Equilibrium Asset Pricing with Rational and Irrational Investors

(Authors’ names blinded for peer review)

We study a multi-period equilibrium asset pricing model with a rational investor and an irrational investor. The rational investor maximizes recursive utility of her consumption and the irrational investor receives additional utility of gains and losses evaluated by cumulative prospect theory. In a special case in which the elasticity of intertemporal substitution and the relative risk aversion degree are one for both the rational and the irrational investors, we prove the existence and uniqueness of the equilibrium price. In addition, we derive a stock return index and show that the irrational investor holds more equities than the rational investor if and only if her loss aversion degree is lower than this index. We also find that the rational investor dominates the market in the long run. Finally, we solve the equilibrium numerically in the general case.

Key words: equilibrium asset pricing; rational and irrational investors; stock return index; market dominance

1. Introduction

Neoclassical finance seeks to understand financial markets by assuming individuals to be rational: first, framing doesn’t matter, i.e., individuals’ decisions are not affected by changing descriptions of decision problems; secondly, individuals apply Bayes’ rule to make inferences and predictions; and thirdly, individuals employ expected utility theory (EUT) to evaluate random payoffs. However, numerous evidences have revealed that individuals are irrational: first, framing matters; secondly, individuals’ inferences and predictions can systematically violate Bayes’ rule; and thirdly, individuals’ preferences for random payoffs are not represented by EUT. In recent years, behavioral finance has been an emerging field which bases its analysis of financial decisions on the ground that individuals are irrational. See the discussion of neoclassical finance and behavioral finance in the survey by Barberis and Thaler (2003).

On the one hand, we have witnessed the recent development of portfolio selection models with irrational investors in different settings, examples being Gomes (2005), Barberis and Xiong (2009), Bernard and Ghossoub (2010), He and Zhou (2011, 2014), Barberis and Huang (2008a, 2009),
Barberis et al. (2006), De Giorgi and Legg (2012), He and Zhou (2014), De Giorgi et al. (2004), Benartzi and Thaler (1995), Berkelaar et al. (2004), and Jin and Zhou (2008). On the other hand, several representative-agent equilibrium pricing models with irrational agents have been proposed for the purposes of understanding a variety of empirical asset pricing facts that cannot be explained by models in neoclassical finance; see for instance Levy and Levy (2004), Del Vigna (2013), Barberis and Huang (2008b, 2001), Barberis et al. (2001). However, a fundamental question remains unanswered: what are the asset pricing implications when both rational and irrational investors live in the market? For example, do irrational investors take less or more risk than rational investors? Do rational investors dominate the market in the long run so that irrational investors cannot affect asset prices? The latter question is extremely important in addressing the criticism of behavioral asset pricing that irrational investors will be driven out of the market by rational investors and thus have no price impact. In this paper, we propose an equilibrium asset pricing model with one rational investor and one irrational investor to answer these questions.

We consider a multi-period consumption-based asset pricing model. There are two assets in the market: a stock paying out dividends with unit supply and a risk-free asset with zero supply. There are two investors in the market: one is rational and the other is irrational. We assume these two investors share the same belief of the market, but differ in their preferences. We model the rational investor’s preferences by recursive utility of consumption, a classical model of rational preferences. In addition to utility of consumption, the irrational investor also receives utility of gains and losses, which is modeled by cumulative prospect theory. In particular, the irrational investor is loss averse. Therefore, the irrationality in our model means the sentiment of experiencing trading gains and losses which does not concern a rational agent. This model of irrationality has been widely used in the literature (Barberis and Huang 2008a, 2009, Barberis et al. 2006, De Giorgi and Legg 2012, He and Zhou 2014). We consider the equilibrium returns of the stock and the risk-free asset such that both the rational and irrational investors’ returns are optimized and the market is clear.

When both the elasticity of intertemporal substitution (EIS) and the relative risk aversion degree (RRAD) in recursive utility are one, we prove the existence and uniqueness of the equilibrium and the equilibrium can be obtained by solving a single-variate equation. We then define a stock return index in equilibrium, which solely depends on the equilibrium stock return. We show that when the irrational investor’s loss aversion degree is lower than the index, she invests more in the stock than the rational investor and the presence of the irrational investor reduces the equity premium. When the irrational investor’s loss aversion degree is higher than the index, she invests less in the stock than the rational investor and the equity premium increases due to the presence of the irrational investor. We then show that the rational investor dominates the market and the irrational investor has little price impact in the long run. However, our numerical study shows that in a reasonably
long period, e.g., a decade, the irrational investor’s market share can still be significant, so the irrational investor still has a sizeable price impact.

In the general case, i.e., when EIS and RRAD are not necessarily one, we derive equilibrium equations for the stock price-dividend ratio and the risk-free rate. We develop an algorithm to solve equilibrium prices numerically.

The remainder of the paper is organized as follows: in Section 2, we propose the equilibrium asset pricing model. We then solve the equilibrium asset prices in Section 3 in the special case in which EIS and RRAD are one and investigate the properties of the equilibrium. In Section 4, we numerically solve the equilibrium in the general case. Finally, Section 5 concludes the paper. All proofs are placed in the Appendix.

2. Equilibrium Asset Pricing Model

2.1. Market

We consider a discrete-time financial market with one risky asset (e.g., a stock) and one risk-free asset. The net supplies of the stock and the risk-free asset are one and zero, respectively. The stock distributes dividend $D_t$ at each time $t$. We assume that the logarithmic growth rates of the dividend, $Z_{t+1} := \log(D_{t+1}) - \log(D_t)$, $t \geq 0$, are i.i.d and denote $\bar{z} := \operatorname{essinf} Z_{t+1}$ and $-\infty < \operatorname{essinf} Z_{t+1} < 0 < \operatorname{esssup} Z_{t+1} < +\infty$. The (gross) return rates of the stock and of the risk-free asset in period $t$ to $t+1$ are $R_{t+1}$ and $R_{f,t+1}$, respectively. Consider an agent who decides her consumption and investment plans at each time $t$. Denote $W_t$ as the before-consumption wealth of the agent. Suppose the agent consumes $C_t$, invests $\theta_t$ in the stock, and invests whatever remains in the risk-free asset. Then, the evolution of $W_t$ is

$$W_{t+1} = (W_t - C_t)R_{f,t+1} + \theta_t(R_{t+1} - R_{f,t+1}), \quad t \geq 0. \tag{1}$$

2.2. Investors’ Preferences

Suppose there are two investors in the market: investor 0 is rational and investor 1 is irrational. At each time $t$, investor $i$ chooses consumption amount $C_{i,t}$, and the dollar amount invested in the stock $\theta_{i,t}$, $i = 0, 1$.

The rational investor’s preference is modeled by recursive utility (Kreps and Porteus 1978). More precisely, the rational investor’s preferences are represented by the utility process $X_{0,t}$, $t \geq 0$, modeled as

$$X_{0,t} = H(C_{0,t}, M(X_{0,t+1}|\mathcal{F}_t)), \quad t \geq 0, \tag{2}$$

where $\mathcal{F}_t$ stands for the information available at time $t$, i.e., $\mathcal{F}_t = \sigma(Z_s, s \leq t)$, $M(X_{0,t+1}|\mathcal{F}_t)$ is the time-$t$ certainty equivalent of time-$(t+1)$ (random) utility $X_{0,t+1}$, and $H(\cdot)$ is an aggregator.
Following Barberis and Huang (2008a, 2009), Barberis et al. (2006), De Giorgi and Legg (2012), and He and Zhou (2014), we assume the irrational investor’s preferences are represented by the utility process \( X_{1,t}, t \geq 0 \), modeled as
\[
X_{1,t} = H(C_{1,t}, M(X_{1,t+1} | \mathcal{F}_t) + b_t G_t), \quad t \geq 0,
\]
where \( G_t \) stands for the utility of gains and losses and \( b_t \), which is \( \mathcal{F}_t \)-measurable, is the parameter measuring the weight of the utility of gains and losses relative to the utility of consumption.

Compared with the rational investor, the additional term \( G_t \) contributes to the irrational investor’s utility. This term models the irrational investor’s sentiment of experiencing gains and losses in every trading period and \( b_t \) measures how strong this sentiment is. Thus, in our model, the “irrationality” comes from being sensitive to trading gains and losses while a rational agent only cares about her consumption.

In the following, we assume
\[
H(c, z) = \begin{cases} 
[(1-\beta)c^{1-\rho} + \beta z^{1-\rho}]^{\frac{1}{1-\gamma}}, & 0 < \rho \neq 1, \\
\exp((1-\beta) \ln c + \beta \ln z), & \rho = 1,
\end{cases} \quad M(X) = \begin{cases} 
[E(X^{1-\gamma})]^{\frac{1}{1-\gamma}}, & 0 < \gamma \neq 1, \\
E[\ln(X)], & \gamma = 1.
\end{cases}
\]
With this specification, \( \beta \) is the discount factor, \( \gamma \) measures the risk aversion of the agent, and \( 1/\rho \) is the elasticity of intertemporal substitution (Kreps and Porteus 1978). On the other hand, we assume
\[
G_t := E[\nu(W_{1,t+1} - (W_{1,t} - C_{1,t})R_{f,t+1}) | \mathcal{F}_t] = E[\nu(\theta_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t],
\]
with, for some \( K \geq 1 \),
\[
\nu(x) := \begin{cases} 
x, & x \geq 0, \\
Kx, & x < 0.
\end{cases}
\]
Thus, the irrational investor chooses the risk-free payoff of the investment, i.e., \( (W_{1,t} - C_{1,t})R_{f,t+1} \), to be the reference point that distinguishes gains and losses and the utility of gains and losses. \( G_t \) is represented by cumulative prospect theory with a piece-wise utility function and with no probability weighting (Tversky and Kahneman 1992). The parameter \( K \geq 1 \) stands for loss aversion degree, which measures how averse to losses the irrational investor is.

