Markov Regime Switching Stochastic Volatility

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Abstract

This is a project on modeling time-varying volatility of S&P 500 weely return for the years 1990 to 2012 using Bayesian methods. First, MCMC on the log-stochastic volatility (SV) model is implemented with simulation results analyzed. Second, I generalize the SV model to encompass regime-switching properties with the markov switching log-stochastic volatility (MSSV) model, under which, high-volatility regime is able to overlap with economic recession periods. Finally, simulation study is performed.

1 The log-stochastic volatility model (SV)

The SV model with an AR(1) specification for the evolution of the log-variance,

$$y_{t+1} = u + \sqrt{V_t}\epsilon_{t+1}$$
$$\log(V_{t+1}) = \alpha_v + \beta_v \log(V_t) + \sigma_v \epsilon^v_{t+1}$$

where y^T denotes S&P 500 weekly return rate, has been widely used to model US stock price return (See, for example, Hamilton and Susmel 1994).

S&P 500 weekly return data 1990-2012 is plotted as following in table 1, and the sample mean and standard error are 0.14% and 2.4%.

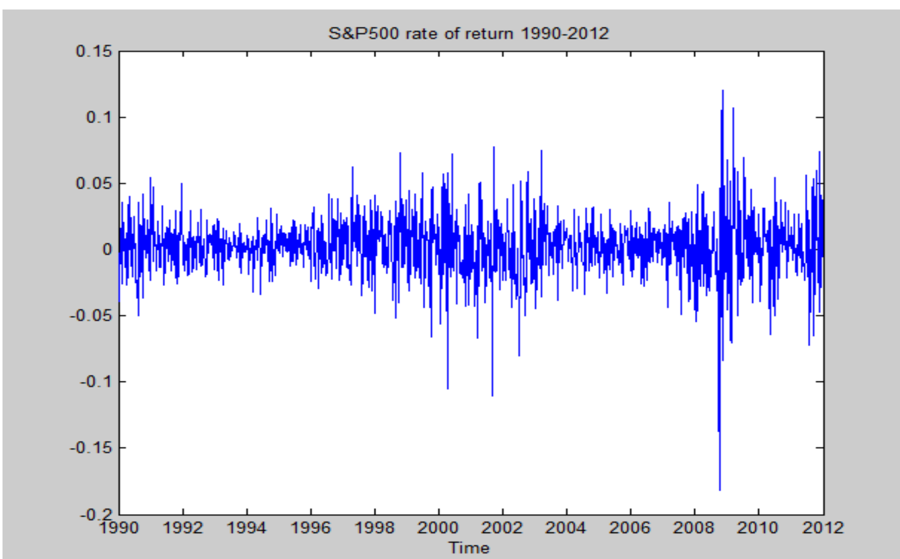

Figure 1. S&P 500 weekly rate of return

Using Gibbs MCMC together with random walk Metropolis, I estimate parameters in the SV model as follows in table 1:

Parameter	Posterior mean	Posterior S. D.	Posterior 90% interval
μ	.0021	.00009	(.00192, .0039)
α_v	-.48	.0003	(-.486, -.474)
β_V	.94	.0003	(.934, .946)
σ_v	.40	.0001	(.398, .402)

Table 1. Reults from the SV model fitting to the S&P 500

Simulate states ($\log(V^T)$) and compare them to volatility index (log (VIX)) as following:

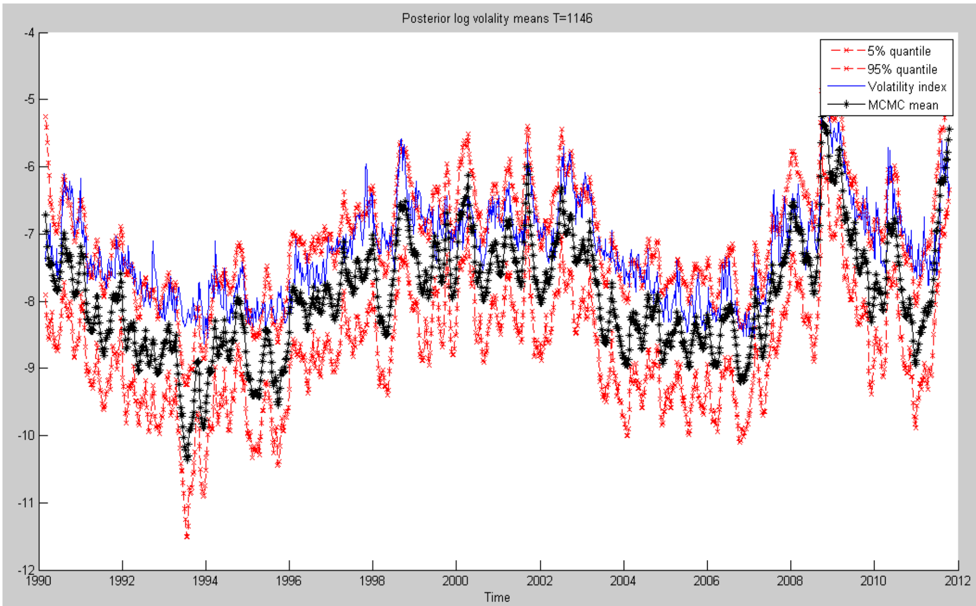

Figure 2. Posterior log-volatility mean & interval V. S. the log VIX data

and plug the volatility comparison as following:

