in an economy with EZ-agents whose preferences for consumption are represented by recursive utility and with LA-agents who experience additional utility of trading gains and losses and are averse to losses. We propose an equilibrium gain-loss ratio for stocks and show that the LA-agents hold less (more) stocks than the EZ-agents if and only if the LA-agents' loss aversion degree is larger (smaller) than this ratio. With myopic EZ- and LA-agents, we prove the existence and uniqueness of the equilibrium and the market dominance of the EZ-agents in the long run. Finally, we find that the equity premiums in this economy and in another economy with a representative agent whose preferences are the average of those of the EZ- and the LA-agents in the former economy are quantitatively similar if the LA-agents participate in the stock market and can be significantly different otherwise.

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1 Introduction

In neoclassical finance, investors are assumed to be rational, i.e., to use Bayes' rule for inference and prediction and to employ expected utility theory (von Neumann and Morgenstern, 1947) and more generally recursive utility theory (Epstein and Zin, 1989) to evaluate wealth and consumption. By contrast, behavioral finance takes the view that investors are irrational; i.e., investors do not follow Bayes' rule or recursive utility theory. In particular, investors' preferences for wealth are represented by cumulative prospect theory (Tversky and Kahneman, 1992). There is a vast literature on asset pricing in behavioral finance; most models in this literature assume a representative agent with irrational preferences. The implications in those models can explain many empirical findings that cannot be explained by models in neoclassical finance.[1]

Asset pricing with both rational and irrational investors, however, is rarely studied. Consequently, the following four questions have not been completely answered: First, does the market equilibrium exist with heterogenous rational and irrational investors? Second, what is the difference between the investment strategies of the rational and irrational investors? Third, will the rational investors dominate the market in the long run? Finally, are the equilibrium asset prices quantitatively similar in an economy with heterogeneous

[1]For instance, Barberis and Huang (2001) and Barberis et al. (2001) assume a representative agent with loss-averse preferences to explain the equity premium puzzle. Barberis et al. (1998) assume a representative agent with wrong belief in stock dividends to explain the under-reaction and over-reaction to public news such as earning announcement. Barberis and Huang (2008b) assume a representative agent whose preferences are represented by the cumulative prospect theory to explain the low risk premium for assets with positively skewed payoffs.

rational and irrational investors and in another economy with a representative investor? In the present paper, we answer these questions based on a multi-period heterogeneous-agent asset pricing model.

We consider an economy with two types of agents: EZ- and LA-agents. A risk-free asset and a risky stock can be traded in discrete time. The EZ-agents' preferences for consumption are represented by the recursive utility theory. The LA-agents are concerned with the utility of trading gains and losses in addition to the consumption utility, and the gain-loss utility is measured by the cumulative prospect theory. In particular, the LA-agents are averse to losses. We assume the EZ- and LA-agents have the same relative risk aversion degree (RRAD) and elasticity of intertemporal substitution (EIS) for the consumption utility, but the LA-agents may have heterogeneous loss aversion degrees (LADs). The return rates of the risk-free asset and the risky stock are to be determined in equilibrium.

We first propose a performance measure for the stock, named equilibrium gain-loss ratio. It is defined as the ratio of the gain and loss of the stock relative to a reference point. This reference point is endogenously determined in equilibrium: it is the equilibrium risk-free rate in an economy with EZ-agents only. We find that the equilibrium gain-loss ratio is always larger than one and the LA-agents have the same consumption-investment plan as the EZ-agents when the former's LAD is equal to this ratio.

Secondly, when the RRAD and EIS are equal to one, i.e., when the EZ- and LA-agents are myopic, we prove the existence and uniqueness of the equilibrium. Moreover, when the economy consists of only one EZ- and one LA-agents, we find that the LA-agent invests less (more) than the EZ-agent in the stock if and only if the LA-agent's LAD is larger (smaller) than the equilibrium gain-loss ratio. Consequently, the equity premium is increasing (decreasing) with respect to the market share of the LA-agent when her LAD is larger (smaller) than the equilibrium gain-loss ratio. Our numerical study suggests that these conclusions still hold for non-myopic agents.

Thirdly, when the economy consists of only one EZ- and one LA-agents with unit RRAD and EIS, we prove that the EZ-agent dominates the market in the long run, i.e., the market share of the EZ-agent converges to one as time goes to infinity. Our simulation results, however, show that the dominance takes effect very slowly: the LA-agent still has significant market share and price impact even after two hundred years. Our numerical study shows that these market dominance results still hold for non-myopic agents.

Fourthly, we compare the equity premiums in an economy with heterogeneous EZ- and LA-agents and in another economy with a representative agent whose preference representation is the average of those of the agents in the first economy. We find that the equity premiums in these two economies are quantitatively close if the LA-agents in the heterogeneous-agent economy participate in the stock market; otherwise, the equity premiums can be significantly different. Indeed, in the latter case the parameters of the LA-agents' preference representation, e.g., the LAD and the parameter measuring how much the LA-agents are concerned with the gain-loss utility, do not affect the price impact of the LA-agents in the heterogeneous-agent economy, and thus do not affect the equity premium. By contrast, the equity premium in the representative-agent economy is sensitive to these parameters because they decide the preferences of the representative agent. Consequently, the equity premiums in these two economies can be significantly different.

2 Literature Review

The preference representation of the LA-agents in our model is similar to Barberis and Huang (2001, 2008a, 2009), Barberis et al. (2001, 2006), De Giorgi and Legg (2012), and He and Zhou (2014). These papers either consider asset pricing with a representative agent or consider portfolio selection of one agent, but the present paper considers asset pricing with heterogeneous agents.

There are several studies on asset pricing with heterogeneous rational and irrational agents. Del Vigna (2013) consider a single-period asset pricing model in which the agents have heterogeneous preferences represented by expected utility theory (EUT) and cumulative prospect theory (CPT). Assuming the asset returns follow a multi-variate normal distribution, the authors prove the existence of the equilibrium. De Giorgi et al. (2011) and De Giorgi and Hens (2006) consider a similar model and study the equilibrium asset returns. De Giorgi et al. (2010) consider a single-period complete market with heterogeneous agents with CPT preferences and show that the equilibrium does not always exist. Xia and Zhou (2013) study the equilibrium in a single-period complete market in which the agents' preferences are represented by rank-dependent expected utility (Quiggin, 1982). The authors assume heterogeneous utility functions but homogeneous probability weighting functions for the agents and prove the existence of the equilibrium. All the aforementioned papers assume single-period setting and assume either normally distributed asset returns or complete markets. The present paper uses multi-period setting and the dividend growth rate can follow any distribution.

There are two closely related papers. Easley and Yang (2014) consider equilibrium asset pricing with one EZ- and one LA-agents, and their setting is similar to ours. The authors conduct numerical studies to show that the EZ-agent dominates the market in the long run when she has the same RRAD and EIS as the LA-agent. When the EZ- and LA-agents have different RRAD and EIS, however, the LA-agent may dominate the market. Compared to Easley and Yang (2014), we prove the existence and uniqueness of the equilibrium and the dominance of the EZ-agent theoretically when the RRAD and EIS are equal to one. In addition, we propose the equilibrium gain-loss ratio and study the trading strategies of the EZ- and LA-agents.

Chapman and Polkovnichenko (2009) consider a single-period asset pricing model with two agents: one agent has EUT preferences and the other has non-EUT preferences. The authors consider several types of non-EUT preferences such as rank-dependent utility, disappointment aversion, ambiguity aversion, and reference dependent preferences. These preferences exhibit first-order risk aversion (Segal and Spivak, 1990) and are thus appealing models to explain the equity premium puzzle in the representative-agent setting. Chapman and Polkovnichenko (2009) show that the equilibrium equity premium with one EUT and one non-EUT agents is significantly lower than the equity premium in an economy with a representative agent whose preferences are the average of those of the EUT and non-EUT agents. The present paper differs from Chapman and Polkovnichenko (2009) in two aspects: First, we consider a multi-period model. Second, we also consider heterogeneity in the LADs of the LA-agents while Chapman and Polkovnichenko (2009) assume homogeneous non-EUT agents.

3 Equilibrium Asset Pricing Model

3.1 The Market

We consider a discrete-time financial market with one risky asset (e.g., a stock) and one risk-free asset. The net supplies of the stock and the risk-free asset are one and zero, respectively. The stock distributes dividend D_t at each time t. The (gross) return rates of the stock and the risk-free asset in period t to $t+1$ are R_{t+1} and $R_{f,t+1}$, respectively. The following assumption about the dividend process is imposed throughout of the paper:

Assumption 1 *The dividend growth rates $Z_{t+1} := D_{t+1}/D_t, t = 0, 1, \dots$ are i.i.d. Furthermore, $z_0 := \text{essinf}\, Z_t > 0$ and $\mathbb{E}\,(Z_t) < \infty$.*

Assuming an i.i.d. dividend process is common in many asset pricing models. On the other hand, assumption $\text{essinf}\, Z_t > 0$ is necessary to induce heterogeneity in stock holding; otherwise, any agent in the market cannot invest more than her wealth in the stock.

Consider an agent who decides her consumption and investment plans at each time t. Denote W_t as the pre-consumption wealth of the agent. Suppose the agent consumes C_t, invests θ_t in the stock, and invests the remaining wealth in the risk-free asset. Then, the dynamics of $\{W_t\}$ is $W_{t+1} = (W_t - C_t)R_{f,t+1} + \theta_t(R_{t+1} - R_{f,t+1})$, $t \geq 0$.

3.2 Investors

Suppose there are m agents in the market. At each time t, agent i chooses consumption amount $C_{i,t}$ and the dollar amount $\theta_{i,t}$ invested in the stock. Following Barberis and Huang (2008a, 2009), Barberis et al. (2006), De Giorgi and Legg (2012), and He and Zhou (2014), we assume agent i's preferences are represented by the utility process $\{X_{i,t}\}$ modeled as

$$X_{i,t} = H(C_{i,t}, M(X_{i,t+1}|\mathscr{F}_t) + b_{i,t}G_{i,t}), \quad t \geq 0. \tag{1}$$

Here, $\mathscr{F}_t$ stands for the information available at time t, $M(X_{i,t+1}|\mathscr{F}_t)$ is time-t's certainty equivalent of agent i's utility at $t+1$, $G_{i,t}$ represents the utility of trading gains and losses, $b_{i,t}$ measures the degree to which agent i is concerned with the gain-loss utility, and H aggregates the current-period consumption, the gain-loss utility in the current period, and the total utility in the following periods.

We apply cumulative prospect theory (CPT) proposed by Tversky and Kahneman (1992) to model $G_{i,t}$. More precisely, we assume

$$G_{i,t} := \mathbb{E}\left[\nu_i(W_{i,t+1} - (W_{i,t} - C_{i,t})R_{f,t+1})|\mathscr{F}_t\right] = \mathbb{E}\left[\nu_i\left(\theta_{i,t}(R_{t+1} - R_{f,t+1})\right)|\mathscr{F}_t\right], \tag{2}$$

where $\nu_i(x) := x\mathbf{1}_{x\geq 0} + K_i x\mathbf{1}_{x<0}$ for some $K_i \geq 1$. In other words, agent i sets the risk-free payoff $(W_{i,t} - C_{i,t})R_{f,t+1}$ to be her *reference point* to distinguish gains and losses for her investment, and then the investment gain and loss are evaluated by CPT with piece-wise linear utility function ν_i and with no probability weighting (Tversky and Kahneman, 1992). The parameter $K_i \geq 1$, named *loss aversion degree* (LAD), models the empirical finding that individuals tend to be more sensitive to losses than to comparable gains. The model of $G_{i,t}$ is the same as in Barberis and Huang (2008a, 2009), Barberis et al. (2006), and De Giorgi and Legg (2012)

When $b_{i,t} \equiv 0$, (1) becomes recursive utility (Epstein and Zin, 1989, Kreps and Porteus, 1978). In this case, we call agent i an *EZ-agent*. When $b_{i,t} > 0$, agent i's aggregate utility includes two components: consumption utility and gain-loss utility, and the agent is loss averse. In this case, we call agent i an *LA-agent*, and $b_{i,t}$ measures how much the LA-agent is concerned with the gain-loss utility.

The recursive utility is considered to be a rational theory of preference representation, so the EZ-agents can be regarded as rational investors. On the other hand, the LA-agents experience additional utility of trading gains and losses, and are loss averse. Such preference representation has been used in many asset pricing models in behavioral finance such as Barberis and Huang (2001, 2008a) and Barberis et al. (2001), so the LA-agents can be regarded as irrational investors.

Following Kreps and Porteus (1978), we assume

$$H(c,z) := \begin{cases} [(1-\beta)c^{1-\rho} + \beta z^{1-\rho}]^{\frac{1}{1-\rho}}, & 0 < \rho \neq 1, \\ e^{(1-\beta)\ln c + \beta \ln z}, & \rho = 1, \end{cases} \qquad M(X) := \begin{cases} [\mathbb{E}(X^{1-\gamma})]^{\frac{1}{1-\gamma}}, & 0 < \gamma \neq 1, \\ e^{\mathbb{E}[\ln(X)]}, & \gamma = 1. \end{cases} \tag{3}$$

In this specification, β is the discount factor, γ is the relative risk aversion degree (RRAD), and $1/\rho$ is the elasticity of intertemporal substitution (EIS).