### 2.3. Optimal Portfolio Selection

Next, we discuss the optimal portfolios of the rational and irrational investors. Because the rational investor’s utility can be regarded as a special case of the irrational investor’s utility by setting \( b_t = 0 \), we focus our attention to the optimal portfolio of the irrational investor in the following.
Finally, when \( \rho \) can be treated by sending the following Bellman equation

\[
\max_{\{c_{1,s},s\ge t; \theta_{1,s}\}_{s\ge t}} \begin{align*}
\text{Maximize:} & \quad X_{1,t} \\
\text{Subject to:} & \quad W_{1,s+1} = (W_{1,s} - C_{1,s})R_{f,s+1} + \theta_{1,s} (R_{s+1} - R_{f,s+1}), \quad s \ge t.
\end{align*}
\]  \tag{6}

We use the methodology in Barberis and Huang (2009) and He and Zhou (2014) to solve problem (6). Denote by \( V_{1,t} \) the optimal value of (6), i.e., the irrational investor’s optimal utility at time \( t \).

Thanks to the recursive nature of utility (3), dynamic programming principle immediately yields the following Bellman equation

\[
V_{1,t} = \max_{c_{1,t},\theta_{1,t}} \left\{ (1 - \beta)C_{1,t}^{1 - \rho} + \beta \left[ \left( E(V_{1,t+1}^{1 - \gamma} | \mathcal{F}_t) \right)^{\frac{1}{1 - \gamma}} + b_{t} E[\nu(\theta_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t]) \right]^{1 - \rho} \right\}^{\frac{1}{\rho}}. \tag{7}
\]

Thanks to the homogeneity of the aggregator \( H \), certainty equivalent \( M \), and CPT utility function \( \nu \), we are able to simplify the Bellman equation (7) by defining \( c_{1,t} := C_{1,t}/W_{1,t}, w_{1,t} := \theta_{1,t}/(W_{1,t} - C_{1,t}), \) and \( \Psi_{1,t} := V_{1,t}/W_{1,t} \), which are the percentage consumption, the percentage allocation to the stock, and the optimal utility per unit wealth, respectively. Plugging these variables into (7), we obtain

\[
\Psi_{1,t} = \max_{c_{1,t},w_{1,t}} \left\{ (1 - \beta)c_{1,t}^{1 - \rho} + \beta (1 - c_{1,t})^{1 - \rho} \left[ \left( E((\Psi_{1,t+1}R_{1,t+1})^{1 - \gamma} | \mathcal{F}_t) \right)^{\frac{1}{1 - \gamma}} + b_{t} E[\nu(w_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t]) \right]^{1 - \rho} \right\}^{\frac{1}{\rho}}. \tag{8}
\]

where \( R_{1,t+1} := R_{f,t+1} + w_{1,t}(R_{t+1} - R_{f,t+1}) \) is the gross return of the irrational investor’s portfolio.

We implicitly assume \( \rho \neq 1 \) and \( \gamma \neq 1 \) in the derivation of (8). The case in which \( \rho = 1 \) or \( \gamma = 1 \) can be treated by sending \( \rho \) or \( \gamma \) to one in (8). Consequently, when \( \rho = 1 \) and \( \gamma \neq 1 \), we have

\[
\Psi_{1,t} = \max_{c_{1,t},w_{1,t}} \exp \left\{ (1 - \beta) \ln c_{1,t} + \beta \ln (1 - c_{1,t}) \right. \\
+ \left. \beta \ln \left[ \left( E((\Psi_{1,t+1}R_{1,t+1})^{1 - \gamma} | \mathcal{F}_t) \right)^{\frac{1}{1 - \gamma}} + b_{t} E[\nu(w_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t]) \right] \right\}. \tag{9}
\]

When \( \rho \neq 1 \) and \( \gamma = 1 \), we have

\[
\Psi_{1,t} = \max_{c_{1,t},w_{1,t}} \left\{ (1 - \beta)c_{1,t}^{1 - \rho} + \beta (1 - c_{1,t})^{1 - \rho} \left[ \exp \left( E(\ln \Psi_{1,t+1} | \mathcal{F}_t) + E(\ln R_{1,t+1} | \mathcal{F}_t) \right) \right. \\
+ \left. b_{t} E[\nu(w_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t]) \right]^{1 - \rho} \right\}^{\frac{1}{\rho}}. \tag{10}
\]

Finally, when \( \rho = \gamma = 1 \), we have

\[
\Psi_{1,t} = \max_{c_{1,t},w_{1,t}} \exp \left\{ (1 - \beta) \ln c_{1,t} + \beta \ln (1 - c_{1,t}) \right. \\
+ \beta \ln \left[ \exp \left( E(\ln \Psi_{1,t+1} | \mathcal{F}_t) + E(\ln R_{1,t+1} | \mathcal{F}_t) \right) \right. \\
+ \left. b_{t} E[\nu(w_{1,t}(R_{t+1} - R_{f,t+1})) | \mathcal{F}_t]) \right] \right\}. \tag{11}
\]
Note that in (8), (9), (10), and (11), the maximization over $c_{1,t}$ and $w_{1,t}$ is separated.

Finally, we specify $b_t$. Consider two specifications: First, following Barberis and Huang (2008a, 2009), Barberis et al. (2006), De Giorgi and Legg (2012), and He and Zhou (2014), we set

$$b_t = b^0, \quad t \geq 0$$

(12)

for some constant $b^0 \geq 0$. Secondly, we choose $b_t$ to be a constant proportion to the certainty equivalent of the optimal utility per unit wealth $\Psi_{1,t+1}$ at time $t+1$. More precisely, we set

$$b_t = \bar{b} M(\Psi_{1,t+1}|\mathcal{F}_t) = \bar{b} \times \begin{cases} \left( \frac{\mathbb{E}(\Psi_{1,t+1})^{1-\gamma}}{|\mathcal{F}_t|} \right)^{\frac{1}{1-\gamma}}, & \gamma \neq 1, \\ \exp \left( \frac{\ln \Psi_{1,t+1}}{|\mathcal{F}_t|} \right), & \gamma = 1, \end{cases}$$

(13)

for some constant $\bar{b} \geq 0$. Compared to specification (12), $b_t$ in (12) is scaled by the certainty equivalent of the optimal utility per unit wealth at time $t+1$. This scaling is useful to obtain tractable equilibrium analysis as we will see later on. Furthermore, numerical studies in Section 4 show that these two specifications of $b_t$ do not differ significantly in the resulting equilibrium asset prices. We further note that Barberis and Huang (2001) and Barberis et al. (2001) also use a scaled version of $b_t$ in their representative-agent model of asset pricing, although the scaling factor therein is different from ours.

### 2.4. Equilibrium

Denote $P_t$ as the ex-dividend price of the risky asset at time $t$. Then, the gross return of the stock is

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{D_t}, \quad t \geq 0.$$

Equilibrium asset pricing is the analysis of finding asset prices such that all investors in the market are optimized and all assets in the market are cleared. Formally, we take the following definition from Yan (2008).

**Definition 1.** A competitive equilibrium is a price system $\{R_{f,t+1}, P_t\}_{t=0}^\infty$ and optimal consumption-investment stream $\{C_{i,t}, \theta_{i,t}\}_{t=0}^\infty, i = 0, 1$ that satisfy

(i) **Individual Optimality:** $\{C_{i,t}, \theta_{i,t}\}_{t=0}^\infty$ is the optimal consumption-investment process with corresponding wealth process $\{W_{i,t}\}_{t=0}^\infty, i = 0, 1$. 

\(^1\)This scaling also makes the two terms, $M(X_{1,t+1}|\mathcal{F}_t)$ and $b_t G_t$, in the aggregation of the investor’s utility comparable. Indeed, with a piece-wise linear utility function $\nu$, the natural unit of $G_t$ is dollar. On the other hand, $M(X_{1,t+1}|\mathcal{F}_t)$ is equal to the wealth multiplied by the certainty equivalent of utility per unit wealth, so it has unit dollar multiplied utility. If we choose $b_{1,t}$ to be constant, $M(X_{1,t+1}|\mathcal{F}_t)$ and $b_t G_t$ do not have the same unit. However, if we choose $b_t$ to be proportional to the certainty equivalent of the optimal utility per unit wealth, these two components have the same unit.
(ii) **Clearing of Consumption:**

\[ C_{0,t} + C_{1,t} = D_t. \]  

(14)

(iii) **Clearing of the stock:**

\[ \theta_{0,t} + \theta_{1,t} = P_t. \]  

(15)

(iv) **Clearing of the risk-free asset:**

\[ (W_{0,t} - C_{0,t} - \theta_{0,t}) + (W_{1,t} - C_{1,t} - \theta_{1,t}) = 0. \]  

(16)

**Proposition 1.** A price system \( \{R_{f,t+1}, P_t\}_{t=0}^\infty \) and a consumption-investment stream \( \{C_{i,t}, \theta_{i,t}\}_{t=0}^\infty \) with corresponding wealth process \( \{W_{i,t}\}_{t=0}^\infty \), \( i = 0, 1 \) constitute a competitive equilibrium if and only if the consumption rate \( c_{i,t} := C_{i,t}/W_{i,t} \) and the percentage allocation to the stock \( w_{i,t} := \theta_{i,t}/(W_{i,t} - C_{i,t}) \) are the optimal choice of investor \( i, i = 0, 1 \) and satisfy

\[
\frac{c_{0,t}}{1 - c_{0,t}} Y_t + \frac{c_{1,t}}{1 - c_{1,t}} (1 - Y_t) = D_t/P_t, \\
w_{0,t} Y_t + w_{1,t} (1 - Y_t) = 1,
\]  

(17) (18)

where \( Y_t := (W_{0,t} - C_{0,t})/P_t \) stands for the rational investor’s (investor 0’s) after-consumption market share at time \( t \).