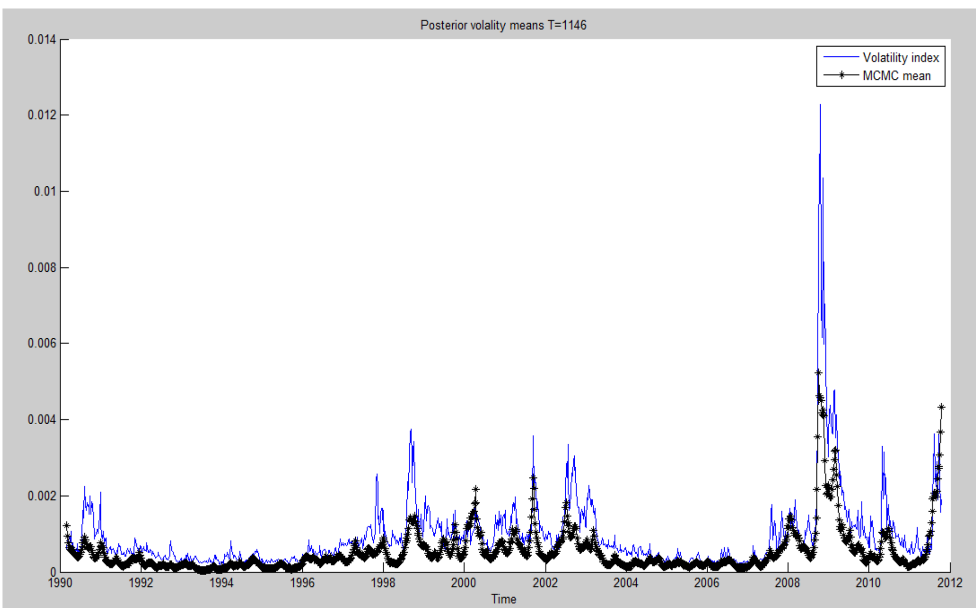

Figure 3. Posterior volatility mean V.S. VIX data

A typical finding of posterior parameter estimation of the SV model is that the posterior mean of β_v is close to 1 (.94), which indicates that under the SV model, volatility shows strong persistence. This is a common conclusion in the ARCH and the SV literature (See, for example, So, Lam and Li 1998). Another notable simulation result is that posterior return rate mean $E[u|y^T]$ is larger than its sample mean (.021% versus .014%). The difference can be explained by the influence of noise, as the weekly return of S&P 500 has fairly large variance (2.4%).

The posterior state mean $E[V^T|y^T]$ are generally close to the volatility index VIX (correlation = .89). The SV model gives a fairly satisfying posterior construction of state space V^T. However, as shown in figure 3, posterior volatility means are almost always slightly smaller than the VIX data. This finding has economic meaning, as VIX index may be overly priced with risk premium. And it can also be explained by MCMC the "smoothing" process as it smoothes the spikes out and renders overall estimation lower.

One notable problem of the SV model is that it fails to capture the sharp volatility spikes during crisis periods, as most spikes generated in posterior volatility mean are far less than the VIX ones, which may contribute to, as a result of smoothing, the underestimation of posterior volatility during those periods. We will discuss another model to tackle this problem in next section.

2 The markov switching log-stochastic volatility model (MSSV)

As stated in the previous section, the SV models fails to model the sharp spikes of volatility, thus a new model, MSSV, is proposed (Hamilton and Susmel 1994).

$$y_{t+1} = u + \sqrt{V_t}\epsilon_{t+1}$$
$$\log(V_{t+1}) = \alpha_{S_{t+1}} + \beta_v \log(V_t) + \sigma_v \epsilon^v_{t+1}$$

where $S_{t+1}$is an unobserved discrete Markov process with domain {1, 2, ..., K} and transition probability matrix:

$$\begin{matrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ . & . & \cdots & . \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{matrix}$$

and $\alpha_{S_{t+1}} = \sum_{i=1}^{S_{t+1}} \tau_i = a_v + \sum_{i=2}^{S_{t+1}} \tau_i$, where a_v is the baseline intercept.