In our model, we assume the agents in the market are homogeneous in the aggregator H and in the certainty equivalent M, so these two quantities are not indexed by the agents' identities. However, the agents are heterogeneous in $b_{i,t}$ and in K_i. In other words, we assume homogeneous EZ-agents and heterogeneous LA-agents and assume all the agents have the same RRAD and EIS.

3.3 Optimal Portfolio Selection

For simplicity, we assume short-selling is not allowed. With preference representation (1), agent i's decision problem at t can be formulated as

$$\begin{aligned} \underset{\{C_{i,s}\}_{s\geq t},\{\theta_{i,s}\}_{s\geq t}}{\text{Max}} \quad & X_{i,t} \\ \text{Subject to} \quad & W_{i,s+1} = (W_{i,s} - C_{i,s})R_{f,s+1} + \theta_{i,s}\left(R_{s+1} - R_{f,s+1}\right), \\ & \theta_{i,s} \geq 0,\ W_{i,s} \geq 0, \quad s \geq t. \end{aligned} \tag{4}$$

We follow Barberis and Huang (2009) and He and Zhou (2014) to solve problem (4). Denote $V_{i,t}$ as the optimal value of (4), i.e., the agent's optimal utility at time t. Thanks to the recursive nature of utility (1), the dynamic programming principle immediately yields the following Bellman equation

$$V_{i,t} = \max_{C_{i,t},\theta_{i,t}} H(C_{i,t}, M(V_{i,t+1}|\mathscr{F}_t) + b_{i,t}G_{i,t}). \tag{5}$$

Thanks to the homogeneity of the aggregator H, certainty equivalent M, and CPT utility function ν_i, we are able to simplify the Bellman equation (5) by defining $c_{i,t} := C_{i,t}/W_{i,t}$, $w_{i,t} := \theta_{i,t}/(W_{i,t} - C_{i,t})$, and $\Psi_{i,t} := V_{i,t}/W_{i,t}$, which are the percentage consumption, percentage allocation to the stock, and optimal utility per unit wealth, respectively. Plugging these variables into (5), we obtain

$$\Psi_{i,t} = \max_{c_{i,t},w_{i,t}} H(c_{i,t}, (1-c_{i,t})\left(M(\Psi_{i,t+1}R_{i,t+1}|\mathscr{F}_t) + b_{i,t}\mathbb{E}\left[\nu_i\left(w_{i,t}(R_{t+1}-R_{f,t+1})\right)|\mathscr{F}_t\right]\right)), \tag{6}$$

where $R_{i,t+1} := R_{f,t+1} + w_{i,t}(R_{t+1} - R_{f,t+1})$ is the gross return of the agent's portfolio. Here, the implicit restrictions on $c_{i,t}$ and $w_{i,t}$ are that $c_{i,t} \geq 0$, $w_{i,t} \geq 0$, and $R_{i,t+1} \geq 0$.

3.4 Equilibrium

Denote P_t as the ex-dividend price of the stock at time t. Then, the gross return of the stock is $R_{t+1} = (P_{t+1} + D_{t+1})/P_t, t \geq 0$. We follow the standard definition of competitive equilibria in the literature; see for instance Yan (2008).

Definition 1 A *competitive equilibrium* is a price system $\{R_{f,t+1}, P_t\}_{t=0}^{\infty}$ and consumption-investment plans $\{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$ with the corresponding wealth processes $\{W_{i,t}\}_{t=0}^{\infty}$, $i = 1, 2, \ldots, m$ that satisfy

(i) *individual optimality*: for each $i = 1, 2, \ldots, m$, $\{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$ is the optimal consumption-investment plan of agent i;

(ii) *clearing of consumption*: $\sum_{i=1}^{m} C_{i,t} = D_t$;

(iii) *clearing of the stock*: $\sum_{i=1}^{m} \theta_{i,t} = P_t$; and

(iv) *clearing of the risk-free asset*: $\sum_{i=1}^{m} (W_{i,t} - C_{i,t} - \theta_{i,t}) = 0$.

As illustrated in Section 3.3, it is natural to consider the percentage consumption and allocation to the stock of each agent, so we reformulate the equilibrium conditions in Definition 1 in terms of these quantities.

Proposition 1 *A price system* $\{R_{f,t+1}, P_t\}_{t=0}^{\infty}$ *and consumption-investment plans* $\{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$ *with corresponding wealth process* $\{W_{i,t}\}_{t=0}^{\infty}$, $i = 1, 2, \dots, m$ *constitute a competitive equilibrium if and only if the percentage consumption* $c_{i,t} := C_{i,t}/W_{i,t}$ *and the percentage allocation to the stock* $w_{i,t} := \theta_{i,t}/(W_{i,t} - C_{i,t})$ *are the optimal choice of agent* i, $i = 1, 2, \dots, m$ *and satisfy*

$$\sum_{i=1}^{m} \frac{c_{i,t}}{1 - c_{i,t}} Y_{i,t} = D_t/P_t, \quad \sum_{i=1}^{m} w_{i,t} Y_{i,t} = 1, \tag{7}$$

where $Y_{i,t} := (W_{i,t} - C_{i,t})/P_t$ *stands for agent* i*'s post-consumption market share at time* t, *and* $(c_{i,t}/(1 - c_{i,t}))Y_{i,t}$ *is defined as* $c_{i,t}\tilde{Y}_{i,t}$ *with* $\tilde{Y}_{i,t} = W_{i,t}/P_t$ *when* $c_{i,t} = 1$.

3.5 Data and Parameter Values

One of the main tasks of the present paper is to compare an economy with heterogeneous EZ- and LA-agents with an economy with a representative agent whose preferences are the average of those of the agents in the first economy. To compare our results to those in Chapman and Polkovnichenko (2009), we use the same dividend data as in that paper. We reproduce in Table 1 the distribution of the dividend growth rate used in Chapman and Polkovnichenko (2009).

Table 1: Distribution of the dividend growth rate. The distribution is assumed to be the same as in Table I of Chapman and Polkovnichenko (2009), which is obtained using the historical gross consumption growth from 1949 to 2006.

State	1	2	3	4	5	6	7	8	9
Outcome	0.976	0.993	1.002	1.011	1.019	1.028	1.037	1.045	1.054
Probability	0.03	0.03	0.10	0.16	0.24	0.19	0.13	0.09	0.03

For RRAD γ and EIS ρ, we assume $\gamma = \rho = 1$ in Section 5 where we study the equilibrium theoretically. In Section 6, we consider other values of γ and ρ ranging from

1.5 to 5; these values are also considered in Barberis and Huang (2008a, 2009) and Barberis et al. (2006).

For the LAD of the LA-agents, we use experimental estimates in the literature. The range of the estimates is between 1.5 and 3.5; see for instance the literature summary in He and Kou (2014). In addition, we use 2.25, which is the estimate obtained by Tversky and Kahneman (1992), as a benchmark value.

Finally, we assume $b_{i,t}$ to take the following parametric form:

$$b_{i,t} = b_i \times M(\Psi_{i,t+1}|\mathscr{F}_t), \quad t \geq 0 \tag{8}$$

for some constant $b_i \geq 0$. In other words, $b_{i,t}$ in the current period is a constant proportion of the certainty equivalent of the optimal utility per unit wealth in the following periods. Our setting of $b_{i,t}$ is different from those in Barberis and Huang (2008a, 2009), Barberis et al. (2006), De Giorgi and Legg (2012), and He and Zhou (2014), where the authors set

$$b_{i,t} = \bar{b}_i, \quad t \geq 0 \tag{9}$$

for some constant $\bar{b}_i \geq 0$. Our setting seems less natural than theirs, but it leads to tractability in the equilibrium analysis in Section 5. Furthermore, numerical studies in Section 6 reveal that these two settings yield nearly identical asset pricing results. In addition, $b_{i,t}$ in (8) resembles the models in Barberis and Huang (2001) and Barberis et al. (2001).[2]

Barberis and Huang (2008a, 2009) and Barberis et al. (2006) formulate $b_{i,t}$ as in (9) and consider values from 0 to 0.5 for $\bar{b}$. Moreover, they consider asset pricing with a representative LA-agent. We calibrate parameter b in (8) to $\bar{b}$ ranging from 0 to 0.5; i.e.,

[2] In these papers, the authors set $b_{i,t}$ to be a constant proportion of a power transformation of the aggregate consumption in the market at time t.

Table 2: Values of b corresponding to values of $\bar{b}$. For each value of $\bar{b}$, $b = \bar{b}/\Psi$, where Ψ is the certainty equivalent of the optimal utility per unit wealth in an economy with one LA-agent only. The distribution of the dividend growth rate is given as in Table 1, $K = 2.25$, and γ and ρ are 1, 3, or 5.

$\bar{b}$	0.005	0.01	0.02	0.03	0.04	0.05	0.1	0.2	0.3	0.4	0.5
$\gamma = 1, \rho = 1$	0.09	0.19	0.39	0.60	0.81	1.03	2.13	4.37	6.63	8.89	11.14
$\gamma = 1, \rho = 3$	0.15	0.29	0.60	0.90	1.21	1.52	3.07	6.19	9.31	12.44	15.56
$\gamma = 1, \rho = 5$	0.17	0.34	0.69	1.03	1.38	1.73	3.49	7.00	10.52	14.04	17.56
$\gamma = 3, \rho = 1$	0.09	0.19	0.39	0.60	0.81	1.02	2.11	4.32	6.55	8.77	11.00
$\gamma = 3, \rho = 3$	0.15	0.30	0.60	0.90	1.21	1.52	3.06	6.17	9.27	12.38	15.49
$\gamma = 3, \rho = 5$	0.17	0.34	0.69	1.03	1.38	1.73	3.48	6.99	10.50	14.00	17.51
$\gamma = 5, \rho = 1$	0.09	0.19	0.39	0.60	0.81	1.02	2.10	4.28	6.48	8.68	10.87
$\gamma = 5, \rho = 3$	0.15	0.30	0.60	0.90	1.21	1.51	3.05	6.14	9.24	12.33	15.42
$\gamma = 5, \rho = 5$	0.17	0.34	0.69	1.03	1.38	1.73	3.47	6.97	10.47	13.97	17.47

for each value of $\bar{b}$ in this range, we calculate b accordingly by setting $b \times M(\Psi_{i,t+1}|\mathscr{F}_t) = \bar{b}$, where $\Psi_{i,t+1}$ is the certainty equivalent of the optimal utility per unit wealth in an economy with one LA-agent only.[3] In the calculation, we set $K = 2.25$ and consider γ and ρ to be one of 1, 3, and 5, and the results are shown in Table 2. We can see that the values of b range from 0 to 17.5, so in the following we assume b takes values from 0 to 20.

4 Equilibrium Gain-Loss Ratio

Theorem 2 *Assume $\beta(M(Z_{t+1}))^{1-\rho} < 1$. Then, $R_{t+1} = \alpha Z_{t+1}$ for certain constant α and $R_{f,t+1} = \mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$ are the equilibrium return rates of the stock and the risk-free asset, respectively, if the market consists of EZ-agents only. Moreover, the equilibrium remains the same if the market consists of not only EZ-agents but also LA-agents with*

[3]In this economy, $\Psi_{i,t+1}$ is a constant Ψ; see for instance Guo and He (2015). Consequently, we can solve $b = \bar{b}/\Psi$.

LAD equal to

$$K^* := \frac{\mathbb{E}\left[\left(R_{t+1} - \left(\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\right)\right)_+\right]}{\mathbb{E}\left[\left(R_{t+1} - \left(\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)\right)\right)_-\right]}, \tag{10}$$

and in this case the EZ- and LA-agents have the same consumption-investment plan.

Condition $\beta(M(Z_{t+1}))^{1-\rho} < 1$ in Theorem 2 is to ensure that the recursive utility of consuming the stock dividend is well-defined; see for instance Epstein and Zin (1989), Marinacci and Montrucchio (2010), and Hansen and Scheinkman (2012) for conditions in general settings.[4]

Theorem 2 provides the equilibrium asset returns when the market consists of EZ-agents only. More importantly, Theorem 2 shows that the presence of LA-agents in the market does not change the equilibrium if their LAD is equal to K^*. In addition, the EZ- and LA-agents have the same consumption-investment strategy in this case. Intuitively, the LA-agents invest less in the stock when becoming more loss averse, so we conjecture that they hold less (more) stocks than the EZ-agents if their LAD is larger (smaller) than K^*. This conjecture is proved in the case $\rho = \gamma = 1$ (Section 5) and numerically confirmed in the general case (Section 6).

We can see that K^* is the ratio of the expected gain and loss of the stock with the reference point being $\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$. Therefore, K^* is similar to the gain-loss ratios proposed by Bernardo and Ledoit (2000) and Cherny and Madan (2009). Note that the reference point $\mathbb{E}\left(R_{t+1}^{1-\gamma}\right)/\mathbb{E}\left(R_{t+1}^{-\gamma}\right)$ is not arbitrarily chosen; it is the equilibrium risk-free rate in an economy with EZ-agents only and thus is endogenously determined in equilibrium. This marks the difference of K^* from the gain-loss ratios in Bernardo and Ledoit (2000) and Cherny and Madan (2009) where the reference points are exogenously given. Therefore, we name K^* *equilibrium gain-loss ratio.*

[4] For instance, with i.i.d. dividend growth rates, condition (c) in Hansen and Scheinkman (2012, Proposition 6) is equivalent to $\beta(M(Z_{t+1}))^{1-\rho} < 1$ and other assumptions in that Proposition are automatically satisfied.