3. **Equilibrium Analysis When Investors are Myopic**

In this section, we analyze the competitive equilibrium when \( \rho = \gamma = 1 \) and \( b_t \) is specified as in (13). In this case, the rational investor becomes an expected utility maximizer with the logarithmic utility, so she is **myopic**. The irrational investor differs from the rational investor in that the former is also sensitive to the experience of trading gains and losses.

3.1. **Existence and Uniqueness of Equilibrium**

**Theorem 1.** Suppose \( \rho = \gamma = 1 \) and \( b_t \) is given as in (13), and \( Z_{t+1} \) has finitely many atoms. Denote \( X := e^{Z_{t+1}}/\beta \) and

\[
f_0(w,a) := \begin{cases} 
\exp \left[ \mathbb{E} \left( \ln(a + w(X - a)) \right) \right], & \mathbb{E} \left( \ln(a + w(X - a)) \right) > -\infty, \\ 0, & \text{o.w.,} 
\end{cases} \]  

(19)

\[
f_1(w,a) := f_0(w,a) + w \mathbb{E} \left[ \nu (X - a) \right]. \]  

(20)

For each \( a > 0 \), denote \( \varphi_i(a) \) as the unique maximizer of \( f_i(w,a) \) with respect to \( w \) in its domain, \( i = 0, 1 \). Then, the competitive equilibrium exists with the equilibrium price dividend ratio

\[
\frac{P_t}{D_t} \equiv \beta \frac{1}{1 - \beta}, \quad t \geq 0,
\]  

(21)
and equilibrium risk-free rate \( R_{f,t+1} \) uniquely determined by
\[
\varphi_0(R_{f,t+1})Y_t + \varphi_1(R_{f,t+1})(1 - Y_t) = 1, \tag{22}
\]
where \( Y_t \) stands for the rational investor’s market share at time \( t \). Moreover, \( R_{f,t+1} \) is a continuous and monotone function of \( Y_t \).

Because \( \rho = \gamma = 1 \) in our setting, both the rational and irrational investor consumes the same constant fraction of their wealths. As a result, to clear the consumption, the equilibrium price-dividend ratio must be a constant. Consequently, the equilibrium stock return
\[
R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}/D_{t+1} + 1}{P_t/D_t} \beta e^{\gamma_{t+1}}
\]
is i.i.d. over time. The risk-free rate is determined by (22), which is a clearing condition for the stock market. We can see that the equilibrium risk-free rate depends on the market shares of the rational and irrational investors (\( Y_t \) and \( 1 - Y_t \), respectively).

### 3.2. A Stock Return Index

Define the following index of the stock return:
\[
K^* = \frac{\mathbb{E}(R_{t+1} - R_{f,b})^+}{\mathbb{E}(R_{t+1} - R_{f,b})^-}, \tag{23}
\]
which is the ratio of the expected gain and expected loss of the stock with the reference point being
\[
R_{f,b} := \frac{1}{\mathbb{E}(1/R_{t+1})}. \tag{24}
\]
Therefore, this index can be regarded as a measure of the performance of the stock.

**Theorem 2.** Let \( w^*_0,t \) and \( w^*_1,t \) be the optimal percentage allocation to the stock of the rational and irrational investors, respectively.

(i) If \( K = K^* \), then \( R_{f,t+1} = R_{f,b} \), \( \mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t] = 0 \), and \( w^*_0,t = w^*_1,t = 1 \).

(ii) If \( K < K^* \), then \( R_{f,t+1} > R_{f,b} \), \( \mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t] > 0 \), and \( 0 < w^*_0,t < 1 < w^*_1,t \).

(iii) If \( K > K^* \), then \( R_{f,t+1} < R_{f,b} \), \( \mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t] < 0 \), and \( w^*_0,t > 1 > w^*_1,t \geq 0 \).

It is well-known that \( R_{f,b} \) is the equilibrium risk-free rate when there is only one investor in the market who maximizes her expected logarithmic utility of consumption. Theorem 2 shows that when \( K < K^* \), i.e., when the irrational investor is insufficiently loss averse, the equilibrium risk-free rate \( R_{f,t+1} \) is lower than the benchmark \( R_{f,b} \) whatever the market shares of the rational and irrational investors are. In addition, the utility of gains and losses of the irrational investor \( G_t = w_{1,t}\mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t] > 0 \) for any allocation \( w_{1,t} > 0 \). As a result, the irrational investor is more willing to buy the stock than the rational investor because doing so can give her additional positive
utility of gains and losses. Consequently, the optimal allocation to the stock of the irrational investor is higher than the rational investor. On the other hand, when $K > K^*$, i.e., when the irrational investor is sufficiently loss averse, the equilibrium risk-free rate is higher than the benchmark $R_{f,b}$, the irrational investor receives negative utility of gains and losses when investing in the stock, and thus she invests less in the stock than the rational investor.

### 3.3. Equity Premium

The conditional *equity premium* is the expected return of the stock in excess of the risk-free rate, i.e.,

$$\text{EP}_t := \mathbb{E}[R_{t+1} | \mathcal{F}_t] - R_{f,t+1}, \quad t \geq 0.$$ 

When the irrational investor is absent, the equilibrium risk-free rate is $R_{f,b}$, so the equity premium is $\text{EP}_{t,b} = \mathbb{E}[R_{t+1} | \mathcal{F}_t] - R_{f,b}$.

**Theorem 3.** (i) If $K < K^*$, then $\text{EP}_t < \text{EP}_{t,b}$, and $\text{EP}_t$ is strictly increasing with respect to the market share of the rational investor.

(ii) If $K > K^*$, then $\text{EP}_t > \text{EP}_{t,b}$, and $\text{EP}_t$ is strictly decreasing with respect to the market share of the rational investor.

Theorem 3 shows that the presence of insufficiently loss averse irrational investors reduces the equity premium and the presence of sufficiently loss averse investors increases the equity premium. Furthermore, if $\varphi_1$ is decreasing, i.e., the irrational investor’s allocation to the stock is decreasing with respect to the risk-free rate, then with higher market share of the irrational investor, the equity premium is lower if the irrational investor is insufficiently loss averse and is higher if the irrational investor is sufficiently loss averse.

Intuitively, it should be the case that $\varphi_1$ is decreasing, i.e., the irrational investor’s allocation to the stock is decreasing with respect to the risk-free rate. In all our numerical studies, this condition is satisfied, although it turns out to be difficult to prove it. However, we can show that $\varphi_0$, the allocation to the stock by the rational investor, is decreasing.

When the stock return $R_{t+1}$ follows binomial distribution

$$R_{t+1} = \begin{cases} X_1, & \text{with probability } p, \\ X_2, & \text{with probability } 1 - p, \end{cases}$$

with $X_1 < X_2$, one can compute that

$$K^* = \frac{X_2}{X_1}.$$ 

Therefore, for a more positively skewed stock return, i.e., for a stock return with larger $X_1$ and smaller $1 - p$, $K^*$ is larger. Consequently, the irrational investor is more likely to be insufficiently
loss averse, and the equity premium is reduced. In other words, stocks with more positively skewed returns ask for less equity premium. This observation is consistent with the conclusion in Barberis and Huang (2008b).

3.4. Market Dominance

An important question is whether the rational investor will dominate the market in the long run. We define the survival or extinction of the rational and irrational investors as following:

**Definition 2.** The rational investor becomes *extinct* if

$$\lim_{t \to \infty} Y_t = 0, \quad a.s.,$$

*survives* if extinction does not occur, and *dominates* the market if

$$\lim_{t \to \infty} Y_t = 1, \quad a.s.$$

The extinction, survival, and dominance of the irrational investor are defined similarly. The existing literature, (e.g., Yan (2008)) use the consumption $C_i,t/\sum_{i=0}^{1} C_i,t$ to measure market dominance. Since $c_i,t = C_i,t/W_i,t$ is a constant, $i = 0, 1$, the two measures are equivalent in this setting.

**Theorem 4.** If $K \neq K^*$, then the rational investor dominates the market. Furthermore,

$$\lim_{t \to \infty} w_{0,t}^* = 1, \quad \lim_{t \to \infty} R_{f,t} = R_{f,b}, \quad \lim_{t \to \infty} w_{1,t}^* = \phi_1(R_{f,b}) \quad a.s.$$

Theorem 4 shows that the rational investor, who maximizes the expected log utility of consumption, dominates the irrational investor. This result is not surprising because the rational investor maximizes the long-run wealth accumulation (Blume and Easley 1992). Because the rational investor dominates the irrational investor in the long run, the equilibrium asset prices in the long run are determined by the rational investor. In particular, the long-run risk-free rate is $R_{f,b}$.

We set the dividend process such that the resulting equilibrium stock return matches the historical quarterly S&P 500 index returns in period 1990–2013. We then simulate the dividend process and calculate the corresponding optimal wealth processes of the rational and irrational investors. The plots of the wealth processes are shown in Figures 1 and 2, which correspond to the case in which $K > K^*$ and $K < K^*$, respectively. We can see that the dominance of the irrational investor is not clear even in 23 years; see Figure 2. Therefore, although in theory the irrational investor will be driven out of the market in the long run, in a reasonably long period, the irrational investor can still survive. Consequently, the irrational investor still affects asset prices in the market.

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2 S&P 500 data is downloaded from Yahoo Data Download.
Figure 1  Simulated wealth process of the rational investor (heavy solid line), simulated wealth process of the irrational investor (dashed line), and benchmark wealth process (solid line). The benchmark wealth process is computed by consuming $1 - \beta$ portion of wealth and investing the remaining fully in the index in every period, i.e., the strategy taken by the rational investor if the irrational investor were absent. The loss aversion of the irrational investor is chosen to be high, i.e., $K > K^*$. 