To implement the MCMC simulation, full conditional distribution for volatility regimes S_n should be specified.

$$f(S_n|\theta) = \prod_{t=2}^{n} p_{s_{t-1}s_t} \pi_{s_1}$$

where $\pi_i = P[s_1 = i]$, $i = 1, 2, ..., K$.

In my simulation, I specify $K = 2$ and $\tau_2 = 1$, which indicates that state $S = 1$ and $S = 2$ represent "low volatility" and "high volatility" period, respectively. Moreover, The prior distributions of $\{\pi_j\}$and $\{p_{ij}\}$, $\{j, i = 1, 2, ..., K\}$ are given as Dirichlet distributions.

Parameter	Posterior mean	Posterior S. D.	Posterior 90% interval
μ	.0021	.00009	(.00020, .0022)
α_v	-5.52	.0052	(-5.6, -5.4)
β_V	.42	.0006	(.419, .42)
σ_v	.93	.0008	(.928, .932)
p_{11}	.68	.0001	(.681,.683)
p_{22}	.29	.0002	(.285,.29)

Table 2. Reults from the MSSV model fitting to the S&P 500

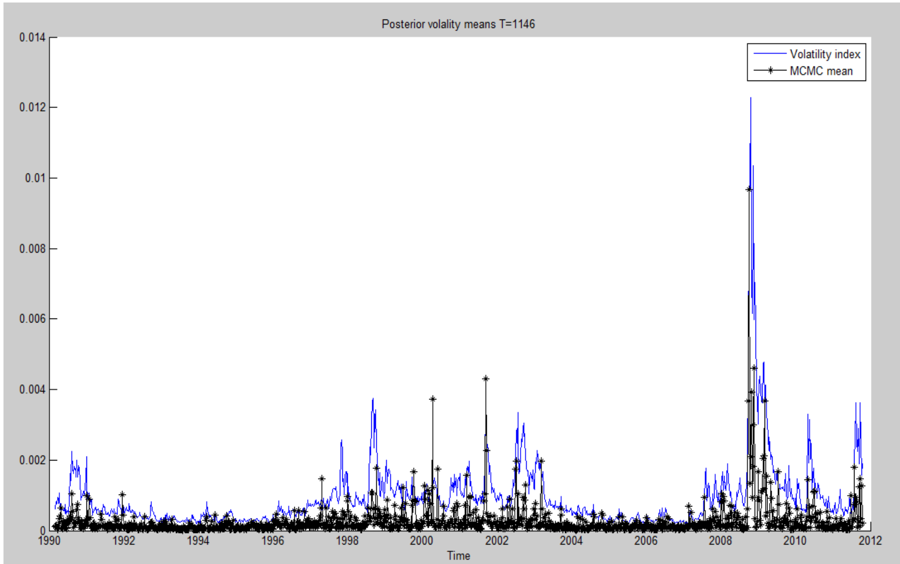

Figure 4. Posterior volatility mean V. S. the log VIX data

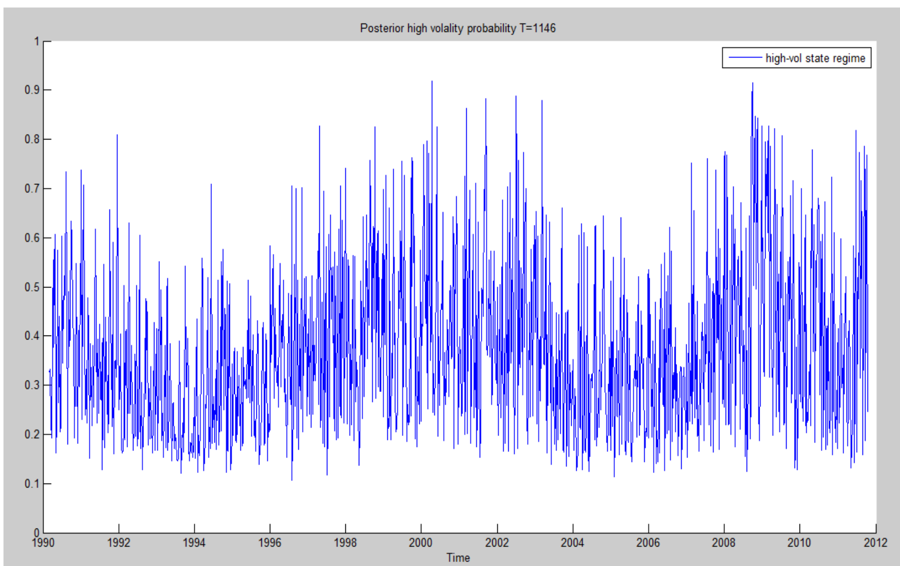

Figure 5. high-volatility posterior probability

Figure 4 displays a plot of simulated volatility state versus the VIX index data. Under the MSSV model, the volatility spikes during recession periods are better discribed as two-state intercepts of underlining volatility is applied. Moreover, figure 5 shows high-volatility state regime posterior probability over time, where during US economic recession or unstable periods of early and late 1990s and late 2000s, high-volatility state has larger probability (.7 to .8 or over .9). Here the high-volatility regime overlaps with the historical periods where stock return volatilities are high.