Note that K^* depends on γ, the RRAD of the agents in the market, but does not depend on $1/\rho$, the EIS of the agents. Moreover, the following Proposition shows that K^* is always larger than 1 and is strictly increasing in γ. Therefore, the more risk averse the agents are, the larger the equilibrium gain-loss ratio is.

Proposition 3 *The equilibrium gain-loss ratio K^* is strictly increasing in $\gamma \geq 0$ and is equal to 1 when $\gamma = 0$.*

We can see that K^* is invariant to scaling: replacing R_{t+1} in (3) with CR_{t+1} for any constant $C > 0$ does not change the value of K^*. Consequently, because $R_{t+1} = \alpha Z_{t+1}$, we can rewrite K^* as

$$K^* = \frac{\mathbb{E}\left[\left(X - \left(\mathbb{E}\left(X^{1-\gamma}\right)/\mathbb{E}\left(X^{-\gamma}\right)\right)\right)_+\right]}{\mathbb{E}\left[\left(X - \left(\mathbb{E}\left(X^{1-\gamma}\right)/\mathbb{E}\left(X^{-\gamma}\right)\right)\right)_-\right]},$$

where X is identically distributed as Z_{t+1}. Table 3 shows the values of K^* with respect to different values of γ, assuming the dividend growth-rate follows the distribution as specified in Table 1. Because typically LAD K is larger than 1.5, so Table 3 shows that $K^* < K$. Consequently, LA-agents with typical LAD hold less stocks than EZ-agents.

Table 3: Equilibrium gain-loss ratio K^* with respect to RRAD γ. The distribution of the dividend growth rate Z_{t+1} is assumed as in Table 1.

γ	0.5	0.9	.95	1.05	1.1	1.5	2	3	4	5
K^*	1.021	1.038	1.040	1.044	1.046	1.063	1.086	1.132	1.182	1.234

Although, the equilibrium gain-loss ratio is defined as a result of equilibrium analysis, it can also be used empirically: for each stock in the market, one can compute its equilibrium gain-loss ratio. In this sense, the equilibrium gain-loss ratio can be used as a measure for stock selection: the larger the equilibrium gain-loss ratio of a stock is, the better the stock is. For example, when the stock return R_{t+1} follows Bernoulli distribution, i.e., $\mathbb{P}(R_{t+1} = x_1) = 1 - \mathbb{P}(R_{t+1} = x_2) = p$ for some $x_1 > x_2$ and $p \in (0, 1)$, we can compute that

$K^* = (x_1/x_2)^\gamma$. If the return of a stock becomes more positively skewed, i.e., x_1 becomes larger and and p becomes smaller, K^* becomes larger. In other words, stocks with more positively skewed returns have larger equilibrium gain-loss ratios.

5 Equilibrium Analysis with Myopic Agents

In this section, we study the competitive equilibrium when $\rho = \gamma = 1$. In this case, each agent in the market becomes myopic; i.e., in each period her optimal consumption and investment depend only on the asset returns in that period. We first study the investment problem of a typical agent in the market. Then, we establish the existence and uniqueness of the equilibrium. Afterwards, we study the equilibrium with one EZ-agent and one LA-agent. Finally, we compare the equity premium in this heterogeneous-agent economy with the one in a representative-agent economy.

5.1 Optimal Portfolio

A typical myopic agent in the market solves the following single-period portfolio choice problem

$$\begin{aligned} \max_{w \geq 0} \quad & \exp\left[\mathbb{E}(\ln(a + w(X - a)))\right] + wb\mathbb{E}\left[\nu\left(X - a\right)\right] \\ \text{subject to} \quad & a + w(X - a) \geq 0, \end{aligned} \tag{11}$$

where a stands for the risk-free rate and X stands for the gross return of the stock. Function $\nu(x) := x\mathbf{1}_{x\geq 0} + Kx\mathbf{1}_{x<0}$ for some $K \geq 1$ is used to evaluate gains and losses and parameter $b \geq 0$ measures how much the agent is concerned with the gain-loss utility.

Proposition 4 *Assume* $\underline{x} := \operatorname{essinf} X > 0$ *and* $\mathbb{E}[X] < \infty$*. Suppose* $a > \underline{x}$*. Then* (11) *admits unique optimal solution* $\varphi(a; b, K)$*. Furthermore,* *(i) There exist $\underline{a} < \overline{a}$ in $[\underline{x}, \mathbb{E}(X)]$ such that $\varphi(a; b, K) = \bar{w}(a) := a/(a - \underline{x})$ if and only if $a \in (\underline{x}, \underline{a}]$ and $\varphi(a; b, K) = 0$ if and only if $a \in [\overline{a}, +\infty)$.*

(ii) For fixed $b \geq 0$ and $K \geq 1$, $\varphi(a; b, K)$ is decreasing and continuous in $a \in (\underline{x}, +\infty)$, is strictly decreasing in a when $\varphi(a; b, K) > 0$, and satisfies $\lim_{a \downarrow \underline{x}} \varphi(a; b, K) = +\infty$.

(iii) For fixed $b \geq 0$ and $a > \underline{x}$, $\varphi(a; b, K)$ is decreasing in K.

(iv) For fixed $K \geq 1$ and $a > \underline{x}$, $\varphi(a; b, K)$ is increasing in b if $\mathbb{E}[\nu(X - a)] \geq 0$ and is decreasing in b if $\mathbb{E}[\nu(X - a)] \leq 0$.

Note that $\bar{w}(a)$ is the maximum percentage allocation to the stock because of the no-bankruptcy constraint $a + w(X - a) \geq 0$. Therefore, Proposition 4-(i) shows that the agent does not invest in the stock if the risk-free rate is sufficiently high and takes the maximum leverage if the risk-free rate is sufficiently low. Proposition 4-(ii) shows that the optimal allocation to the stock is continuous and decreasing in the risk-free rate. Proposition 4-(iii) shows that the more loss averse the agent is, the less she invests in the stock. Finally, Proposition 4-(iii) reveals that when the agent becomes more concerned with the gain-loss utility, i.e., when b becomes larger, whether she invests more in the stock depends on whether she experiences positive or negative gain-loss utility of investing in the stock: if positive utility is experienced, she invests more in the stock; otherwise, she invests less.

5.2 Existence and Uniqueness of Equilibrium

Theorem 5 *Suppose $\rho = \gamma = 1$ and $b_{i,t}$ is given as in (8). Then, the competitive equilibrium exists and is unique. The equilibrium price-dividend ratio $P_t/D_t \equiv \beta/(1-\beta), t \geq 0$ and the equilibrium stock return $R_{t+1} = Z_{t+1}/\beta$. The equilibrium risk-free rate $R_{f,t+1}$ is* *uniquely determined by*

$$\sum_{i=1}^{m} \varphi_i(R_{f,t+1}) Y_{i,t} = 1, \tag{12}$$

where $Y_{i,t} \in [0,1]$ *is the market share of agent* i*'s post-consumption wealth at time* t *and* $\varphi_i(a)$ *is the optimal solution to* (11) *with* $X = R_{t+1}$, $b = b_i$, *and* $K = K_i$. *Furthermore,* $\underline{x} < R_{f,t+1} < \mathbb{E}[R_{t+1}]$ *where* $\underline{x} := \operatorname{essinf} R_{t+1}$.

Because $\rho = \gamma = 1$, every agent in the market consumes the same constant fraction of her wealth. As a result, the equilibrium price-dividend ratio must be a constant and the equilibrium stock return

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}/D_{t+1} + 1}{P_t/D_t} \frac{D_{t+1}}{D_t} = \frac{1}{\beta} Z_{t+1}.$$

Furthermore, because Z_{t+1}'s are i.i.d., so are R_{t+1}'s. The risk-free rate is determined by (12), which is a clearing condition for the stock. We can see that the equilibrium risk-free rate depends on the market shares of the agents.

5.3 Equilibrium Analysis with EZ- and LA-Agents

In this subsection, we consider the case in which there are only two agents in the market: one is an EZ-agent and the other is an LA-agent. We index the EZ-agent with $i = 0$ and set $b_0 = 0$, and index the LA-agent with $i = 1$ and set $b_1 = b > 0$ and $K_1 = K \geq 1$. In addition, we denote $\nu(x) = x 1_{x \geq 0} + K x \mathbf{1}_{x<0}$ as the utility function of the LA-agent for trading gains and losses.

5.3.1 Optimal investment

Theorem 6 *Let $w^*_{0,t}$ and $w^*_{1,t}$ be the optimal percentage allocation to the stock of the EZ- and LA-agents, respectively.*

(i) If $K = K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t\right] = 0$ and $w^*_{0,t} = w^*_{1,t} = 1$.

(ii) If $K < K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t\right] > 0$ and $0 < w^*_{0,t} < 1 < w^*_{1,t}$.

(iii) If $K > K^*$, then $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t\right] < 0$, and $w^*_{0,t} > 1 > w^*_{1,t} \geq 0$.

Theorem 6-(i) shows that the LA-agent invests the same amount in the stock as the EZ-agent when her LAD is K^*. This result is consistent with Theorem 2. Theorem 6-(ii) shows that the gain-loss utility of holding the stock, $\mathbb{E}\left[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t\right]$, is positive if the LA-agent's LAD is strictly smaller than K^*. In this case, the LA-agent is willing to hold more stocks than the EZ-agent. This result reveals that an LA-agent, though being loss averse, can take more risk than an EZ-agent who is not concerned with trading gains and losses. If $K > K^*$, the gain-loss utility of holding the stock is negative and thus the LA-agent holds less stocks than the EZ-agent. Table 3 shows that the LAD of a typical LA-agent is larger than K^*, so she holds less stocks than EZ-agents.

We also observe that the EZ-agent always holds some stocks in equilibrium, but the LA-agent may choose not to hold any stocks.

5.3.2 Equity Premium

The conditional *equity premium* is the expected return of the stock in excess of the risk-free rate, i.e., $\mathrm{EP}_t := \mathbb{E}[R_{t+1}|\mathscr{F}_t] - R_{f,t+1}$, $t \geq 0$. Because $R_{t+1} = Z_{t+1}/\beta$, $t \geq 0$ are i.i.d., the variation of the equity premium caused by the market shares of the LA- and EZ-agents is fully determined by the variation of the risk-free rate. The following theorem reveals the dependence of the risk-free rate and thus the equity premium on the market share of the EZ-agent.

Theorem 7 *Denote* $\underline{x} := \text{essinf } R_{t+1}$*. Then,* $R_{f,b} := 1/\mathbb{E}(1/R_{t+1}) \in (\underline{x}, \mathbb{E}[R_{t+1}])$ *is the equilibrium risk-free rate when the EZ-agent's market share* $Y_t = 1$*. On the other hand,* $R_{f,a}$*, which is the unique* y *solving* $e^{\mathbb{E}(\ln R_{t+1})}(1 - y\mathbb{E}(1/R_{t+1})) + b\mathbb{E}[\nu(R_{t+1} - y)] = 0$ *in* $(\underline{x}, \mathbb{E}[R_{t+1}])$*, is the equilibrium risk-free rate when* $Y_t = 0$*. Furthermore,*

(i) If $K = K^*$*, then* $R_{f,t+1} \equiv R_{f,b}$ *and* $EP_t = \mathbb{E}[R_{t+1}] - R_{f,b}$*.*

(iii) If $K > K^*$*, then* $R_{f,t+1}$ *is continuous and strictly increasing with respect to* $Y_t \in [0, 1]$*. Consequently,* EP_t *is continuous and strictly decreasing with respect to* $Y_t \in [0, 1]$*.*

(i) If $K < K^*$*, then* $R_{f,t+1}$ *is continuous and strictly decreasing with respect to* $Y_t \in [0, 1]$*. Consequently,* EP_t *is continuous and strictly increasing with respect to* $Y_t \in [0, 1]$*.*

Theorem 7 shows that when the LA-agent's LAD K happens to be K^*, she behaves the same as the EZ-agent, so the equity premium is the same as in an economy with the EZ-agent only. When $K > K^*$, the LA-agent invests less than the EZ-agent in the stock, so the larger the market share of the LA-agent has, the larger the equity premium is. When $K < K^*$, the LA-agent invests more than the EZ-agent in the stock, so the larger the market share of the LA-agent has, the smaller the equity premium is.

5.3.3 Market dominance

An important question in behavioral finance is whether rational investors will dominate the market. In our model, the EZ-agent is considered to be rational and the LA-agent is considered to be irrational, so we study whether the EZ-agent will dominate the market in the long run. Recall that Y_t is the market share of the EZ-agent. The EZ-agent *becomes extinct* if $\lim_{t\to\infty} Y_t = 0$ almost surely, *survives* if extinction does not occur, and *dominates* the market if $\lim_{t\to\infty} Y_t = 1$ almost surely. The extinction, survival, and dominance of the LA-agent are defined similarly.