Figure 2  Simulated wealth process of the rational investor (heavy solid line), simulated wealth process of the irrational investor (dashed line), and benchmark wealth process (solid line). The benchmark wealth process is computed by consuming $1 - \beta$ portion of wealth and investing the remaining fully in the index in every period, i.e., the strategy taken by the rational investor if the irrational investor were absent. The loss aversion of the irrational investor is chosen to be low, i.e., $K < K^*$. 

4. Equilibrium Analysis When Investors are Non-Myopic

In this section, we study equilibrium asset pricing for a general $\rho$ and a general $\gamma$.

**Theorem 5.** Define

$$F(c,x) = (1 - \beta)c^{1-\rho} + \beta(1-c)^{1-\rho}x^{1-\rho}, \quad x > 0, c > 0,$$
\[
\phi(x) := \frac{1}{1 + \left(\frac{\beta}{1-\gamma}\right)^{1/\rho} x^{(1-\rho)/\rho}}, \quad x > 0
\]

and recall the certainty equivalent function \(M(\cdot)\) defined as in (4). Suppose there exist functions \(\varphi_i(y, a), \Psi_i(y)\), \(R_f(y)\), \(g(y)\), \(R(y, z)\), and \(h(y, z)\), \(i = 0, 1\), such that

\[
\varphi_0(y, a) = \arg \max_w \{ M[\Psi_0(h(y, Z_{t+1}))](a + w(R(y, Z_{t+1}) - a)) \}, \quad (25)
\]

\[
\varphi_1(y, a) = \arg \max_w \{ M[\Psi_1(h(y, Z_{t+1}))](a + w(R(y, Z_{t+1}) - a)) + bM[\Psi_1(h(y, Z_{t+1}))]|E[w(R(y, Z_{t+1})) - a)] \}, \quad (26)
\]

\[
\Theta_0(y, a) := \max_w \{ M[\Psi_0(h(y, Z_{t+1}))](a + w(R(y, Z_{t+1}) - a)) \}, \quad (27)
\]

\[
\Theta_1(y, a) := \max_w \{ M[\Psi_1(h(y, Z_{t+1}))](a + w(R(y, Z_{t+1}) - a)) + bM[\Psi_1(h(y, Z_{t+1}))]|E[w(R(y, Z_{t+1})) - a)] \}, \quad (28)
\]

\[
\Psi_i(\cdot) = F(\phi(\Theta_i(y, R_f(y))), \Theta_i(y, R_f(y))), \quad i = 0, 1, \quad (29)
\]

\[
1 - \varphi_0(y, R_f(y)) y + \varphi_1(y, R_f(y))(1 - y), \quad (30)
\]

\[
\frac{1}{g(y)} = \frac{\phi(\Theta_0(y, R_f(y)))}{1 - \phi(\Theta_0(y, R_f(y)))} y + \frac{\phi(\Theta_1(y, R_f(y)))}{1 - \phi(\Theta_1(y, R_f(y)))}(1 - y), \quad (31)
\]

\[
h(y, z) = 1 / \left[ 1 + \left( \frac{1}{y} - 1 \right) \frac{R_f(y) + \varphi_1(y, R_f(y)) \cdot (R(y, z) - R_f(y))}{R_f(y) + \varphi_0(y, R_f(y)) \cdot (R(y, z) - R_f(y))} \right], \quad (32)
\]

\[
R(y, z) = \frac{g(h(y, z))}{g(y)} + 1 e^{-x}. \quad (33)
\]

Then, the equilibrium exist and the equilibrium stock price and risk-free rate are

\[
P_t = g(Y_t) D_t, \quad R_{f,t+1} = R_f(Y_t),
\]

respectively. Furthermore, the optimal consumption rate and percentage allocation to the stock of investor \(i\) are

\[
c^*_i = \phi(\Theta_i(Y_t, R_f(Y_t))), \quad w^*_i = \varphi_i(Y_t, R_f(Y_t)), \quad i = 0, 1.
\]

Moreover, \(Y_t\) is the market share of the rational investor (investor 0) at time \(t\) and evolves according to

\[
Y_{t+1} = h(Y_t, Z_{t+1}).
\]

In the following, we numerically solve the equilibrium equations in Theorem 5 by employing the algorithm in Appendix B. Following Barberis and Huang (2009), we set \(K = 2.75, \beta = .98, \gamma = .95\). We choose \(b = .01\) so that the average of \(b_t\) matches the value of \(b\) used in Barberis and Huang (2009). In order to match the first three moments of historical quarterly S&P 500
index returns in period 1990-2013, we assume the log dividend growth rate to follow Bernoulli distribution:

\[
Z = \begin{cases} 
\ln(1.0976), & \text{with probability 0.6060 (high state)}, \\
\ln(0.9125), & \text{with probability 0.3940 (low state)}. 
\end{cases} 
\] (34)

Figure 3 depicts the price-dividend ratio, consumption rate of the rational and irrational investor, (expected) return rates of the stock and of the risk-free asset, equity premium, percentage allocations to the stock of the rational and irrational investors, and the transition of the rational investor’s market share. We observe that the irrational investor always invests less in the stock than the rational, the same as the conclusion in Theorem 2. The price-dividend and the stock returns in the good market scenario and in the bad market scenario are insensitive to the rational investor’s market share. The equity premium, which is represented by the gap between the expected stock return and risk-free rate, varies with respect to the rational investor’s market share. Moreover, the equity premium increases along with the rational investor’s market share, and becomes largest as the rational investor dominates the market. Finally, the rational investor’s market share increases, i.e., \( Y_{t+1} > Y_t \), in the good market scenario because she invests more in the stock than the irrational investor. Similarly, the rational investor’s market share decreases in the bad market scenario.

![Figure 3](image-url)

**Figure 3**  
Price-dividend ratio, percentage allocations to the stock of the rational and irrational investors, (expected) return rates of the stock and of the risk-free asset, and the transition of the rational investor’s market share. \( b_t \) is specified as in (13).
Figure 4  Price-dividend ratio, percentage allocations to the stock of the rational and irrational investors, (expected) return rates of the stock and of the risk-free asset, and the transition of the rational investor’s market share. $b_t$ is specified as in (12).

Furthermore, we show that the two specifications of $b_t$ in (12) and (13) do not differ significantly in determining the equilibrium prices. Indeed, Figure 4 plots the price-dividend ratio, percentage allocations to the stock of the rational and irrational investors, (expected) return rates of the stock and of the risk-free asset, and the transition of the rational investor’s market share with $b_t \equiv b^0 = 0.3$ while keeping other parameters the same as used in Figure 3. We can see that the plots in Figures 3 and 4 are very similar.

**Theorem 6.** Define the following index of the stock return for the general case:

\[
K^* = \frac{\mathbb{E}(X) - \mathbb{E}\left(\frac{X^{1-\gamma}}{-\gamma} + \frac{X^{1-\gamma}}{-\gamma}\right)}{\mathbb{E}(X) - \frac{\mathbb{E}(X^{1-\gamma})}{-\gamma}}
\]

where $X \sim e^Z$. If $K = K^*$, then $R = R_b$, $R_f = R_{f,b}$ and $\varphi_0(y) = \varphi_1(y) = 1$ is one solution set to the equilibrium system (25)-(33), where $R_b$ and $R_{f,b}$ are equilibrium risky and risk-free rate when there is only rational investor in the market.

Applying our numeric algorithm, we also have the following empirical results similar to Theorem 2:
If $K < K^*$, then $0 < \varphi_0(y) < \varphi_1(y) < 1$, $\forall$ $0 < y < 1$.

(iii) If $K > K^*$, then $\varphi_0(y) > \varphi_1(y) \geq 0$, $\forall$ $0 < y < 1$.

$K^*$ only depends on $\gamma$ and the distribution of $Z$. Table 1 shows different $K^*$ values with respect to different relative risk aversion degree $\gamma$, where $Z$ follows Bernoulli distribution specified in (34).

| Table 1 Stock Return Index $K^*$ for General Case |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\gamma$  | 0.5   | 0.9   | .95   | 1.05  | 1.1   | 1.5   | 2     | 3     | 4     |
| $K^*$     | 1.0967| 1.1808| 1.1918| 1.214 | 1.2253| 1.3192| 1.4468| 1.7403| 2.0934| 2.5180|

Figure 5 depicts the price-dividend ratio, consumption rate of the rational and irrational investor, (expected) return rates of the stock and of the risk-free asset, equity premium, percentage allocations to the stock of the rational and irrational investors, and the transition of the rational investor’s market share when $K = 1.01 < K^*$. On the contrary, here the irrational investor always invests more in the stock than the rational. The equity premium, which is represented by the gap between the expected stock return and risk-free rate, varies with respect to the rational investor’s market share. Moreover, the equity premium increases along with the market share the rational investor, and becomes largest as the rational investor dominates the market. Finally, the rational investor’s market share decreases in good market scenario and decreases in bad market scenario.

5. Conclusions

In this paper, we have proposed a multi-period consumption-based equilibrium pricing model with one rational investor and one irrational investor. The rational investor’s preferences are represented by recursive utility of consumption while an additional utility of trading gains and losses, which is modeled by cumulative prospect theory, enters into the irrational investor’s preferences. When the EIS and RRAD of both the rational and the irrational investors are one, we have proved the existence of the equilibrium. Furthermore, we have shown that the irrational investor holds more stock than the rational investor if and only if her loss aversion degree is lower than a stock return index. The equity premium increases in the presence of sufficiently loss averse irrational investors but decreases in the presence of insufficiently loss averse irrational investors. We have also shown that the rational investor dominates the market in the long run. However, we have illustrated that in a reasonably long time period, the irrational investor still survives and has a price impact. Finally, we solve the equilibrium numerically when EIS or RRAD is not necessarily one.