Parameter estimation results are shown in table 2. In contrast to those in the SV model, the volatility persistence factor $\beta_v$is much smaller (.42 versus .94), which is consistent with the arguement in existing literature (See, for example, So, Lam and Li 1998) that the apparent persistence of variance may in fact be a result of structural shift in the model (Markov swithcing drift). Further analysis regarding applying the method to simulated data is presented in next section.

3 Simulation Study

I now apply the MCMC method discribed in the previous section. Here I simulate a log-volatility model with Markov-switching drift model discribed in section 2 with $u = .001$, $\alpha_v = -5$, $\beta_v = .5$, $\sigma_v = .22$, and $P = [\begin{matrix} .7 & .3 \\ .7 & .3 \end{matrix}]$with data size T=1000.

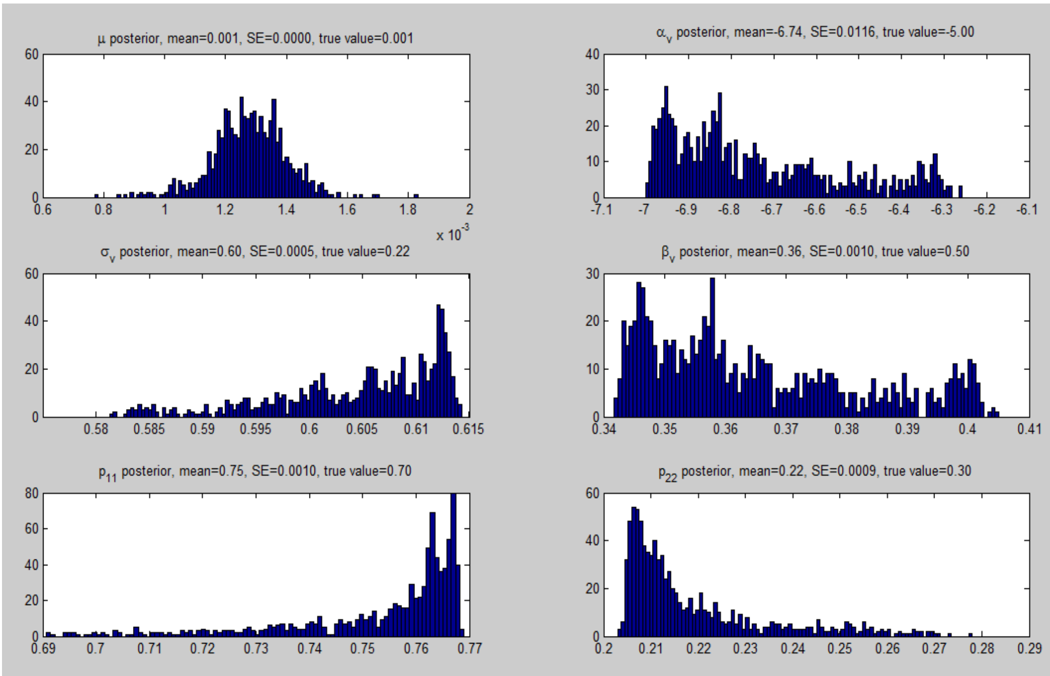

Figure 6. Parameter esitmation results for the simulated data

The results of MCMC smapling esimation are summarized in figure 6. With a relatively small simulation sample size $N = 1000$, my MCMC method is able to give a satisfying estimation of transition probability matrix P. However, most posterior parameter distributions are asymmetric, except the one of return rate drift u. To have better posterior distribution shapes, one may want to increase simulation steps N, or, possibly more effectively, to propose a reasonable prior distribution on state-related drifts τ_i, $i = 2, 3, ..., K$ and run the MCMC again.

4 Conclusion

This article performs Bayesian MCMC simulation on the SV and the MSSV model to S&P 500 weekly returns for the years 1990 to 2012. MCMC on the SV model gives a good construction of underlining volatility when compared to the VIX index, and simulated log-volatility persistence is strong under this model. However, the SV model fails to generate sharp volatility spikes as they appear in the VIX index. MSSV gives a better description of volatilty spikes, and lower volatility persistence, which is consistent with the theorems proposed by existing literatures. To verify my MCMC method, a simulation study is implemented, where transition probabilities are well estimated with a relatively small simulation sample size.

References

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