There is a vast literature on market dominance; see for instance the surveys by Blume and Easley (2009, 2010). In this literature, an alternative definition of market dominance is based on the consumption share instead of market share. This alternative definition is the same as the one used here because the consumption rates of the EZ- and LA-agents are the same constant.

Theorem 8 *Recall $R_{f,a}$ in Theorem 7 as the equilibrium risk-free rate when the market consists of the LA-agent only. Suppose one of the following two conditions is satisfied: (i) $K < K^*$; and (ii) $K > K^*$ and $\lim_{c\downarrow 0} \mathbb{E}[1/(c + X - \underline{x})] \in (1/(R_{f,a} - \underline{x}), +\infty]$. Then, the EZ-agent dominates the market.*

Theorem 8 shows that the EZ-agent, who maximizes the expected logarithmic utility of consumption, drives the LA-agent, who receives additional gain-loss utility, out of the market. This result, though nontrivial to prove, is not surprising because an expected utility maximizer with logarithmic utility function maximizes the long-run wealth accumulation (Blume and Easley, 1992).

The existent market dominance literature assumes that all market participants are expected utility maximizers or more generally EZ-agents with possibly heterogeneous RRAD, EIS, and beliefs; see for instance Blume and Easley (2009, 2010), Borovička (2013), and the references therein. The only work addressing both EZ- and LA-agents is Easley and Yang (2014), in which the authors illustrate numerically that the EZ-agents dominate the market when they have the same RRAD and EIS as the LA-agents. We are the first one to prove the dominance when the RRAD and EIS of the EZ- and LA-agents are one.

Because the EZ-agent dominates the market in the long run, the equilibrium asset prices in the long run are determined by the EZ-agent only, i.e., the LA-agent has no price impact in the long run. In particular, the long-run risk-free rate is $R_{f,b}$. However, although theoretically the LA-agent becomes distinct in the market and has zero price impact after infinite number of years, it is unclear whether this agent is negligible after a sufficiently long but finite time period, e.g., after 50 years. In the following, we assume the dividend growth rate distribution as in Table 1, set $\rho = \gamma = 1$ and $\beta = 0.98$, and use simulation to compute the market share $1 - Y_t$ and the price impact of the LA-agent for $t = 5$, 20, 50, and 200. Here, we define the price impact of the LA-agent as $(\mathrm{EP}_t - \mathrm{EP}_{t,b})/\mathrm{EP}_{t,b}$, where $\mathrm{EP}_{t,b}$ is the equity premium when the market consists of the EZ-agent only. With equal market shares at the beginning, we simulate a thousand paths along each of which the market share and price impact of the LA-agent are computed at each time, e.g., at 5, 20, 50, and 200. The mean and standard error (in bracket) of these two quantities are reported in Table 4. We find that the market share and price impact of the LA-agent remain significant even after 200 years. Therefore, the LA-agent survives and has large price impact even after a long time. This result indicates that irrational investors such as those with loss aversion cannot be ignored in asset pricing. Easley and Yang (2014) conduct similar numerical tests and reach the same conclusion.

5.4 Representative-Agent models

In nearly all asset pricing models in behavioral finance, a representative agent is assumed. The preferences of the representative agent are assumed to be the average of the preferences of all market participants. For instance, Barberis et al. (2001) assume a representative agent who receives consumption utility and gain-loss utility, and the LAD of the agent is chosen to be 2.25, the average of LADs found in experimental studies. A natural question then arise: are the asset prices in an economy with heterogeneous agents the same as those in an economy with a representative agent whose preferences are the average of the preferences

Table 4: Market share $1 - Y_t$ and price impact $(\mathrm{EP}_t - \mathrm{EP}_{t,b})/\mathrm{EP}_{t,b}$ of the LA-agent. Parameters are set to be $\beta = .98$, $b = 1$, $\rho = 1$, and $\gamma = 1$. The dividend growth rate distribution is given as in Table 1. The initial market share of the LA-agent is 0.5. A thousand scenarios of the stock return series are simulated. In each scenario, the EZ- and LA-agents follow their optimal portfolios and their market share and price impact in 5, 20, 50, and 200 years are computed. The mean and standard error (in bracket) of the market share and price impact of the LA-agent are reported.

$K = 1 < K^*$				
Years	5	20	50	200
Market Share	0.5 [5.45E-05]	0.5 [1.17E-04]	0.499 [1.91E-04]	0.498 [4.11E-04]
Price Impact	-0.32 [7.77E-13]	-0.32 [8.98E-04]	-0.30 [8.07E-04]	-0.25 [1.40E-03]
$K = 2.25 > K^*$				
Years	5	20	50	200
Market Share	0.499 [1.50E-03]	0.497 [1.80E-03]	0.492 [2.70E-03]	0.469 [6.00E-03]
Price Impact	1.14 [1.66E-04]	1.18 [3.70E-04]	1.17 [5.80E-04]	1.17 [1.10E-03]

of the agents in the first economy?

Chapman and Polkovnichenko (2009) study a single-period asset pricing model with two agents: one has EUT preferences and the other has non-EUT preferences. One example of the non-EUT preferences is $V(c) = \alpha\mathbb{E}[u(c)] + (1-\alpha)\mathbb{E}[\nu(u(c) - u(r))]$ where c stands for the consumption in the single period, u is the utility function, r is a reference consumption level, $\nu(x) := x^\gamma \mathbf{1}_{x\geq 0} - K(-x)^\gamma \mathbf{1}_{x<0}$ for some $\gamma \in (0,1]$, and $\alpha \in [0,1]$. In other words, the agent's preference for c is a combination of the classical expected utility of consumption and the gain-loss utility (in terms of the utility of consumption).[5] Note that this preference representation is similar to the one of the LA-agent in our model: the agent receives additional utility of gains and losses. Moreover, one can observe from the single-period portfolio choice problem (11) that parameter b in our model plays a similar role as $(1-\alpha)/\alpha$ in the preference representation in Chapman and Polkovnichenko (2009).

[5]This preference representation was proposed by Kőszegi and Rabin (2006, 2007).

Chapman and Polkovnichenko (2009) consider a two-state economy and compute the equity premium with equally weighted EUT (i.e., with $\alpha = 1$) and non-EUT (i.e., with $\alpha < 1$) agents. Then, the authors compute the equity premium in another two-state economy with a representative agent whose preferences are the average of the EUT and non-EUT agents (i.e., whose α is the average of the α's of the EUT and non-EUT agents). The dividend growth rate is calibrated to the distribution in Table 1. Three values of α are used: 0.25, 0.5, and 0.75, and K is set to be 1.5, 2.0, or 2.5. The authors find that the asset prices in the heterogeneous-agent economy and in the representative-agent economy are significantly different.

We test whether the conclusion in Chapman and Polkovnichenko (2009) still holds in our multi-period asset pricing model. Suppose the economy consists of two equally weighted agents with parameters b_i and K_i, $i = 1, 2$, respectively. According to Theorem 5, the equilibrium risk-free rate $R_{f,t+1}$ is determined by $0.5\varphi(R_{f,t+1}; b_1, K_1) + 0.5\varphi(R_{f,t+1}; b_2, K_2) = 1$. Consider another economy with a representative agent whose preferences follow (1) with parameter b and K. Then, the equilibrium risk-free rate is determined by $\varphi(R_{f,t+1}; b, K) = 1$.

In the following, for chosen values of b_i and K_i, $i = 1, 2$, we set $b = 0.5(b_1 + b_2)$ and $K = 0.5(K_1 + K_2)$; i.e., the representative agent's references are the average of the preferences of the EZ- and LA-agents in the heterogeneous-agent economy. We then compute the equilibrium risk-free rates and thus the equity premiums in the two economies. The dividend growth rate is given as in Table 1.[6] Other parameters are set as follows: $\beta = 0.98$ and $\rho = \gamma = 1$.

Table 5 presents the equity premiums in the heterogeneous-agent and representative-agent economies and their difference when the two agents in the first economy are an

[6]We also used the two-state setting as in Chapman and Polkovnichenko (2009) and obtained very similar results.

EZ-agent (i.e., $b_1 = 0$) and an LA-agent (i.e., $b_2 > 0$). We can see that the equity premiums differ significantly for most parameter values, which is consistent with the findings in Chapman and Polkovnichenko (2009). We also report the stock holding of the LA-agent in the heterogenous-agent economy. We observe that when the LA-agent participates in the stock market, the equity premiums in the heterogeneous-agent and representative-agent economies are nearly identical. When the LA-agent does not participate in the stock market, however, the equity premiums in the two economies differ significantly. This observation can be explained as follows: The LA-agent's optimal allocation to the stock is decreasing with respect to b_2 when she experiences negative gain-loss utility of investing in the stock. Consequently, the LA-agent does not hold the stock when b_2 is larger than some threshold, e.g, when $b_2 \geq 0.2$. In this case, any values of b_2 larger than 0.2 lead to the same equity premium in the heterogeneous-agent economy because the LA-agent does not hold the stock regardless the value of b_2 and thus has the same price impact. On the other hand, the representative agent's preferences are the average of those of the EZ- and LA-agents and thus depend on b_2. Moreover, the representative agent must hold all the stock in equilibrium and thus the equity premium is sensitive to b_2. Therefore, the equity premiums in the heterogeneous-agent and representative-agent economies can be totally different especially when b_2 is large. Indeed, we observe from Table 5 that the larger b_2 is, the larger the difference in the equity premiums in these two economies.

Table 6 presents the equity premiums in the heterogeneous-agent and representative-agent economies when the two agents in the first economy are LA-agents with different LADs. Similar to the case of heterogeneous b, when the LA-agents participate in the stock market, the equity premiums in these two economies are nearly identical; otherwise they can differ significantly.

To summarize, in an economy with heterogeneous EZ- and LA-agents, if nearly all the agents participate in the stock market, the equity premium in this economy is similar to the one in an economy with a representative agent whose preferences are the average of the preferences of the EZ- and LA-agents. When a considerable fraction of agents in the heterogeneous-agent economy do not participate in the stock market, it is likely that the equity premium in this economy is much smaller than in the representative-agent economy. In other words, assuming representative-agent models significantly over-estimates the equity premium in this case, and this result is consistent with the findings in the single-period model studied in Chapman and Polkovnichenko (2009).

6 Equilibrium Analysis with Non-Myopic Agents

When the agents are non-myopic, i.e., when one of ρ and γ is not equal to one, we are unable to compute the equilibrium asset prices in closed form. Similarly, when $b_{i,t}$ is given as in (9), we do not have closed-form solutions either. In this case, we compute the equilibrium asset prices numerically.

In general, both the risk-free rate $R_{f,t+1}$ and the price-dividend ratio of the stock are functions of the market shares of the agents in the market. In particular, when the market consists of one EZ- and one LA-agents, these two quantities are functions of the market share of the EZ-agent. These two functions can be solved numerically from the equilibrium condition (7). Our algorithm is similar to the one used in Easley and Yang (2014), so we opt not to provide the details here.

First, we investigate whether the formulation of $b_{i,t}$ as in (8) and in (9) leads to significantly different equilibrium asset prices. Consider two economies, each of which consists of one EZ- and one LA-agents. Assume $\gamma = \rho = 1$ and $\beta = 0.98$ for the EZ- and LA-agents in these two economies. The LA-agents in these two economies have the same LAD $K = 2.25$, but $b_{i,t}$ is formulated differently: in the first economy, $b_{i,t}$ is specified as in (8) with $b = 0.19$ and in the second economy, $b_{i,t}$ is specified as in (9) with $\bar{b} = 0.01$; see Table 2. Figure

Table 5: Comparison of the equity premiums EP_{homo} and EP_{hetero} in a heterogeneous-agent and in a representative-agent economies, respectively. The heterogeneous-agent economy consists of one EZ-agent and one LA-agent with preferences represented by (1), and these two agents have equal market share. The representative-agent consists of only one agent whose preferences are the average of the agents in the heterogeneous-agent economy. The dividend growth rate follows distribution as specified in Table 1, $\beta = 0.98$, and $\gamma = \rho = 1$. Diff stands for $EP_{homo}/EP_{hetero} - 1$ and w_2^* is the percentage allocation of the LA-agent to the stock.

b_1	b_2	$K_1 = K_2$	EP_{homo} (%)	EP_{hetero} (%)	Diff (%)	w_2^*
0	0.10	1.5	0.046	0.046	0	0.721
0	0.10	2	0.061	0.061	0	0.217
0	0.10	2.5	0.069	0.061	13	0.000
0	0.33	1.5	0.069	0.061	13	0.000
0	0.33	2	0.108	0.061	76	0.000
0	0.33	2.5	0.147	0.061	140	0.000
0	1.00	1.5	0.116	0.061	89	0.000
0	1.00	2	0.201	0.061	228	0.000
0	1.00	2.5	0.271	0.061	342	0.000
0	3.00	1.5	0.178	0.061	190	0.000
0	3.00	2	0.309	0.061	405	0.000
0	3.00	2.5	0.424	0.061	594	0.000
0	10.00	1.5	0.224	0.061	266	0.000
0	10.00	2	0.394	0.061	543	0.000
0	10.00	2.5	0.531	0.061	769	0.000
0	20.00	1.5	0.246	0.061	302	0.000
0	20.00	2	0.434	0.061	609	0.000
0	20.00	2.5	0.583	0.061	853	0.000

Table 6: Comparison of the equity premiums $\mathrm{EP}_{\mathrm{homo}}$ and $\mathrm{EP}_{\mathrm{hetero}}$ in a heterogeneous-agent and in a representative-agent economies, respectively. The heterogeneous-agent economy consists of two LA-agents with LAD K_1 and K_2, respectively, and these two agents have equal market share. The representative-agent consists of only one agent whose preferences are the average of the agents in the heterogeneous-agent economy. The dividend growth rate follows distribution as specified in Table 1, $\beta = 0.98$, and $\gamma = \rho = 1$. Diff stands for $\mathrm{EP}_{\mathrm{homo}}/\mathrm{EP}_{\mathrm{hetero}} - 1$ and w_2^* is the percentage allocation to the stock of the LA-agent with the larger LAD.