For future work, we may investigate the equilibrium when there are heterogeneous irrational investors with different loss aversion degrees. In particular, we can study which irrational investor has the biggest price impact.
Appendix A: Proofs

Proof of Proposition 1 On the one hand, if \( \{R_{f,t+1}, P_t\}_{t=0}^{\infty} \) and \( \{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty} \) constitute a competitive equilibrium, then \( c_{i,t} \) and \( w_{i,t}, t \geq 0 \) are optimal consumption rate and percentage allocation to the stock, respectively. Adding (15) and (16), we obtain

\[
(W_{0,t} - C_{0,t}) + (W_{1,t} - C_{1,t}) = P_t.
\]

Then, dividing both sides of (14) and (15) by \( P_t \), we obtain (17) and (18).

One the other hand, suppose \( c_{i,t} \) and \( w_{i,t}, t \geq 0 \) are the optimal consumption rate and percentage allocation to the stock, respectively, and satisfy equations (17) and (18) for some \( Y_t \geq 0, i = 0, 1 \). Define \( W_{0,t} := Y_t P_t / (1 - c_{0,t}), W_{1,t} := (1 - Y_t) P_t / (1 - c_{1,t}), C_{i,t} := c_{i,t} W_{i,t}, \) and \( \theta_{i,t} := w_{i,t} W_{i,t}, i = 0, 1 \). Then, it is straightforward to check that \( \{C_{i,t}, \theta_{i,t}\}_{t=0}^{\infty} \) is the optimal consumption-investment process with corresponding wealth process \( \{W_{i,t}\}_{t=0}^{\infty} \) and equations (14)–(16) are satisfied. □

Proof of Theorem 1 Denote \( \underline{\mu} := \text{essinf } X \). Because we assume essinf \( Z_{t+1} \in (-\infty, 0) \), we have \( \underline{\mu} \in (0, 1) \). For each \( a > 0 \), denote \( \bar{w} := a / (a - \underline{\mu}) \) if \( a > \underline{\mu} \) and denote \( \bar{w} := +\infty \) otherwise. Then, the domain of \( f_i(w, a) \) with respect to \( w \) is contained in \([0, \bar{w}]\). Furthermore, it is obvious that \( f_0(w,a) \) is positive and continuous in \( w \in [0, \bar{w}] \) due to the dominated convergence theorem. Moreover, \( f_0(w,a) = \exp{\left[ \mathbb{E}(\ln(a + w(X - a)) 1_{\{X > a\}}) \right]} \exp{\left[ \mathbb{E}(\ln(a + w(X - a)) 1_{\{X < a\}}) \right]} \). By monotone convergence theorem we have \( \lim_{w \uparrow \bar{w}} f_0(w,a) = f_0(\bar{w},a) \). Therefore, \( f_0 \) and \( f_1 \) are continuous in \( w \in [0, \bar{w}] \), so their maximizers must exist.
Dominated convergence theorem shows that $f_0$ is twice continuously differentiable in $w \in (0, \bar{w})$ and
\[
\frac{\partial}{\partial w} f_0(w, a) = f_0(w, a) \mathbb{E} \left( \frac{X - a}{a + w(X - a)} \right)
\]
\[
\frac{\partial^2}{\partial w^2} f_0(w, a) = f_0(w, a) \left[ -\mathbb{E} \left( \frac{X - a}{a + w(X - a)} \right)^2 + \left( \mathbb{E} \left( \frac{X - a}{a + w(X - a)} \right) \right)^2 \right].
\]

Jensen's inequality shows that $\frac{\partial^2}{\partial w^2} f_0(w, a) < 0$, so $f_0$ is strictly concave in $w$. Consequently, $f_1$ is also strictly concave in $w$. As a result, $\varphi_i(a)$ is well-defined, $i = 0, 1$. In addition, $\varphi_i(a) = 0$ if $\lim_{w \uparrow 0} \frac{\partial}{\partial w} f_i(w, a) \leq 0$, $\varphi_i(a) = \bar{w}$ if $\lim_{w \downarrow 0} \frac{\partial}{\partial w} f_i(w, a) \geq 0$, and $\varphi_i(a)$ is determined by the first-order condition:
\[
\frac{\partial}{\partial w} f_i(w, a) = 0 \quad (35)
\]
otherwise.

By dominated convergence theorem, one can see that $\frac{\partial}{\partial w} f_i(w, a)$ is continuous in $(w, a)$ and strictly increasing in $w$. Moreover, for $0 < w < \epsilon < \bar{w}$, $\frac{|X - a|}{a + w(X - a)} = \frac{1}{\epsilon} \left( 1 - \frac{X}{a} \right)$ where $X - a \geq X - a \left( 1 - \frac{1}{\bar{w}} \right) = \bar{X} \geq 0$. Therefore, monotone convergence theorem gives that
\[
\lim_{w \uparrow 0} \frac{\partial}{\partial w} f_0(w, a) = f_0(0, a) \mathbb{E}(X - a) = \mathbb{E}(X - a). \quad (36)
\]

And $\frac{X - a}{a + w(X - a)} = \frac{1}{\bar{w}} \left( 1 - \frac{w}{X - a} \right)$ where $X - a \geq X - a \left( 1 - \frac{1}{\bar{w}} \right) = \bar{X} \geq 0$. Therefore, monotone convergence theorem gives that
\[
\lim_{w \downarrow 0} \frac{\partial}{\partial w} f_0(w, a) = f_0(w, a) \left( 1 - \frac{1}{\bar{w}} \frac{\mathbb{E}(X - a)}{\bar{X}} \right). \quad (37)
\]
Equation (36) and (37) guarantees the continuity of $\frac{\partial}{\partial w} f_0(w, a)$ and $\frac{\partial}{\partial w} f_1(w, a)$ in $w$ on $[0, \bar{w}]$.

From equation (36) we have
\[
\varphi_0(a) \geq 0 \iff a \leq a_0 := \mathbb{E}X,
\]
where equality holds iff. $a = \mathbb{E}X$.

Similarly, because $\frac{\partial}{\partial w} f_1(0, a) = \mathbb{E}(X - a) + b \mathbb{E}[\nu(X - a)]$ is a decreasing function of $a$, there exists a unique solution $\underline{a}_1$ to $\frac{\partial}{\partial w} f_1(0, a) = 0$ such that
\[
\varphi_1(a) \geq 0 \iff a \leq \underline{a}_1,
\]
where equality holds iff. $a = \underline{a}_1$.

When $\mathbb{E}\frac{1}{\bar{X} - \underline{x}} = \infty$, equation (37) shows that $\lim_{w \downarrow 0} \frac{\partial}{\partial w} f_0(w, a)/f_0(w, a) = -\infty$ for $a > \underline{x}$. Therefore $\varphi_i(a)$ is determined by equation (35) when $x < a \leq \underline{a}_i$. It can be shown that $\lim_{a \downarrow \underline{a}_i} \varphi_i(a) = \infty$. Otherwise, there exist $\{w(a_n)\}$ and $M$ such that $w(a_n) < M$ as $a_n \downarrow \underline{x}$, thus $a_n + w(a_n)(X - a_n) > a_n + M(x - a_n) > \epsilon > 0$ when $a_n$ is sufficiently close to $\underline{x}$. However, taking $n \to \infty$ in $\frac{\partial}{\partial w} f_0(w(a_n), a_n) = f_0(w(a_n), a_n) \mathbb{E}(X - a_n)/\mathbb{E}X$, by Fatou’s lemma, $\lim_{n \to \infty} \frac{\partial}{\partial w} f_0(w(a_n), a_n)/f_0(w(a_n), a_n) \geq \mathbb{E} \lim_{n \to \infty} \frac{X - a_n}{w(a_n)(X - a_n)} \geq \mathbb{E} \frac{X - \underline{x}}{\epsilon} > 0$, which contradicts $\frac{\partial}{\partial w} f_0(w(a_n), a_n) = 0$. Moreover, $b \mathbb{E}[\nu(X - \underline{x})] > 0$ gives $\varphi_1(a) > \varphi_0(a) \to \infty$ as $a \downarrow \underline{x}$. Therefore,
\[
\varphi_i(a) \to \infty, \quad a \downarrow \underline{x}.
\]

In this case we define $\underline{a}_i = \underline{x}$, $i = 0, 1$. 

Authors’ names blinded for peer review
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When $\mathbb{E} \frac{1}{X-a} < \infty$, equation (36) shows that
\[
\varphi_0(a) \neq \bar{w}(a) \Leftrightarrow a > \bar{a}_0 := \left(\mathbb{E} \frac{X}{X-a}\right) / \left(\mathbb{E} \frac{1}{X-a}\right).
\]
Moreover, simple calculation shows that $\frac{\partial^2}{\partial w^2} f_0(\bar{w}, a) = f_0(\bar{w}, a) \left[ \frac{1}{w} \mathbb{E} a \left[ \mathbb{E} \left( \frac{1}{X-a} \right)^2 - (\mathbb{E} \frac{1}{X-a})^2 \right] - \frac{1}{w} \mathbb{E} \frac{1}{X-a} \right] < 0$ as $\bar{w} > 1$. The exchangeability of operators $\frac{\partial^2}{\partial w^2}$ and $\mathbb{E}$ is guaranteed by the continuity of $\frac{X-a}{(a+w(X-a))^2}$ when $w \neq \bar{w}$ and $\mathbb{E} \left[ \frac{X-a}{(a+w(X-a))^2} \right] = \lim_{w \to 1} a \mathbb{E} \left[ \frac{X-a}{(a+w(X-a))^2} \right]$, which is guaranteed by monotone convergence theorem similar as shown above. Therefore, $\frac{\partial}{\partial a} f_1(\bar{w}, a) = \frac{\partial}{\partial a} f_0(\bar{w}, a) + b \mathbb{E} [\nu(X-a)]$ is a decreasing function of $a$. There exists a unique $\bar{a}_1$ solution to $\frac{\partial}{\partial a} f_1(\bar{w}, a) = 0$ such that
\[
\varphi_1(a) \neq \bar{w}(a) \Leftrightarrow a > \bar{a}_1,
\]

When $a \leq \bar{a}$, by definition we have $\varphi_i = +\infty$. Therefore $\varphi_i$ is continuous at $(0, \bar{a}]$.