$b_1 = b_2$	K_1	K_2	$\mathrm{EP}_{\mathrm{homo}}$ (%)	$\mathrm{EP}_{\mathrm{hetero}}$ (%)	Diff (%)	w_2^*
0.10	1.50	2.00	0.069	0.069	0	0.557
0.10	1.50	3.00	0.092	0.085	9	0.000
0.10	1.50	4.00	0.116	0.085	37	0.000
0.10	2.00	2.50	0.092	0.100	-8	0.789
0.33	1.50	2.00	0.131	0.116	13	0.000
0.33	1.50	3.00	0.194	0.116	67	0.000
0.33	1.50	4.00	0.248	0.116	114	0.000
0.33	2.00	2.50	0.193	0.186	4	0.000
3.00	1.50	2.00	0.288	0.218	32	0.000
3.00	1.50	3.00	0.438	0.218	101	0.000
3.00	1.50	4.00	0.556	0.218	155	0.000
3.00	2.00	2.50	0.436	0.374	17	0.000
10.00	1.50	2.00	0.337	0.243	39	0.000
10.00	1.50	3.00	0.502	0.250	101	0.000
10.00	1.50	4.00	0.635	0.244	160	0.000
10.00	2.00	2.50	0.501	0.434	15	0.000
20.00	1.50	2.00	0.346	0.247	40	0.000
20.00	1.50	3.00	0.512	0.247	107	0.000
20.00	1.50	4.00	0.646	0.247	161	0.000
20.00	2.00	2.50	0.512	0.435	18	0.000

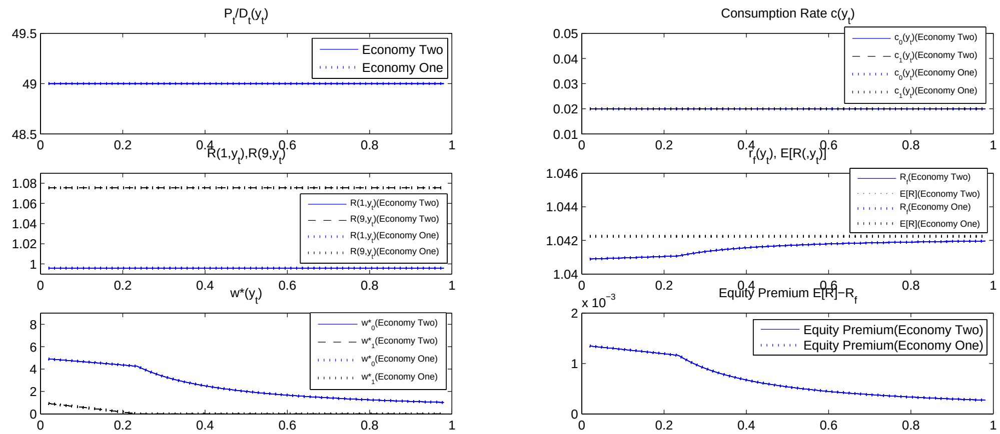

Figure 1: Price-dividend ratio, optimal consumption of the EZ- and LA-agents, stock return rate (in states 1 and 9), risk-free rate, conditional expected stock return, optimal portfolios of the two agents, and conditional equity premium as functions of the market share of the EZ-agent in two economies: in economy one, $b_{i,t}$ is specified as in (8) with $b = 0.19$ and in economy two it is specified as in (9) with $\bar{b} = 0.01$. The dividend growth rate distribution is given as in Table 1, $\beta = 0.98$, $\gamma = \rho = 1$, and $K = 2.25$.

1 presents the price-dividend ratio, optimal consumption of the EZ- and LA-agents, stock return (in states 1 and 9), risk-free rate, conditional expected stock return, optimal portfolios of the two agents, and conditional equity premium as functions of the market share of the EZ-agent in these two economies, and we observe that those quantities are almost the same in these two economies. We have the same observation for other values of γ, ρ, and β, and for the case in which the market consists of multiple agents. Therefore, we conclude that in our model the setting of $b_{i,t}$ as in (8) or as in (9) has little impact on the equilibrium. Because setting (8) leads tractability, we use it in the theoretical analysis in Section 5.

Secondly, we numerically compute the market share of the EZ-agent in the long run for general γ and ρ. We find the same conclusion as in Easley and Yang (2014): the EZ-agent dominates the market in the long run. However, just as in the case of $\rho = \gamma = 1$, the dominance takes effect very slowly.

Finally, following the analysis in Section 5.4, we compare the equity premiums in two economies for general ρ and γ: one with heterogeneous EZ- and LA-agents and the other with a representative agent whose preferences are the average of those of the agents in the first economy. The results are qualitatively the same as in Section 5.4.

7 Conclusions

In this paper, we have proposed a multi-period equilibrium asset pricing model with EZ- and LA-agents. The EZ-agents' preferences for consumption are represented by recursive utility and the LA-agents are concerned with the utility of trading gains and losses in addition to the consumption utility. We have defined an equilibrium gain-loss ratio and shown that the LA-agents hold less (more) stocks than the EZ-agents if and only if the LAD of the former agents is larger (smaller) than this ratio. With unit RRAD and EIS, we have proved the existence and uniqueness of the equilibrium. When the market consist of one EZ- and one LA-agents, we have found that the equity premium is increasing (decreasing) with respect to the market share of the LA-agent when the LAD of this agent is larger (smaller) than the equilibrium gain-loss ratio. We have also proved that the EZ-agent dominates the market in the long run and demonstrated that the dominance takes effect slowly in time. Finally, we have compared the equity premiums in our heterogeneous-agent economy and in another economy with a representative agent whose preferences are the average of those of the agents in the first economy. We have found that the equity premiums are quantitatively similar when the LA-agents in the first economy hold stocks and can be significantly different otherwise.

A Proofs

Proof of Proposition 1 On the one hand, if $\{R_{f,t+1}, P_t\}_{t=0}^{\infty}$ and $\{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$, $i = 1, 2, \ldots, m$ constitute a competitive equilibrium, then $c_{i,t}$ and $w_{i,t}$, $t \geq 0$ are the optimal percentage consumption and allocation to the stock, respectively, of agent i. Combining the clearing conditions for the stock and for the risk-free asset, we obtain $\sum_{i=1}^{m}(W_{i,t} - C_{i,t}) = P_t$, and consequently (7) follows.

One the other hand, suppose $c_{i,t}$ and $w_{i,t}$, $t \geq 0$ are the optimal percentage consumption and allocation to the stock, respectively, of agent i and satisfy (7). With $W_{i,t} := Y_{i,t}P_t/(1 - c_{0,t})$, $C_{i,t} := c_{i,t}W_{i,t}$, and $\theta_{i,t} := w_{i,t}W_{i,t}$, $i = 1, 2, \ldots, m$, one can check that $\{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$ is the optimal consumption-investment plan with the corresponding wealth process $\{W_{i,t}\}_{t=0}^{\infty}$ for agent i and the clearing conditions in Definition 1 are satisfied. $\square$

Proof of Theorem 2 The equilibrium analysis in an economy with EZ-agents only is standard in the literature; see for instance Epstein and Zin (1989, 1991) and Guo and He (2015).

Now, suppose the LA-agents also exist in the market with LAD equal to K^*. We can verify that $\mathbb{E}[\nu_i(R_{t+1} - R_{f,t+1})] = 0$, so the LA-agents' gain-loss utility is zero. Consequently, the LA-agents' optimal portfolio, optimal consumption, and optimal utility are the same as the EZ-agents, so the market's equilibrium does not change. $\square$

Proof of Proposition 3 We only need to show that $\mathbb{E}\left(X^{1-\gamma}\right)/\mathbb{E}\left(X^{-\gamma}\right)$ is strictly decreasing in $\gamma \geq 0$.

We first show that $h(t) := \ln \mathbb{E}\left(e^{tY}\right)$ is strictly convex in $t \leq 1$, where $Y := \ln X$. Because essinf $X > 0$ and $\mathbb{E}(X) < \infty$, $\mathbb{E}\left(e^{tY}\right)$ is continuous and well defined for $t \leq 1$ and is twice continuously differentiable in $t < 1$. Furthermore, its first- and second-order derivatives can be computed by interchanging the differential and expectation operators. Then, $h(t)$ is twice continuously differentiable and

$$h''(t) = \frac{\left(\mathbb{E}[Y^2e^{tY}]\right)\left(\mathbb{E}[e^{tY}]\right) - \left(\mathbb{E}[Ye^{tY}]\right)^2}{(\mathbb{E}[e^{tY}])^2} = \tilde{\mathbb{E}}[Y^2] - (\tilde{\mathbb{E}}[Y])^2,$$

where $\tilde{\mathbb{E}}$ is the expectation associated with $\tilde{\mathbb{P}}$ defined as $d\tilde{\mathbb{P}}/d\mathbb{P} = e^{tY}/\mathbb{E}[e^{tY}]$. Jensen's inequality immediately yields $h''(t) > 0$, so $h(t)$ is strictly concave in $t \le 1$.

Now, for any $0 \le \gamma_1 < \gamma_2$, we have

$$\mathbb{E}\left(X^{1-\gamma_1}\right)/\mathbb{E}\left(X^{-\gamma_1}\right) = e^{h(1-\gamma_1)-h(-\gamma_1)} > e^{h(1-\gamma_2)-h(-\gamma_2)} = \mathbb{E}\left(X^{1-\gamma_2}\right)/\mathbb{E}\left(X^{-\gamma_2}\right),$$

where the inequality is the case because $-\gamma_1 > -\gamma_2$ and h is strictly convex.

Finally, it is obvious that $\mathbb{E}\left(X^{1-\gamma}\right)/\mathbb{E}\left(X^{-\gamma}\right) = \mathbb{E}[X]$ when $\gamma = 0$, so K^* becomes 1 when $\gamma = 0$. $\square$

Proof of Proposition 4 For fixed $b \ge 0$ and $K \ge 1$, denote

$$\begin{aligned} h(w,a) &:= \mathbb{E}[\ln(a + w(X-a))], \quad g(a) := \mathbb{E}[\nu(X-a)], \\ f_0(w,a) &:= e^{h(w,a)}, \quad f(w,a) := f_0(w,a) + wbg(a). \end{aligned} \tag{13}$$

Then, the optimal solution to (11) is the maximizer of $f(w,a)$ in $w \in [0, \bar{w}(a)]$. For notational simplicity, we denote the optimal solution as $\varphi(a)$. Here and hereafter, we use convention that $\ln(0) = -\infty$ and $1/0 = +\infty$. Then, h, f_0, and f are well-defined at $w = \bar{w}(a)$.