Finally, we will show that $\varphi_i(a)$ is determined by first order condition (35) in $[\bar{a}, \bar{a}_1]$ gives that $\varphi_i(a)$ is continuous in $[\bar{a}, \bar{a}_1]$. We will just prove the case $\lim_{a \to 0} \varphi_0(a) = \varphi_0(\bar{a}_0)$ for $\bar{a}_0 \neq \bar{a}$. The other parts are similar. Otherwise, there exist $\epsilon > 0$ and $a_n \downarrow \bar{a}_0$, s.t. $\varphi_0(a_n) < \bar{a}_0 - \epsilon$. By Fatou’s lemma, we have $\lim_{a \to 0} \frac{\partial}{\partial a} f_0(\varphi_0(a_n), a_n) / f_0(\varphi_0(a_n), a_n) \geq \mathbb{E} \lim_{n \to \infty} \frac{X-a_n}{\varphi_0(a_n)(X-a_n)} > 0$, which contradicts $\frac{\partial}{\partial a} f_0(\varphi_0(\bar{a}_0), \bar{a}_0) = 0$.

Combing our results above, we conclude that $\varphi_i(a)$ is continuous in $a$ where $\varphi_i(a) = \infty$ as $a = \bar{a}$.

One can see that $\varphi_i(a) = \bar{w} > 1$ for any $a \leq \bar{a}$, and $\varphi_i(a) = 0$ for any $a \geq \bar{a}$.

Now, we show the existence of the equilibrium. First, it is straightforward to see from (11) that the optimal consumption rate of investor $i$ is $c_{i,t} = 1 - \beta$, $i = 0, 1$. As a result, (17) leads to the equilibrium price dividend ratio $P_{t+1}/D_{t+1} \equiv \frac{\beta}{1+\beta}$. Consequently, the stock return in equilibrium is
\[
R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}/D_{t+1} + 1}{P_t/D_t} \frac{D_{t+1}}{D_t} = \frac{1}{\beta} \nu_{i,t+1},
\]
which is i.i.d. over time. As a result, given a risk-free rate $R_f,i,t+1$ observed at time $t$, investor $i$’s optimal portfolio $w_{i,t}$ solved from problem (11) is exactly $\varphi_i(R_{f,i,t+1})$. As a result, (18) implies (22), which defines the equilibrium risk-free rate because $\varphi_i$ is continuous and $\varphi_i(a) = \bar{w} > 1$ for any $a \leq \bar{a}$, and $\varphi_i(a) = 0$ for any $a \geq \bar{a}$.

We will show the decreasingness of $\varphi_0(a)$ and $\varphi_1(a)$ in $a$. Furthermore, $\varphi_0(a)$ is strictly decreasing in $a$ on $[\bar{a}, \bar{a}_0]$.

When $a \in (\bar{a}, \bar{a}_0]$, the decreasingness of $\varphi_0(a)$ is guaranteed by the decreasingness of $\bar{w}(a)$. Therefore, we only need to show the case then $\varphi_0(a)$ is an interior point determined by equation (35). Taking partial derivative with respect to $a$ on both sides, we have
\[
0 = \frac{\partial^2}{\partial w^2} f_0(\varphi_0,a) \frac{d\varphi_0}{da} + \frac{\partial^2}{\partial w \partial a} f_0(\varphi_0,a).
\]
Notice that we have:
\[
\frac{\partial^2}{(\partial w)^2} f_0(w, a) = -f_0(w, a) E \frac{(X - a)^2}{[a + w(X - a)]^2} < 0, \quad \frac{\partial^2}{\partial w \partial a} f_0(w, a) = -f_0(w, a) E \frac{X}{[a + w(X - a)]^2} < 0.
\]
Thus we have that
\[
d_{\varphi_0} \frac{da}{da} < 0,
\]
i.e., the strict decreasingness of \(\varphi_0(a)\) on \(\{x, \bar{a}_0\}\).

Further we show the decreasingness of \(\varphi_1\). Since \(\varphi_1\) is non-increasing in \(\{x, \bar{a}_1\}\) and \([\bar{a}_1, \bar{a}_0\] simliarly, we only need to show the case when \(\varphi_1\) is determined by equation (35), i.e.
\[
f_0(\varphi_1, a) E \left[ \frac{X - a}{a + \varphi_1(X - a)} \right] + g(a) = 0. \tag{39}
\]
or equivalently,
\[
E \left[ \frac{X - a}{a + \varphi_1(X - a)} \right] + f_0(\varphi_1, a)^{-1} g(a) = 0. \tag{40}
\]
where \(g(a) = \hat{b} E[\nu(X - a)]\) is differentiable a.e., and \(g'(a) = -\hat{b} [\bar{F}_X(a) + K \bar{F}_X(a)] < 0\) at differentiable points.

When \(\varphi_1 \geq 1\), we take derivatives with respect to \(a\) at differentiable point on both sides of equation (39) and have
\[
0 = \frac{\partial^2}{(\partial w)^2} f_0(\varphi_1, a) \frac{d\varphi_1}{da} + \frac{\partial^2}{\partial w \partial a} f_0(\varphi_1, a) + g'(a).
\]
We have proved that \(\frac{\partial^2}{(\partial w)^2} f_0(\varphi_1, a) < 0\) and \(g'(a) < 0\). And hence we are to show
\[
\frac{\partial^2}{\partial w \partial a} f_0(\varphi_1, a) = E \frac{X}{[a + \varphi_1(X - a)]^2} + E \frac{X - a}{a + \varphi_1(X - a)} \cdot E \frac{1 - \varphi_1}{a + \varphi_1(X - a)} =: h(\varphi_1, a) \leq 0.
\]
Denote \(Y = \frac{1}{a + \varphi_1(X - a)} > 0\), then \(X = \frac{1}{a + \varphi_1(X - a)}\), and
\[
h(\varphi_1, a) = -E Y^2 \frac{1}{\varphi_1} \left[ \frac{1}{Y} - (1 - \varphi_1) a \right] + (1 - \varphi_1) E Y E \left[ \frac{1}{\varphi_1} \left( \frac{1}{Y} - a \right) \right] = \frac{1 - \varphi_1}{\varphi_1} \left( E Y^2 - (E Y)^2 \right) - E Y < 0.
\]
When \(\varphi_1 < 1\), equilibrium equation (22) gives that \(\varphi_0 > 1 > \varphi_1\), hence \(\frac{\partial}{\partial w} f_0(\varphi_1, a) > \frac{\partial}{\partial w} f_0(\varphi_0, a) \geq 0\), and we have \(g(a) < 0\). We take derivatives with respect to \(a\) at differentiable point on both sides of equation (39) and have
\[
0 = -E \frac{(X - a)^2}{(a + \varphi_1(X - a))^2} f_0(\varphi_1, a) \frac{d\varphi_1}{da} - E \frac{X}{(a + \varphi_1(X - a))^2} + f_0^{-1} g(a) \left[ \frac{g'(a)}{g(a)} - E \left( \frac{1 - \varphi_1}{a + \varphi_1(X - a)} \right) \right].
\]
Since \(-E \frac{(X - a)^2}{(a + \varphi_1(X - a))^2} f_0(\varphi_1, a)^{-1} g(a) \frac{\partial}{\partial w} f_0(\varphi_1, a) < 0\) and \(-E \frac{X}{(a + \varphi_1(X - a))^2} < 0\), we are to show
\[
\frac{g'(a)}{g(a)} - E \left( \frac{1 - \varphi_1}{a + \varphi_1(X - a)} \right) < 0.
\]
Notice that
\[
g(a) = \int_{(x, \infty)} (x - a) dF(x) + K \int_{(-\infty, a)} (x - a) dF(x) = \int_{(a, \infty)} x dF(x) + K \int_{(-\infty, a]} x dF(x) + g'(a) a =: A + g'(a) a.
\]
where \(A > 0\).
Therefore,\[ \frac{g' (a)}{g (a)} = \frac{1}{a} + \frac{1}{a} > 0 \]

And
\[ \frac{\partial}{\partial w} \mathbb{E} \left( \frac{1 - w}{a + w(X - a)} \right) = - \mathbb{E} \frac{x}{|a + \varphi_1(X - a)|^2} < 0, \]

hence \( \mathbb{E} \left( \frac{1 - w}{a + w(X - a)} \right) \) is a decreasing function in \( w \). Therefore, \[ \mathbb{E} \left( \frac{1 - \varphi_1}{a + \varphi_1(X - a)} \right) \geq \mathbb{E} \left( \frac{1 - \varphi_1}{a + \varphi_1(X - a)} \right) \bigg|_{\varphi_1 = 0} = \frac{1}{a}, \]

Therefore, we have
\[ \frac{d \varphi_1}{da} < 0 \]
when \( \varphi_1 (a) \) is an interior point.