We first compute the derivatives of f. Recalling the assumption $\mathbb{E}[X] < \infty$ and applying the dominated convergence theorem, one can prove that $\frac{\partial}{\partial w}h$ and $\frac{\partial}{\partial a}h$ exist for $w \in [0, \bar{w}(a)), a \in (\underline{x}, +\infty)$, and h, $\frac{\partial}{\partial w}h$, and $\frac{\partial}{\partial a}h$ are continuous in (w,a) in the same region. Similarly, $\frac{\partial^2}{\partial w^2}h$ and $\frac{\partial^2}{\partial w \partial a}h$ exist and are continuous in (w,a) for $w \in (0, \bar{w}(a)), a \in (\underline{x}, +\infty)$. Furthermore, the derivatives of h with respect to w and to a can be computed by interchanging the expectation and differential; i.e.,

$$\begin{aligned} &\frac{\partial}{\partial w}h(w,a) = \mathbb{E}\left[\frac{X-a}{a+w(X-a)}\right], \quad \frac{\partial}{\partial a}h(w,a) = \mathbb{E}\left(\frac{1-w}{a+w(X-a)}\right), \\ &\frac{\partial^2}{\partial w^2}h(w,a) = -\mathbb{E}\left[\frac{(X-a)^2}{(a+w(X-a))^2}\right], \quad \frac{\partial^2}{\partial w \partial a}h(w,a) = -\mathbb{E}\left[\frac{X}{(a+w(X-a))^2}\right]. \end{aligned}$$

As a result, $\frac{\partial}{\partial w} f$ exists and is continuous in (w,a) for $w \in [0, \bar{w}(a)), a \in (\underline{x}, +\infty)$, and

$$\frac{\partial}{\partial w} f(w,a) = f_0(w,a)\mathbb{E}\left[\frac{X-a}{a+w(X-a)}\right] + bg(a). \tag{14}$$

Similarly, $\frac{\partial^2}{\partial w^2} f$ exists and is continuous in (w,a) for $w \in (0, \bar{w}(a)), a \in (\underline{x}, +\infty)$, and

$$\frac{\partial^2}{\partial w^2} f(w,a) = f_0(w,a)\left[-\mathbb{E}\left(\frac{X-a}{a+w(X-a)}\right)^2 + \left(\mathbb{E}\left(\frac{X-a}{a+w(X-a)}\right)\right)^2\right] < 0, \tag{15}$$

where the inequality follows from Jensen's inequality. Finally, because $K \geq 1$, ν is concave. Consequently, g is concave and thus absolutely continuous. As a result, $\frac{\partial}{\partial w} f(w,a)$ is also absolutely continuous in a and

$$\begin{aligned}\frac{\partial^2}{\partial w \partial a} f(w,a) = f_0(w,a)\Big[\mathbb{E}\left(\frac{X-a}{a+w(X-a)}\right)\mathbb{E}\left(\frac{1-w}{a+w(X-a)}\right) \\ -\mathbb{E}\left(\frac{X}{(a+w(X-a))^2}\right)\Big] + bg'(a).\end{aligned} \tag{16}$$

We have shown that f is continuous in $w \in [0, \bar{w}(a))$. Furthermore, noting that

$$h(w,a) = \mathbb{E}[\ln(a+w(X-a))\mathbf{1}_{X \geq a}] + \mathbb{E}[\ln(a+w(X-a))\mathbf{1}_{X<a}]$$

and applying the monotone convergence theorem, we conclude that $\lim_{w\uparrow\bar{w}(a)} h(w,a) = h(\bar{w}(a),a)$. Consequently, f is continuous in $w \in [0, \bar{w}(a)]$. Therefore, the maximizer of $f(w,a)$ in w exists. Furthermore, (15) shows that f is strictly concave in w and thus the maximizer is unique. In the following, denote $\varphi(a)$ as the maximizer.

Because f is strictly concave in w, we conclude that $\varphi(a) = 0$ if and only if $\frac{\partial}{\partial w} f(0,a) \leq 0$. Straightforward computation yields $\frac{\partial}{\partial w} f(0,a) = \mathbb{E}(X) - a + bg(a)$, which is continuous and strictly decreasing in a. One can see that the root $\overline{a}$ of $\frac{\partial}{\partial w} f(0,a)$ exists and is unique, and $\varphi(a) \leq 0$ if and only if $a \geq \overline{a}$. Moreover, $\frac{\partial}{\partial w} f(0, \mathbb{E}(X)) \leq 0$ because $K \geq 1$, and consequently $\overline{a} \leq \mathbb{E}(X)$.

On the other hand, $\varphi(a) = \bar{w}(a)$ if and only if $\lim_{w\uparrow\bar{w}(a)} \frac{\partial}{\partial w} f(w,a) \geq 0$. Note that this limit must exist and take values in $[-\infty, +\infty)$ because f is strictly concave in w. Straightforward calculation yields

$$\frac{\partial}{\partial w} f(w,a) = e^{\mathbb{E}(\ln(c+X-\underline{x}))} - \frac{a}{w} e^{\mathbb{E}(\ln(c+X-\underline{x}))} \mathbb{E}[1/(c+X-\underline{x})] + bg(a), \tag{17}$$

where $c := (a/w) - (a - \underline{x})$. Note that w increasingly goes to $\bar{w}(a)$ if and only if c decreasingly goes to zero, so

$$\lim_{w \uparrow \bar{w}(a)} \frac{\partial}{\partial w} f(w,a) = e^{\mathbb{E}(\ln(X-\underline{x}))} - (a - \bar{x})\xi + bg(a)$$

where $\xi := \lim_{c \downarrow 0} \{(\exp[\mathbb{E}(\ln(c+X-\underline{x}))]) (\mathbb{E}[1/(c+X-\underline{x})])\} \in [0, +\infty]$. Here, the limit defining ξ exists because $\lim_{w \uparrow \bar{w}(a)} \frac{\partial}{\partial w} f(w,a)$ exists. Furthermore, by Jensen's inequality, we conclude that $\xi \in [1, +\infty]$. If $\xi < +\infty$, $\lim_{w \uparrow \bar{w}(a)} \frac{\partial}{\partial w} f(w,a)$ is continuous and strictly decreasing in a and its value is nonnegative when $a = \underline{x}$ and goes to $-\infty$ as a goes to $+\infty$. Therefore, there exist $\underline{a} \in [\underline{x}, +\infty)$ such that $\lim_{w \uparrow \bar{w}(a)} \frac{\partial}{\partial w} f(w,a) \geq 0$ if and only if $a \leq \underline{a}$. Consequently, $\varphi(a) = \bar{w}(a)$ if and only if $a \leq \underline{a}$. If $\xi = +\infty$, we have $\lim_{w \uparrow \bar{w}(a)} \frac{\partial}{\partial w} f(w,a) = -\infty$, so by setting $\underline{a} := \underline{x}$ in this case, we also conclude that $\varphi(a) = \bar{w}(a)$ if and only if $a \in (\bar{x}, \underline{a}]$.

We have shown that $\varphi(a) = 0$ if and only if $a \in [\overline{a}, +\infty)$ and $\varphi(a) = \bar{w}(a)$ if and only if $a \in (\bar{x}, \underline{a}]$. Therefore, for $a \in (\underline{a}, \overline{a})$, we have $\varphi(a) \in (0, \bar{w}(a))$, and $\varphi(a)$ is uniquely determined by $\frac{\partial}{\partial w} f(\varphi(a), a) = 0$. Because $\frac{\partial}{\partial w} f(w,a)$ is strictly increasing in w and continuous in (w,a) for $0 \leq w < \bar{w}(a), a > \underline{x}$, we conclude that $\varphi(a)$ is continuous in $a \in (\underline{a}, \overline{a})$.

Next, we show that $\varphi(a)$ is strictly decreasing in $a \in (\underline{a}, \overline{a})$. Note from (16) that $\frac{\partial}{\partial w} f(w,a)$ consists of two terms: the first term is a continuous function of (w,a) and the second term is an absolutely continuous function of a. Then, a non-standard implicit function theorem (Ettlinger, 1928, Theorem II) yields that $\varphi(\cdot)$ is absolutely continuous and

$$\frac{\partial^2}{\partial w^2} f(\varphi(a), a)\varphi'(a) + \frac{\partial^2}{\partial w \partial a} f(\varphi(a), a) = 0.$$

Because, $\frac{\partial^2}{\partial w^2} f(w,a) < 0$, we only need to show that $\frac{\partial^2}{\partial w \partial a} f(\varphi(a), a) < 0$.

Fixing $w \geq 1$ and defining $Y = \frac{1}{a+w(X-a)} > 0$, we have

$$
\begin{aligned}
&-\mathbb{E}\frac{X}{[a+w(X-a)]^2} + \mathbb{E}\left[\frac{X-a}{a+w(X-a)}\right] \cdot \mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right] \\
=&\frac{1-w}{w}a[\mathbb{E}(Y)^2 - (\mathbb{E}(Y))^2] - \mathbb{E}[Y] \leq -\mathbb{E}[Y] < 0,
\end{aligned}
$$

where the first inequality is due to Jensen's inequality. Therefore, from (16) we conclude $\frac{\partial^2}{\partial w \partial a} f(\varphi(a), a) < 0$ for $\varphi(a) \geq 1$.

By taking derivative with respect to w, we can verify that $\mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right]$ is decreasing in w. Therefore, for any $w < 1$, we have

$$
0 < \mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right] \leq \frac{1}{a}.
$$

On the other hand, because of the concavity of $g(\cdot)$, we conclude $g(a) \geq g(0) + g'(a)a > g'(a)a$, where the second equality is the case because $g(0) > 0$. Because $g(\cdot)$ is concave and strictly decreasing, we must have $g'(a) < 0$. Consequently, $g(a)/g'(a) < a$. Therefore, we conclude that for any $w < 1$,

$$
-g(a)\mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right] + g'(a) = -g'(a)\left\{\frac{g(a)}{g'(a)}\mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right] - 1\right\} < 0.
$$

Now, for any $a \in (\underline{a}, \overline{a})$ such that $0 < \varphi(a) < 1$, because $\frac{\partial}{\partial w} f(\varphi(a), a) = 0$, we obtain from (16) that

$$
\frac{\partial^2}{\partial w \partial a} f(\varphi(a), a) = -bg(a)\mathbb{E}\left[\frac{1-w}{a+w(X-a)}\right] + bg'(a) - f_0(\varphi(a), a)\mathbb{E}\left[\frac{X}{(a+w(X-a))^2}\right] < 0.
$$

We have shown that $\varphi(a)$ is continuous and strictly decreasing on $(\underline{a}, \overline{a})$, is equal to 0 on $[\overline{a}, +\infty)$, and is equal to $\bar{w}(a)$ on $(\underline{x}, \underline{a}]$. To finish the proof of (ii), we only need to show that $\varphi(a)$ is continuous from the right at $\underline{a}$ if $\underline{a} > \underline{x}$, $\varphi(a)$ is continuous from the left at $\overline{a}$, and $\lim_{a \downarrow \underline{x}} \varphi(a) = +\infty$.

Suppose $\underline{a} > \underline{x}$ and $\varphi(a)$ is not continuous from the right at $\underline{a}$, then there exist $\epsilon_0 > 0$ and a_n's that decreasingly converge to $\underline{a}$ as $n \to \infty$ such that $\varphi(a_n) \le \bar{w}(\underline{a}) - \epsilon_0 < \bar{w}(a_n)$. Because $f(w, a_n)$ is strictly concave in w and $\varphi(a_n)$ is the maximizer of $f(w, a_n)$ in w, we have $\frac{\partial}{\partial w} f(\bar{w}(\underline{a}) - \epsilon_0, a_n) \le 0$. Sending n to infinity and recalling the continuity of $\frac{\partial}{\partial w} f$ in (w, a), we conclude $\frac{\partial}{\partial w} f(\bar{w}(\underline{a}) - \epsilon_0, \underline{a}) \le 0$. On the other hand, by the definition of $\underline{a}$, we have $\lim_{w \uparrow \bar{w}(\underline{a})} \frac{\partial}{\partial w} f(w, \underline{a}) = 0$. Because $f(w, a)$ is strictly concave in w, we conclude that $\frac{\partial}{\partial w} f(\bar{w}(\underline{a}) - \epsilon_0, \underline{a}) > 0$, which is a contradiction. Thus, $\varphi(a)$ is continuous from the right at $\underline{a}$. Similarly, we can show that $\varphi(a)$ is continuous from the left at $\bar{a}$.

Suppose it is not true that $\lim_{a \downarrow \underline{x}} \varphi(a) = +\infty$, then there exist $M \in (1, +\infty)$ and $a_n \downarrow \underline{x}$ such that $\varphi(a_n) \le M < \bar{w}(a_n)$. Then, we must have $\frac{\partial}{\partial w} f(M, a_n) \le 0$, and thus $\limsup_{n \to \infty} \frac{\partial}{\partial w} f(M, a_n) \le 0$. On the other hand, from (14), one can conclude that $\liminf_{a \downarrow \underline{x}} \frac{\partial}{\partial w} f(M, a) > 0$, which is a contradiction. Therefore, we must have $\lim_{a \downarrow \underline{x}} \varphi(a) = +\infty$.

Finally, one can see that $g(a)$ and thus $\frac{\partial}{\partial w} f(w, a)$ are decreasing in K, so we immediately conclude (iii). On the other hand, $\frac{\partial}{\partial w} f(w, a)$ is decreasing in b when $\mathbb{E}[\nu(X - a)] \le 0$ and is increasing in b when $\mathbb{E}[\nu(X - a)] \ge 0$, so (iv) follows immediately. $\square$

Proof of Theorem 5 Suppose $\Psi_{i,t}, t \ge 0$ is well defined for each i. Then, it is straightforward to see from (6) that the optimal percentage consumption of agent i is $c^*_{i,t} = 1 - \beta$, $i = 1, \dots, m$. As a result, (7) leads to the unique equilibrium price dividend ratio $P_t/D_t \equiv \frac{\beta}{1-\beta}$. Consequently, the stock return in equilibrium must be

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}/D_{t+1} + 1}{P_t/D_t} \cdot \frac{D_{t+1}}{D_t} = \frac{1}{\beta} Z_{t+1}.$$

Because Z_{t+1}'s are i.i.d., so are R_{t+1}'s.

Next, recalling $b_{i,t}$ as defined in (8) and Bellman equation (6), and noting that R_{t+1}'s are i.i.d., we immediately conclude that agent i's optimal percentage allocation to the stock at time t is $\varphi_i(R_{f,t+1})$, where $\varphi_i(a)$ is the optimal solution to (11) with $X = R_{t+1}$, $b = b_i$, and $K = K_i$. Therefore, from (7), we conclude that $R_{f,t+1}$ is the equilibrium risk-free rate if and only if it satisfies $\sum_{i=1}^{m} \varphi_i(R_{f,t+1}) = 1$. By proposition 4, $\varphi_i(a)$ is continuous and decreasing in $a \in (\underline{x}, +\infty)$ and is strictly decreasing in a when $\varphi_i(a) > 0$. Furthermore, by Proposition 4, $\varphi_i(a) = 0$ for $a \geq \mathbb{E}(X)$ and $\lim_{a \downarrow \underline{x}} \varphi_i(a) = +\infty$. Therefore, the equilibrium risk-free rate exists, is unique, and lies in $(\underline{x}, \mathbb{E}(X))$.

Finally, we show that $\{\Psi_{i,t}\}$ is indeed well defined, i.e., uniquely exists. Note that R_{t+1}'s are i.i.d. and $R_{f,t+1} = R_f(\mathbf{Y}_t)$ for some function R_f on $\Delta := \{\mathbf{y} := (y_1, \ldots, y_m) \in \mathbb{R}^n | y_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^{m} y_i = 1\}$, where $\mathbf{Y}_t := (Y_{1,t}, Y_{2,t}, \ldots, Y_{m,t})$ stands for the market share vector of the m agents in the market. Therefore, we expect $\Psi_{i,t}$ to be a function of the market share vector as well, i.e., $\Psi_{i,t} = \exp[\psi_i(\mathbf{Y}_t)]$ for some continuous function ψ_i on Δ. Furthermore, given the market share vector $\mathbf{Y}_t$ at time t, the market share vector $\mathbf{Y}_{t+1}$ at time $t+1$ is determined by Z_{t+1}, i.e., there exist a function $h(\mathbf{y}, z)$ from $\Delta \times \mathbb{R}_+$ to Δ such that $\mathbf{Y}_{t+1} = h(\mathbf{Y}_t, Z_{t+1})$. Recall $f(w, a)$ as defined in (13), which is the objective function of problem (11). When $\mathbf{Y}_t = \mathbf{y} \in \Delta$, $R_{f,t+1} = R_f(\mathbf{y})$, so the optimal portfolio $w_{i,t}^* = \varphi_i(R_f(\mathbf{y}))$. In addition, the optimal consumption rate $c_{i,t}^* = 1 - \beta$. Therefore, by taking logarithm on both sides of (6) and noting that $\Psi_{i,t} = \exp[\psi_i(\mathbf{y})]$ and $\Psi_{i,t+1} = \exp[\psi_i(\mathbf{Y}_{t+1})] = \exp[\psi_i(h(\mathbf{y}, Z_{t+1}))]$, we obtain

$$\psi_i(\mathbf{y}) = (1-\beta)\ln(1-\beta) + \beta \ln \beta + f(\varphi_i(R_f(\mathbf{y})), R_f(\mathbf{y})) + \beta \mathbb{E}[\psi_i(h(\mathbf{y}, Z_{t+1}))], \quad \forall \mathbf{y} \in \Delta. \quad (18)$$

It is straightforward to see that the right-hand of (18) is a contract mapping from the space of continuous functions on Δ with maximum norm into the same space. Therefore, (18) admits a unique solution, and consequently, $\{\Psi_{i,t}\}$ is well-defined. □

Proof of Theorem 6 Following the proof of Theorem 5, we conclude that $w_{0,t}^* = \varphi_0(R_{f,t+1})$ where $\varphi_0(a)$ is the optimal solution to (11) with $X = R_{t+1}$ and $b = 0$, and that $w_{1,t}^* = \varphi_1(R_{f,t+1}, K)$ where $\varphi_1(a; K)$ is the optimal solution to (11) with $X = R_{t+1}$, $b > 0$, and $K \geq 1$. In other words, $\varphi_0(a)$ and $\varphi_1(a; K)$ are the maximizers of $f_0(w, a)$ and $f(w, a)$ in w, respectively, where f_0 and f are defined as in (13).

It is straightforward to verify that $\frac{\partial}{\partial w} f_0(1, R_{f,b}) = 0$, so $\varphi_0(R_{f,b}) = 1$. Because $\mathbb{E}[\nu(R_{t+1} -$

$R_{f,b})] = 0$ when $K = K^*$, we have $\frac{\partial}{\partial w} f(1, R_{f,b}) = 0$ in this case. Consequently, $\varphi_1(R_{f,b}; K^*) = 1$. As a result, we have $Y_t\varphi_0(R_{f,b}) + (1 - Y_t)\varphi_1(R_{f,b}; K^*) = 1$, so $R_{f,b}$ is the equilibrium risk-free rate and $w^*_{i,t} = 1, i = 0, 1$ when $K = K^*$.

Next, we consider the case in which $K < K^*$. Because $Y_t\varphi_0(R_{f,b}) + (1 - Y_t)\varphi_1(R_{f,b}; K^*) = 1$, $\varphi_1(a, K)$ is decreasing in K, and φ_i is decreasing in a, $i = 0, 1$, we conclude that the equilibrium risk-free rate $R_{f,t+1}$ when $K > K^*$, which solves $Y_t\varphi_0(R_{f,t+1}) + (1 - Y_t)\varphi_1(R_{f,t+1}; K) = 1$, must be larger than or equal to $R_{f,b}$. We further claim that $R_{f,t+1} > R_{f,b}$. Otherwise, $R_{f,t+1} = R_{f,b}$, which leads to $\varphi_0(R_{f,t+1}) = \varphi_0(R_{f,b}) = 1$. By the market clearing condition, we have $\varphi_1(R_{f,t+1}; K) = \varphi_1(R_{f,b}; K) = 1$. Consequently, $\frac{\partial}{\partial w} f_0(1, R_{f,b}) = \frac{\partial}{\partial w} f(1, R_{f,b}) = 0$. However, this cannot be the case because $K > K^*$. Therefore, we must have $R_{f,t+1} > R_{f,b}$. As a result, $w^*_{0,t} = \varphi_0(R_{f,t+1}) < \varphi_0(R_{f,b}) = 1$. We further claim that $w^*_{0,t} > 0$. Otherwise, $\frac{\partial}{\partial w} f_0(0, R_{f,t+1}) \le 0$, which leads to $R_{f,t+1} \ge \mathbb{E}[R_{t+1}]$. Because $K \ge 1$, we conclude that $\mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t] \le 0$. Consequently, $\frac{\partial}{\partial w} f(0, R_{f,t+1}) \le 0$ and $w^*_{1,t} = 0$. This is a contradiction because the market cannot clear when $w^*_{0,t} = w^*_{1,t} = 0$. Therefore, we must have $w^*_{0,t} > 0$. On the other hand, we must have $w^*_{1,t} > 1$, so

$$b\mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathscr{F}_t] = \frac{\partial}{\partial w} f(1, R_{f,t+1}) - \frac{\partial}{\partial w} f_0(1, R_{f,t+1}) > -\frac{\partial}{\partial w} f_0(1, R_{f,t+1}) > 0,$$

where the first inequality is the case because $f(w, a)$ is strictly concave in w and $w^*_{1,t} > 1$, and the second inequality is the case because $f_0(w, a)$ is strictly concave in w and $0 < w^*_{0,t} < 1$.

The case in which $K > K^*$ can be proved similarly. $\square$

Proof Theorem 7 Denote $\varphi_i(R_{f,t+1})$ as the optimal portfolio of agent $i, i = 0, 1$. We first consider the case in which $K < K^*$. In this case, $\varphi_1(R_{f,t+1}) > 1 > \varphi_0(R_{f,t+1})$, so equilibrium equation (12) yields

$$Y_t = \frac{1 - \varphi_1(R_{f,t+1})}{\varphi_0(R_{f,t+1}) - \varphi_1(R_{f,t+1})} = \frac{1}{1 + \frac{1 - \varphi_0(R_{f,t+1})}{\varphi_1(R_{f,t+1}) - 1}}.$$

Because $\varphi_i(a)$ is decreasing in a, $i = 0, 1$, we conclude that Y_t is strictly decreasing in $R_{f,t+1}$. Consequently, the equilibrium $R_{f,t+1}$ is strictly decreasing in Y_t.

The other two cases can be proved similarly. $\square$

Proof of Theorem 8 Recall the optimal solution to (11) in Proposition 4. Denote $\varphi_0(a) := \varphi(a;0,1)$ and $\varphi_1(a) := \varphi(a;b,K)$. Denote $w^*_{0,t}$ and $w^*_{1,t}$ as the optimal percentage allocation to the stock of the EZ- and LA-agents, respectively. Then, $w^*_{i,t} = \varphi_i(R_{f,t+1})$, $i = 0,1$. Denote X as a random variable that is identically distributed as R_{t+1} and denote $\underline{x}$ as its essential infimum.

When $K < K^*$, Theorem 7 yields $R_{f,t+1} \in [R_{f,b}, R_{f,a}]$ and $\underline{x} < R_{f,b} < R_{f,a} < \mathbb{E}[X]$. Consequently, $w^*_{0,t} = \varphi_0(R_{f,t+1}) \geq \varphi_0(R_{f,a}) > 0$, where the last inequality is the case because $R_{f,a} < \mathbb{E}[R_{t+1}]$ and because $\varphi_0(a) = 0$ if and only if $a \geq \mathbb{E}(X)$. On the other hand, $w^*_{1,t} = \varphi_1(R_{f,t+1}) \leq \varphi_1(R_{f,b}) < +\infty$, where the last inequality is the case because $R_{f,b} > \underline{x}$. Combining with Theorem 6, we conclude that

$$0 < \varphi_0(R_{f,a}) \leq w^*_{0,t} < 1 < w^*_{1,t} \leq \varphi_1(R_{f,b}) < +\infty. \tag{19}$$

When $K > K^*$, Theorem 7 yields $R_{f,t+1} \in [R_{f,a}, R_{f,b}]$. Recall that $\varphi_0(a)$ is the maximizer of $f_0(w,a)$ in $w \in [0, \bar{w}(a)]$, where $f_0(w,a)$ is defined as in the proof of Proposition 4. Furthermore, from (17), we obtain

$$\frac{\partial}{\partial w} f_0(w,a) = e^{\mathbb{E}(\ln(c+X-\underline{x}))}\left\{1-(c+a-\underline{x})\mathbb{E}[1/(c+X-\underline{x})]\right\}$$

where $c := (a/w) - (a-\underline{x})$. Because $\lim_{c\downarrow 0}\mathbb{E}[1/(c+X-\underline{x})] \in (1/(R_{f,a}-\underline{x}), +\infty]$, there exist $\varepsilon_0 > 0$ such that $\mathbb{E}[1/(\varepsilon_0+X-\underline{x})] > 1/(R_{f,a}-\underline{x})$. Consequently, for any $a \geq R_{f,a}$,

$$\begin{aligned}\frac{\partial}{\partial w} f_0\left(\frac{a}{\varepsilon_0+a-\underline{x}}, a\right) &= e^{\mathbb{E}(\ln(\varepsilon_0+X-\underline{x}))}\left\{1-(\varepsilon_0+a-\underline{x})\mathbb{E}[1/(\varepsilon_0+X-\underline{x})]\right\}\\ &< e^{\mathbb{E}(\ln(\varepsilon_0+X-\underline{x}))}\left\{1-(R_{f,a}-\underline{x})\mathbb{E}[1/(\varepsilon_0+X-\underline{x})]\right\}\\ &< 0.\end{aligned}$$

As a result, by the concavity of $f_0(w,a)$ in w, we conclude that $\varphi_0(a) < a/(\varepsilon_0+a-\underline{x})$ for any

$a \geq R_{f,a}$. Combining with Theorem 6, we conclude that

$$0 \leq w^*_{1,t} < 1 < w^*_{0,t} < R_{f,t+1}/(\varepsilon_0 + R_{f,t+1} - \underline{x}) < \bar{w}(R_{f,t+1}) \leq \bar{w}(R_{f,a}) < +\infty. \tag{20}$$

Denote X_t as the post-consumption wealth ratio of the LA- and EZ-agents, i.e., $X_t := (1 - Y_t)/Y_t$. Because $w^*_{0,t} < \bar{w}(R_{f,t+1})$, $\forall t \geq 0$, we must have $Y_t > 0$, $t > 0$, so $\{X_t\}$ is well defined. Straightforward calculation yields

$$\begin{aligned} X_{t+1} &= \frac{(1 - c^*_{1,t+1})W_{1,t+1}}{(1 - c^*_{0,t+1})W_{0,t+1}} = \frac{W_{1,t+1}}{W_{0,t+1}} \\ &= \frac{[R_{f,t+1} + w^*_{1,t}(R_{t+1} - R_{f,t+1})]W_{1,t}}{[R_{f,t+1} + w^*_{0,t}(R_{t+1} - R_{f,t+1})]W_{0,t}} \\ &= \left[1 + (w^*_{1,t} - w^*_{0,t})A_{t+1}\right] X_t, \end{aligned} \tag{21}$$

where

$$A_{t+1} =: \frac{R_{t+1} - R_{f,t+1}}{R_{f,t+1} + w^*_{0,t}(R_{t+1} - R_{f,t+1})}. \tag{22}$$

From (19) and (20), we conclude that $w^*_{1,t} - w^*_{0,t}$ is uniformly bounded. On the other hand, when $K < K^*$, by considering the cases of $R_{t+1} - R_{f,t+1} \geq 0$ and $R_{t+1} - R_{f,t+1} < 0$, respectively, (19) yields

$$|A_{t+1}| \leq \max(1/w^*_{0,t}, R_{f,t+1}/\underline{x}) \leq \max(1/\varphi_0(R_{f,a}), R_{f,a}/\underline{x}) < +\infty.$$

When $K > K^*$, by considering the cases of $R_{t+1} - R_{f,t+1} \geq 0$ and $R_{t+1} - R_{f,t+1} < 0$, respectively, (20) yields

$$\begin{aligned} |A_{t+1}| &\leq \max\left(1, \frac{R_{f,t+1}}{R_{f,t+1} + (R_{f,t+1}/(\varepsilon_0 + R_{f,t+1} - \underline{x}))\,(\underline{x} - R_{f,t+1})}\right) \\ &= \max\left(1, \frac{\varepsilon_0 + R_{f,t+1} - \underline{x}}{\varepsilon_0}\right) \leq \max\left(1, \frac{\varepsilon_0 + R_{f,b} - \underline{x}}{\varepsilon_0}\right) < +\infty. \end{aligned}$$

Therefore, A_t is uniformly bounded. As a result, X_t is integrable for any $t \geq 0$.