Hereby we have shown that \( \varphi_0 \) and \( \varphi_1 \) are non-increasing functions in \( a \), moreover, \( \varphi_0 \) is strictly decreasing in \( a \) in \( (\xi, \mathbb{E}X) \). Therefore, equilibrium risk-free return \( R_{f,t+1} \) is uniquely determined by equilibrium equation (22).

Moreover, since \( \varphi_0 (a) \) and \( \varphi_1 (a) \) are continuous functions in \( a \), \( R_{f,t+1} \) is also a continuous function in \( Y_i \).

Finally, we will show the monotonicity of \( R_{f,t+1} \) in \( Y_i \). When \( \varphi_0 > 1 > \varphi_1 \), equilibrium equation (22) gives that
\[ Y_i = \frac{1 - \varphi_1 (R_{f,t+1})}{\varphi_0 - \varphi_1 (R_{f,t+1})} = \frac{1}{1 + \frac{\varphi_0 (R_{f,t+1}) - 1}{\varphi_1 (R_{f,t+1})}}, \]
Since \( \varphi_0 (R_{f,t+1}) - 1 > 0, 1 - \varphi_1 (R_{f,t+1}) > 0 \) and both \( \varphi_0 (R_{f,t+1}) - 1 \) and \( \frac{1}{\varphi_1 (R_{f,t+1})} \) are increasing function in \( R_{f,t+1}, Y_i \) is a decreasing function in \( R_{f,t+1} \), and hence \( R_{f,t+1} \) is a decreasing function in \( Y_i \). The increasingness of \( R_{f,t+1} \) in \( Y_i \) when \( \varphi_0 < 1 < \varphi_1 \) can be similarly proved. To conclude, we show that \( R_{f,t+1} \) is a monotone function in \( Y_i \).

\[ \square \]

Proof of Theorem 2 According to Theorem 1, \( w_{i,t}^* = \varphi_i (R_{f,t+1}), i = 0, 1 \), where \( \varphi_i (a) \) is the maximizer of \( f_i (w, a) \) with respect to \( w, i = 0, 1 \). Noting that
\[ \frac{\partial}{\partial w} f_i (w, a) = \frac{\partial}{\partial w} f_0 (w, a) + \tilde{b}_2 \mathbb{E} [\nu (X - a)] \]
and
\[ \frac{\partial}{\partial w} f_0 (1, a) = f_0 (1, a) \mathbb{E} \left( \frac{X - a}{a + 1 \cdot (X - a)} \right) = \exp [\mathbb{E} (\ln X)] \mathbb{E} \left( \frac{X - a}{X} \right) \],
which is strictly decreasing in \( a \), we conclude that \( \frac{\partial}{\partial w} f_i (1, a) \) is strictly decreasing in \( a, i = 0, 1 \).

It is easy to check that \( \frac{\partial}{\partial w} f_0 (1, R_{f,b}) = 0 \). When \( K = K^* \), it is obvious that \( \mathbb{E} [\nu (X - R_{f,b})] = 0 \). Therefore, in this case \( \frac{\partial}{\partial w} f_i (1, R_{f,b}) = 0, i = 0, 1 \). Because \( \frac{\partial}{\partial w} f_i (1, a) \) is strictly decreasing in \( a \), we have \( \frac{\partial}{\partial w} f_i (1, a) < 0 \), leading to \( w_{i,t}^* < 1 \), when \( a > R_{f,b} \) and \( \frac{\partial}{\partial w} f_i (1, a) > 0 \), leading to \( w_{i,t}^* > 1 \), when \( a < R_{f,b} \), \( i = 0, 1 \). Because of the clearing condition (18), the equilibrium risk-free rate must be \( R_{f,b} \), and \( w_{0,t}^* = w_{1,t}^* = 1 \).

When \( K < K^* \), we have \( \mathbb{E} [\nu (X - R_{f,b})] > 0 \). As a result, \( \frac{\partial}{\partial w} f_0 (1, R_{f,b}) = 0 \) and \( \frac{\partial}{\partial w} f_i (1, R_{f,b}) > 0 \). When \( a < R_{f,b} \), we have \( \frac{\partial}{\partial w} f_i (1, a) > 0 \), leading to \( \varphi_i (a) > 1, i = 0, 1 \). Therefore, we must have the equilibrium
risk-free rate $R_{f,t+1} > R_{f,b}$. Consequently, $\partial w_0 f_0(1, R_{f,t+1}) < 0$, showing that $w_{0,t} < 1$. To clear the market, we must have $w_{1,t}^\ast > 1$. Because $w_{1,t}^\ast = \varphi_1(R_{f,t+1})$ and $\partial w_0 f_0(w, a) = \partial w f_0(w, a) + bE[\nu(X - a)]$, we must have $\mathbb{E}[\nu(X - R_{f,t+1})] > 0$.

The case in which $K > K^*$ can be proved similarly. Denote $R_{f,\infty}$ such that $\varphi_0(R_{f,\infty}) = \varphi_1(R_{f,\infty})$. Then when $K < K^*$, $R_{f,b} < R_{f,t+1} < R_{f,\infty}$, and when $K > K^*$, $R_{f,\infty} < R_{f,t+1} < R_{f,b}$.

Finally, we show that $w_{0,t}^\ast > 0$. We know that $w_{0,t}^\ast = 0$ if and only if $\partial w f_0(0, R_{f,t+1})$, i.e., $\mathbb{E}R_{t+1} \leq R_{f,t+1}$. In this case, $\mathbb{E}[\nu(R_{t+1} - R_{f,t+1})|\mathcal{F}_t] \leq 0$ because $K > 0$. Consequently, $\partial w f_0(0, R_{f,t+1}) \leq 0$, showing that $w_{1,t}^\ast = 0$. The market cannot be in equilibrium in this case because the clearing condition (17) cannot be satisfied. Thus, we must have $w_{0,t}^\ast > 0$ and $R_{f,t+1} < \mathbb{E}R_{t+1}$. □

**Proof Theorem 3** We first consider the case in which $K < K^*$. Theorem 2 shows that $R_{f,t+1} > R_{f,b}$, so $\mathbb{E}P_t < \mathbb{E}P_{t,b}$. As shown in the proof of Theorem 1, $R_{f,t+1}$ is strictly decreasing in $Y_t$. As a result, $\mathbb{E}P_t$ is strictly increasing in $Y_t$.

The case in which $K > K^*$ can be proved similarly. □

**Proof of Theorem 4** Denote $X_t$ as the wealth ratio of the irrational and rational investors, i.e., $X_t := (1 - Y_t)/Y_t$. Then,

$$X_{t+1} = \frac{(1 - c_{1,t+1}^\ast)W_{1,t+1}}{(1 - c_{0,t+1}^\ast)W_{0,t+1}} = \frac{W_{1,t+1}}{W_{0,t+1}} = \frac{R_{f,t+1} + w_{1,t}^\ast(R_{t+1} - R_{f,t+1})}{R_{f,t+1} + w_{0,t}^\ast(R_{t+1} - R_{f,t+1})} W_{0,t}

= \left[1 + (w_{1,t}^\ast - w_{0,t}^\ast) R_{t+1} - R_{f,t+1} \right] W_{0,t}

= X_t + X_t (w_{1,t}^\ast - w_{0,t}^\ast) A_{t+1},

(41)$$

where

$$A_{t+1} = \frac{R_{t+1} - R_{f,t+1}}{R_{f,t+1} + w_{0,t}^\ast(R_{t+1} - R_{f,t+1})},

(42)$$

$A_{t+1}$ is well defined as

$$w_{0,t}^\ast = \varphi_0(R_{f,t+1}) = \arg\max_{w \in [0, w(R_{f,t+1})]} \left\{ \exp[\mathbb{E}[\ln(R_{f,t+1} + w(R_{t+1} - R_{f,t+1}))]] \right\},

(43)$$

which guarantees that $R_{f,t+1} + w_{0,t}^\ast(R_{t+1} - R_{f,t+1}) \neq 0, \ a.s.$

Notice that

$$\mathbb{E}[A_{t+1}|\mathcal{F}_t] = \mathbb{E} \left[ \frac{R_{t+1} - R_{f,t+1}}{R_{f,t+1} + w_{0,t}^\ast(R_{t+1} - R_{f,t+1})} \big| \mathcal{F}_t \right]

= \mathbb{E} \left[ \frac{\partial}{\partial w} \left[ \ln(R_{f,t+1} + w(R_{t+1} - R_{f,t+1})) \right] \big| w = w_{0,t}^\ast \right| \mathcal{F}_t

= \frac{\partial}{\partial w} \mathbb{E} \left[ \ln(R_{f,t+1} + w(R_{t+1} - R_{f,t+1})) \big| w = w_{0,t}^\ast \right] \mathcal{F}_t

= \frac{\partial}{\partial w} f_0(w_{0,t}^\ast, R_{f,t+1}) \cdot \frac{1}{f_0(w_{0,t}^\ast, R_{f,t+1})}.