Because $0 < w^*_{0,t} < \bar{w}(R_{f,t+1})$, from (14), we conclude that

$$\mathbb{E}[A_{t+1}|\mathscr{F}_t] = \mathbb{E}[A_{t+1}|R_{f,t+1}] = \frac{\partial}{\partial w} f_0(w^*_{0,t}, R_{f,t+1})(f_0(w^*_{0,t}, R_{f,t+1}))^{-1} = 0,$$

where the last equality is the result of the first-order condition of the optimality of $w^*_{0,t}$. Consequently,

$$\mathbb{E}[X_{t+1}|\mathscr{F}_t] = X_t \left[1 + (w^*_{1,t} - w^*_{0,t})\mathbb{E}[A_{t+1}|\mathscr{F}_t]\right] = X_t,$$

showing that $\{X_t\}$ is a positive martingale. By the martingale convergence theorem, X_t converges almost surely and in L^1 to a nonnegative $\mathscr{F}_\infty$-measurable random variable X_∞. Consequently, Y_t converges almost surely to $Y_\infty := \frac{1}{1+X_\infty} \in (0, 1]$. Because $R_{f,t+1}$ is a continuous function of Y_t on $[0, 1]$, $R_{f,t+1}$ converges almost surely, and we denote the limit as $R_{f,\infty}$. Because $R_{f,t+1}$'s are bounded by a_1 and a_2, where $a_1 := \min(R_{f,a}, R_{f,b}) > \underline{x}$ and $a_2 := \max(R_{f,a}, R_{f,b}) < \mathbb{E}(X)$, and $\varphi_i(a)$ is continuous in $a \in (\underline{x}, +\infty)$, we conclude that

$$\lim_{t\to\infty} w^*_{i,t} = \lim_{t\to\infty} \varphi_i(R_{f,t+1}) = \varphi_i(R_{f,\infty}), \quad i = 0, 1$$

almost surely.

We claim that $\varphi_0(R_{f,\infty}) \neq \varphi_1(R_{f,\infty})$. Indeed, when $K < K^*$, $\varphi_0(R_{f,t+1}) < 1 < \varphi_1(R_{f,t+1})$, so we must have $\varphi_0(R_{f,\infty}) \leq 1 \leq \varphi_1(R_{f,\infty})$. Consequently, $\varphi_0(R_{f,\infty}) = \varphi_1(R_{f,\infty})$ if and only if $\varphi_0(R_{f,\infty}) = \varphi_1(R_{f,\infty}) = 1$. A similar argument shows this sufficient and necessary condition is also true when $K > K^*$. Now, suppose $\varphi_0(R_{f,\infty}) = \varphi_1(R_{f,\infty}) = 1$. Then, because $\varphi_i(a)$ is the maximizer of $f_i(w, a)$ in w, the first-order condition yields $\frac{\partial}{\partial w} f_0(1, R_{f,\infty}) = 0$ and $\frac{\partial}{\partial w} f(1, R_{f,\infty}) = 0$. From these two equations, we can derive $K = K^*$, which is a contradiction. Therefore, we must have $\varphi_0(R_{f,\infty}) \neq \varphi_1(R_{f,\infty})$.

Next, we show $\mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$. To this end, it is sufficient to find some $\epsilon > 0$ such that $\mathbb{P}(\cap_{n=1}^\infty \cup_{t=n}^\infty \{|A_t| > \epsilon\}) = 1$, i.e., such that $\lim_{n\to\infty} \mathbb{P}(\cap_{t=n}^\infty\{|A_t| \leq \epsilon\}) = 0$.

Recall $a_1 = \min(R_{f,a}, R_{f,b}) > \underline{x}$ and $a_2 = \max(R_{f,a}, R_{f,b}) < \mathbb{E}(X)$. There exist $\eta > 0$ and $\delta > 0$ such that $\sup_{a\in[a_1,a_2]} \mathbb{P}(X \in [a-\delta, a+\delta]) \le 1-\eta$. For any $a \in [a_1, a_2]$, consider

$$h(a) := \mathbb{P}\left(\left|\frac{X-a}{a+\varphi_0(a)(X-a)}\right| \le \epsilon\right) = \mathbb{P}\left(X \in \left[a - \frac{a\epsilon}{1+\varphi_0(a)\epsilon}, a + \frac{a\epsilon}{1-\varphi_0(a)\epsilon}\right]\right).$$

Choose $\epsilon > 0$ such that

$$\sup_{a\in[a_1,a_2]} \left[\frac{a\epsilon}{1-\varphi_0(a)\epsilon} + \frac{a\epsilon}{1+\varphi_0(a)\epsilon}\right] = \sup_{a\in[a_1,a_2]} \left[\frac{2a\epsilon}{1-(\varphi_0(a)\epsilon)^2}\right] \le \frac{2a_2\epsilon}{1-(\varphi_0(a_1)\epsilon)^2} < \delta.$$

Then, we have $\sup_{a\in[a_1,a_2]} h(a) \le 1-\eta$.

For each $t \ge 0$, because $R_{f,t+1} \in [a_1, a_2]$, we have $\mathbb{P}(|A_{t+1}| \le \epsilon|\mathscr{F}_t) = h(R_{f,t+1}) \le 1-\eta$. Then, for any $1 \le n < N$,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{t=n}^{N}\{|A_t| \le \epsilon\}\right) &= \mathbb{E}\left[\prod_{t=n}^{N} 1_{\{|A_t|\le\epsilon\}}\right] = \mathbb{E}\left[\mathbb{E}\left(\prod_{t=n}^{N} 1_{\{|A_t|\le\epsilon\}}|\mathscr{F}_{N-1}\right)\right] \\
&= \mathbb{E}\left[\left(\prod_{t=n}^{N-1} 1_{\{|A_t|\le\epsilon\}}\right)\mathbb{P}\left(|A_N| \le \epsilon|\mathscr{F}_{N-1}\right)\right] \\
&\le (1-\eta)\mathbb{E}\left[\left(\prod_{t=n}^{N-1} 1_{\{|A_t|\le\epsilon\}}\right)\right] \\
&\le (1-\eta)^{N-n},
\end{aligned}$$

where the last inequality is due to mathematical induction. Sending N to infinity, we obtain $\mathbb{P}\left(\bigcap_{t=n}^{\infty}\{|A_t| \le \epsilon\}\right) = 0$ for each $n \ge 1$, so we conclude $\mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$.

Recalling $\varphi_0(R_{f,\infty}) \ne \varphi_1(R_{f,\infty})$, we conclude from (21) that $\lim_{t\to\infty} X_t > 0$ implies $\lim_{t\to\infty} A_t = 0$. As a result, $\mathbb{P}(\lim_{t\to\infty} X_t > 0) \le \mathbb{P}(\lim_{t\to\infty} A_t = 0) = 0$. Consequently, we conclude $\lim_{t\to\infty} X_t = 0$ almost surely, i.e., $\lim_{t\to\infty} Y_t = 1$ almost surely. $\square$

References

Barberis, N. and Huang, M. (2001). Mental accounting, loss aversion and individual stock returns, *Journal of Finance* **56**(4): 1247–1292.

Barberis, N. and Huang, M. (2008a). The loss aversion/narrow framing approach to the equity premium puzzle, *in* R. Mehra (ed.), *Handbook of the Equity Risk Premium*, Elsevier, North Holland, Amsterdam, pp. 199–236.

Barberis, N. and Huang, M. (2008b). Stocks as lotteries: The implications of probability weighting for security prices, *American Economic Review* **98**(5): 2066–2100.

Barberis, N. and Huang, M. (2009). Preferences with frames: A new utility specification that allows for the framing of risks, *Journal of Economic Dynamics and Control* **33**(8): 1555–1576.

Barberis, N., Huang, M. and Santos, T. (2001). Prospect theory and asset prices, *Quarterly Journal of Economics* **116**(1): 1–53.

Barberis, N., Huang, M. and Thaler, R. H. (2006). Individual preferences, monetary gambles, and stock market participation: A case for narrow framing, *American Economic Review* **96**(4): 1069–1090.

Barberis, N., Shleifer, A. and Vishny, R. (1998). A model of investor sentiment, *Journal of Financial Economics* **49**(3): 307 – 343.

Bernardo, A. E. and Ledoit, O. (2000). Gain, loss, and asset pricing, *Journal of Political Economy* **108**(1): 144–172.

Blume, L. and Easley, D. (1992). Evolution and market behavior, *Journal of Economic Theory* **58**(1): 9–40.

Blume, L. and Easley, D. (2009). Market selection and asset pricing, *in* T. Hens and K. R. Schenk-Hoppé (eds), *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, Amsterdam, pp. 403–437.

Blume, L. and Easley, D. (2010). Heterogeneity, selection, and wealth dynamics, *Annual Review of Economics* **2**: 425–450.

Borovička, J. (2013). Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. Working Paper.

Chapman, D. A. and Polkovnichenko, V. (2009). First-order risk aversion, heterogeneity, and asset market outcomes, *Journal of Finance* **64**(4): 1863–1887.

Cherny, A. and Madan, D. (2009). New measures for performance evaluation, *Review of Financial Studies* **22**(7): 2571–2606.

De Giorgi, E. G. and Legg, S. (2012). Dynamic portfolio choice and asset pricing with narrow framing and probability weighting, *Journal of Economic Dynamics and Control* **36**(7): 951–972.

De Giorgi, E. and Hens, T. (2006). Making prospect theory fit for finance, *Financial Markets and Portfolio Management* **20**(3): 339–360.

De Giorgi, E., Hens, T. and Levy, H. (2011). CAPM equilibria with prospect theory preferences. Unpublished manuscript.
URL: *http://ssrn.com/abstract=420184*

De Giorgi, E., Hens, T. and Rieger, M. O. (2010). Financial market equilibria with cumulative prospect theory, *Journal of Mathematical Economics* **46**(5): 633–651.

Del Vigna, M. (2013). Financial market equilibria with heterogeneous agents: Capm and market segmentation, *Mathematics and Financial Economics* **7**(4): 405–429.

Easley, D. and Yang, L. (2014). Loss aversion, survival and asset prices. Working Paper.

Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* **57**(4): 937–969.

Epstein, L. G. and Zin, S. E. (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis, *Journal of Political Economy* **99**(2): 263–286.

Ettlinger, H. (1928). Existence theorems for implicit functions of real variables, *Bulletin of the American Mathematical Society* **34**: 315–318.

Guo, J. and He, X. D. (2015). Equilibrium analysis with loss-averse investors. Working Paper.

Hansen, L. P. and Scheinkman, J. A. (2012). Recursive utility in a markov environment with stochastic growth, *Proceedings of the National Academy of Sciences* **109**(30): 11967–11972.

He, X. D. and Kou, S. G. (2014). Profit sharing in hedge funds. Working Paper.

He, X. D. and Zhou, X. Y. (2014). Myopic loss aversion, reference point, and money illusion, *Quantitative Finance* **14**(9): 1541–1554.

Kőszegi, B. and Rabin, M. (2006). A model of reference-dependent preferences, *Quarterly Journal of Economics* **121**(4): 1133–1165.

Kőszegi, B. and Rabin, M. (2007). Reference-dependent risk attitudes, *American Economic Review* **97**(4): 1047–1073.

Kreps, D. M. and Porteus, E. L. (1978). Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* **46**(1): 185–200.

Marinacci, M. and Montrucchio, L. (2010). Unique solutions for stochastic recursive utilities, *Journal of Economic Theory* **145**(5): 1776–1804.

Quiggin, J. (1982). A theory of anticipated utility, *Journal of Economic Behavior and Organization* **3**: 323–343.

Segal, U. and Spivak, A. (1990). First order versus second order risk aversion, *Journal of Economic Theory* **51**(1): 111 – 125.

Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty, *Journal of Risk and Uncertainty* **5**(4): 297–323.

von Neumann, J. and Morgenstern, O. (1947). *Theory of games and economic behavior*, Princeton University Press, New Jersey.

Xia, J. and Zhou, X. Y. (2013). Arrow-Debreu equilibria for rank-dependent utilities, *Mathematical Finance* **forthcoming**.

Yan, H. (2008). Natural selection in financial markets: Does it work?, *Management Science* **54**(11): 1935–1950.