The exchangability of partial derivative and expectation in the second line is guaranteed by the continuity of $\frac{\partial}{\partial w} f_0(w, R_{f,t+1})$ on $[0, w(R_{f,t+1})]$, as shown in the proof of theorem 1.
If \( w_{0,i}^* \) is an interior point, \( \frac{\partial}{\partial w} f_0(w_{0,i}^*, R_{f,t+1}) = 0 \). Therefore,
\[
\mathbb{E}[A_{i+1} | \mathcal{F}_t] = \frac{\partial}{\partial w} f_0(w_{0,i}^*, R_{f,t+1}) \cdot \frac{1}{f_0(w_{0,i}^*, R_{f,t+1})} = 0.
\]
If \( w_{0,i}^* = \bar{w}(R_{f,t+1}) \), \( \frac{\partial}{\partial w} f_0(w_{0,i}^*, R_{f,t+1}) \geq 0 \) and \( w_{0,i}^* = \frac{R_{f,t+1}}{\text{esssup}(R_{f,t+1})} > 1 > w_{1,i}^* \). Therefore,
\[
(w_{1,i}^* - w_{0,i}^*) \mathbb{E}[A_{i+1} | \mathcal{F}_t] = (w_{1,i}^* - w_{0,i}^*) \frac{\partial}{\partial w} f_0(w_{0,i}^*, R_{f,t+1}) \cdot \frac{1}{f_0(w_{0,i}^*, R_{f,t+1})} \leq 0.
\]
Therefore,
\[
\mathbb{E}[X_{t+1} | \mathcal{F}_t] = X_t + X_t \cdot (w_{1,i}^* - w_{0,i}^*) \mathbb{E}[A_{i+1} | \mathcal{F}_t] \leq X_t,
\]
showing that \( X_t \) is a positive supermartingale. By martingale convergence theorem, there exists an integrable and nonnegative \( \mathcal{F}_\infty \)-measurable random variable \( X_\infty \), such that
\[
\lim_{t \to \infty} X_t = X_\infty, \quad \text{a.s.}
\]
Therefore, with \( Y_\infty := \frac{1}{1 + X_\infty} \in (0,1] \), we have
\[
\lim_{t \to \infty} Y_t = Y_\infty, \quad \text{a.s.}
\]
Because \( R_{f,t+1} \) is continuous in \( Y_t \) and \( \varphi_i \) is continuous in \( R_{f,t+1} \), we have
\[
\lim_{t \to \infty} R_{f,t+1} = R_{f,\infty}, \quad \lim_{t \to \infty} w_{i,t}^*(R_{f,t+1}) = \varphi_i(R_{f,\infty}), \quad i = 0, 1, \quad \text{a.s.}
\]
for some nonnegative \( \mathcal{F}_\infty \)-measurable random variable \( R_{f,\infty} \).

\( K \neq K^* \) gives that \( \varphi_1(R_{f,\infty}) \neq \varphi_0(R_{f,\infty}) \neq 0 \) a.s. Otherwise, equilibrium equation (18) gives that \( \varphi_1 = \varphi_0 = 1 \). Because \( \bar{w} > 1 \), both \( \varphi_1 \) and \( \varphi_0 \) are interior points. Then we have first order conditions (35)
\[
\frac{\partial}{\partial w} f_0(w,a) \bigg|_{w=1,a=R_{f,\infty}} = f_0(1, R_{f,\infty}) \mathbb{E} \frac{R - R_{f,\infty}}{R} = 0,
\]
\[
\frac{\partial}{\partial w} f_1(w,a) \bigg|_{w=1,a=R_{f,\infty}} = \frac{\partial}{\partial w} f_0(1, R_{f,\infty}) + \tilde{b}(E(R - R_{f,\infty})^+ - KE(R - R_{f,\infty})^-) = 0,
\]
which give \( K = K^* \) as the first equation gives \( R_{f,\infty} = R_{f,\alpha} \) and thus the second gives \( K = K^* \).

Recall the definition of \( A_i \) in (42). We will show that
\[
\mathbb{P}(\lim_{t \to \infty} A_i = 0) = 0, \quad (44)
\]
or sufficiently, for a sufficiently small \( \epsilon > 0 \), we have
\[
\mathbb{P}(E_i \ i.o.) = 1, \quad (45)
\]
where \( E_i := \{|A_i| > \epsilon\} \). Denote
\[
f(a) := \mathbb{P}(|\frac{X - a}{\bar{a} + \varphi_0(a)(X - a)}| > \epsilon) = 1 - \mathbb{P}(X \in [a - \frac{a\epsilon}{1 + \varphi_0(a)\epsilon}, a + \frac{a\epsilon}{1 - \varphi_0(a)\epsilon}])
\]
where \( X \overset{d}{=} R_t \). Since \( X \) has only finitely many atoms, it can be shown that \( f(a) \) is a lower semi-continuous function. Denote \( \bar{a} = \min(R_{f,\infty}, R_{f,\alpha}) \) and \( \bar{a} = \max(R_{f,\infty}, R_{f,\alpha}) \). Therefore, there exists \( a^* \in [\bar{a}, \bar{a}] \) such that
\[
\inf_{a \in [\bar{a}, \bar{a}]} f(a) = f(a^*) > 0, \quad \text{where } f(a^*) > 0 \text{ is guaranteed by } a^* \leq \mathbb{E}X \text{ and } \epsilon \text{ sufficiently small, i.e.}
\]
\[
\mathbb{P}(E_{i+1} | \mathcal{F}_i) = \mathbb{P}(|A_{i+1} > \epsilon | \mathcal{F}_i) \geq \mathbb{P}(|\frac{R_{i+1} - a^*}{a^* + \varphi_0(a^*)(R_{i+1} - a^*)} | > \epsilon) := \mathbb{P}(\tilde{E}_{i+1} > 0) \quad \text{a.s.}
\]
Here we denote $\widetilde{E}_t = \{ |z^{*}\phi_{\beta,\rho}(R_t, a) - a^{*}R_t| > \epsilon \}$. $\{\widetilde{E}_t\}$ are i.i.d. events with positive probability. Therefore $\sum_{t=1}^{\infty} P(\widetilde{E}_t) < \infty$ and Borel-Cantelli lemma gives $P(\widetilde{E}_t \text{ i.o.}) = 1$. Moreover, since $P(E_{t+1}^c | \mathcal{F}_t) \leq P(\widetilde{E}_{t+1}^c) \text{ a.s.}$,

$$P(\bigcup_{t=n}^{N} E_t) = 1 - P(\bigcap_{t=n}^{N} E_t^c) = 1 - E\left(\prod_{t=n}^{N} 1_{E_t^c}\mid \mathcal{F}_t\right)$$

$$\geq 1 - E\left[\prod_{t=n}^{N-1} 1_{E_t^c} P(E_N^c \mid \mathcal{F}_t)\right] = 1 - E\left[\prod_{t=n}^{N-1} 1_{E_t^c} P(E_N^c)\right]$$

$$\geq \ldots \geq 1 - \prod_{t=n}^{N} P(E_t^c) = P(\bigcup_{t=n}^{N} \widetilde{E}_t)$$

First taking $N \rightarrow \infty$ and then $n \rightarrow \infty$, we have

$$P(\bigcap_{t=n}^{\infty} E_t) \geq P(\bigcap_{t=n}^{\infty} \widetilde{E}_t)$$

i.e.

$$P(E_t \text{ i.o.}) \geq P(\widetilde{E}_t \text{ i.o.}) = 1,$$

which finishes the proof of (45) and hence (44).

Taking $t \rightarrow \infty$ in equation (41), and because that $\varphi_1(R_{f,\infty}) - \varphi_0(R_{f,\infty}) \neq 0 \text{ a.s.}$ and that $A_t$ does not converge to $0$, a.s., we have that

$$\lim_{t \rightarrow \infty} X_t = 0, \text{ a.s.}$$

Because $\lim_{t \rightarrow \infty} Y_t = 1$, a.s., (22) implies that $\lim_{t \rightarrow \infty} w_{0,t}^{*} = 1$, a.s. Because $w_{0,t}^{*} = \varphi_0(R_{f,t+1})$ and $\varphi_0(R_{f,b}) = 1$, we conclude that $\lim_{t \rightarrow \infty} R_{f,t+1} = R_{f,b}$. Consequently, $\lim_{t \rightarrow \infty} w_{1,t}^{*} = \varphi_1(R_{f,b})$. \hfill \square

**Proof of Theorem 6** Similar to the stock return index in Section 3.2, we can define the stock return index for general case:

$$K^* = \frac{\mathbb{E}(R_b - R_{f,b})^+}{\mathbb{E}(R_b - R_{f,b})^-}$$

(46)

where $R_b$ and $R_{f,b}$ are equilibrium risky and risk-free rate with only rational investor in the market, which satisfy the following equations together with $\Theta$ and $\Psi$:

$$R_{f,b} = \frac{\mathbb{E}(R_b^{\lambda - \gamma})}{\mathbb{E}(R_b^{\gamma})},$$

(47)

$$R_b = \frac{1}{1 - \phi_{\beta,\rho}(\Theta)} e^z,$$

(48)

$$\Theta = \Psi f(R_b),$$

(49)

$$\Psi = F_{\beta,\rho}(\phi_{\beta,\rho}(\Theta), \Theta).$$

(50)

Similar to the proof of Theorem 3, when $K = K^*$ and $R_f = R_{f,b}$, we have the $\varphi_0 = \varphi_1 = 1$. Furthermore, $\frac{1}{1 - \phi_{\beta,\rho}(\Theta)}$ is a deterministic constant. Plugging (48) into (47) and further into (46), we can see that $\frac{1}{1 - \phi_{\beta,\rho}(\Theta)}$ will be canceled out and we have the expression of $K^*$ in Theorem 6. \hfill \square
Appendix B: Algorithm of Solving Equations (25)–(33)

Here is the recursive algorithm solving equations (25)–(33):

1. Start from initial guess of the risk-free rate $R_f$, price-dividend ratio $g$, and transition of the rational investor’s market share $h$: $R_f^{(0)}$, $g^{(0)}$, and $h^{(0)}$.
2. Compute the stock return $R^{(0)}$ from $R_f^{(0)}$, $g^{(0)}$, and $h^{(0)}$ using (33).
3. Compute $\varphi_i^{(0)}$ and $\Psi_i^{(0)}$ from (25)–(29).
4. Compute the updates of $R_f$, $g$, and $h$ by using equations (31)–(32) to obtain $R_f^{(1)}$, $g^{(1)}$, and $h^{(1)}$.
5. Continue the update until $R_f^{(n)}$, $g^{(n)}$, and $h^{(n)}$.

References


